Discussion Papers In Economics And Business

A Characterization of the Randomized Uniform Rule

Wataru KUREISHI Hideki MIZUKAMI

Discussion Paper 05-20

Graduate School of Economics and
Osaka School of International Public Policy (OSIPP)
Osaka University, Toyonaka, Osaka 560-0043, JAPAN

A Characterization of the Randomized Uniform Rule

Wataru KUREISHI　Hideki MIZUKAMI

Discussion Paper 05-20

July 2005

この研究は「大学院経済学研究科・経済学部記念事業」
基金より援助を受けた、記して感謝する。

Graduate School of Economics and
Osaka School of International Public Policy (OSIPP)
Osaka University, Toyonaka, Osaka 560-0043, JAPAN

A Characterization of the Randomized Uniform Rule*

Wataru KUREISHI[†‡]
kureishi@iser.osaka-u.ac.jp

Hideki MIZUKAMI[§]
mizukami@eco.toyama-u.ac.jp

July 1, 2005

Abstract

We consider the problem of allocating several units of an indivisible object among the agents with single-peaked and risk-averse utility functions. We introduce *equal probability for the best*, and show that the randomized uniform rule is the only randomized rule satisfying strategy-proofness, Pareto optimality, and equal probability for the best. This is an alternative characterization of the result of Ehlers and Klaus (2004).

JEL classification: D63, D71

Keywords: The Randomized Uniform Rule, Single-Peaked Utility Functions, Equal Probability for the Best, Strategy-Proofness, Indivisibility

---

*This paper owes much to thoughtful and helpful comments of Hideshi Itoh, Tatsuyoshi Saijo, Ken-Ichi Shimomura, and Takuma Wakayama. We also thank Shinji Ohseto, participants of the Japanese Economic Association meeting of May 2001, and seminar participants at Osaka University for their useful comments. The earlier version of this paper received the Osaka University Institute of Social and Economic Research Moriguchi Prize in January 2003.

†Graduate School of Economics, Osaka University, Japan.
‡Corresponding author.
§Faculty of Economics, Toyama University, Japan.

1 Introduction

We consider the situation where the manager of a firm allocates several recruited workers to some departments such as sales, production, and so on. Each department has an optimal number of workers, because if the department has fewer workers than the number, it cannot accomplish its task, and if it has more, it needs extra equipment and incurs additional training costs.

The above mentioned problem faced by the manager is an example of the problem of allocating several units of an indivisible and identical object among a finite group of agents with single-peaked, risk-averse, and von-Neumann and Morgenstern utility functions. In this paper, we analyze the problem axiomatically, which was initiated by Sasaki (1997).

When the objects are infinitely divisible, a deterministic rule known as the uniform rule is a solution to the above problem in the sense that it satisfies strategy-proofness, efficiency, and some kind of fairness (Benassy (1982), Sprumont (1991), and Ching (1994)). The uniform rule works as follows: each agent is allowed to choose his/her own peak subject to a common bound which is chosen so as to attain feasibility.

However, the uniform rule cannot be applied to the problem where the objects are not infinitely divisible but indivisible because the common bound that the uniform rule allocates can be a fraction. In order to deal with the difficulty due to indivisibility, we use not deterministic rules but randomized rules, which associate with each profile of the utility functions a probability distribution on feasible allocations. Another justification for considering randomized rules rather than deterministic rules is that it is the simplest way to ensure fairness when the objects are indivisible.

As we consider a probabilistic extension of rules, it is reasonable to consider a probabilistic extension of the uniform rule, which we call *the randomized uniform rule*. The randomized uniform rule is a randomized rule which outputs a probability distribution that has the following marginal distribution: the agent who gets his/her own peak under the uniform rule gets his/her own peak with probability 1, and the rest of the agents get the common bound rounded up and that rounded down with the common probabilities which are chosen so as to equate the expected value of each agent to the common bound.

Sasaki (1997) first considered the randomized extension of the uniform rule. He introduced an equal probability uniform rule, which is a special case of the randomized uniform rule, and showed that it satisfies strategy-proofness, Pareto optimality, and anonymity. He also pointed out the importance of risk-averseness of the utility functions. If risk-loving von-Neumann and Morgenstern utility functions are allowed, the randomized uniform rule does not satisfy Pareto optimality.

In this paper, we characterize the randomized uniform rule by strategy-proofness, Pareto optimality, and *equal probability for the best*. Equal probability for the best is the axiom relating to fairness. Given an agent and a probability distribution, let us call the number of the object which gives him/her the highest utility with a strictly positive probability the best for the agent. Then, equal probability for the best requires the randomized rules to output the probability distribution which has the following marginal distribution: if two agents have the same utility function, then they should get *the same best* with *the same probability*. For example, suppose that the manager has to allocate three recruited workers to two departments, A and B. Further, we assume that both departments have the same utility function and want more than three workers, that is, their peaks are three. Let us consider the case in which a randomized rule outputs a probability distribution which gives department A the marginal distribution that places probability zero on getting three workers, and probability $1/3$ on getting two, one, and no worker(s) respectively. We can say that getting two workers is the best for department A in this case. Then, this axiom requires that the probability distribution should give department B the marginal distribution where getting two workers is the best for it and is placed probability $1/3$ on. However, the probability placed on getting one worker and that on getting no worker do not need to be equal between the departments.

The contributions of this paper are as follows. First, our paper is the first to introduce the axiom relating to fairness that is indigenous to indivisibility or randomization. The axioms used in previous papers on randomization of the uniform rule, such as equal treatment of equals, symmetry, anonymity, and so on, are just the randomized version of those used in the deterministic environment. However, equal probability for the best has no counterpart in the deterministic environment. In addition, equal probability for the best is a weak axiom in the sense that it requires that only a part of the marginal distribution should be equal between the agents with the same utility function, whereas some other axioms require the whole part should be.

Second, we succeed in characterizing the randomized uniform rule by strategy-proofness, Pareto optimality, and equal probability for the best. This result is an alternative characterization of Ehlers and Klaus' (2004).

They succeeded in characterizing the randomized uniform rule by strategy-proofness, Pareto optimality, and equal treatment of equals. This implies that in the presence of strategy-proofness and Pareto optimality, equal probability for the best and equal treatment of equals are identical, whereas equal probability for the best is logically independent of equal treatment of equals.

This paper is organized as follows: Section 2 describes the model. Several axioms and the randomized uniform rule are introduced, and our theorem is presented in Section 3. The proof of our theorem is similar to that of Ching (1994).

2 The Model

We consider the problem of allocating several units $X \in \mathbb{Z}_{++}$ of *an indivisible object* among a finite group of agents, $N = \{1, 2, \ldots, n\}$. The objects are identical. We denote the set of the indivisible objects by $C = \{0, 1, 2, \ldots, X\}$. Each agent $i \in N$ has a von-Neumann and Morgenstern utility function $u_i : C \to \mathbb{R}$ which is *single-peaked* and *risk-averse.* u_i is *single-peaked* if there is *a unique peak* $p(u_i) \in C$ such that for all $k, k' \in C$ with $p(u_i) \leq k < k'$ or $k' < k \leq p(u_i)$, $u_i(p(u_i)) \geq u_i(k) > u_i(k')$. u_i is *risk-averse* if for any $k \in C \backslash \{0, X\}$, $u_i(k) - u_i(k-1) > u_i(k+1) - u_i(k)$. We assume that the utility functions are only privately known. Let U be the set of all single-peaked, risk-averse, and von-Neumann and Morgenstern utility functions. $u = (u_1, u_2, \ldots, u_n) \in U^n$ is called a profile of the utility functions. u_{-i} stands for $(u_1, \ldots, u_{i-1}, u_{i+1}, \ldots, u_n) \in U^{n-1}$. (N, u, X) is called a problem. We say that the problem is in excess demand if $\sum_{i \in N} p(u_i) \geq X$. If $\sum_{i \in N} p(u_i) < X$, the problem is in excess supply. A feasible allocation is a vector $x = (x_1, x_2, \ldots, x_n) \in C^n$ with $\sum_{i \in N} x_i = X$.[1] Let $A(X)$ be the set of all feasible allocations. f is a probability distribution on $A(X)$ such that $f : A(X) \to [0, 1]$ and $\sum_{x \in A(X)} f(x) = 1$. Let $S(X)$ be the set of all probability distributions on $A(X)$. *A randomized rule* is a function $\phi : U^n \to S(X)$. We define the probability with which a probability distribution f gives k units of the indivisible object to agent i by $f_i(k) \equiv \sum_{x \in \{x \in A(X) | x_i = k\}} f(x)$. This means that f_i is *the marginal distribution* of f with respect to the number of the object that the agent i obtains. Given a probability distribution f, we call $\sum_{k \in C} f_i(k) u_i(k)$ the expected utility of agent i. We abuse notation,

[1]Free-disposability is not allowed.

and write $\phi(u)(x)$ as $\phi(x;u)$ and $\phi_i(u)(k)$ as $\phi_i(k;u)$.[2]

The following example demonstrates that two distinct probability distributions have the same marginal distribution.

Example 1. Let $n = 4$ and $X = 2$. Let $f = [\frac{1}{6} \circ (1,1,0,0), \frac{1}{6} \circ (1,0,1,0), \frac{1}{6} \circ (1,0,0,1), \frac{1}{6} \circ (0,1,1,0), \frac{1}{6} \circ (0,1,0,1), \frac{1}{6} \circ (0,0,1,1)]$ and $f' = [\frac{1}{2} \circ (1,1,0,0), \frac{1}{2} \circ (0,0,1,1)]$. Both f and f' give each agent i the same marginal distribution $f_i(k) = f'_i(k) = \frac{1}{2}$ for $k = 0,1$ and $f_i(2) = f'_i(2) = 0$.

3 Axioms and the Randomized Uniform Rule

We are interested in the following axioms. The first is the requirement that no agent ever benefits from misrepresenting his/her utility function, which is only privately known.

Strategy-proofness. For any $i \in N$, any $u_i, \hat{u}_i \in U$, and any $u_{-i} \in U^{n-1}$, $\sum_{k \in C} \phi_i(k;u)u_i(k) \geq \sum_{k \in C} \phi_i(k;\hat{u}_i,u_{-i})u_i(k)$.

A standard axiom that needs no further explanation is Pareto optimality.

Pareto optimality. For any $u \in U^n$, there is no $f \in S(X)$ such that for all $i \in N$, $\sum_{k \in C} f_i(k)u_i(k) \geq \sum_{k \in C} \phi_i(k;u)u_i(k)$ and for some $j \in N$, $\sum_{k \in C} f_j(k)u_j(k) > \sum_{k \in C} \phi_j(k;u)u_j(k)$.

Pareto optimality implies same-sidedness, which requires that for any agent every number of the object which he/she obtains with a strictly positive probability should be equal to or less (more) than his/her own peak when the problem is in excess demand (supply).

Same-sidedness. For any $u \in U^n$ and any $i \in N$,
(i) when $\sum_{j \in N} p(u_j) \geq X$, for all $k \in C, \phi_i(k;u) > 0$ implies $k \leq p(u_i)$, and
(ii) when $\sum_{j \in N} p(u_j) < X$, for all $k \in C, \phi_i(k;u) > 0$ implies $k \geq p(u_i)$.

Lemma 1. If ϕ is Pareto optimal, then it is same-sided.

Proof of Lemma 1. See Sasaki (1997). □

For the converse statement of Lemma 1 to be true we need property B, which is presented in the Appendix.

The final is related to fairness. We call it *equal probability for the best.* Given a probability distribution and an agent, there exists a number of the object

[2] $\phi(x;u)$ is the probability which the randomized rule ϕ places on the feasible allocation $x \in A(X)$ when the reported utility profile is $u \in U^n$. $\phi_i(k;u)$ is the probability with which agent i obtains k units of the object under the randomized rule ϕ when the reported utility profile is u.

which gives him/her the highest utility with a strictly positive probability. The number is called the best. It should be noted that since every utility function is single-peaked the agent has at most two bests: one is on the left side of his/her peak and the other is on the right side. Also note that each agent has only one best if the randomized rule satisfies same-sidedness. Then, equal probability for the best requires the randomized rule to output the probability distribution which has the following marginal distribution: if two agents have the same utility function, then at least one of the bests and the corresponding probabilities should be same between the agents. Note that the marginal distribution of each agent does not need to be same between the agents.

In order to define equal probability for the best formally, for any $i \in N$ and any $f \in S(X)$, we need B^i_f: the set of all pairs consisting of both the number of the object which gives agent i the highest utility with a strictly positive probability and the corresponding probability of the number under the probability distribution f. Note that B^i_f is a singleton or a doubleton.

For any probability distribution $f \in S(X)$, any $i \in N$, and any $u_i \in U$, let $B^i_f \equiv \{(b, f_i(b)) \in C \times (0,1] \mid u_i(b) \geq u_i(k)$ for all $k \in C$ such that $f_i(k) > 0\}$.

Using B^i_f, equal probability for the best can be represented as follows.

Equal probability for the best. For any $u \in U^n$ and any $i, j \in N$, if $u_i = u_j$, then $B^i_{\phi(u)} \cap B^j_{\phi(u)} \neq \emptyset$.

Equal treatment of equals says that if two agents' utility functions are same, their expected utilities should also be same.

Equal treatment of equals. For any $u \in U^n$ and any $i, j \in N$, if $u_i = u_j$, then $\sum_{k \in C} \phi_i(k) u_i(k) = \sum_{k \in C} \phi_j(k) u_j(k)$.

Remark 1. Both equal probability for the best and equal treatment of equals are implied by symmetry,[3] which is weaker than anonymity.[4] It is easy to check that equal probability for the best and equal treatment of equals are logically independent.

When objects are infinitely divisible, a well-known deterministic rule satisfying strategy-proofness is the uniform rule (Benassy (1982)). It is the

[3]For any $u \in U^n$ and any $i, j \in N$, if $u_i = u_j$, then $\phi_i(k) = \phi_j(k)$ for all $k \in C$.

[4]For any $u \in U^n$, any permutation π of N, any $i \in N$, and any $k \in C$, $\phi_i(k; u^\pi) = \phi_{\pi(i)}(k; u)$ where $u^\pi = (u_{\pi(1)}, u_{\pi(2)}, \ldots, u_{\pi(n)}) \in U^n$.

function $\Upsilon : U^n \to \mathbb{R}^n_+$ such that for any $u \in U^n$,

$$\Upsilon_j(u) \equiv \begin{cases} \min\{p(u_j), \lambda\} & \textit{if } \sum_{i\in N} p(u_i) \geq X, \\ \max\{p(u_j), \lambda\} & \textit{otherwise}, \end{cases}$$

for all $j \in N$, where $\lambda \in \mathbb{R}_+$ solves $\sum_{i\in N} \Upsilon_i(u) = X$.

We must note that the common bound λ can be a fraction so the uniform rule cannot be used to solve the problems with the indivisible objects. As an alternative rule, we propose a probabilistic extension of the uniform rule, *the randomized uniform rule*, which is a particular randomized rule.

Definition 1. *The randomized uniform rule* $\Phi : U^n \to S(X)$ is the function defined as follows: for any $u \in U^n$,
(i) when $\sum_{i\in N} p(u_i) \geq X$,

$$\begin{cases} \Phi_j(p(u_j); u) = 1 & \text{if } p(u_j) \leq \lambda, \\ \left.\begin{array}{ll} \Phi_j(\mu+1; u) & = \lambda - \mu \\ \Phi_j(\mu; u) & = 1 - (\lambda - \mu) \end{array}\right. & \text{if } p(u_j) > \lambda, \end{cases}$$

(ii) when $\sum_{i\in N} p(u_i) < X$,

$$\begin{cases} \Phi_j(p(u_j); u) = 1 & \text{if } p(u_j) \geq \lambda, \\ \left.\begin{array}{ll} \Phi_j(\mu+1; u) & = \lambda - \mu \\ \Phi_j(\mu; u) & = 1 - (\lambda - \mu) \end{array}\right. & \text{if } p(u_j) < \lambda, \end{cases}$$

for all $j \in N$, where $\lambda \in \mathbb{R}_+$ solves $\sum_{i\in N} \sum_{k\in C} \Phi_i(k)k = X$ and $\mu \equiv \lfloor \lambda \rfloor$.[5][6]

The randomized uniform rule associates with each profile of the utility functions a probability distribution that induces the marginal distribution of the following form. When the problem is in excess demand (supply), the agent whose peak is equal to or less (more) than the common bound λ gets his/her own peak with probability 1. The rest of the agents get $\mu+1$ with probability $\lambda - \mu$ and μ with probability $1 - \lambda + \mu$ respectively. μ is obtained through rounding down the common bound λ.

The randomized uniform rule outputs "a" probability distribution which induces each agent's marginal distribution defined as (i) and (ii) of Definition 1. Hence, the randomized uniform rule uniquely determines *a marginal*

[5] For any $a \in \mathbb{R}$, let $\lfloor a \rfloor = n$ be the integer such that $n \leq a < n+1$.

[6] More than one probability distribution can have the marginal distribution defined as (i) and (ii) of Definition 1, as we pointed out in Example 1. The randomized uniform rule chooses one of such probability distributions arbitrarily.

distribution for each agent, but it does not uniquely determine *a probability distribution* (although it uniquely chooses a probability distribution) because, as we pointed out in Example 1, more than one probability distribution can have an identical marginal distribution.

A special case of the randomized uniform rule is the equal probability uniform rule.

Example 2. The equal probability uniform rule $\Psi : U^n \to S(X)$ is defined as follows: for any $u \in U^n$,
(i) $\Psi_i(u) = \Phi_i(u)$ for any $i \in N$, and
(ii) $\Psi(x; u) = \Psi(x'; u)$ for any $x, x' \in A(X)$ such that $\Psi(x; u) > 0$ and $\Psi(x'; u) > 0$.

The equal probability uniform rule outputs the probability distribution (i) which has the same marginal distribution as that of the randomized uniform rule, and (ii) where any feasible allocations each of which is given a strictly positive probability have an equal probability. Sasaki (1997) introduced a randomized uniform rule which is a particular equal probability uniform rule, which uniquely determines not only *a marginal distribution* for each agent but also *a probability distribution* (see Sasaki (1997) for details).

3.1 An Alternative Characterization of the Randomized Uniform Rule

The randomized uniform rule can be characterized by strategy-proofness, Pareto optimality, and equal probability for the best.

Theorem. The randomized uniform rule Φ is the only randomized rule satisfying strategy-proofness, Pareto optimality, and equal probability for the best.

Proof of the Theorem. Proof of the Theorem is in the Appendix. □

The main feature of this Theorem is that our characterization is conducted with the axiom, equal probability for the best with respect to the axiom relating to fairness. This axiom is indigenous to indivisibility or randomization so that it has no counterpart in the deterministic environment. On the other hand, in previous papers on randomization of the uniform rule, all axioms on fairness, such as equal treatment of equals, symmetry, anonymity, and so on, are just the randomized version of those used in the deterministic environment. Hence, we can say that our characterization is the first one that is not a mere randomized version of the results obtained in the deterministic environment.

Ehlers and Klaus (2004) showed that the class of the randomized uniform rules is characterized by Pareto optimality, strategy-proofness, and equal treatment of equals. Hence, by our Theorem we succeed in giving an alternative characterization of Ehlers and Klaus' (2004). This implies that in the presence of strategy-proofness and Pareto optimality, equal probability for the best and equal treatment of equals are identical, whereas equal probability for the best is logically independent of equal treatment of equals. Moreover, by Remark 1 anonymity or symmetry implies equal probability for the best respectively, and it is obvious that the randomized uniform rule satisfies anonymity and symmetry. These indicate that equal probability for the best, equal treatment of equals, symmetry, and anonymity are identical in the presence of strategy-proofness and Pareto optimality.

References

Benassy, J. P. (1982) *The Economics of Market Disequilibrium*, New York: Academic Press.

Ching, S. (1994) "An Alternative Characterization of the Uniform Rule", *Social Choice and Welfare*, Vol. 11, No. 2, pp. 131-136.

Ehlers, L. and B. Klaus (2004) "Probabilistic Assignments of Identical Indivisible Objects and Uniform Probabilistic Rule", *Review of Economic Design*, Vol. 8, No. 3, pp. 249-268.

Sasaki, H. (1997) "Randomized Uniform Allocation Mechanism and Single-Peaked Preferences of Indivisible Good", Waseda University, mimeo.

Sprumont, Y. (1991) "The Division Problem with Single-Peaked Preferences: A Characterization of the Uniform Allocation Rule", *Econometrica*, Vol. 59, No. 2, pp. 509-519.

Appendix

Before proceeding to prove the Theorem, we need some following lemmas. Lemma 2 says that Pareto optimality implies property B, which requires that randomized rules assign the probability distribution where each agent receives at most two numbers of the object which are adjacent with each other.

Property B (Sasaki (1997)). For any $u \in U^n$ and any $i \in N$, there exists $\alpha_i \in C$ with $\phi_i(\alpha_i + 1; u) > 0$ such that $\phi_i(\alpha_i; u) + \phi_i(\alpha_i + 1; u) = 1$.

Lemma 2 (Sasaki (1997)). If ϕ satisfies Pareto optimality, then it satisfies property B.

Proof of Lemma 2. See Sasaki (1997). □

If we assume that ϕ satisfies same-sidedness, then we can show the converse statement of Lemma 2.

Lemma 3. If ϕ satisfies same-sidedness and property B, then it satisfies Pareto optimality.

Proof of Lemma 3. Suppose the contrary. There exists $f \in S(X)$ such that for all $i \in N$, $\sum_{k \in C} f_i(k) u_i(k) \geq \sum_{k \in C} \phi_i(k; u) u_i(k)$ and for some $j \in N$, $\sum_{k \in C} f_j(k) u_j(k) > \sum_{k \in C} \phi_j(k; u) u_j(k)$. Without loss of generality, we assume that $\sum_{i \in N} p(u_i) \geq X$. Let $\lambda_i \equiv \sum_{k \in C} f_i(k) k$ and $\mu_i \equiv \lfloor \lambda_i \rfloor$ for all $i \in N$. From risk-averseness, we have for all $i \in N$, $(1 - \lambda_i + \mu_i) u_i(\mu_i) + (\lambda_i - \mu_i) u_i(\mu_i + 1) \geq \sum_{k \in C} f_i(k) u_i(k)$. From property B, for all $i \in N$, there exists $\alpha_i \in C$ such that $\phi_i(\alpha_i; u) + \phi_i(\alpha_i + 1; u) = 1$. Therefore, we have $\sum_{k \in C} \phi_i(k; u) u_i(k) = \phi_i(\alpha_i; u) u_i(\alpha_i) + \phi_i(\alpha_i + 1; u) u_i(\alpha_i + 1)$ for all $i \in N$. Hence, we have $(1 - \lambda_i + \mu_i) u_i(\mu_i) + (\lambda_i - \mu_i) u_i(\mu_i + 1) \geq \phi_i(\alpha_i; u) u_i(\alpha_i) + \phi_i(\alpha_i + 1; u) u_i(\alpha_i + 1)$ for all $i \neq j$ and $(1 - \lambda_j + \mu_j) u_j(\mu_j) + (\lambda_j - \mu_j) u_j(\mu_j + 1) > \phi_j(\alpha_j; u) u_j(\alpha_j) + \phi_j(\alpha_j + 1; u) u_j(\alpha_j + 1)$.

From same-sidedness, we have for all $i \in N$, $p(u_i) \geq \sum_{k \in C} \phi_i(k; u) k$. Hence, single-peakedness implies that for all $i \neq j$, $(1 - \lambda_i + \mu_i) \mu_i + (\lambda_i - \mu_i)(\mu_i + 1) \geq \phi_i(\alpha_i; u) \alpha_i + \phi_i(\alpha_i + 1; u)(\alpha_i + 1)$ and $(1 - \lambda_j + \mu_j) \mu_j + (\lambda_j - \mu_j)(\mu_j + 1) > \phi_j(\alpha_j; u) \alpha_j + \phi_j(\alpha_j + 1; u)(\alpha_j + 1)$.

Summing up gives us $\sum_{i \in N} (1 - \lambda_i + \mu_i) \mu_i + \sum_{i \in N} (\lambda_i - \mu_i)(\mu_i + 1) > \sum_{i \in N} \phi_i(\alpha_i; u) \alpha_i + \sum_{i \in N} \phi_i(\alpha_i + 1; u)(\alpha_i + 1)$. The left hand side becomes $\sum_{i \in N} \lambda_i = \sum_{i \in N} \sum_{k \in C} f_i(k) k = X$. The last equality follows from feasibility. Moreover, the right hand side also becomes $\sum_{i \in N} \sum_{k \in C} \phi_i(k; u) k = X$ by feasibility. This is a contradiction. □

Property B is the key property in the sense that the problem of allocating X units of the identical and indivisible object among n individuals with single-peaked and risk-averse utility functions reduces to that of allocating X amount of the infinitely divisible object among them, as explained in Ehlers and Klaus (2004).

The rest closely follows Ching (1994). Lemma 4 says that Pareto optimality and strategy-proofness imply own peak monotonicity, which requires that the expected value of the marginal distribution of each agent is monotonic with respect to his/her reported peak.

Own peak monotonicity. For all $i \in N$, all $u_i, u_i' \in U$, and all $u_{-i} \in U^{n-1}$, if $p(u_i') \leq p(u_i)$, then $\sum_{k\in C} \phi_i(k; u_i', u_{-i})k \leq \sum_{k\in C} \phi_i(k; u)k$.

Lemma 4. If ϕ satisfies Pareto optimality and strategy-proofness, then it is own peak monotonic.

Proof of Lemma 4. Suppose the contrary. We have $p(u_i') \leq p(u_i)$ and $\sum_{k\in C} \phi_i(k; u_i', u_{-i})k > \sum_{k\in C} \phi_i(k; u)k$. By property B, there exist $\alpha_i, \alpha_i' \in C$ such that $\phi_i(\alpha_i; u)+\phi_i(\alpha_i+1; u) = 1$ and $\phi_i(\alpha_i'; u_i', u_{-i})+\phi_i(\alpha_i'+1; u_i', u_{-i}) = 1$. Hence, we have $\sum_{k\in C} \phi_i(k; u)k = \phi_i(\alpha_i; u)\alpha_i + \phi_i(\alpha_i + 1; u)(\alpha_i + 1)$ and $\sum_{k\in C} \phi_i(k; u_i', u_{-i})k = \phi_i(\alpha_i'; u_i', u_{-i})\alpha_i' + \phi_i(\alpha_i' + 1; u_i', u_{-i})(\alpha_i' + 1)$.

If $\sum_{j\in N} p(u_j) \leq X$, by same-sidedness, $p(u_i) \leq \sum_{k\in C} \phi_i(k; u)k$. Then, we have $p(u_i') \leq \sum_{k\in C} \phi_i(k; u)k = \phi_i(\alpha_i; u)\alpha_i + \phi_i(\alpha_i + 1; u)(\alpha_i + 1) < \sum_{k\in C} \phi_i(k; u_i', u_{-i})k = \phi_i(\alpha_i'; u_i', u_{-i})\alpha_i' +\phi_i(\alpha_i'+1; u_i', u_{-i})(\alpha_i'+1)$. By single-peakedness, we have $\phi_i(\alpha_i; u)u_i'(\alpha_i)+\phi_i(\alpha_i+1; u)u_i'(\alpha_i+1) > \phi_i(\alpha_i'; u_i', u_{-i})u_i'(\alpha_i')+\phi_i(\alpha_i'+1; u_i', u_{-i})u_i'(\alpha_i'+1)$. Hence, $\sum_{k\in C} \phi_i(k; u)u_i'(k) > \sum_{k\in C} \phi_i(k; u_i', u_{-i})u_i'(k)$, contradicting strategy-proofness.

If $\sum_{j\in N} p(u_j) > X$, then for all $j \in N$, $\sum_{k\in C} \phi_j(k; u)k \leq p(u_j)$. There are two cases: (i) If $\sum_{j\neq i} p(u_j) + p(u_i') \geq X$, then, by same-sidedness, we have $\sum_{k\in C} \phi_i(k; u_i', u_{-i})k \leq p(u_i')$. Then, $\sum_{k\in C} \phi_i(k; u)k = \phi_i(\alpha_i; u)\alpha_i + \phi_i(\alpha_i + 1; u)(\alpha_i+1) < \sum_{k\in C} \phi_i(k; u_i', u_{-i})k = \phi_i(\alpha_i'; u_i', u_{-i})\alpha_i'+\phi_i(\alpha_i'+1; u_i', u_{-i})(\alpha_i'+1) \leq p(u_i') \leq p(u_i)$. Single-peakedness implies that $\phi_i(\alpha_i; u)u_i(\alpha_i) + \phi_i(\alpha_i + 1; u)u_i(\alpha_i+1) < \phi_i(\alpha_i'; u_i', u_{-i})u_i(\alpha_i')+\phi_i(\alpha_i'+1; u_i', u_{-i})u_i(\alpha_i'+1)$. This means that $\sum_{k\in C} \phi_i(k; u)u_i(k) < \sum_{k\in C} \phi_i(k; u_i', u_{-i})u_i(k)$.
(ii) If $\sum_{j\neq i} p(u_j) + p(u_i') < X$, then $p(u_i') < \sum_{k\in C} \phi_i(k; u)k$ (otherwise, $p(u_i') \geq \sum_{k\in C} \phi_i(k; u)k$. Since for all $j \in N, p(u_j) \geq \sum_{k\in C} \phi_j(k; u)k$, we have $X = \sum_{j\in N} \sum_{k\in C} \phi_j(k; u)k \leq p(u_i') + \sum_{j\neq i} p(u_j) < X$). Hence, we have $p(u_i') < \sum_{k\in C} \phi_i(k; u)k = \phi_i(\alpha_i; u)\alpha_i+\phi_i(\alpha_i+1; u)(\alpha_i+1) < \sum_{k\in C} \phi_i(k; u_i', u_{-i})k = \phi_i(\alpha_i'; u_i', u_{-i})\alpha_i'+\phi_i(\alpha_i'+1; u_i', u_{-i})(\alpha_i'+1)$. Single-peakedness implies $\phi_i(\alpha_i; u)u_i'(\alpha_i)+ \phi_i(\alpha_i + 1; u)u_i'(\alpha_i + 1) > \phi_i(\alpha_i'; u_i', u_{-i})u_i'(\alpha_i') + \phi_i(\alpha_i' + 1; u_i', u_{-i})u_i'(\alpha_i' + 1)$.

This implies that $\sum_{k\in C}\phi_i(k;u)u_i'(k) > \sum_{k\in C}\phi_i(k;u_i',u_{-i})u_i'(k)$.

In both cases, we obtain a contradiction to strategy-proofness. $\square$

Uncompromisingness requires that given u_{-i}, the expected value of agent i's marginal distribution does not change if his/her initial peak is less (more) than his/her initial expected value and his/her new peak is equal to or less (more) than his/her initial expected value. Lemma 5 shows that Pareto optimality and strategy-proofness imply uncompromisingness.

Uncompromisingness. For all $i \in N$, all $u_i, u_i' \in U$, and all $u_{-i} \in U^{n-1}$, if $[p(u_i) < \sum_{k\in C}\phi_i(k;u)k$ and $p(u_i') \leq \sum_{k\in C}\phi_i(k;u)k]$ or $[p(u_i) > \sum_{k\in C}\phi_i(k;u)k$ and $p(u_i') \geq \sum_{k\in C}\phi_i(k;u)k]$, then $\sum_{k\in C}\phi_i(k;u)k = \sum_{k\in C}\phi_i(k;u_i',u_{-i})k$.

Lemma 5. If ϕ satisfies Pareto optimality and strategy-proofness, then it is uncompromising.

Proof of Lemma 5. Suppose the contrary. Then, $[p(u_i) < \sum_{k\in C}\phi_i(k;u)k$ and $p(u_i') \leq \sum_{k\in C}\phi_i(k;u)k]$ or $[p(u_i) > \sum_{k\in C}\phi_i(k;u)k$ and $p(u_i') \geq \sum_{k\in C}\phi_i(k;u)k]$ and $\sum_{k\in C}\phi_i(k;u)k \neq \sum_{k\in C}\phi_i(k;u_i',u_{-i})k$. Without loss of generality, we assume that $p(u_i) < \sum_{k\in C}\phi_i(k;u)k$ and $p(u_i') \leq \sum_{k\in C}\phi_i(k;u)k$. By property B, there exist $\alpha_i, \alpha_i' \in C$ such that $\phi_i(\alpha_i;u) + \phi_i(\alpha_i+1;u) = 1$ and $\phi_i(\alpha_i';u_i',u_{-i}) + \phi_i(\alpha_i'+1;u_i',u_{-i}) = 1$. Hence, we have $\sum_{k\in C}\phi_i(k;u)k = \phi_i(\alpha_i;u)\alpha_i + \phi_i(\alpha_i+1;u)(\alpha_i+1)$ and $\sum_{k\in C}\phi_i(k;u_i',u_{-i})k = \phi_i(\alpha_i';u_i',u_{-i})\alpha_i' + \phi_i(\alpha_i'+1;u_i',u_{-i})(\alpha_i'+1)$.

There are two cases: (i) If $\sum_{k\in C}\phi_i(k;u)k < \sum_{k\in C}\phi_i(k;u_i',u_{-i})k$, then $p(u_i') \leq \phi_i(\alpha_i;u)\alpha_i + \phi_i(\alpha_i+1;u)(\alpha_i+1) < \phi_i(\alpha_i';u_i',u_{-i})\alpha_i' + \phi_i(\alpha_i'+1;u_i',u_{-i})(\alpha_i'+1)$. Single-peakedness implies that $\phi_i(\alpha_i;u)u_i'(\alpha_i) + \phi_i(\alpha_i+1;u)u_i'(\alpha_i+1) > \phi_i(\alpha_i';u_i',u_{-i})u_i'(\alpha_i') + \phi_i(\alpha_i'+1;u_i',u_{-i})u_i'(\alpha_i'+1)$. This implies that $\sum_{k\in C}\phi_i(k;u)u_i'(k) > \sum_{k\in C}\phi_i(k;u_i',u_{-i})u_i'(k)$. (ii) If $\sum_{k\in C}\phi_i(k;u)k > \sum_{k\in C}\phi_i(k;u_i',u_{-i})k$, let $u_i'' \in U$ be such that $p(u_i'') = p(u_i)$ and $\phi_i(\alpha_i';u_i',u_{-i})u_i''(\alpha_i') + \phi_i(\alpha_i'+1;u_i',u_{-i})u_i''(\alpha_i'+1) > \phi_i(\alpha_i;u)u_i''(\alpha_i) + \phi_i(\alpha_i+1;u)u_i''(\alpha_i+1)$. By property B, there exists $\alpha_i'' \in C$ such that $\phi_i(\alpha_i'';u_i'',u_{-i}) + \phi_i(\alpha_i''+1;u_i'',u_{-i}) = 1$. Hence, $\sum_{k\in C}\phi_i(k;u_i'',u_{-i})k = \phi_i(\alpha_i'';u_i'',u_{-i})\alpha_i'' + \phi_i(\alpha_i''+1;u_i'',u_{-i})(\alpha_i''+1)$. By Lemma 4, we have $\sum_{k\in C}\phi_i(k;u)k = \sum_{k\in C}\phi_i(k;u_i'',u_{-i})k$. This implies that $\alpha_i = \alpha_i''$ and $\phi_i(\alpha_i;u) = \phi_i(\alpha_i'';u_i'',u_{-i})$. Hence, we have $\phi_i(\alpha_i;u)u_i''(\alpha_i) + \phi_i(\alpha_i+1;u)u_i''(\alpha_i+1) = \phi_i(\alpha_i'';u_i'',u_{-i})u_i''(\alpha_i'') + \phi_i(\alpha_i''+1;u_i'',u_{-i})u_i''(\alpha_i''+1)$. Therefore, we have $\phi_i(\alpha_i';u_i',u_{-i})u_i''(\alpha_i') + \phi_i(\alpha_i'+1;u_i',u_{-i})u_i''(\alpha_i'+1) > \phi_i(\alpha_i'';u_i'',u_{-i})u_i''(\alpha_i'') + \phi_i(\alpha_i''+1;u_i'',u_{-i})u_i''(\alpha_i''+1)$. This gives us that $\sum_{k\in C}\phi_i(k;u_i',u_{-i})u_i''(k) > \sum_{k\in C}\phi_i(k;u_i'',u_{-i})u_i''(k)$.

In both cases, we obtain a contradiction to strategy-proofness. $\square$

Proof of the Theorem. The randomized uniform rule obviously satisfies equal probability for the best. Since it satisfies same-sidedness and property B, Lemma 3 implies that it satisfies Pareto optimality. An argument similar to Sprumont (1991) can be used to show strategy-proofness.

Let ϕ be a randomized rule satisfying strategy-proofness, Pareto optimality, and equal probability for the best. Suppose the contrary. Then, $\phi(u)$ violates (i) and (ii) of Definition 1 for some $u \in U^n$. Hence, we have $(\phi_1(u), \ldots, \phi_n(u)) \neq (\Phi_1(u), \ldots, \Phi_n(u))$, or equivalently there exists some agent $h \in N$ such that $\phi_h(u) \neq \Phi_h(u)$. This implies that $\sum_{k \in C} \phi_h(k; u)k \neq \sum_{k \in C} \Phi_h(k; u)k$ since each of $\sum_{k \in C} \phi_h(k; u)k$ and $\sum_{k \in C} \Phi_h(k; u)k$ is obtained as a convex combination of two integers by property B and Definition 1 respectively. Without loss of generality, we assume that $\sum_{j \in N} p(u_j) \geq X$ and $p(u_1) \leq \cdots \leq p(u_n)$.

Suppose $u = (u_n, \ldots, u_n)$. Note that $\sum_{i \in N} p(u_i) = n \cdot p(u_n)$. By property B, for any agent $i \in N$, there exists $\alpha_i \in C$ with $\phi_i(\alpha_i + 1; u) > 0$ such that $\phi_i(\alpha_i; u) + \phi_i(\alpha_i + 1; u) = 1$. From same-sidedness, we have $p(u_n) = p(u_i) \geq \alpha_i + 1 > \alpha_i$ for all $i \in N$. Hence, the best for agent i is $\alpha_i + 1$. Therefore, equal probability for the best gives us that for any $i, j \in N, \alpha_i = \alpha_j, \phi_i(\alpha_i; u) = \phi_j(\alpha_j; u)$, and $\phi_i(\alpha_i + 1; u) = \phi_j(\alpha_j + 1; u)$. Hence, $\sum_{k \in C} \phi_i(k; u)k = \phi_i(\alpha_i; u)\alpha_i + \phi_i(\alpha_i + 1; u)(\alpha_i + 1) = \phi_j(\alpha_j; u)\alpha_j + \phi_j(\alpha_j + 1; u)(\alpha_j + 1) = \sum_{k \in C} \phi_j(k; u)k$ for any $i, j \in N$. This implies that $\sum_{k \in C} \phi_h(k; u)k = X/n$ from feasibility. On the other hand, the expected value of the marginal distribution induced from the randomized uniform rule $\sum_{k \in C} \Phi_i(k; u)k$ is equal to $(\lambda - \mu)(\mu + 1) + (1 - \lambda + \mu)\mu = \lambda$ for all $i \in N$ when $\sum_{i \in N} p(u_i) > X$. When $\sum_{j \in N} p(u_j) = X$, any agent $i \in N$ receives $p(u_n) = \lambda$ with probability 1. Hence, feasibility implies that $\sum_{k \in C} \Phi_h(k; u)k = X/n$. These contradict $\sum_{k \in C} \phi_h(k; u)k \neq \sum_{k \in C} \Phi_h(k; u)k$. If $u \neq (u_n, \ldots, u_n)$, we proceed as follows.

Step 1. Since $\sum_{k \in C} \phi_h(k; u)k \neq \sum_{k \in C} \Phi_h(k; u)k$, there exists $m \in N$ (perhaps $m = h$) such that $\sum_{k \in C} \phi_m(k; u)k > \sum_{k \in C} \Phi_m(k; u)k$ (otherwise, we have $\sum_{k \in C} \phi_i(k; u)k \leq \sum_{k \in C} \Phi_i(k; u)k$ for all $i \in N$. If $\sum_{k \in C} \phi_h(k; u)k > \sum_{k \in C} \Phi_h(k; u)k$, then we have a contradiction. If $\sum_{k \in C} \phi_h(k; u)k < \sum_{k \in C} \Phi_h(k; u)k$, then $X = \sum_{i \neq h} \sum_{k \in C} \phi_i(k; u)k + \sum_{k \in C} \phi_h(k; u)k < \sum_{i \neq h} \sum_{k \in C} \Phi_i(k; u)k + \sum_{k \in C} \Phi_h(k; u)k = X$ from feasibility, which is a contradiction). Same-sidedness implies that $p(u_m) \geq \sum_{k \in C} \phi_m(k; u)k$. Let $u'_m \equiv u_n$. Note that $p(u'_m) = p(u_n) \geq p(u_m)$ and $\sum_{i \neq m} p(u_i) + p(u'_m) \geq \sum_{i \in N} p(u_i) \geq X$. By Lemma 4, $p(u'_m) \geq p(u_m)$ implies that $\sum_{k \in C} \phi_m(k; u'_m, u_{-m})k \geq \sum_{k \in C} \phi_m(k; u)k$. By Definition 1, $\sum_{k \in C} \Phi_m(k; u'_m, u_{-m})k = \sum_{k \in C} \Phi_m(k; u)k$. This is because agent m obtains not his/her own peak with certainty but μ

and $\mu+1$ with the common probabilities when his/her utility function is u_m since $p(u_m) \geq \sum_{k\in C}\phi_m(k;u)k > \sum_{k\in C}\Phi_m(k;u)k$. Altogether, we have $\sum_{k\in C}\phi_m(k;u'_m,u_{-m})k > \sum_{k\in C}\Phi_m(k;u'_m,u_{-m})k$.

Suppose $(u'_m,u_{-m}) = (u_n,\dots,u_n)$. Note that $\sum_{i\neq m}p(u_i)+p(u'_m) = n\cdot p(u_n)$. By property B, for all $i\in N$ there exists $\alpha'_i\in C$ with $\phi_i(\alpha'_i+1;u'_m,u_{-m})>0$ such that $\phi_i(\alpha'_i;u'_m,u_{-m})+\phi_i(\alpha'_i+1;u'_m,u_{-m})=1$. Since $\sum_{i\neq m}p(u_i)+p(u'_m)\geq X$, same-sidedness implies that $p(u_i)\geq\alpha'_i+1>\alpha'_i$ for all $i\neq m$ and $p(u'_m)\geq\alpha'_m+1>\alpha'_m$. Hence, the best for agent i is α'_i+1. Therefore, equal probability for the best gives us that for any $i,j\in N, \alpha'_i=\alpha'_j, \phi_i(\alpha'_i;u'_m,u_{-m})=\phi_j(\alpha'_j;u'_m,u_{-m})$, and $\phi_i(\alpha'_i+1;u'_m,u_{-m})=\phi_j(\alpha'_j+1;u'_m,u_{-m})$. Hence, we have $\sum_{k\in C}\phi_i(k;u'_m,u_{-m})k=\phi_i(\alpha'_i;u'_m,u_{-m})\alpha'_i+\phi_i(\alpha'_i+1;u'_m,u_{-m})(\alpha'_i+1)=\phi_j(\alpha'_j;u'_m,u_{-m})\alpha'_j+\phi_j(\alpha'_j+1;u'_m,u_{-m})(\alpha'_j+1)=\sum_{k\in C}\phi_j(k;u'_m,u_{-m})k$ for any $i,j\in N$. This implies that $\sum_{k\in C}\phi_m(k;u'_m,u_{-m})k = X/n$ from feasibility. On the other hand, the expected value of the marginal distribution induced from the randomized uniform rule $\sum_{k\in C}\Phi_i(k;u'_m,u_{-m})k$ is equal to $(\lambda'-\mu')(\mu'+1)+(1-\lambda'+\mu')\mu'=\lambda'$ for all $i\in N$ when $\sum_{i\neq m}p(u_i)+p(u'_m)>X$. When $\sum_{i\neq m}p(u_i)+p(u'_m)=X$, any agent $i\in N$ receives $p(u_n)=\lambda'$ with probability 1. Hence, feasibility implies that $\sum_{k\in C}\Phi_m(k;u'_m,u_{-m})k=X/n$. These contradict $\sum_{k\in C}\phi_m(k;u'_m,u_{-m})k > \sum_{k\in C}\Phi_m(k;u'_m,u_{-m})k$. If $(u'_m,u_{-m})\neq(u_n,\dots,u_n)$, we proceed to Step 2.

Step 2. Since $u'_m=u_n$, equal probability for the best gives us $\alpha'_m=\alpha'_n, \phi_m(\alpha'_m;u'_m,u_{-m})=\phi_n(\alpha'_n;u'_m,u_{-m})$, and $\phi_m(\alpha'_m+1;u'_m,u_{-m})=\phi_n(\alpha'_n+1;u'_m,u_{-m})$. Hence, we have $\sum_{k\in C}\phi_m(k;u'_m,u_{-m})k=\sum_{k\in C}\phi_n(k;u'_m,u_{-m})k$. Definition 1 gives us that $\sum_{k\in C}\Phi_m(k;u'_m,u_{-m})k=\sum_{k\in C}\Phi_n(k;u'_m,u_{-m})k$. Hence, we have $\sum_{k\in C}\phi_i(k;u'_m,u_{-m})k>\sum_{k\in C}\Phi_i(k;u'_m,u_{-m})k$ for $i=m,n$ (perhaps $m=n$), which gives us that there exists $l\neq m,n$ such that $\sum_{k\in C}\phi_l(k;u'_m,u_{-m})k<\sum_{k\in C}\Phi_l(k;u'_m,u_{-m})k$. (otherwise, we have that $\sum_{k\in C}\phi_i(k;u'_m,u_{-m})k\geq\sum_{k\in C}\Phi_i(k;u'_m,u_{-m})k$ for all $i\in N$. Then, we have $X=\sum_{i\neq m,n}\sum_{k\in C}\phi_i(k;u'_m,u_{-m})k+\sum_{i=m,n}\sum_{k\in C}\phi_i(k;u'_m,u_{-m})k > \sum_{i\neq m,n}\sum_{k\in C}\Phi_i(k;u'_m,u_{-m})k+\sum_{i=m,n}\sum_{k\in C}\Phi_i(k;u'_m,u_{-m})k=X$ from feasibility, which is a contradiction). Definition 1 gives us $\sum_{k\in C}\Phi_l(k;u'_m,u_{-m})k\leq p(u_l)$. Let $u'_l\equiv u_n$. Note that $p(u'_l)=p(u_n)\geq p(u_l)$ and $\sum_{i\neq m,l}p(u_i)+p(u'_m)+p(u'_l)\geq\sum_{i\in N}p(u_i)\geq X$. By Lemma 5, $p(u_l)>\sum_{k\in C}\phi_l(k;u'_m,u_{-m})k$ and $p(u'_l)>\sum_{k\in C}\phi_l(k;u'_m,u_{-m})k$ imply that $\sum_{k\in C}\phi_l(k;u'_m,u'_l,u_{-m,l})k=\sum_{k\in C}\phi_l(k;u'_m,u_{-m})k$. By Definition 1, we have $\sum_{k\in C}\Phi_l(k;u'_m,u'_l,u_{-m,l})k\geq\sum_{k\in C}\Phi_l(k;u'_m,u_{-m})k$. Hence, we have that $\sum_{k\in C}\phi_l(k;u'_m,u'_l,u_{-m,l})k<\sum_{k\in C}\Phi_l(k;u'_m,u'_l,u_{-m,l})k$.

Suppose $(u'_m,u'_l,u_{-m,l})=(u_n,\dots,u_n)$. Then, the same argument as the latter part of Step 1 leads to a contradiction. If $(u'_m, u'_l, u_{-m,l}) \neq (u_n, \ldots, u_n)$, we apply Step 1 to the utility profile $(u'_m, u'_l, u_{-m,l})$.

At each step, we replace the utility function of an agent by u_n. Since n is finite, such replacements lead to a contradiction. $\square$