Discussion Papers In Economics And Business

Random Correlation Matrix and De-Noising

Ken-ichi Mitsui and Yoshio Tabata

Discussion Paper 06-26

Graduate School of Economics and
Osaka School of International Public Policy (OSIPP)
Osaka University, Toyonaka, Osaka 560-0043, JAPAN

Random Correlation Matrix and De-Noising

Ken-ichi Mitsui and Yoshio Tabata

Discussion Paper 06-26

September 2006

この研究は「大学院経済学研究科・経済学部記念事業」
基金より援助を受けた、記して感謝する。

Graduate School of Economics and
Osaka School of International Public Policy (OSIPP)
Osaka University, Toyonaka, Osaka 560-0043, JAPAN

Random Correlation Matrix and De-Noising

Ken-ichi Mitsui *and Yoshio Tabata†

Abstract

In Finance, the modeling of a correlation matrix is one of the important problems. In particular, the correlation matrix obtained from market data has the noise. Here we apply the de-noising processing based on the wavelet analysis to the noisy correlation matrix, which is generated by a parametric function with random parameters. First of all, we show that two properties, i.e. symmetry and ones of all diagonal elements, of the correlation matrix preserve via the de-noising processing and the efficiency of the de-nosing processing by numerical experiments. We propose that the de-noising processing is one of the effective methods in order to reduce the noise in the noisy correlation matrix.

JEL classification: C51, C61, C63, G32

keyword correlation matrix, calibration, rank reduction, de-noising, wavelet analysis

*Doctor Candidate of Osaka University, E-mail:cfatk406@jtw.zaq.ne.jp
†Graduate School of Business Administration, Nanzan Univeristy, E-mail:tabata@nanzan-u.ac.jp

1 Introduction

One of the important problems in finance and risk management is how to specify a correlation matrix. For instance, calibrating BGM model which is also called LIBOR forward rate model (LFM), one must estimate the correlation matrix of forward rate of each index.

It is one of the methods to specify the correlation matrix to fit the (noisy) elements of the market correlation matrix by means of some parametric function. In De Jong et al. (2004), using USD historical interest rate data they estimated the following function ρ of correlation matrix:

$$\rho(T_i, T_j) = \exp\left\{ -\gamma_1|T_i - T_j| - \frac{\gamma_2|T_i - T_j|}{\max(T_i, T_j)^{\gamma_3}} - \gamma_4|\sqrt{T_i} - \sqrt{T_j}|\right\} \tag{1}$$

where $\gamma_1, \gamma_3, \gamma_4 > 0$, and T_i is the expiry date.

There exists another method to specify the correlation matrix based on the decomposition using eigenvalues and eigenvectors of matrices. This is often called rank-reduction in finance and is implemented by some researchers; "zeroing eigenvalues"(Spectral decomposition, also known as principle component analysis) in Rebonato and Jäckel (1999), "2 and 4 rank-reduction" in Brigo (2002), "eigenvalue zeroing by iteration" in Morini and Webber (2006), application to multi-factor lognormal forward rate model in Rebonato (1999).

Moreover, Grubišić and Pietersz (2005) considered the problem to find *nearest-low-rank correlation matrix* and made numerical comparisons with several algorithms where the correlation matrix was generated by (1) with γ-parameters assumed to be distributed normally (capped at 0 for $\gamma_1, \gamma_3, \gamma_4$) with mean and standard errors[1] estimated by De Jong et al. (2004).

Let $\mathcal{S}_n$ be the set of $n \times n$ $\mathbb{R}$-valued symmetric matrices and X^* be the transpose of X. For $\mathbf{X} \in \mathcal{S}_n$, denote by $\mathbf{X} \succeq 0$ that $\mathbf{X}$ is positive semidefinite. Denote $\mathbf{C} \in \mathcal{S}_n$ a correlation matrix, e.g. that of observations in market. Then the problem to specify the correlation matrix by k rank-reduction is given by the following optimization problem;

[1] The estimated values; $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ (standard errors in parentheses) are 0.000 (-), 0.480 (0.099), 1.511 (0.289), 0.186 (0.127), respectively.

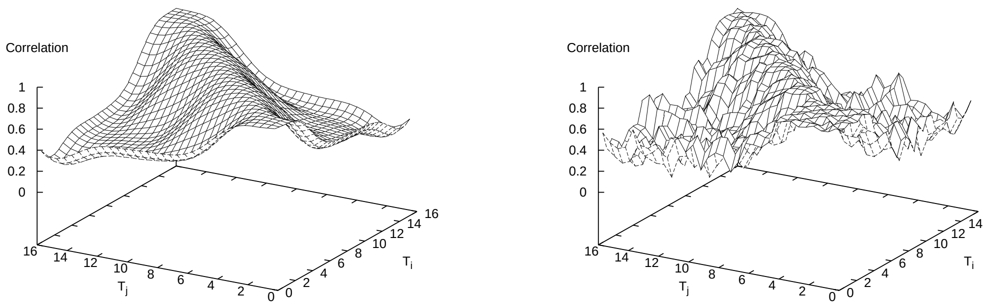

Figure 1: 4 reduced-rank angle parameterization.

$$
\begin{aligned}
&\text{Find } \mathbf{X} \in \mathcal{S}_n \\
&\text{such that} \quad \min_{\mathbf{X}} \tfrac{1}{2}\|\mathbf{C}-\mathbf{X}\|^{\mathbf{2}} \\
&\qquad\qquad\quad \text{subject to } \operatorname{rank}(\mathbf{X}) = k; \{X_{ii}\} = 1, i \in \{1, \ldots, n\}; \\
&\qquad\qquad\quad \mathbf{X} \succeq 0 \ (\mathbf{X} = \mathbf{B}\mathbf{B}^*, \mathbf{B} \in \mathbb{R}^{n\times k}).
\end{aligned}
\tag{2}
$$

In particular, Rebonato (1999), Rebonato and Jäckel (1999) suggested the reduced-rank angle parameterization that the matrix $\mathbf{B}$ in (2) is approximated by means of trigonometric functions. We will briefly view the procedure.

Let us implement the reduced-rank angle parameterization.[2] As generating the correlation matrix $\mathbf{C}$ by (1) with constant γ-parameters estimated by De Jong et al. (2004), which we call the *non-noisy* correlation matrix, and applying the reduced-rank angle parameterization of 4 rank-reduction, we have the calibrated $\mathbf{X}$ such as Figure 1(left). And Figure 1(right) is in the random elements case where γ-parameters assumed to be distributed normally (capped at 0 for $\gamma_1, \gamma_3, \gamma_4$) with mean and standard errors estimated by De Jong et al. (2004), which we call the *noisy* correlation matrix.

From these figures, we can see the calibrated matrix $\mathbf{X}$ preserving noise through the reduced-rank angle parameterization of the 4 rank-reduction. It is natural for us to wish to have a calibrated correlation matrix of which the noise is reduced. Then in this paper we propose a procedure of rank-reduction method with a noise reduction technique for a noisy

[2] In (1) we set $T_i = \{0.5, 1.0, \ldots, 16.0\}$ and the dimension of the correlation matrix is 32×32. Through Section 1-3, we implicitly use this setting.

correlation matrix, which is called "de-noising" given by Donoho (1995), Donoho (1993), Donoho and Johnstone (1994) based on Wavelet analysis. To put it more precisely, we solve the optimization problem (2) after applying "de-noising" to a noisy correlation matrix $\mathbf{C}$.

The procedure of "de-noising", which will be explained in Section 2, is $(i)$to apply $\mathbf{C} \in \mathcal{S}_n$ to the wavelet transform, (ii) to implement the noise reduction procedure called "soft-thresholding" for high frequency signal and (iii) to take the inverse wavelet transform.

Then two questions arise: let $\bar{\mathbf{C}}$ be $\mathbf{C}$ to which the processing of "de-noising" is achieved,

(I) $\bar{\mathbf{C}} \in \mathcal{S}_n$?

(II) diag$\{\bar{\mathbf{C}}\}$=$\{1,\ldots,1\}$?

As discussing the two questions below, we previously state the answers: The answer to (I) is yes, that is, the property of symmetry of the correlation matrix preserves through the "de-noising" processing. As for (II), by a numerical example we will show that all diagonal elements of the correlation matrix processed do not feature ones. And we will deal with this problem by means of some procedures.

This paper is organized as follows: In Section 2 we give preliminaries; rank-reduction, wavelet analysis (multi-resolution analysis), and de-noising. In Section 3, we discuss above two questions and propose a procedure of rank-reduction method with a noise reduction technique. Section 4 contains numerical experiments. Finally in Section 5 we conclude.

2 Preliminaries

In this section we briefly review the rank-reduction method, the wavelet analysis (multiresolution analysis) and the "de-noising" procedure.

2.1 Rank-reduction

Let $\mathbf{C}$ be a correlation matrix in $\mathbb{R}^{n\times n}$. If one estimates the correlation matrix, the number of parameters to be estimated is $n(n-1)/2$. Since the number of parameters of the correlation matrix depends on n, the estimation problem becomes more burdensome as n is increasing.

For the large number of n, Rebonato and Jäckel (1999), Rebonato (1999) suggested a rank-reduction method. To see the procedure of the rank-reduction, assume that $\mathbf{C} \in \mathcal{S}_n$

and $\mathbf{C} \succeq 0$, i.e. $\mathbf{C}$ is a positive semi-definite symmetric matrix. Denote the eigenvectors and eigenvalues of $\mathbf{C}$ by $\mathbf{\Lambda}$ and $\mathbf{P}$, respectively. Then

$$\mathbf{C} = \mathbf{P}\mathbf{\Lambda}\mathbf{P}^*$$

where $\mathbf{P}$ is a real orthogonal matrix, i.e. $\mathbf{P}\mathbf{P}^* = \mathbf{P}\mathbf{P} = I$.

As denoting $\mathbf{\Lambda} := \mathbf{A}^*\mathbf{A}$ and $\mathbf{B} := \mathbf{P}\mathbf{A}$, we have

$$\mathbf{C} = \mathbf{P}\mathbf{A}\mathbf{A}^*\mathbf{P}^* \;=\; \mathbf{B}\mathbf{B}^*. \tag{3}$$

$\{\lambda_i^2\}_{i=1}^n$ denotes the elements of the eigenvalues $\mathbf{\Lambda}$ such that $\lambda_1^2 > \lambda_2^2 > \ldots > \lambda_n^2$. If one choose k rank-reduction ($k < n$), then he only has to set the eigenvales as $\lambda_{k+1}^2 = \lambda_{k+2}^2 = \cdots = \lambda_n^2 := 0$.

Using the trigonometric function, Rebonato (1999) specified the i-th row vector of $\mathbf{B} \in \mathbb{R}^{n\times k}$ by

$$\begin{aligned}
b_{i,1} &= \cos(\theta_{i,1}) \\
b_{i,k} &= \cos(\theta_{i,k})\prod_{l=1}^{k-1}\sin(\theta_{i,l}), \text{ for } 1 < k < n, \\
b_{i,n} &= \prod_{l=1}^{k-1}\sin(\theta_{i,l}).
\end{aligned}$$

There exists the advantage in that the correlation matrix $\mathbf{C}$ can be expressed by the angles θ of $n \times k$ elements. That is, we can reduce the number of the parameters estimated from $n \times (n-1)/2$ to $n \times (k-1)$ for $(n+1)/2 > k$.

Let $\bar{\mathbf{X}}(\theta)$ be the approximated correlation matrix of $\mathbf{C}$ by the trigonometric function with angles θ. We can obtain the angles θ by solving the optimization problem

$$\min_\theta \sum_{i,j=1}^n |\{\bar{\mathbf{X}}(\theta)\}_{ij} - \{\mathbf{C}\}_{ij}|^2.$$

However in order to estimate the angles θ, for convenience we here iteratively calibrate $\theta_{i,j}$ from $\theta_{i,j-1}, \theta_{i,j-2}, \ldots, \theta_{i,1}$, $j < k$ using the decomposed matrix $\mathbf{B}$.[3] The point to notice

[3]Since we take such procedure in order to estimate angles θ, the case where the elements of $\mathbf{B}$ do not

is that it is not necessary to feature ones in all diagonal elements of $\mathbf{C}$ in (3) via above decomposition, while all diagonal elements of the correlation matrix constructed by $b_{i,k}$ are ones (Rebonato (1999)). Then in order to feature ones in diagonal we process the rescaling

$$\{c_{rescaling}\}_{ij} := \frac{c_{ij}}{\sqrt{c_{ii}c_{jj}}} \tag{4}$$

where $\{c_{rescaling}\}_{ij}$ denotes (i,j) element of rescaled $\mathbf{C}$, and c_{ij} is that of $\mathbf{C}$ (Brigo (2002)).

2.2 Wavelet analysis

In order to view the concept of the wavelet analysis, first of all we start with the multiresolution analysis. Subsequently in accordance with the non-standard form (see pp153-157 of Dahmen et al. (1997)), we explain two-dimensional multiresolution analysis.

We introduce multiresolution analysis, which is the important concept. Let V_j be a vector space indexed by $j \in \mathbf{Z}$. Then a *multiresolution analysis* of $\mathbb{L}^2(\mathbb{R})$ is a collection of V_j such that

1. $\cdots V_2 \subset V_1 \subset V_0 \subset V_{-1} \subset \cdots$,
2. $\cap_{j=-\infty}^{\infty} V_j = 0$, $\cup_{j=-\infty}^{\infty} V_j$ is dense in $\mathbb{L}^2(\mathbb{R})$,
3. $f(x) \in V_j \Leftrightarrow f(2x) \in V_{j-1}$,
4. $f(x) \in V_0 \Leftrightarrow f(x+k) \in V_0$,
5. $\exists$ a scaling function $\phi \in V_0$ such that $\{\phi(x-k)\}_{k\in\mathbf{Z}}$ is a Riesz basis of V_0.

As denoting the orthogonal space of V_j by W_j, then we have $V_j \oplus W_j = V_{j-1}$. We define two operators: $P_j : \mathbb{L}^2(\mathbb{R}) \to V_j,\ Q_j : \mathbb{L}^2(\mathbb{R}) \to W_j$.

We give a concrete (discrete transform) example of the multiresolution analysis of order 1 (at resolution level 1), which is Mallat Algorithm (see e.g. Resnikoff and Wells (1998), Press et al. (1992)).

Example 2.1 (4-coefficient case) *Let $\{h_i\}_{i=0,1,2,3}$ be the filter coefficients*[4] *and define $\{g_i\}_{i=0,1,2,3}$ such that $g_0 = h_3,\ g_1 = -h_2,\ g_2 = h_1$ and $g_3 = -h_0$. Let $\tilde{n} = 2^n,\ n \in \mathbb{N}$.*

enclose the range of $b_{i,k}$ by any angle θ happens. We skip such case in numerical experiments of Section 4.

[4] *This is the case where the genus is 4.*

Define the transformation matrices $H, \bar{H} \in \mathbb{R}^{2^{n-j}\times\tilde{n}}$ *such that*

$$H := \begin{bmatrix} h_0 & h_1 & h_2 & h_3 & & & & & & \\ & & h_0 & h_1 & h_2 & h_3 & & & & \\ & & & & & & \cdots & & & \\ h_2 & h_3 & & & & & & & h_0 & h_1 \end{bmatrix},$$

$$G := \begin{bmatrix} g_0 & g_1 & g_2 & g_3 & & & & & & \\ & & g_0 & g_1 & g_2 & g_3 & & & & \\ & & & & & & \cdots & & & \\ g_2 & g_3 & & & & & & & g_0 & g_1 \end{bmatrix},$$

where $j(\in \mathbb{N})$ *is the number of the resolution level.*

Let $f(x)$ *be a function in* $\mathbb{L}^2(\mathbb{R})$ *and* $\tilde{f}(x)$ *be the corresponding discrete data in* $\mathbb{R}^{\tilde{n}}$*. Then the relationship of the multiresolution analysis and the discrete wavelet transform of order 1, i.e.* $j = 1$ *is*

$$\begin{array}{ccccc} \tilde{f}(x) \in \mathbb{R}^{\tilde{n}} & \longrightarrow & H\tilde{f}(x) \in \mathbb{R}^{\tilde{n}/2} & \oplus & G\tilde{f}(x) \in \mathbb{R}^{\tilde{n}/2} \\ \wr\!\| & & \wr\!\| & & \wr\!\| \\ f(x) & \longrightarrow & (P_1 f)(x) & \oplus & (Q_1 f)(x) \\ \cap & & \cap & & \cap \\ V_0 & & V_1 & \oplus & W_1. \end{array}$$

Now define the Non-Standard form (NS-form for short) for two-dimensional multiresolution analysis. We refer to Dahmen et al. (1997).

Let P_j and Q_j be projectors on subspaces V_j and W_j, respectively. Define an operator T of the NS-form by

$$T = \sum_{j\in\mathbf{Z}} (Q_j T Q_j + Q_j T P_j + P_j T Q_j).$$

If resolution level J is finite, then

$$T_0 = \sum_{j=1}^{J} (Q_j T Q_j + Q_j T P_j + P_j T Q_j) + P_J T P_J.$$

For convenience, denote $T_0 = \{\{A_j, B_j, \Gamma_j\}_{j=1}^{J} T_J\}$ such that

$$
\begin{aligned}
A_j &= Q_j T Q_j \ : \ W_j \to W_j \\
B_j &= Q_j T P_j \ : \ V_j \to W_j \\
\Gamma_j &= P_j T Q_j \ : \ W_j \to V_j \\
T_J &= P_J T P_J \ : \ V_J \to V_J.
\end{aligned}
$$

And we denote the operator of the two-dimensional discrete wavelet transform by $\mathcal{T}_0$.

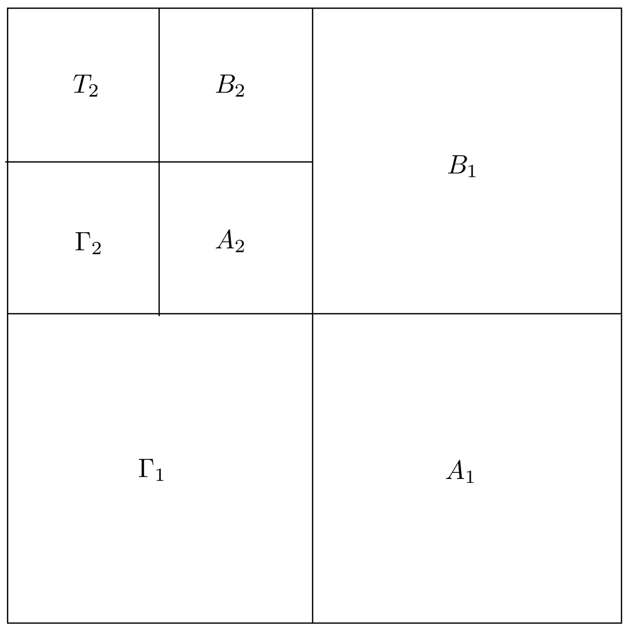

Figure 2: NS-form at resolution level 2.

Figure 2 shows two-dimensional multiresolution analysis by the operator T_0 with $J = 2$. Notice that though the structure of elements in this figure is different from that in Figure 5 of pp.155 in Dahmen et al. (1997), the elements of each NS-form are identical. The difference is in the arrangement of elements and the dimensions of the NS-forms: If the dimensions of the original data matrix are $\tilde{n} \times \tilde{n}$, the dimension of our structure is same to the original matrix but that in Dahmen et al. (1997) is $2\tilde{n} \times 2\tilde{n}$.

Let us clarify the discrete two-dimensional wavelet transform through an example.

Example 2.2 *Let us consider the 4-coefficient case same as in Example 2.1. As applying the two-dimensional wavelet transform of order 1 to* $f(\mathbf{x}) \in \mathbb{L}^2(\mathbb{R}^2)$*, we have* $(T_0)f(\mathbf{x}) = \{(A_1)f(\mathbf{x}), (B_1)f(\mathbf{x}), (\Gamma_1)f(\mathbf{x}), (T_1)f(\mathbf{x})\}$*. And let* $\mathbf{C}$ *be a discrete data matrix in* $\mathbb{R}^{\tilde{n}\times\tilde{n}}$ *corresponding to* $f(\mathbf{x})$*. Using the transform matrices of* H *and* G*, we have* $(\mathcal{T}_0)\mathbf{C} = \{G\mathbf{C}G^*, H\mathbf{C}G^*, G\mathbf{C}H^*, H\mathbf{C}H^*\}$ *and the following relationship;*

$$
\begin{aligned}
&(A_1)f(\mathbf{x}) \cong G\mathbf{C}G^*, && (B_1)f(\mathbf{x}) \cong H\mathbf{C}G^*, \\
&(\Gamma_1)f(\mathbf{x}) \cong G\mathbf{C}H^*, && (T_1)f(\mathbf{x}) \cong (\mathcal{T}_1)\mathbf{C} \;=\; H\mathbf{C}H^*.
\end{aligned}
$$

2.3 De-noising

Donoho (1995), Donoho (1993) and Donoho and Johnstone (1994) proposed a method to extract an unknown function $f(t_i)$ on $[0, 1]$ from noisy data

$$d_i = f(t_i) + \sigma z_i,\ i = 0, \ldots, \tilde{n} - 1 \tag{5}$$

where $t_i = i/\tilde{n}$, z_i is a Gaussian White noise and σ is a noise level.

In accordance with Donoho (1995), the processing of "de-noising" is to find $\hat{f}$ which is at least as smooth as f with high probability by minimizing the mean square error

$$\tilde{n}^{-1} \sum_{i=0;}^{\tilde{n}-1} \mathbb{E}(\hat{f}(i/\tilde{n}) - f(i/\tilde{n}))^2.$$

Let us apply the de-noising processing to $\{d_i\}_{i=0}^{\tilde{n}-1}$. Assume $\tilde{n} = 2^n, n \in \mathbb{N}$. First, applying the discrete wavelet transform at resolution level 1 to $\mathbf{d} := \{d_i\}_{i=0}^{\tilde{n}-1} \in \mathbb{R}^{\tilde{n}}$, we have

$$\mathbf{d} \to H\mathbf{d} + G\mathbf{d}.$$

Secondly implement the following "soft-thresholding" to $H\mathbf{d} := \{Hd_i\}_{i=0}^{(\tilde{n}-1)/2}$;

$$
\{H\mathbf{d}\}_i^{new} := \begin{cases}
\{Hd\}_i - t_0 & \text{for } d_i > t_0 \\
0 & \text{for } |d_i| \leqslant t_0 \\
\{Hd\}_i + t_0 & \text{for } d_i < -t_0
\end{cases}
$$

where $t_0 := \mathrm{MAD}\sqrt{2\log\tilde{n}}$ and MAD is the median absolute deviation

$$\mathrm{MAD} := \frac{median(||H\mathbf{d}| - median(|H\mathbf{d}|)|)}{0.6745}.$$

Here $median(\mathbf{d})$ is the median of $\{d_i\}_{i=0}^{\tilde{n}-1}$. Thirdly processing the inverse discrete wavelet transform with $\{H\mathbf{d}\}_{new} := \{\{Hd\}_i^{new}\}_{i=0}^{(\tilde{n}-1)/2}$ in place of $\{Hd_i\}_{i=0}^{(\tilde{n}-1)/2}$, i.e.

$$\mathbf{d}_{new} \leftarrow \{H\mathbf{d}\}_{new} + G\mathbf{d},$$

we obtain the de-noised data $\mathbf{d}_{new}$.

Similarly we can apply the de-noising processing to the two-dimensional data structure. Let us consider the NS-form in Figure 2. This data is processed via the discrete wavelet transform at resolution level 2. In this case, all we have to do is to process the soft-thresholding to $A_1, B_1, \Gamma_1, A_2, B_2$ and Γ_2, respectively. And with these data, the de-noised correlation matrix can be obtained via the inverse discrete wavelet transform.

3 Properties of the correlation matrix processed by de-noising

In this section, we discuss the two questions as mentioned above. To begin with, we prove $\bar{\mathbf{C}} \in \mathcal{S}_n$. Therefore we first show a symmetric matrix preserves symmetry of the processed matrix through both the wavelet transform and the inverse wavelet transform. Secondly the de-noising processing is confirmed not to ruin symmetry of the matrix. Then as for question (II), we give a concrete example and observe that all diagonal elements of a processed symmetric matrix are not ones. For this problem, we make ones all diagonal elements of the matrix by means of rescaling. Finally we apply the rank-reduction method with de-noising to a noisy correlation matrix.

Let $\mathbf{C}$ be a noisy correlation matrix in $\mathcal{S}_{\tilde{n}}$, $\tilde{n} = 2^n, n \in \mathbb{N}$. $\mathcal{T}_0$ denotes the operator of the two-dimensional discrete wavelet transform defined in Section 2 and we denote $\overline{(\mathcal{T}_0)\mathbf{C}}$ that it is obtained by the processing of the soft-thresholding to $(\mathcal{T}_0)\mathbf{C}$. And let $\overline{\mathbf{C}}$ be the de-noised correlation matrix obtained by the processing of the inverse discrete wavelet transform to

$\overline{(\mathcal{T}_0)\mathbf{C}}$.

Figure 3 shows the series of flow of de-noising.

$$\begin{array}{ccc} \mathbf{C} & \underset{\text{wavelet transform}}{\longrightarrow} & (\mathcal{T}_0)\mathbf{C} \\ & & \downarrow \quad \text{soft} - \text{thresholding} \\ \bar{\mathbf{C}} & \underset{\text{inverse wavelet transform}}{\longleftarrow} & \overline{(\mathcal{T}_0)\mathbf{C}} \end{array}$$

Figure 3: The series of flow of the de-nosing processing to a correlation matrix.

In accordance with Figure 3, question (I) is whether each processing; "wavelet transform", "soft-thresholding" and "inverse wavelet transform", preserves symmetry of the input data, or not. So we divide two parts in order to prove $\bar{\mathbf{C}} \in \mathcal{S}_{\tilde{n}}$: Part 1 is that the wavelet transform and inverse wavelet transform preserve symmetry, i.e. (a)$\mathbf{C} \in \mathcal{S}_{\tilde{n}} \Rightarrow (\mathcal{T}_0)\mathbf{C} \in \mathcal{S}_{\tilde{n}}$ and (b) $\overline{(\mathcal{T}_0)\mathbf{C}}\mathcal{S}_{\tilde{n}} \Rightarrow \bar{\mathbf{C}} \in \mathcal{S}_{\tilde{n}}$. And part 2 is that the processing of soft-thresholding preserves symmetry of the matrix, i.e. $(\mathcal{T}_0)\mathbf{C} \in \mathcal{S}_{\tilde{n}} \Rightarrow \overline{(\mathcal{T}_0)\mathbf{C}}\mathcal{S}_{\tilde{n}}$.

We give the two lemmas that these processes preserve symmetry of the matrix.

Lemma 3.1 *Let $\mathbf{C}$ be in $\mathcal{S}_{\tilde{n}}$. $\mathcal{T}_0$ denotes the operator of the discrete wavelet transform and $\mathcal{T}_0^{-1}$ denotes that of the inverse discrete wavelet transform. Then $(\mathcal{T}_0)\mathbf{C} \in \mathcal{S}_{\tilde{n}}$ where $(\mathcal{T}_0)\mathbf{C}$ has the NS-form as in Figure 2. Moreover $(\mathcal{T}_0^{-1})\mathbf{C} \in \mathcal{S}_{\tilde{n}}$.*

Proof. Let $(\mathcal{T}_0)\mathbf{C}$ be in the NS-form. We prove $(\mathcal{T}_0)\mathbf{C} \in \mathcal{S}_{\tilde{n}}$ for 4-coefficient case of the discrete wavelet transform at resolution level 1 as in Example 2.1. We use notation in Example 2.1. For another coefficient case the property of symmetry of the matrix can be shown by similar way. For the higher resolution level, we can iteratively show $(\mathcal{T}_j)\mathbf{C}$, for $j > 1$ with $(\mathcal{T}_j)\mathbf{C} \in \mathcal{S}_{\tilde{n}}$. And for the inverse discrete wavelet transform, it can be shown with recalling $H^* = H^{-1}$ and $G^* = G^{-1}$.

$\{c_{ij}\}_{i,j=1}^{\tilde{n}}$ and $\{G\mathbf{C}G^*\}_{ij}, ij = \{1, \ldots, \tilde{n}/2\}$ denote the elements of $\mathbf{C} \in \mathcal{S}_n$ and $G\mathbf{C}G^*$, respectively. Then

$$\{G\mathbf{C}G^*\}_{k,l} \quad = \quad \sum_{j=0}^{3} g_j \sum_{i=0}^{3} c_{(i+1)+2(l-1),(j+1)+2(k-1)} g_i, \text{ for } k, l = \{1, \ldots, \tilde{n}/2 - 1\},$$

$$\begin{aligned}
\{G\mathbf{C}G^*\}_{k,\tilde{n}/2} &= \sum_{i=0}^{1} g_i \sum_{j=0}^{3} c_{(i+1)+(\tilde{n}-2),(j+1)+2(k-1)} g_i + \sum_{i=2}^{3} g_i \sum_{j=0}^{3} c_{(i+1)-2,(j+1)+2(k-1)} g_i, \\
&\qquad \text{for } k = \{1, \ldots \tilde{n}/2\}, \\
\{G\mathbf{C}G^*\}_{\tilde{n}/2,l} &= \sum_{j=0}^{1} g_j \sum_{i=0}^{3} c_{(i+1)+2(l-1),(j+1)+(\tilde{n}-2)} g_i + \sum_{j=2}^{3} g_j \sum_{i=0}^{3} c_{(i+1)+2(l-1),(j+1)-2} g_i, \\
&\qquad \text{for } l = \{1, \ldots \tilde{n}/2\}.
\end{aligned}$$

Therefore by $c_{ij} = c_{ji}$, $G\mathbf{C}G^* \in \mathcal{S}_{\tilde{n}/2}$. As for $H\mathbf{C}H^*$, the property of symmetry holds by the similar way.

It is easy to show $G\mathbf{C}H^* = (H\mathbf{C}G^*)^*$. Since $\mathbf{C} \in \mathcal{S}_{\tilde{n}}$, i.e. $\mathbf{C} = \mathbf{C}^*$, this equality holds from

$$\begin{aligned}
(H\mathbf{C}G^*)^* &= G\mathbf{C}^* H^* \\
&= G\mathbf{C}H^*
\end{aligned}$$

Thus we have that $\mathbf{C} \in \mathcal{S}_{\tilde{n}} \Rightarrow (\mathcal{T}_0)\mathbf{C} \in \mathcal{S}_{\tilde{n}}$. □

Lemma 3.2 *Let $\mathbf{C}$ be in $\mathcal{S}_{\tilde{n}}$. $\mathcal{T}_0$ denotes the operator of the discrete wavelet transform and $\overline{(\mathcal{T}_0)\mathbf{C}}$ denotes $(\mathcal{T}_0)\mathbf{C}$ to which the soft-thresholding is processed. Then $\overline{(\mathcal{T}_0)\mathbf{C}} \in \mathcal{S}_{\tilde{n}}$.*

Proof. Let $(\mathcal{T}_0)\mathbf{C}$ be in the NS-form and $\overline{(\mathcal{T}_0)\mathbf{C}}$ denotes $(\mathcal{T}_0)\mathbf{C}$ processed via soft-thresholding. Since $\mathbf{C} \in \mathcal{S}_{\tilde{n}}$, $(\mathcal{T}_0)\mathbf{C} \in \mathcal{S}_{\tilde{n}}$ by Lemma 3.1. Since the result at resolution level 1 can be iteratively applied to the higher resolution level case, we will prove for the case where the resolution level is 1. $\{G\mathbf{C}G^*\}_{ij}$, $\{G\mathbf{C}H^*\}_{ij}$ and $\{H\mathbf{C}G^*\}_{ij}$ denote the elements of $G\mathbf{C}G^*$, $G\mathbf{C}H^*$ and $H\mathbf{C}G^*$. Note that $\{G\mathbf{C}G^*\}_{ij} = \{G\mathbf{C}G^*\}_{ji}$, $\{G\mathbf{C}H^*\}_{ij} = \{(H\mathbf{C}G^*)^*\}_{ij}$.

As processing soft-thresholding to $\{G\mathbf{C}G^*\}_{ij}$, then we have

$$\{G\mathbf{C}G^*\}_{ij}^{new} := \begin{cases} \{G\mathbf{C}G^*\}_{ij} - t_0^{G\mathbf{C}G^*}, & \text{for } \{G\mathbf{C}G^*\}_{ij} > t_0^{G\mathbf{C}G^*} \\ 0, & \text{for } |\{G\mathbf{C}G^*\}_{ij}| \leqslant t_0^{G\mathbf{C}G^*} \\ \{G\mathbf{C}G^*\}_{ij} + t_0^{G\mathbf{C}G^*}, & \text{for } \{G\mathbf{C}G^*\}_{ij} < t_0^{G\mathbf{C}G^*} \end{cases}$$

where $t_0^{G\mathbf{C}G^*}$ is $\mathrm{MAD}_{G\mathbf{C}G^*}\sqrt{2\log(\tilde{n}/2 \times \tilde{n}/2)}$ and $\mathrm{MAD}_{G\mathbf{C}G^*}$ is the median absolute deviation

$$\mathrm{MAD}_{G\mathbf{C}G^*} := \frac{median(||G\mathbf{C}G^*| - median(|G\mathbf{C}G^*|)|)}{0.6745}.$$

Since $\{G\mathbf{C}G^*\}_{ij} = \{G\mathbf{C}G^*\}_{ji}$, it is obvious that $\{G\mathbf{C}G^*\}_{ij}^{new} = \{G\mathbf{C}G^*\}_{ji}^{new}$.

It is clear that $\{G\mathbf{C}H^*\}_{ij}^{new} = \{(H\mathbf{C}G^*)^*\}_{ij}^{new}$ holds because of $\{G\mathbf{C}H^*\}_{ij} = \{(H\mathbf{C}G^*)^*\}_{ij}$. Thus we have $\overline{(\mathcal{T}_0)\mathbf{C}} \in \mathcal{S}_{\tilde{n}}$. $\square$

Thus we have the following proposition from Lemmas 3.1 and 3.2.

Proposition 3.1 *Let* $\mathbf{C}$ *be in* $\mathcal{S}_{\tilde{n}}$. *As applying the de-noising processing to* $\mathbf{C} \in \mathcal{S}_{\tilde{n}}$, *then we have* $\bar{\mathbf{C}} \in \mathcal{S}_{\tilde{n}}$.

By Proposition 3.1, $\mathbf{C} \in \mathcal{S}_{\tilde{n}} \Rightarrow \bar{\mathbf{C}} \in \mathcal{S}_{\tilde{n}}$ of question (I) holds, i.e. the de-nosing processing preserves symmetry of the correlation matrix.

Now we will consider question (II): Are all of the diagonal elements of the correlation matrix via de-noising ones? Let us consider the example of Figure 1(right) in Section 1 again. This figure is an implementation of the 4-rank reduction of angle parameterization. The noisy correlation matrix to be calibrated is generated by (2) with γ-parameters estimated by De Jong et al. (2004) which are normally distributed. Figure 4(left) shows such a noisy correlation matrix.

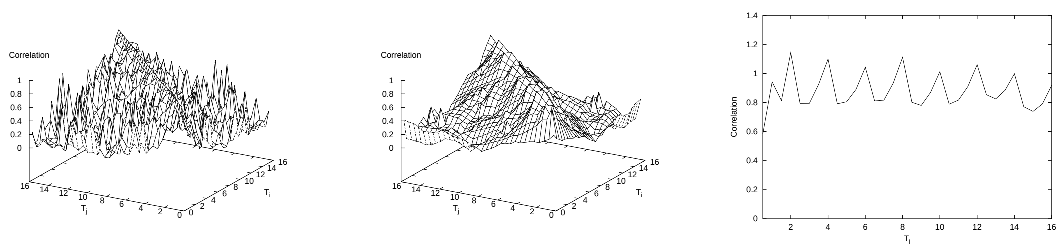

Figure 4: Noisy correlation matrix, de-noising and diagonal.

Applying the processing of de-noising, where we use genus 2 of Daubechies and take resolution lavel 2, to the noisy matrix we have the noise reduced matrix, Figure 4(center), and the right of this figure shows the diagonal elements of it. Though this example is one of many cases, it is clear that the processing of de-noising loses the property that all of diagonal elements of the correlation matrix are ones.

Then we may hit upon the rescaling as in Section 2, i.e.

$$\rho_{ij} := \frac{\bar{c}_{ij}}{\sqrt{\bar{c}_{ii}\bar{c}_{jj}}} \tag{6}$$

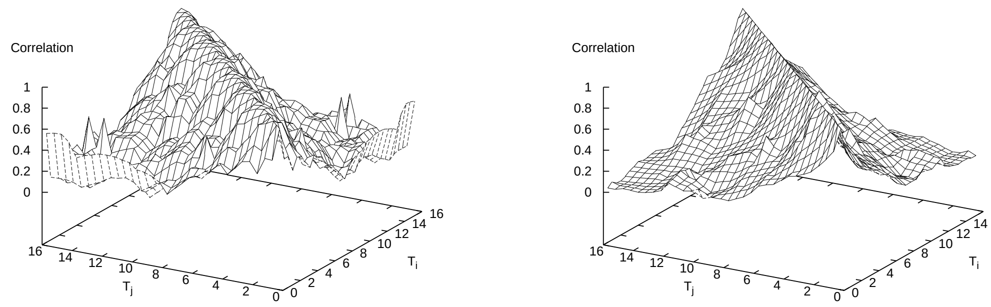

Figure 5: De-noised correlation matrices.

where $\bar{c}_{ij}$ is the (i, j)-element of the correlation matrix $\bar{\mathbf{C}}$ applied de-noising to $\mathbf{C}$, and ρ_{ij} is the (i, j)-element of the correlation matrix by the rescaling process (6). Figure 5(left) shows the rescaled correlation matrix of that in Figure 4(center).

We propose the following procedure in place of the rescaling for the property of non-ones of diagonal elements in the correlation matrix. Our proposing procedure is based on taking into account the form (1) and (5).

To begin with, taking the logarithm of the noisy correlation matrix $\mathbf{C}$ we have from (1)

$$\log c_{ij} = -\gamma_1 |T_i - T_j| - \frac{\gamma_2 |T_i - T_j|}{\max(T_i, T_j)^{\gamma_3}} - \gamma_4 |\sqrt{T_i} - \sqrt{T_j}|. \tag{7}$$

Then, comparing (7) with (5), the term corresponding to the noise level in (7) depends on T_i and T_j while that in (5) is constant. It should be divided by some function of T_i and T_j, e.g. $|T_i - T_j|$, $|\sqrt{T_i} - \sqrt{T_j}|$ and so on. However from (7) it is difficult to specify a function to adjust (7). We leave the matter open for further research. For convenience we here divide (7) by $|T_i - T_j|$, i.e.

$$\log c_{ij\,ad} := \begin{cases} \dfrac{\log c_{ij}}{|T_i - T_j|} & \text{for } i \neq j \\ 0 & \text{for } i = j. \end{cases} \tag{8}$$

We apply the soft-thresholding processing to $\log c_{ij\,ad}$. Denoting the elements of de-noised $\log c_{ij\,ad}$ by $\overline{\log c_{ij\,ad}}$, we again adjust $\overline{\log c_{ij\,ad}}$ using $|T_i - T_j|$, i.e.

$$\bar{c}_{ij_{ad}} := \begin{cases} \overline{\log c_{ij_{ad}}} \times |T_i - T_j| & \text{for } i \neq j \\ 0 & \text{for } i = j. \end{cases} \tag{9}$$

Finally taking the exponential of $\bar{c}_{ij_{ad}}$, we obtain the de-noised correlation matrix.

The important point to note is that the exponential values of $\bar{c}_{ij_{ad}}$ for $i = j$ are ones since $\bar{c}_{ij_{ad}}$ are set to equal zeros for $i = j$ in above procedure. Therefore the problem of non-zero diagonal elements of the de-noised correlation matrix is solved. We furthermore would like to lay special emphasis on achieving more effective in the processing of de-noising. The effectiveness of de-noising comes from the adjustment in $\log c_{ij_{ad}}$, i.e. reducing the dependence of T_i and T_j in the noise level. Figure 5(right) shows the de-noised correlation matrix via our proposing procedure. The following table is the results of these errors of the de-noised correlation matrices; by the rescaling process (6) (Figure 5(right)) and by our proposing procedure (Figure 5(left)). Here the error is defined by

$$\text{Error} := \sum_{i>j} (\{c_{non-noise}\}_{ij} - \{\bar{c}_{noise}\}_{ij})^2 \tag{10}$$

where $\{c_{non-noise}\}_{ij}$ is (i,j)-element of the correlation matrix generated by (1) with constant γ-parameters by De Jong et al. (2004), and $\{\bar{c}_{noise}\}_{ij}$ is (i,j)-element of the correlation matrix, which is similarly generated but γ-parameters are assumed to be normally distributed with mean and standard error estimated by De Jong et al. (2004), processed via each the de-noising procedure. We will use the error defined as (10) in the next section.

Table 1: Comparison of errors of de-noised correlation matrices.

	Rescaling processing	Our proposing procedure
Error	7.691851274	1.362358664

Finally we give Figure 6 that the de-noising processing (genus 2 of Daubechies and resolution level 2) with 4 rank-reduction of angle parameterization is applied to a noisy correlation matrix; Figure 5(left).

Comparing Figure 6 with Figure 1(right), we can find it that the noise of the correlation matrix with 4 rank-reduction is reduced by the de-noising processing. In the next section,

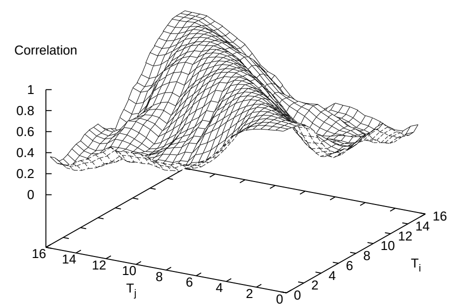

Figure 6: The de-noised correlation matrix with 4 rank-reduction of angle parameterization.

we investigate accuracy of the de-noised correlation matrix by numerical experiments.

4 Numerical results

We experiment on the de-noising processing for the numerical efficiency with different cases of dimensions of the correlation matrix (We will select 32×32 and 64×64. For 64×64, $T_i = \{0.5, 0.25, \ldots, 16\}$.), wavelet basis, resolution levels and the number of reduced rank. Before proceeding, we confirm the arbitrary parameters.

First, it is the kind of wavelet bases. Wavelet bases used in numerical experiments are Daubechies 2-10, Coiflet 2-10(only the even number) and Symlet 4-10, where the numbers are genus of wavelet bases. Secondly, resolution levels. Resolution levels depend on the dimension of data implemented the processing of the discrete wavelet transform. Thirdly, for rank reduction of angle parameterization we can choose the reduction number. It goes without saying that as the number of reduction increases, the rank reduced correlation matrix becomes closer to original one.

Tables 2, 3 and 4 show the *average* error, which is the average of errors defined as (10) by ten trials, of between the non-noisy correlation matrix and the noisy one implemented the de-noising processing. We attempt the de-noising processing of both our proposed procedure and the processing of soft-thresholding directly to the noisy matrix with rescaling (the errors of this procedure are in parentheses), respectively.

From these tables we can see that each wavelet basis has an effect on reduction of noise

Table 2: The average errors of ten trials in the case of Daubechies

Dimension		32×32			64×64			
Resolution level		1	2	3	1	2	3	4
Genus	2	3.44358	1.36874	0.751105	13.7606	5.05773	3.24893	3.18586
		(6.5766)	(7.00861)	(14.0139)	(20.6087)	(15.7337)	(21.4448)	(41.5408)
	3	3.50705	1.18642	0.858654	13.8425	5.08308	3.10786	2.88396
		(6.38463)	(6.84171)	(13.7677)	(21.1469)	(14.8616)	(20.6742)	(55.5278)
	4	3.64409	1.14975	0.832121	12.6055	4.91367	2.79003	13.5351
		(6.5261)	(6.62765)	(12.037)	(21.5777)	(15.0325)	(18.7512)	(38.7879)
	5	3.53333	1.42193	0.868147	13.9286	4.89595	2.86826	3.08182
		(6.39687)	(5.66469)	(10.6696)	(22.0682)	(13.8777)	(18.4925)	(32.0768)
	6	3.78049	1.32183	0.862698	13.6346	4.58636	3.09941	2.8359
		(5.80856)	(5.72751)	(11.3211)	(20.4376)	(13.3408)	(19.879)	(40.6603)
	7	3.52203	1.17397	0.816016	13.8312	4.84761	3.08698	2.99891
		(5.94017)	(6.28963)	(11.8559)	(20.442)	(15.8285)	(20.7714)	(42.1737)
	8	3.388	1.32895	0.896223	16.5219	5.11483	3.15518	2.96394
		(6.38468)	(6.72878)	(12.1746)	(20.8376)	(15.8776)	(20.8797)	(44.2803)
	9	3.55838	1.71375	1.65457	13.309	6.76205	5.94225	7.30971
		(5.90228)	(8.22615)	(15.2378)	(21.4901)	(18.4647)	(25.0892)	(50.9098)
	10	3.38783	1.24019	0.860728	13.9113	4.93779	3.02772	2.92374
		(6.65682)	(5.8396)	(9.44729)	(22.4168)	(14.353)	(17.853)	(36.9585)

Notes. The average errors of ten trials without de-noising are 15.5953 for 32×32 dimensions and 59.822 for 64×64 dimensions.

Table 3: The average errors of ten trials in the case of Symlet

Dimension		32×32			64×64			
Resolution level		1	2	3	1	2	3	4
Genus	4	3.86259	1.16994	0.895216	13.6503	4.70693	3.01925	2.81157
		(6.17135)	(6.50023)	(14.3212)	(21.0281)	(15.2748)	(21.5359)	(48.5764)
	5	3.37395	1.35071	0.926237	22.146	4.68016	2.87207	2.58169
		(5.75714)	(5.57946)	(9.95477)	(20.1559)	(13.2243)	(17.6088)	(30.6046)
	6	3.45378	1.2953	0.865213	14.3042	4.85847	2.77971	2.61081
		(6.61443)	(6.33027)	(12.4705)	(21.7418)	(16.3727)	(20.7341)	(44.5585)
	7	3.79725	1.24548	0.752502	13.7907	4.64732	3.03287	2.93934
		(5.48011)	(6.32576)	(12.0127)	(20.9631)	(14.5937)	(20.0962)	(36.0471)
	8	3.19487	1.39074	0.944772	17.7634	4.53851	2.73749	2.46935
		(6.48661)	(5.48471)	(11.4298)	(20.8045)	(14.0669)	(17.8917)	(40.9124)
	9	3.49675	1.26362	0.77321	13.0209	4.79031	3.02266	3.14536
		(6.59085)	(6.56399)	(11.0917)	(21.5547)	(14.6114)	(19.5484)	(36.4282)
	10	3.6747	1.3378	0.828756	15.0832	4.92973	2.94068	2.76981
		(6.44662)	(6.70892)	(12.0674)	(22.215)	(15.8075)	(19.2987)	(42.6612)

Notes. The average errors of ten trials without de-noising are 15.5953 for 32×32 dimensions and 59.822 for 64×64 dimensions.

Table 4: The average errors of ten trials in the case of Coiflet

Dimension		32×32			64×64			
Resolution level		1	2	3	1	2	3	4
Genus	2	3.69033	1.27274	0.705419	14.3172	4.70438	2.89604	2.52319
		(6.54103)	(7.32041)	(19.1372)	(21.0537)	(16.0893)	(25.0673)	(63.1392)
	4	3.20396	1.23853	0.838013	14.0103	4.49569	2.86704	2.63752
		(6.58319)	(6.35604)	(12.5084)	(20.8765)	(14.8778)	(19.9739)	(46.1159)
	6	3.74978	1.23459	0.904902	14.5603	4.80749	2.91008	2.5838
		(6.28458)	(6.4922)	(12.0739)	(21.1872)	(14.4249)	(18.3925)	(37.9046)
	7	3.64536	1.34903	0.857927	13.56	4.8745	2.77451	2.50775
		(6.49138)	(6.30138)	(12.305)	(20.9527)	(14.8684)	(17.6072)	(40.4683)
	10	3.59507	1.22305	0.788097	14.1545	4.82791	2.92645	2.76616
		(6.25314)	(5.85984)	(11.4895)	(21.6836)	(14.1711)	(19.306)	(44.7241)

Notes. The average errors of ten trials without de-noising are 15.5953 for 32×32 dimensions and 59.822 for 64×64 dimensions.

in the correlation matrix. Moreover, as increasing the resolution level we can achieve the effect more. The exceptions however exist, e.g. the case of Daubechies 4 and 5 (64×64) at resolution level 3-4 in Table 2. Since the errors without the de-noising processing of our proposed procedure or the processing of soft-thresholding with rescaling are apparently lager than those with de-noising (see the notes of each table), there is no need to go into details about the case of no de-nosing.

Table 5: The errors of one trial in the case of Daubechies(32×32)

Number of reduction		2		4	
Resolution level		1	2	1	2
Genus	2	3.47839	1.2304	3.47839	1.24315
		(7.80334)	(7.47017)	(7.80334)	(5.9989)
	3	3.35458	1.43929	3.43044	1.26945
		(5.54594)	(5.3571)	(7.79298)	(5.66786)
	4	3.53995	1.46991	3.22274	1.24773
		(7.11109)	(8.02536)	(8.11481)	(5.58232)
	5	4.00682	1.09067	3.35526	1.70721
		(7.21688)	(6.41524)	(7.89056)	(5.84447)
	6	3.87462	1.47681	3.43223	1.15183
		(6.07728)	(4.64816)	(6.15129)	(6.35335)
	7	3.40078	1.22714	4.10106	1.27902
		(5.50847)	(6.62746)	(7.599)	(6.03748)
	8	3.42811	1.28955	3.8106	1.29159
		(6.18565)	(5.29226)	(7.34069)	(5.49672)
	9	3.13907	1.66327	3.10539	1.73053
		(6.58181)	(7.43516)	(7.89338)	(6.70895)
	10	2.90535	1.103	3.09497	1.26236
		(7.14704)	(5.77488)	(7.60268)	(5.38444)

Notes. The errors of one trial without de-noising are 88.3573 for 2 rank-reduction and 27.966 for 4 rank-reduction.

Table 6: The errors of one trial in the case of Coiflet(32×32)

Number of reduction		2		4	
Resolution level		1	2	1	2
Genus	2	3.3737	1.20694	2.94696	1.12896
		(4.82535)	(7.32102)	(10.6183)	(8.77185)
	4	3.13909	0.901501	2.89812	1.59106
		(5.26267)	(5.3324)	(10.4358)	(6.25513)
	6	5.42238	1.32268	3.08198	0.988084
		(6.39225)	(6.11669)	(7.0152)	(7.10367)
	8	3.33172	1.0402	3.49045	0.941876
		(5.80168)	(6.94001)	(7.23875)	(4.41349)
	10	3.72734	1.56455	3.39523	0.880426
		(5.85368)	(7.111)	(6.40654)	(6.43521)

Notes. The errors of one trial without de-noising are 88.3573 for 2 rank-reduction and 27.966 for 4 rank-reduction.

Let us now attempt to apply the de-noising processing with rank reduction of angle parameterization. Tables 5-10 show the error of one trial for 2 and 4 rank reduction of angle parameterization via each de-noising processing; our proposing procedure and the processing

Table 7: The errors of one trial in the case of Symlet(32×32)

Number of reduction		2		4	
Resolution level		1	2	1	2
Genus	4	3.62293	1.85502	2.91825	1.09082
		(7.09927)	(6.63074)	(10.8838)	(7.78382)
	5	3.71419	1.44007	3.55601	1.254
		(3.86964)	(6.48222)	(6.60093)	(4.46325)
	6	3.30247	1.38123	3.47745	1.291
		(5.58069)	(6.64417)	(8.54499)	(7.13329)
	7	3.33233	1.66218	3.50047	1.39193
		(6.07235)	(7.43886)	(8.39405)	(5.7028)
	8	3.6064	1.57959	2.73993	1.31866
		(5.39474)	(5.78264)	(6.69617)	(4.5321)
	9	3.73318	1.17182	3.53883	1.25953
		(5.47919)	(7.4134)	(8.53637)	(6.96826)
	10	2.82371	1.0937	3.50745	1.23799
		(5.24888)	(6.94403)	(8.69077)	(7.0935)

Notes. The errors of one trial without de-noising are 88.3573 for 2 rank-reduction and 27.966 for 4 rank-reduction.

Table 8: The errors of one trial in the case of Daubechies(64×64)

Number of reduction		2			4		
Resolution level		1	2	3	1	2	3
Genus	2	12.1616	5.26446	2.82076	12.1616	4.91042	3.10275
		(19.5914)	(14.7071)	(21.5356)	(19.5914)	(14.0582)	(24.5382)
	3	15.6802	4.35574	2.98269	15.6802	4.7495	3.76557
		(22.2374)	(13.8672)	(22.2439)	(22.2374)	(12.5364)	(21.2842)
	4	12.8896	4.9642	3.02964	11.0168	5.45062	3.53037
		(22.0133)	(13.7415)	(16.6519)	(20.1757)	(15.2537)	(21.7249)
	5	12.8131	4.79937	3.03124	10.9582	4.81322	2.324
		(17.852)	(13.472)	(17.3104)	(20.3416)	(14.1903)	(20.164)
	6	14.4926	4.70052	2.48094	14.1549	3.96311	4.09055
		(21.2014)	(14.9421)	(23.9511)	(18.5432)	(11.532)	(20.0147)
	7	40.1463	4.35982	3.25772	11.4282	5.78058	3.4782
		(17.5169)	(16.0471)	(24.6019)	(20.0715)	(15.8102)	(21.2178)
	8	17.0413	4.83928	2.44311	11.7313	5.24397	2.60227
		(19.9906)	(18.1415)	(21.8668)	(20.1699)	(15.8513)	(24.1446)
	9	13.5226	6.51037	5.194	21.6994	5.71822	7.44931
		(21.5648)	(21.1974)	(21.1485)	(22.6609)	(14.5948)	(24.2116)
	10	14.1428	4.98524	2.57266	11.7691	5.12459	2.65957
		(23.8334)	(15.1974)	(17.7665)	(20.4305)	(13.339)	(19.8569)

Notes. The errors of one trial without de-noising are 345.772 for 2 rank-reduction and 122.603 for 4 rank-reduction.

Table 9: The errors of one trial in the case of Coiflet(64×64)

Number of reduction		2			4		
Resolution level		1	2	3	1	2	3
Genus	2	13.5877	4.35398	3.14338	15.3794	4.46501	3.51686
		(19.9902)	(17.6652)	(25.7925)	(19.7249)	(13.1525)	(26.8393)
	3	14.3766	5.44463	3.22989	11.2677	4.92094	2.41073
		(21.611)	(14.4066)	(17.3562)	(20.7702)	(15.409)	(19.2783)
	4	13.6058	3.97281	2.72442	14.2765	4.25495	3.62265
		(22.789)	(12.4538)	(21.9992)	(18.8921)	(10.8864)	(17.196)
	5	14.7043	5.37858	3.00026	11.1599	4.92286	2.54106
		(21.6336)	(15.3365)	(18.3604)	(20.5655)	(14.9866)	(17.1398)
	6	13.952	4.38408	2.4456	14.1915	4.21693	3.67346
		(19.1076)	(11.9616)	(18.7356)	(19.0094)	(11.0611)	(16.135)

Notes. The errors of one trial without de-noising are 345.772 for 2 rank-reduction and 122.603 for 4 rank-reduction.

Table 10: The errors of one trial in the case of Symlet(64×64)

Number of reduction	2			4		
Resolution level	1	2	3	1	2	3
4	15.1263	4.26181	3.50585	11.1341	5.89072	3.53155
	(19.6266)	(18.3206)	(21.7973)	(20.9973)	(14.4947)	(21.7264)
5	13.7677	5.19445	1.89293	11.531	4.80997	2.3099
	(17.8774)	(17.5095)	(17.2179)	(19.8187)	(13.5594)	(20.6842)
6	14.1219	4.35096	4.10913	14.549	4.36362	3.72636
	(21.8168)	(14.0008)	(17.502)	(19.5489)	(11.7588)	(20.5307)
7	14.168	4.71341	2.92352	10.9073	5.13088	2.88013
	(19.929)	(17.7445)	(22.6889)	(20.871)	(15.3294)	(19.9687)
8	13.0946	4.60801	3.40682	15.8596	4.21399	3.72223
	(20.1913)	(12.0684)	(18.8404)	(18.8492)	(10.7254)	(16.4405)
9	14.6731	5.20236	2.35333	11.1207	5.09365	2.71046
	(20.4017)	(15.8373)	(20.3934)	(20.4596)	(16.4008)	(18.5191)
10	13.3157	4.60539	2.99442	15.1549	4.30419	3.72316
	(19.6267)	(17.2603)	(18.3559)	(19.6054)	(12.2258)	(17.9967)

Notes. The errors of one trial without de-noising are 345.772 for 2 rank-reduction and 122.603 for 4 rank-reduction.

of soft-thresholding with rescaling (in parentheses). Tables 5-7 are for 32×32 dimensions, Tables 8-10 for 64×64 dimensions, respectively. We take resolution level to 2 for 32×32 dimensions and 3 for 64×64 dimensions. Since there exist exceptions of the increasing error at resolution level 3 in Table 2, we omit the case of resolution level 4 for 64×64 dimensions and 3 for 32×32 while such exceptions occur only for Daubechies case of 64×64.

The results of the numerical experiments for 2 and 4 rank reduction of angle parameterization with the de-noising processing are excellent as expected it from Tables 2-4. There exists no apparent difference in errors by the kind of wavelet bases and genus. And the decreasing tendency in errors exists as the resolution level increases. However the exception of this tendency arises for Daubechies coefficient (64×64, resolution level 9 and 4 rank-reduction) case in Table 8.

5 Conclusion

In this paper, we proposed a simple procedure based on the de-noising processing using wavelet analysis in order to reduce the noise in the noisy correlation matrix. And supposing the elements of the noisy correlation matrix as (1) with random γ-parameters, we made numerical experiments on the reduction of the noise by this procedure. The result of numerical experiments showed good performance. Subsequently, the rank reduction method with de-noising was also effective in modeling the approximated correlation matrix. Though it depends on the assumption of the distribution of noise in the correlation matrix, these numerical results lead to us that our proposed procedure of de-noising works well in modeling the correlation matrix from the noisy correlation data.

Lastly we remark the further research on improving the efficiency of de-noising and the application to finance.

First as mentioned in Section 3, the problem is how to adjust in (8) and (9). We divided (7) simply by $|T_i - T_j|$. This means that from the noisy correlation data we substantially extract the unknown $\hat{f}_{ij}$ such that

$$\{\text{correlation data}\}_{ij} = \exp\{\hat{f}_{ij} + \sigma|T_i - T_j|z_{ij}\} \tag{11}$$

where $z_{ij} \sim^{iid} N(0,1)$ and σ is the noise level. Therefore we presumed (11) instead of (1), alternatively. Then we should establish the processing of de-noising or soft-thresholding to (1) in order to improve the effectiveness of de-noising, if (1) is the good parametric function for the market correlation. If not so, we should consider another de-noising or soft-thresholding for better parametric function.

Secondly, it is the practical application to the calibration of LFM. The calibration of LFM is usually accomplished as follows (Refer to, e.g. London (2004)): Let $\sigma_i(a,b,c,d,\theta)$ be a volatility model. v_{market} denotes the market volatilities. Then the parameters a,b,c,d,θ are estimated by solving

$$\min_{a,b,c,d,\theta} \sum_i \left(\frac{v_{market} - \sigma_i^2(a,b,c,d,\theta)}{v_{market}} \right)^2.$$

Using the Newton-Raphson, gradient-decent or Brent's method solves the minimization problem. For instance, in such iterative calculus one can apply the de-noising processing as follows: Implement the de-noising processing to the angles obtained by the iterative routine. Set the de-noised angles as the initial values and execute the iteration again. Though we will have to discuss the convergence by such procedure, it is likely to become very effective to solve the minimization problem in the case of clarifying of effectiveness of it.

References

Brigo, D. A note on correlation and rank reduction. Working paper. available at www.damianobrigo.it, 2002.

Dahmen, W., Kurdila, A. and Oswald, P. (1997) Multiscale Wavelet Methods for Partial Differential Equations, (Academic Press).

De Jong, F., Driessen, J. and Pelsser, A. (2004) On the Information in the Interest Rate Term Structure and Option Prices, Review of Derivatives Research 7(2), pp. 99-127.

Donoho, D.L. (1995) De-Noising via Soft Thresholding, IEEE Trans. Inform. Theory 41, pp. 613-627.

Donoho, D. L. (1993) Progress in wavelet analysis and WVD: a ten minute tour, in Progress in wavelet analysis and applications, Y. Meyer, S. Roques, 109-128. Frontieres Ed.

Donoho, D.L. and Johnstone, I.M. (1994) Ideal spatial adaptation by wavelet shrinkage, Biometrika 81, pp. 425-455.

Grubišić, I. and Pietersz, R. Efficient Rank Reduction of Correlation Matrices. preprint, Utrecht University, 2005.

London, J. (2004) Modeling Derivatives in C++, (John Wiley and Sons Inc.).

M. Morini and N. Webber, An EZI method to reduce the rank of a correlation matrix in financial modelling, To appear in Applied Mathematical Finance, 2006.

Press, W. H., Flannery, B. P., Teukolsky S. A. and Vetterling, W. T. (1992) Numerical Recipes in C: The Art of Scientific Computing, (Cambridge University Press).

Rebonato, R. (1999) On the simultaneous calibration of multi-factor log-normal interest-rate models to Black volatilities and to the correlation matrix, Journal of Computational Finance 2(4), pp. 5-27.

Rebonato, R. and Jäckel, P. The Most General Methodology to Create a Valid Correlation Matrix for Risk Management and Option Pricing Purposes. QUARC preprint,1999.

Resnikoff, H. L. and Wells Jr, R.O. (1998) Wavelet Analysis: The Scalable Structure of Information (Springer-Velag. New York).