# Discussion Papers In Economics And Business


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Discussion Paper 04-19



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# The Comparative Statics of Equilibrium Derivative Prices*

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**Abstract**

We examine the conditions for preferences and risks that guarantee monotonicity of equilibrium derivative prices. In a Lucas economy with a derivative, we derive the equilibrium derivative price under expectation with respect to risk–neutral probability, and analyze comparative statics on the equilibrium derivative price based on the risk–neutral probability.

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# 1 Introduction

One of the most important questions for optimal portfolio problems is what conditions on preferences and risks guarantee monotonicity of optimal portfolios. The analysis has been extended to equilibrium asset prices in pure exchange economies by some studies such as Gollier and Schlesinger (2002) and Ohnishi and Osaki (2004) because they are consequences of investor portfolio optimization. For details on these topics, Gollier (2001) provided an excellent survey. It is needles to say that examination of these effects on equilibrium derivative prices is necessary because of the importance of derivatives from both academic and practical viewpoints. However, to our best knowledge, there has been no formal analysis examining them. The goal of this paper is to examine them.

Our analysis much owes to the previous literature on comparative statics of optimal portfolios. In particular, it has a close relation to Gollier and Schlesinger (1996) and Kijima and Ohnishi (1996). Gollier and Schlesinger (1996) showed that the addition of noise risk to portfolio risk lead to the unambiguous comparative static result on the optimal portfolio with some restrictions on preference. Kijima and Ohnishi (1996) determined that two special classes of First–order Stochastic Dominance (FSD) guarantee the desirable comparative static result for decision problem by a different way from previous studies such as Landsberger and Meilijson (1990) and Eeckhoudt and Gollier (1995). Cleary, our analysis differs from these previous literatures, since its concern is not with optimal portfolios but equilibrium derivative prices. Further, our analysis follows from the previous study of comparative statics of equilibrium asset prices. In particular, our analysis has a close relation to comparative statics based on risk–neutral probability such as Milgrom (1981) and Ohnishi and Osaki (2004). Our results are also related to Gollier and Schlesinger's recent analysis (2002) in which they made comparative statics based on excess demand functions.[1)] Our results are a generalization of results obtained in these previous literatures because of analyzing assets with non–linear payoffs and/or obtaining them under weaker conditions.

This paper is organized as follows. In Sec. 2, we derive an equilibrium derivative price in a pure–exchange economy with homogeneous investors, and rewrite it using risk–neutral probability. In Sec. 3, we show that shifts in the sense of two special classes of the FSD have monotone effects on the equilibrium derivative price. We examine the effects of additional noise risks on the equilibrium derivative price in Sec. 4. In the conclusion, we summarize the results, and give some comments on

[1)] Gollier and Schlesinger (2002) discuss some stochastic dominances that guarantee the monotonicity of equilibrium asset prices based on the central dominance introduced by Gollier (1995). However, these stochastic dominances can be justified, only when its parameter satisfies a certain condition. We can obtain the results in Sec. 3 without that restriction.

future research.

## 2 Equilibrium Derivative Price

Let us consider a static version of a Lucas (1978) economy except for the introduction of a derivative, that is, a two–date pure exchange economy with homogeneous investors. Every investor has an identical expected utility representation with a strictly increasing, strictly concave, and sufficiently smooth von Neumann–Morgenstern utility function (vN–M function) $u$, which means that all required higher order derivatives are assumed to be exist. Every investor is endowed with $w$ units of a risk–free asset, one unit of a risky asset and with one unit of a derivative written on it. Let us define that the risk–free asset is the numeraire in the economy, and the gross risk–free rate is normalized to one. The risky asset payoff at the final date is a random variable $\tilde{x}$ with a Cumulative Distribution Function (CDF) $F$. The CDF $F$ of $\tilde{x}$ has a bounded support $[a, b]$ and is assumed to be differentiable, that is, the Probability Density Function (PDF) $f = F'$ exists. We consider an economy in which the one–fund separation theorem holds, and therefore the risky asset can be viewed as the market portfolio. The payoff of the derivative is defined as a function of the risky asset payoff $x$ and is denoted by $p$. The payoff function of the derivative is assumed so that the final wealth in equilibrium given by $w + x + p(x)$, is an increasing function of $x$. An economic interpretation of this assumption is given as follows. Since the supply of the risky asset which is endowed with one unit for each investor, is considered as the norm for quantity, the supply of the derivative is represented by the slope of its payoff function. If the slope of the payoff function is sufficiently small relative to the risky asset payoff, the assumption is satisfied. When the payoff function is differentiable, the condition turns out to be $p'(x) \geq -1$ for all $x \in [a, b]$. Because supplies of derivatives written on market portfolios are sufficiently small compared to those for market portfolios in actual financial asset markets, this assumption is permissible.

The investor buys the portfolio $(\alpha, \beta, \gamma)$ to maximize his or her expected utility from final wealth, where $(\alpha, \beta, \gamma)$ is the portfolio for the risk–free asset, the risky asset and the derivative respectively. Let us represent the price of the risky asset by $m$ and the price of the derivative by $q$. The investor problem is given as follows:[2)]

$$\mathbf{P}: \quad \max_{(\alpha,\beta,\gamma)} \; \mathbb{E}[u(\alpha + \beta\tilde{x} + \gamma p(\tilde{x}))] \quad \text{s.t. } \alpha + \beta m + \gamma q = w + m + q. \tag{1}$$

Define the Lagrangian $\mathcal{L}(\alpha, \beta, \gamma; \lambda) := \mathbb{E}[u(\alpha + \beta\tilde{x} + \gamma p(\tilde{x}))] - \lambda(\alpha + \beta m + \gamma q -$

[2)] The constraint can be considered to be an equality since the objective function is strictly increasing.

$w-m-q)$, where $\lambda$ is the Lagrange multiplier. Because the objective function is a strictly concave function and the constraint is linear, the first–order–conditions meet the necessary and sufficient conditions for the optimality. By the homogeneity of investors, the demand of the assets are equal to the endowment in equilibrium: $\alpha = w,\ \beta = 1, \gamma = 1$, that is, a no–trade equilibrium occurs. The solution of investor's problem in equilibrium is given as follows:

$$\frac{\partial \mathcal{L}}{\partial \alpha} = \mathbb{E}[u'(z(\tilde{x}))] - \lambda = 0 \tag{2}$$

$$\frac{\partial \mathcal{L}}{\partial \beta} = \mathbb{E}[\tilde{x}u'(z(\tilde{x}))] - \lambda m = 0 \tag{3}$$

$$\frac{\partial \mathcal{L}}{\partial \gamma} = \mathbb{E}[p(\tilde{x})u'(z(\tilde{x}))] - \lambda q = 0 \tag{4}$$

where $z(x)$ is the final wealth in equilibrium defined by $z(x) := w + x + p(x)$, and is an increasing function of $x$. By Eqs. (2) and (4), the equilibrium derivative price is given as follows:

$$q = \frac{\mathbb{E}[p(\tilde{x})u'(z(\tilde{x}))]}{\mathbb{E}[u'(z(\tilde{x}))]}. \tag{5}$$

Let us define the function

$$\hat{f}(x : u, f) := \frac{u'(z(x))f(x)}{\mathbb{E}[u'(z(\tilde{x}))]}, \quad x \in [a, b]. \tag{6}$$

Since $\hat{f}(x : u, f) \geq 0$ for all $x \in [a, b]$ and $\int_a^b \hat{f}(t : u, f)\mathrm{d}t = 1$, we can regard $\hat{f}(x : u, f)$ as a PDF defined on the bounded support $[a, b]$. By taking the expectation with respect to the PDF $\hat{f}$, the equilibrium derivative price can be rewritten as

$$q = \hat{\mathbb{E}}[p(\tilde{x})], \tag{7}$$

where $\hat{\mathbb{E}}$ denotes the expectation operator with respect to the PDF $\hat{f}$. The probability $\hat{F}(x : u, f) := \int_a^x \hat{f}(t : u, f)\mathrm{d}t,\ x \in [a, b]$ induced by the PDF $\hat{f}$, is called risk–neutral probability, since asset prices become equal to the expected values of their payoffs under risk–neutral probability.

# 3 The First–order Stochastic Dominance

Let us consider two different economies, say economy 1 and 2. The payoff of the risky asset in economy $i$ $(= 1, 2)$, is represented by the random variable $\tilde{x}(i)$, and these random variables are ordered with respect to the First–order Stochastic Dominance (FSD). We examine the effect of FSD changes in risk on equilibrium derivative prices using comparative static analysis.

In this section, we consider the two special classes of FSD: the Monotone Likelihood Ratio Dominance (MLRD) and Monotone Probability Ratio Dominance (MPRD).[3) ]Since these stochastic dominances imply the FSD, they can be viewed as the special classes of FSD.

## 3.1 The Monotone Likelihood Ratio Dominance

The definition of MLRD is given as follows:

**Definition 3.1.** $\tilde{x}(2)$ dominates $\tilde{x}(1)$ in the sense of MLRD if $f(y,2)/f(y,1) \geq f(x,2)/f(x,1)$ holds for all $y \geq x$. We denote it as $\tilde{x}(2) \geq_{\text{MLRD}} \tilde{x}(1)$. □

According to Kijima and Ohnishi (1996), we can obtain the following inequality:

$$\frac{\hat{f}(y:u,f(2))}{\hat{f}(y:u,f(1))} = \frac{\mathbb{E}[u'(z(\tilde{x}(1))]f(y,2)}{\mathbb{E}[u'(z(\tilde{x}(2))]f(y,1)} \geq \frac{\mathbb{E}[u'(z(\tilde{x}(1))]f(x,2)}{\mathbb{E}[u'(z(\tilde{x}(2))]f(x,1)} = \frac{\hat{f}(x:u,f(2))}{\hat{f}(x:u,f(1))} \tag{8}$$

holds for all $y \geq x$ by the definition of MLRD. Eq. (8) means that risk–neutral probability $\hat{F}(2)$ dominates $\hat{F}(1)$ in the sense of MLRD. Noting that the MLRD is stronger than the FSD, we can obtain

$$q(1) = \hat{\mathbb{E}}[p(\tilde{x}(1))] \leq (\geq)\ \hat{\mathbb{E}}[p(\tilde{x}(2))] = q(2) \tag{9}$$

for derivatives whose payoff functions are increasing (decreasing).

We summarize the above discussion as the following proposition:

**Proposition 3.1.** Let us consider two economies with risky asset payoffs by $\tilde{x}(1)$ and $\tilde{x}(2)$, and denote the equilibrium prices of derivatives written on them by $q(1)$ and $q(2)$. If $\tilde{x}(2) \geq_{\text{MLRD}} \tilde{x}(1)$, then $q(2) \geq (\leq)\ q(1)$ holds for all derivatives with increasing (decreasing) payoff functions. □

## 3.2 The Monotone Probability Ratio Dominance

The definition of MPRD is given as follows:

**Definition 3.2.** $\tilde{x}(2)$ dominates $\tilde{x}(1)$ in the sense of MPRD if $F(y,2)/F(y,1) \geq F(x,2)/F(x,1)$ holds for all $y \geq x$. We denote it as $\tilde{x}(2) \geq_{\text{MPRD}} \tilde{x}(1)$. □

Note that the MPRD is a stochastic dominance that is weaker than the MLRD but stronger than the FSD, that is, the MLRD implies the MPRD, and the MPRD implies the FSD, see Eeckhoudt and Gollier (1995) for the proof. By the definition of MPRD, we have $f(x,2)/F(x,2) \geq f(x,1)/F(x,1)$ for all $x \in [a,b]$. We will show

[3)] The MPRD is also known as the reversed hazard rate dominance.

that risk–neutral probability $\hat{F}(2)$ dominates $\hat{F}(1)$ in the sense of MPRD, that is, we have to obtain the following inequality:[4)] for all $x \in [a, b]$,

$$\frac{\hat{f}(x,2)}{\hat{F}(x,2)} = \frac{u'(z(x))f(x,2)}{\int_a^x u'(z(t))f(t,2)\mathrm{d}t} \geq \frac{u'(z(x))f(x,1)}{\int_a^x u'(z(t))f(t,1)\mathrm{d}t} = \frac{\hat{f}(x,1)}{\hat{F}(x,1)}. \tag{10}$$

Whitt (1980) proved that the following statements are equivalent:

- $\tilde{x}(2)$ dominates $\tilde{x}(1)$ in the sense of MPRD;
- $[\tilde{x}(2) \mid \tilde{x}(2) \leq x]$ dominates $[\tilde{x}(2) \mid \tilde{x}(2) \leq x]$ in the sense of FSD for all $x \in [a, b]$.

Since $u'(z(x))$ is a decreasing function of $x$,

$$\begin{aligned} \mathbb{E}[u'(z(\tilde{x}(2))) \mid \tilde{x}(2) \leq x] &= \int_a^x \frac{1}{F(x,2)} u'(z(t))f(t,2)\mathrm{d}t \\ &\leq \int_a^x \frac{1}{F(x,1)} u'(z(t))f(t,1)\mathrm{d}t = \mathbb{E}[u'(z(\tilde{x}(1))) \mid \tilde{x}(1) \leq x] \end{aligned} \tag{11}$$

holds for all $x \in [a, b]$. It follows from Eq. (11) and $f(x,2)/f(x,1) \geq F(x,1)/F(x,1)$ that

$$\int_a^x u'(z(t))f(t,2)\mathrm{d}t \leq \frac{F(x,2)}{F(x,1)} \int_a^x u'(z(t))f(t,1)\mathrm{d}t \leq \frac{f(x,2)}{f(x,1)} \int_a^x u'(z(t))f(t,1)\mathrm{d}t \tag{12}$$

holds for all $x \in [a, b]$. Eq. (12) means Eq. (10), that is, risk–neutral probability $\hat{F}(2)$ dominates $\hat{F}(1)$ in the sense of MPRD. We can obtain the following proposition by an argument similar to that in the previous subsection:

**Proposition 3.2.** Let us consider two economies with the risky asset payoffs by $\tilde{x}(1)$ and $\tilde{x}(2)$, and denote the equilibrium prices of derivatives written on them by $q(1)$ and $q(2)$. If $\tilde{x}(2) \geq_{\mathrm{MPRD}} \tilde{x}(1)$, then $q(2) \geq (\leq)\ q(1)$ holds for all derivatives with increasing (decreasing) payoff functions. □

**Remark 3.1.** It should noted that the concavity of $u$ explicitly used in the proof of Prop. 3.2, whereas it does not appear in the proof of Prop. 3.1. This means that Prop. 3.2 implicitly holds under more restrictive conditions than Prop. 3.1, and this requirement is consistent with the fact that the MPRD is weaker than the MLRD. □

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[4)] Kijima and Ohnishi (1996) obtained this inequality in a different manner and applied it to the decision problem. However, we give a simpler proof for the completeness of our paper.

# 4 The Addition of Noise Risk

Let us consider the random variables $\tilde{\epsilon}$ such that the following two conditions are satisfied

- the expectations are equal to zero: $\mathbb{E}[\tilde{\epsilon}] = 0$;
- they are independent from the risky asset payoffs.

We call these random variables noise risk. We examine the effects of additional noise risk on equilibrium derivative prices using comparative static analysis. In this section, we consider two cases of additional noise risk: the addition of noise risk to the endowment and to the risky asset payoff.

## 4.1 The Addition of Noise Risk to the Endowment

In this subsection, the investor endows the non–tradable component except for the endowment previously considered. The non–tradable component is the noise risk which is a random variable $\tilde{\epsilon}$. The objective function of investor's problem considered in Sec. 2 can be written with (non–tradable) noise risk to the endowment as:

$$\mathbb{E}[u(\alpha + \beta\tilde{x} + \gamma p(\tilde{x}) + \tilde{\epsilon})]. \tag{13}$$

Let us define the derived utility function by $v(x) := \mathbb{E}[u(x + \tilde{\epsilon})]$ (Kihlstrom et. al., 1981; Nachman, 1982), and rewrite Eq. (13) as:

$$\mathbb{E}[v(\alpha + \beta\tilde{x} + \gamma p(\tilde{x}))]. \tag{14}$$

This means that we can view the investor's problem under the addition of noise risk to the endowment as the problem of investor with preference $v$. The equilibrium derivative price can be written by using risk–neutral probability:

$$q(v) = \hat{\mathbb{E}}_v[p(\tilde{x})], \tag{15}$$

where $\hat{\mathbb{E}}_v$ is the expectation operator with respect to the CDF $\hat{F}(x : v, f)$.

Kimball (1993) introduced the notion of Standard Risk–Aversion (SRA) concerning vN–M functions, which is the property that both their risk–aversion and prudence are decreasing functions, and proved that derived utility functions induced zero–mean risks are more risk–averse than the original function under SRA, that is, $\mathcal{A}(v) = -v''/v' \geq -u''/u' = \mathcal{A}(u)$ holds, where prudence is defined by $\mathcal{P}(u) := -u'''/u''$. An equivalent condition for this inequality is given by the condition that there exists an increasing and concave function $g$ such that $v = g \circ u$

(Pratt, 1964). Differentiating the above equation yields that $v'/u'$ is an increasing function. Therefore, by a discussion similar to Sec. 3.1,

$$\frac{\hat{f}(y:u,f)}{\hat{f}(y:v,f)} = \frac{\mathbb{E}[v'(z(\tilde{x}))]u'(y)}{\mathbb{E}[u'(z(\tilde{x}))]v'(y)} \geq \frac{\mathbb{E}[v'(z(\tilde{x}))]u'(x)}{\mathbb{E}[u'(z(\tilde{x}))]v'(x)} = \frac{\hat{f}(x:u,f)}{\hat{f}(x:v,f)} \tag{16}$$

holds for all $y \geq x$. This means that risk–neutral probability $\hat{F}(x:u,f)$ dominates $\hat{F}(x:v,f)$ in the sense of MLRD.

Following to Sec. 3.1, we can obtain:

$$q(u) = \hat{\mathbb{E}}_u[p(\tilde{x})] \geq (\leq)\ \hat{\mathbb{E}}_v[p(\tilde{x})] = q(v) \tag{17}$$

for the derivatives whose payoff functions are increasing (decreasing). We summarize the result of this subsection as the following proposition:

**Proposition 4.1.** Assume that investor preferences display the SRA. Additions of noise risk to endowments decrease (increase) equilibrium derivative prices, whenever their payoff functions are increasing (decreasing). □

## 4.2 The Addition of Noise Risk to the Risky Asset Payoff

We examine the effect of additional noise risk to the payoffs of risky assets on equilibrium derivative prices in this subsection. The addition of noise risk to the payoff of the risky asset is represented by $\tilde{x}+\tilde{\epsilon}$. Rothschild and Stiglitz (1970, 1971) introduced the notion of Second–order Stochastic Dominance (SSD) that is defined via concave functions. One equivalent condition to SSD is given by the addition of noise risk such that the conditional expectation is equal to zero. This means that the additional noise risk considered in this subsection is a special case of the SSD, since the SSD does not require the condition of independence.

By the addition of noise risk to the risky asset payoff, the equilibrium price of the derivative written on it is given as follows in a discussion similar to Sec. 2 :

$$\begin{aligned} q(\epsilon) &= \frac{\mathbb{E}[p(\tilde{x}+\tilde{\epsilon})u'(w+(\tilde{x}+\tilde{\epsilon})+p(\tilde{x}+\tilde{\epsilon}))]}{\mathbb{E}[u'(w+(\tilde{x}+\tilde{\epsilon})+p(\tilde{x}+\tilde{\epsilon}))]} \\ &= \frac{\mathbb{E}_{\tilde{x}}(\mathbb{E}_{\tilde{\epsilon}}[(p(\tilde{x})+\tilde{\epsilon})u'(w+(\tilde{x}+\tilde{\epsilon})+p(\tilde{x}+\tilde{\epsilon}))\mid\tilde{x}])}{\mathbb{E}_{\tilde{x}}(\mathbb{E}_{\tilde{\epsilon}}[u'(w+(\tilde{x}+\tilde{\epsilon})+p(\tilde{x}+\tilde{\epsilon}))\mid\tilde{x}])}. \end{aligned} \tag{18}$$

Assuming that the payoff function is differentiable, the following inequality is obtained for a sufficiently small noise risk in the case of increasing payoff functions:

$$\begin{aligned} q(\epsilon) &\simeq \frac{\mathbb{E}_{\tilde{x}}(\mathbb{E}_{\tilde{\epsilon}}[(p(\tilde{x})+p'(\tilde{x})\tilde{\epsilon})u'(w+\tilde{x}++p(\tilde{x})+(1+p'(\tilde{x}))\tilde{\epsilon}) \mid \tilde{x})}{\mathbb{E}_{\tilde{x}}(\mathbb{E}_{\tilde{\epsilon}}[u'(w+\tilde{x}+p(\tilde{x})+(1+p'(\tilde{x}))\tilde{\epsilon}) \mid \tilde{x}])} \\ &= \frac{\mathbb{E}_{\tilde{x}}(\mathbb{E}_{\tilde{\epsilon}}[p(\tilde{x})u'(w+\tilde{x}+p(\tilde{x})+(1+p'(\tilde{x}))\tilde{\epsilon} \mid \tilde{x}])}{\mathbb{E}_{\tilde{x}}(\mathbb{E}_{\tilde{\epsilon}}[u'(w+\tilde{x}+p(\tilde{x})+(1+p'(\tilde{x})\tilde{\epsilon}) \mid \tilde{x})])} \\ &\qquad + \frac{\mathbb{E}_{\tilde{x}}(p'(\tilde{x})\mathbb{E}_{\tilde{\epsilon}}[\tilde{\epsilon}u'(w+\tilde{x}+p(\tilde{x})+(1+p'(\tilde{x}))\tilde{\epsilon}) \mid \tilde{x}])}{\mathbb{E}_{\tilde{x}}(\mathbb{E}_{\tilde{\epsilon}}[u'(w+\tilde{x}+p(\tilde{x})+(1+p'(\tilde{x}))\tilde{\epsilon} \mid \tilde{x}])} \\ &\leq \frac{\mathbb{E}_{\tilde{x}}(\mathbb{E}_{\tilde{\epsilon}}[p(\tilde{x})u'(w+\tilde{x}+p(\tilde{x})+(1+p'(\tilde{x}))\tilde{\epsilon}) \mid \tilde{x}])}{\mathbb{E}_{\tilde{x}}(\mathbb{E}_{\tilde{\epsilon}}[u'(w+\tilde{x}+p(\tilde{x})+(1+p'(\tilde{x})\tilde{\epsilon}) \mid \tilde{x}])}, \end{aligned} \tag{19}$$

where the inequality follows from the covariance inequality:[5)]

$$\mathbb{E}_{\tilde{\epsilon}}[\tilde{\epsilon}u'(w+\tilde{x}+p(\tilde{x})+(1+p'(\tilde{x}))\tilde{\epsilon}) \mid \tilde{x}] \leq \mathbb{E}[\tilde{\epsilon}]\mathbb{E}_{\tilde{\epsilon}}[u'(w+\tilde{x}+p(\tilde{x})+(1+p'(\tilde{x}))\tilde{\epsilon}) \mid \tilde{x}] = 0. \tag{20}$$

Using the derived utility function $\mathbb{E}[v(\tilde{x})] := \mathbb{E}_{\tilde{x}}(\mathbb{E}_{\tilde{\epsilon}}[u(x+(1+p'(\tilde{x}))\tilde{\epsilon}) \mid \tilde{x}])$, we can rewrite Eq. (19) by

$$q(\epsilon) \leq \frac{\mathbb{E}[p(\tilde{x})v'(w+\tilde{x}+p(\tilde{x}))]}{\mathbb{E}[v'(w+\tilde{x}+p(\tilde{x}))]}. \tag{21}$$

Assuming that preferences display SRA, we have the following inequality from a manner similar to the previous subsection:

$$q(\epsilon) \leq \frac{\mathbb{E}[p(\tilde{x})v'(w+\tilde{x}+p(\tilde{x}))]}{\mathbb{E}[v'(w+\tilde{x}+p(\tilde{x}))]} \leq \frac{\mathbb{E}[p(\tilde{x})u'(w+\tilde{x}+p(\tilde{x}))]}{\mathbb{E}[u'(w+\tilde{x}+p(\tilde{x}))]} = q. \tag{22}$$

The following inequality holds for the case of decreasing payoff functions in a similar discussion except for changing the sign:

$$q(\epsilon) \geq \frac{\mathbb{E}[p(\tilde{x})v'(w+\tilde{x}+p(\tilde{x}))]}{\mathbb{E}[v'(w+\tilde{x}+p(\tilde{x}))]} \geq \frac{\mathbb{E}[p(\tilde{x})u'(w+\tilde{x}+p(\tilde{x}))]}{\mathbb{E}[u'(w+\tilde{x}+p(\tilde{x}))]} = q, \tag{23}$$

where the derived utility function is defined by $\mathbb{E}[v(\tilde{x})] = \mathbb{E}_{\tilde{x}}(\mathbb{E}_{\tilde{\epsilon}}[u(x+(1-p'(x))\tilde{\epsilon}) \mid \tilde{x}])$. Assuming the differentiability of payoff functions in this subsection, we have $1-p'(x) \geq 0$, for all $x \in [a,b]$ by the assumption in Sec. 2. We summarize the result as the following proposition:

**Proposition 4.2.** Assume that payoff functions of derivatives are differentiable. We also assume that noise risk is sufficiently small and investor preferences display the SRA. Additions of noise risk to risky asset payoffs decrease (increase) equilibrium derivative prices, whenever their payoff functions are increasing (decreasing). □

[5)] The covariance inequality (Theorem 4.1 in McEntire, 1984) claims the following statement: if both $f$ and $g$ are increasing functions, then $\mathbb{E}[f(\tilde{x})g(\tilde{x})] \geq \mathbb{E}[f(\tilde{x})]\mathbb{E}[g(\tilde{x})]$ holds for every random variable $\tilde{x}$.

# 5 Conclusion

Using comparative static analysis, we have shown that equilibrium derivative prices have some monotone properties for shifts of risky asset payoffs with respect to two sub–classes of the FSD (Sec. 3), and for additions of noise risks under some restrictions on investor preferences (Sec. 4). These results are a generalization of the previous studies, such as Gollier and Schlesinger (2002).

We give two comments on future research. First, the analysis of Sec. 4.2. should weaken the restrictions on noise risks and payoff functions. Although piecewise linear functions are not differentiable, they are arbitrarily approximated by smooth functions. Therefore, the result of Sec. 4.2 holds for derivatives with piecewise linear functions, which constitute an important class of derivatives because they include most types of derivatives traded in actual financial asset markets, e.g. vanilla types of call and put options. Second, we have to analyze the economy where the raison d'être of derivatives is justified.[6] Despite a standard setting, risks cannot be transferred among investors by the derivatives, since the investors do not trade derivatives in equilibrium. This means that the roles which derivatives play, are not clear in our economy.

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[6] In a recent paper, Franke. et. al (1998) justified the raison d'être of derivatives by inducing a non–linear risk sharing rule among investors being faced with heterogeneous background risks, even if the investors have the linear risk tolerance with an identical slope. Our analysis is different from theirs since they did not provide any qualitative analysis for equilibrium derivative prices.

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