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Abstract

We study the Full Bayesian Updating rule for convex capacities. Following a route suggested by Jaffray (1992), we define some properties one may want to impose on the updating process, and identify the classes of (convex and strictly positive) capacities that satisfy these properties for the Full Bayesian Updating rule. This allows us to characterize two parametric families of convex capacities: $(\varepsilon, \delta)-$contaminations (which were introduced, in a slightly different form, by Huber (1981)) and $\varepsilon-$contaminations.

1 Introduction

Non additive measures (capacities) have proved to be useful for (i) describing the available information in situations of uncertainty (statistical perspective) and (ii) representing individuals' behavior in situations of complete uncertainty, especially through Schmeidler's (1989) Choquet Expexted Utility model (decision-theoretic perspective). However, although a large body of research has been successfully devoted to the investigation of the properties of capacities, there is still no consensus on how they should be updated as new information arrives. This question is crucial from a statistical and a decision-theoretic point of view. On one hand, as pointed out by Gilboa and Schmeidler (1993), "it may be viewed as the problem statistical inference is trying to solve" (p.35); on the other hand, a consistent theory of updating is needed in order to use the Choquet Expected Utility model in a dynamic setting.

As noted by Eichberger, Grant, and Kelsey (2007), there have been two main approaches to this problem in the literature. The first takes the statistical point of view, and focus on the effects that different updating rules may have on capacities (see e.g. Dempster (1967), Dempster (1968), Wasserman and Kadane (1990), Fagin and Halpern (1991), Walley (1991), Jaffray (1992)). The second approach aims at deriving updating rules from patterns of individuals' unconditional and conditional preferences (see, e.g., Gilboa and Schmeidler (1993), Eichberger, Grant, and Kelsey (2007), Wang (2003)). Both approaches have delivered important results. It is thus surprising that – to the best of our knowledge – these two strands of literature remained separated. This can be explained, of course, by the fact that capacities viewed as a way to represent objective information, and capacities used to represent a decision maker's subjective attitudes towards uncertainty are very different from a conceptual point of view. Nevertheless, one may expect some gain from confronting the statistical and the decision-theoretic approach.

This suggests a third and complementary approach, initiated by Jaffray (1992), that has received little attention. Assume that we want (i) to use an *a priori* given upating rule and (ii) that the updating process satisfies some desirable properties. It might be the case that the updating rule does not satisfy these properties for *all* capacities. Does this mean that we should abandon the updating rule? A less extreme solution consists in identifying the domain on which the rule can be safely applied. We will then have a coherent theory of updating, on a smaller domain. Note that, actually, classical updating rules of capacities only apply to particular (namely, convex) capacities. Assume that, moreover, we are able to provide a behavorial characterization (say, for uncertainty averse Choquet Expected Utility maximizers) of decision makers that use this updating rule in such a way that it always satisfy the desirable properties. This would force the capacities used in the representation of the decision maker's preference to belong to a specific class. We would thus obtain a more precise representation of individuals' preference, that might be tractable for applications.

This is precisely the route we follow in this paper. In the first part of the paper, we focus on the so-called Full Bayesian Updating rule (FBU) proposed, among others, by Dempster (1967). To any such capacity, one can associate its core, which is the set of probability measures that dominate the capacity. A convex capacity is the lower envelope of its core. In this sense, a convex capacity *represents* its core. The FBU rule consists in first updating all the probability measures in the core of the unconditional capacity, and then taking the lower envelope of the updated core. This defines a new capacity, which is the updated capacity. A natural question is whether the updated capacity represents the updated core. If such is the case for all non-null event, we say that the capacity satisfies the *regular updating property for the FBU rule*. Observe that this property is important insofar it implies that the updating process does not entail any loss of information. Jaffray (1992) showed that this property is not always satisfied, and identified necessary and sufficient conditions for beliefs to satisfy it. We go a step further, and identify the subset of strictly positive and convex capacities which satisfy this property. If the state space is finite, the set of regular strictly positive and convex positive capacities is the set of $(\varepsilon, \delta)-$contaminations, a parametric class of capacities introduced by Huber (1981) in a slightly different form, and that have, to the best of our knwoledge, not been studied from a decision theoretic point of view. Moreover any $(\varepsilon, \delta)-$contaminations satisfies the regular updating property for the FBU rule. If the state space is infinite countable, the set of strictly positive, convex and regular capacities reduces to the well-known $\varepsilon-$contaminations (which are special cases of $(\varepsilon, \delta)-$contaminations).

As we noticed, there is no consensus on how capacities should be updated.

Beside the FBU rule, the most popular updating rule is probably the so called Dempster-Shafer updating (DSU) rule, that has initially be proposed for beliefs (see e.g. Dempster (1967), Dempster (1968), Shafer (1976)), but can be extended to convex capacities (see Gilboa and Schmeidler (1993)). We may want *not to choose* between the DSU and the FBU rules. We will say that a convex capacity satisfies the *Dempster-Shafer Consistency property for the FBU rule* if for any non-null event, the updated capacities obtained by applying the FBU and the DSU rules coincide. It turns out that, if the state space is finite, the set of strictly positive, convex and Dempster-Shafer Consistent capacities is the set of $\varepsilon-$contaminations.

The rest of the paper is organized as follows. Notations and preliminary results are presented in section 2. Section 3 is devoted to regular bayesian updating. In Section 4, we identify the set of convex and strictly positive capacities for which full bayesian and Dempster-Shafer updating rules coincides. Section 5 concludes.

2 Setup and preliminaries

When facing uncertainty, it is often the case that the available information is not precise enough to infer a unique probability measure on possible events. Such is the case, for instance, if one relies on large-scale sampling with incomplete information. Another classical example is the collection of probabilistic opinions given by experts to the decision maker. The available information may then be summarized by a subset $\mathscr{P}$ of $\mathbb{P}$,the set of all simply additive probability measures on some measurable space (S, Σ). One can associate to $\mathscr{P}$ its lower envelope ν:

$$\nu(A) = \inf_{P \in \mathscr{P}} P(A), \, \forall A \in \Sigma.$$

Observe that we have $\nu(\emptyset) = 0$, $\nu(S) = 1$ and $\nu(A) \geq \nu(B)$ whenever $B \subseteq A$. Such a set function is called a capacity. Formally,

Definition 1. *A capacity is a mapping $\nu : \Sigma \to [0, 1]$ satisfying $\nu(\emptyset) = 0$, $\nu(S) = 1$ and $\nu(A) \leq \nu(B)$ for all $A, B \in \Sigma$ such that $A \subseteq B$. Moreover, a capacity is said to be:*

(i) convex *if for all $A, B \in \Sigma$, $\nu(A \cup B) + \nu(A \cap B) \geq \nu(A) + \nu(B)$;*

(ii) *a* belief function *if for all $n \geq 2$ and $A_1, \dots, A_n \in \Sigma$,*

$$\nu\left(\bigcup_{i=1}^{n} A_i\right) \geq \sum_{\{I|\emptyset \neq I \subseteq \{1,\dots,n\}\}} (-1)^{|I|+1} \nu\left(\bigcap_{i \in I} A_i\right);$$

(iii) strictly positive *if for all $A \in \Sigma$, $A \neq \emptyset$, $A \neq S$, $1 > \nu(A) > 0$* [1]*;*

(iv) weakly lower continuous *if for all $A \neq S$, and any non-decreasing sequence $\{A_n\}_n$ such that $A_n \uparrow A$, $\nu(A_n) \uparrow \nu(A)$.*

Conversely, one can associate to the capacity ν a (possibly empty) set of probability measures, called the *core* of ν:

$$\text{core}(\nu) = \{P \in \mathbb{P} | P(A) \geq \nu(A), \forall A \in \Sigma\}.$$

In general, a set of probability measures $\mathscr{P}$ is not representable by its lower envelope ν, that is core$(\nu) \neq \mathscr{P}$ (see, e.g., Huber (1981)). Nevertheless, in many situations, ν represents $\mathscr{P}$. Such is the case, for instance, if $\mathscr{P}$ is compatible with data generated by a random set. Actually, such set of probability measures can even be represented by a belief. More generaly, any convex capacity represents its core.

A particular class of capacities, called $(\varepsilon, \delta)-$contamination will play a crucial role in the sequel. They are defined as follows.

Definition 2. *A capacity ν on Σ is an $(\varepsilon, \delta)-$contamination of a probability measure $P_0 \in \mathbb{P}$ if $\nu(\emptyset) = 0$, $\nu(S) = 1$ and*

$$\forall A \in \Sigma,\ A \neq S,\ \nu(A) = \max\left((1-\varepsilon)P_0(A) - \delta, 0\right),$$

where $\delta \in [0, 1)$ and $\varepsilon \in [-\delta, 1]$.

Note that usual $\varepsilon-$contaminations are special cases of $(\varepsilon, \delta)-$contaminations, with $\delta = 0$ and $\varepsilon \in [0, 1]$.

Remark 1. *If a capacity ν is an $(\varepsilon, \delta)-$contamination, then ν is convex.*

[1] Note that a convex capacity ν is strictly positive if and only if for all $A \in \Sigma$, $A \neq \emptyset$, $A \neq S$, $\nu(A) > 0$.

Proof. Let ν be an $(\varepsilon, \delta)-$contamination. Observe first that, by construction, $\nu(\emptyset) = 0$, $\nu(S) = 1$ and $0 \leq \nu(A) \leq 1$. We first show that ν is monotone.

Let $A, B \in \Sigma$ be such that $A \subseteq B$. If $\nu(A) = 0$ or $\nu(B) = 1$, we obviously have $\nu(A) \leq \nu(B)$. If $\nu(A) > 0$ and $\nu(B) < 1$, then $\nu(A) = (1-\varepsilon)P_0(A) - \delta \leq (1-\varepsilon)P_0(B) - \delta = \nu(B)$. Thus $\nu(A) \geq \nu(B)$, i.e., ν is monotone.

It remains to check that ν is convex. Let $A, B \in \Sigma$. If $\nu(A) = 0$, then $\nu(A \cap B) = 0$, and monotonicity of ν implies $\nu(A \cup B) \geq \nu(B)$. Therefore we have $\nu(A \cap B) + \nu(A \cup B) = \nu(A \cup B) \geq \nu(A) + \nu(B)$. Assume now that $\nu(A) > 0$ and $\nu(B) > 0$. If either $A = S$ or $B = S$, we obviously have $\nu(A \cap B) + \nu(A \cup B) = \nu(A) + \nu(B)$. If $A \neq S$ and $B \neq S$, we have:

$$\begin{aligned} \nu(A) + \nu(B) &= (1-\varepsilon)P_0(A) - \delta + (1-\varepsilon)P_0(B) - \delta \\ &= (1-\varepsilon)P_0(A \cup B) - \delta + (1-\varepsilon)P_0(A \cap B) - \delta. \end{aligned}$$

Clearly $A \cap B \neq S$, hence $(1-\varepsilon)P_0(A \cap B) - \delta \leq \nu(A \cap B)$. Two cases are possible:

(a) $A \cup B \neq S$. We then have $(1-\varepsilon)P_0(A \cup B) - \delta \leq \nu(A \cup B)$.

(b) $A \cup B = S$. We then have $(1-\varepsilon)P_0(A \cup B) - \delta = 1 - \varepsilon - \delta \leq 1 = \nu(A \cup B)$.

Hence, we have for all $A, B \in \Sigma$, $\nu(A \cup B) + \nu(A \cap B) \geq \nu(A) + \nu(B)$, i.e., ν convex. □

Because an $(\varepsilon, \delta)-$contaminations is convex, it represents its core. Let us define:

Definition 3. *An $(\varepsilon, \delta)-$contamination of a probability measure $P_0 \in \mathbb{P}$ is the set*

$$\mathscr{P}(P_0, \varepsilon, \delta) = \{P \in \mathbb{P} | P(A) \geq (1-\varepsilon)P_0(A) - \delta\},$$

where $\delta \in [0, 1)$ and $\varepsilon \in [-\delta, 1]$.

One can easily check that:

Remark 2. *If a capacity ν is an $(\varepsilon, \delta)-$contamination, then* $\mathrm{core}(\nu) = \mathscr{P}(P_0, \varepsilon, \delta)$.

Remark 3. *If a strictly positive capacity ν is an $(\varepsilon, \delta)-$ contamination, then $\nu(A) = (1-\varepsilon)P_0(A) - \delta$ for all $A \in \Sigma \setminus \{\emptyset, S\}$.*

Remark 4. *If S is finite, with $|S| \geq 2$, a strictly positive $(\varepsilon, \delta)-$contamination is characterized by P_0 strictly positive, $\delta \in [0, \frac{\alpha}{1-\alpha})$ where $\alpha = \min_{s \in S} P_0(\{s\})^2$, and $\varepsilon \in \left[-\delta, 1 - \frac{\delta}{min_{s \in S} P_0(\{s\})}\right)$.*

3 Bayesian Regularity

We will focus, in this section, on the properties of a particular updating rule for capacities – the so-called Full Bayesian Updating (FBU) rule –, from a statistical point of view. Assume we want (i) to use the FBU rule and (ii) that the updating process satisfies some natural properties. It might be the case that the FBU rule does not satisfy these properties for *all* capacities. Then, requiring (i) and (ii) will imply some constraints on the domain on which the FBU rule should be applied. The question we address is: can we precisely identify the domain on which the FBU rule can be safely applied (with respect to some desirable properties)?

Let S be a nonempty set of states of the world, and $\Sigma = 2^S$ be an algebra of events on it. We assume that $|S| \geq 4$, where $|A|$ denotes the cardinal of set A. Given an event E, we denote by Σ^E the subalgebra $\Sigma^E = \{A \in \Sigma | A \subseteq E\}$, and by $\mathbb{P}^E$ the set of probability measures on (E, Σ^E). Assume that the initial available information can be represented by a convex capacity ν on Σ, and that we learn some event $E \subset S$. We thus have to update our information. Several update rules exist. A very natural one is the so-called Full Bayesian Update rule, that consists in apllying the classical bayesian update rule to each probability measure that belongs to the core of ν, and then considering the lower envelope of the resulting set. Formally, for all probability measure P on (S, Σ) and $E \in \Sigma$ be an event such that $P(E) > 0$. We denote by P^E the conditional probability measure P given E, defined as:

$$P^E(A) = \frac{P(A)}{P(E)}, \forall A \in \Sigma^E.$$

We extend this definition to sets of probability measures as follows. Let $\mathscr{P}$ be a set of probability measures and $E \in \Sigma$ be such that $P(E) > 0$ for all $P \in \mathscr{P}$.

[2]Note that necessarily $\frac{\alpha}{1-\alpha} \leq 1$.

We denote by $\mathscr{P}^E$ the conditional probability set $\mathscr{P}$ given E, defined as:

$$\mathscr{P}^E = \{P^E \in \mathbb{P}^E | P \in \mathscr{P}\}.$$

We will refer to $\mathscr{P}^E$ as the *full bayesian update* of $\mathscr{P}$ given E.

Let ν be a convex capacity, and $E \in \Sigma$ be such that $\nu(E) > 0$, i.e., such that $P(E) > 0$ for all $P \in \text{core}(\nu)$. The *full bayesian update* of ν given E is the capacity ν^E defined as:

$$\nu^E(A) = \inf\{P^E(A) | P \in \text{core}(\nu)\}, \, \forall A \in \Sigma^E.$$

Such a capacity can also be written as:

$$\nu^E(A) = \frac{\nu(A)}{\nu(A) + 1 - \nu(A \cup E^c)}, \, \forall A \in \Sigma^E,$$

where $E^c = S \setminus E$. This rule has been studied, among others, by Fagin and Halpern (1991), Jaffray (1992), Wasserman and Kadane (1990). Note that, as proved by several authors (see e.g. Chateauneuf and Jaffray (1995)), ν^E is convex. As shown by Jaffray (1992), it is in general *not* the case that ν^E represents $(\text{core}(\nu))^E$, as shown by the following example provided in Jaffray (1992)).

Example 1. *Let* $S = \{1,2,3,4\}$, $E = \{1,2,3\}$ *and* $\mathscr{P} = \{p = (\frac{1}{12}, \frac{1}{4}, \frac{2}{3} - \alpha, \alpha), \, \alpha \in [0, \frac{2}{3}]\}$. *Define the capacity* ν *as follows:* $\nu(A) = \inf_{p \in \mathscr{P}} p(A)$. *Then* $(\text{core}(\nu))^E = \{p \in \Delta(\{1,2,3\}) | p = (\alpha, 3\alpha, 1 - 4\alpha), \, \frac{1}{12} \le \alpha \le \frac{1}{4}\}$ *and* $core(\nu)^E = \{p \in \Delta(\{1,2,3\}) | p(\{1\}) \ge \frac{1}{12}, \, p(\{2\}) \ge \frac{1}{4}, \, p(\{1,3\}) \ge \frac{1}{4}, \, p(\{2,3\}) \ge \frac{3}{4}\}$. *The following picture shows* $\mathscr{P}^E = core(\nu)^E$ *(in blue) and* $\text{core}(\nu^E)$ *(in green).*

In other words, applying the FBU rule to a convex capacity (which represents its core) may entail some lost of information. If such is not the case, that is, if ν^E represents $(\text{core}(\nu))^E$ for any non-null event E, we will say that the updating is *regular*. Regularity is clearly a desirable property for an updating rule. For future reference, we state formally this property.

Property 1 (Regular updating)**.** *A convex capacity* ν *is said to satisfy the*

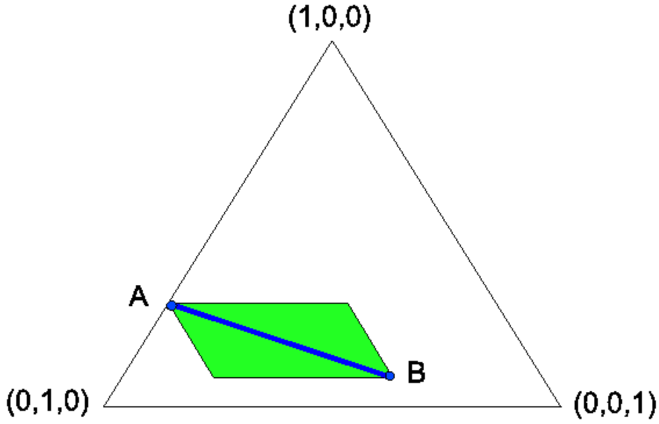

Figure 1: $\text{core}(\nu)^E \subsetneq core(\nu^E)$

regular updating property for FBU rule *if for all* $E \in \Sigma$ *such that* $\nu(E) > 0$*,*

$$\text{core}(\nu^E) = (\text{core}(\nu))^E.$$

We will try, in the sequel, to identify the class of strictly positive and convex capacities that can be regularly updated by the FBU rule. We will succesively consider the case of a finite and an infinite countable state-space.

3.1 The finite case

Jaffray (1992) has shown that when S is finite, a belief function ν satisfies the regular updating property iff

$$\left[\begin{array}{l} A, B \in \Sigma \setminus \{\emptyset, S\} \\ 0 < \nu(A \cap B),\, \nu(A \cup B) < 1 \end{array}\right] \Rightarrow [\nu(A \cap B) + \nu(A \cup B) = \nu(A) + \nu(B)]\,. \tag{$\star$}$$

We first show that this results actually holds true for any convex capacity when S is finite.

Proposition 1. *Let S be finite. Then a convex capacity ν on (S, Σ) satisfies the regular updating property for the FBU rule iff condition $(\star)$ holds.*

Proof. See the Appendix. □

A consequence of the regular updating property is given by the following corollary, which is a generalization of a result proved in Jaffray (1992) for belief functions.

Corollary 1. *Let S be finite, and ν be a convex capacity defined on Σ. The following statements are equivalent:*

(i) *ν satisfies the regular updating property for the FBU rule;*

(ii) *For all $E_1, E_2 \in \Sigma$ such that $\nu(E_1 \cap E_2) > 0$, $\nu^{E_1 \cap E_2} = \left(\nu^{E_1}\right)^{E_2}$*

Proof. See the Appendix. □

Is it possible to provide a characterization of the set of capacities that satisfy condition $(\star)$? The following proposition provides an answer in the case where S is finite and one focus on strictly positive convex capacities.

Proposition 2. *Let S be finite, and ν be a strictly positive convex capacity defined on Σ. The following statements are equivalent:*

(i) *Condition $(\star)$ holds;*

(ii) *ν satisfies the regular updating property for the FBU rule;*

(iii) *ν is an $(\varepsilon, \delta)-$contamination.*

Remark 5. *Eichberger, Grant, and Lefort (2009) independently proposed a characterization of convex capacities for which Condition $(\star)$ holds, under the additional assumption that the capacity satisfies $\nu(A) = 0$ iff $\nu(A^c) = 1$. Considering a different domain, they obtained a different characterization.*

Proof. See the Appendix. □

Thus, the only strictly positive capacities that satisfy the regular updating property for the FBU rule are $(\varepsilon, \delta)-$contaminations. A natural question is then: how looks the full bayesian update of a strictly positive $(\varepsilon, \delta)-$contamination? The answer is: it is a strictly positive $(\varepsilon, \delta)-$contamination.

Proposition 3. *If ν is a strictly positive $(\varepsilon, \delta)-$contamination, then ν^E is also a strictly positive $(\varepsilon, \delta)-$contamination.*

Proof. Let ν be a strictly positive $(\varepsilon,\delta)-$contamination, i.e., $\nu(\{s\}) = (1-\varepsilon)P_0(\{s\}) - \delta > 0$, for all $s \in S$. Observe that this implies $P_0(E) > 0$ for all $E \neq \emptyset$. Let $E \in \Sigma$ be such that $E \neq \emptyset$, $E \neq \{s\}$ and $E \neq S$ (the cases $E = \{s\}$ and $E = S$ are trivial). For all $A \neq \emptyset$, $A \subseteq E$, we have:

$$\begin{aligned}
\nu^E(A) &= \frac{\nu(A)}{\nu(A) + 1 - \nu(A \cup E^c)} \\
&= \frac{(1-\varepsilon)P_0(A) - \delta}{(1-\varepsilon)P_0(A) - \delta + 1 - (1-\varepsilon)P_0(A \cup E^c) + \delta} \\
&= \frac{(1-\varepsilon)P_0(A) - \delta}{(1-\varepsilon)P_0(E) + \varepsilon} \\
&= \left(1 - \frac{\varepsilon}{\varepsilon + (1-\varepsilon)P_0(E)}\right)\frac{P_0(A)}{P_0(E)} - \frac{\delta}{\varepsilon + (1-\varepsilon)P_0(E)}.
\end{aligned}$$

Therefore, $\nu^E(A) = (1-\varepsilon')P_0^E(A) - \delta'$, with $\varepsilon' = \frac{\varepsilon}{\varepsilon+(1-\varepsilon)P_0(E)}$ and $\delta' = \frac{\delta}{\varepsilon+(1-\varepsilon)P_0(E)}$.

It is straightforward to check that ν^E is strictly positive. It thus remains to check that $\delta' \in [0,1)$ and $\varepsilon' \in [-\delta', 1]$.

Since $1 \geq (1-\varepsilon)P_0(E) + \varepsilon > 0$, we have $\delta' \geq \delta$, hence $\delta' \geq 0$, and $\varepsilon' \geq -\delta'$. Because ν^E is strictly positive, we have $(1-\varepsilon')P_0^E(\{S\}) - \delta' > 0$ for all $s \in E$. Therefore $\varepsilon' < 1 - \frac{\delta'}{\min_{s \in E} P_0^E(\{s\})}$, and thus $\varepsilon' < 1$.

It remains to check that $\delta' < 1$. Let us first assume that $|E| \geq 3$. Consider $s_1, s_2 \in E$ with $s_1 \neq s_2$. Because ν^E is a strictly positive capacity, we have $\nu^E(\{s_1,s_2\}) - \nu^E(\{s_1\}) - \nu^E(\{s_2\}) < 1$. But $\nu^E(\{s_1,s_2\}) - \nu^E(\{s_1\}) - \nu^E(\{s_2\}) = (1-\varepsilon')P_0^E(\{s_1,s_2\}) - (1-\varepsilon')P_0^E(\{s_1\}) - (1-\varepsilon')P_0^E(\{s_2\}) + \delta'$. Hence $\nu^E(\{s_1,s_2\}) - \nu^E(\{s_1\}) - \nu^E(\{s_2\}) = \delta'$ and therefore $\delta' < 1$.

Assume finally that $|E| = 2$, and let $E = \{s_1, s_2\}$. Since ν^E is strictly positive, we have $(1-\varepsilon')\min_{s_i} P_0^E(\{s_i\}) - \delta' > 0$. Hence $\varepsilon' < 1 - \frac{\delta'}{\alpha'}$, where $\alpha' = \min_i P_0^E(\{s_i\})$. Observe that $-\delta' \leq \varepsilon'$ implies $-\delta' < 1 - \frac{\delta'}{\alpha'}$, and thus $\delta' < \frac{\alpha'}{1-\alpha'}$. But $\frac{\alpha'}{1-\alpha'} \leq 1$, i.e., $\alpha' \leq \frac{1}{2}$, and therefore $\delta' < 1$, which completes the proof. □

Note that Proposition 2 is not true if ν is not stricly positive, as shown by the following example.

Example 2. *Let $S = \{1,2,3,4\}$ and ν be defined on 2^S by: $\nu(\{1,2\}) = \nu(\{3,4\}) = \nu(\{1,2,3\}) = \nu(\{2,3,4\}) = \nu(\{1,3,4\}) = \nu(\{1,2,4\}) = \frac{1}{8}$ and $\nu(A) = 0$ otherwise. It is easy to check that ν is convex and satisfies condition*

$(\star)$. *However, ν cannot be an $(\varepsilon, \delta)-$contamination. Indeed, if it were the case, we should have:*

$$\nu(\{1,2\}) = \frac{1}{8} = (1-\varepsilon)P_0(\{1,2\}) - \delta.$$

Therefore, $\varepsilon \neq 1$.

Since $\nu(\{1,2\}) = \nu(\{1,2,3\}) > 0$, one gets $P_0(\{3\}) = 0$. A similar argument applied to $\nu(\{1,2\}) = \nu(\{1,2,4\})$, $\nu(\{3,4\}) = \nu(\{1,2,4\})$ and $\nu(\{3,4\}) = \nu(\{2,3,4\})$ gives $P_0(\{4\}) = P_0(\{1\}) = P_0(\{2\}) = 0$, hence $P_0 = 0$, which is impossible.

This example shows that if we relax the strict positivity assumption, some capacities that are not $(\varepsilon, \delta)-$capacities can be regularly updated by the FBU rule. Although we do not know what is exactly the class of convex capacities satisfying the regularity property for the FBU rule, we can see that it contains at least the set of *all* $(\varepsilon, \delta)-$capacities.

Proposition 4. *Let S be finite. If ν is an $(\varepsilon, \delta)-$contamination, then it satisfies the regular updating property for the FBU rule.*

Proof. By Proposition 1, it is enough to prove that any $(\varepsilon, \delta)-$contamination satisfies condition $(\star)$. Let ν be an $(\varepsilon, \delta)-$contamination with respect to probability measure P_0, and $A, B \in \Sigma \backslash \{\emptyset, S\}$ such that $0 < \nu(A \cap B), \nu(A \cup B) < 1$. Because ν is monotone, we then have $\nu(A) > 0$ and $\nu(B) > 0$. Thus,

$$\begin{aligned}
\nu(A \cup B) + \nu(A \cap B) &= ((1-\varepsilon)P_0(A \cup B) - \delta) + ((1-\varepsilon)P_0(A \cap B) - \delta) \\
&= (1-\varepsilon)P_0(A) + (1-\varepsilon)P_0(B) - (1-\varepsilon)P_0(A \cap B) \\
&\quad -\delta + (1-\varepsilon)P_0(A \cap B) - \delta \\
&= (1-\varepsilon)P_0(A) - \delta + (1-\varepsilon)P_0(B) - \delta \\
&= \nu(A) + \nu(B).
\end{aligned}$$

Therefore, ν satisfies condition $(\star)$. $\square$

3.2 The infinite countable case

We now consider the case of a countable state space. Then, the strict positivity assumption will have important consequences.

Proposition 5. *Let ν be a strictly positive, weakly lower continuous, convex capacity on $2^{\mathbb{N}}$. The following statements are equivalent:*

(i) ν satisfies the regular updating property;

(ii) ν is an $\varepsilon-$contamination with $\varepsilon \in [0, 1)$ and P_0 is $\sigma-$additive.

Proof. See the Appendix □

4 Dempster-Shafer Consistency

If the use of the bayesian rule is uncontroversial when one has to update probability measure, the question of how it should be extended to convex capacities is matter of debate. Some authors (e.g. Dempster (1967), Dempster (1968), Gilboa and Schmeidler (1993), Shafer (1976)) advocated for the so-called Dempster-Shafer updating (DSU) rule[3]. The Dempster-Shafer update of a convex capacity ν with respect to E, with $\nu(E) > 0$ is a capacity ν_E defined on Σ^E by:

$$\nu_E(A) = \frac{\nu(A \cup E^c) - \nu(E^c)}{1 - \nu(E^c)},\ A \in \Sigma^E$$

A property that might be desirable is that the FBU rule coincide with the DSU rule. We thus state the:

Property 2 (Dempster-Shafer Consistency)**.** *A convex capacity ν satisfies the Dempster-Shafer Consistency property for the FBU rule if for all $E \in \Sigma$ such that $\nu(E) > 0$,*

$$\nu^E = \nu_E.$$

Focusing again on strictly positive convex capacities defined on a finite algebra, we have the following characterization of the Dempster-Shafer Property.

Proposition 6. *Let S be finite, and ν be a strictly positive convex capacity on Σ. The following statements are equivalent:*

(i) ν satisfies the Dempster-Shafer Consistency Property for the FBU rule;

[3]Although the DSU rule was initally defined for beliefs, it can be extended to convex capacities.

(ii) ν *is an* $\varepsilon-$*contamination with* $\varepsilon \in [0,1)$.

Proof. That (ii) implies (i) is easily checked. Let us show that (i) implies (ii). Let $A, B \in \Sigma \setminus \emptyset$ be such that $A \cap B = \emptyset$ and $A \cup B \neq S$. Define $E = B^c$. Observe that since $B \neq S$, we have $B^c \neq \emptyset$ and thus $\nu(E) > 0$. Since $A \subseteq E$, we have by assumption

$$\frac{\nu(A)}{\nu(A) + 1 - \nu(A \cup E^c)} = \frac{\nu(A \cup E^c) - \nu(E^c)}{1 - \nu(E^c)},$$

i.e.,

$$\nu(A)(1 - \nu(B)) = (\nu(A) + 1 - \nu(A \cup B))(\nu(A \cup B) - \nu(B)),$$

or equivalently

$$(1 - \nu(A \cup B))(\nu(A \cup B) - \nu(A) - \nu(B)) = 0. \tag{1}$$

Since ν is strictly positive, $\nu(A \cup B) < 1$. Thus equation (1) implies $\nu(A \cup B) = \nu(A) + \nu(B)$. Hence, for all $A \in \Sigma \setminus \{\emptyset, S\}$, $\nu(A) = \sum_{s \in A} \nu(\{s\})$. Define $a = \sum_{s \in S} \nu(\{s\})$. Since ν is strictly positive, $a > 0$. Since ν is convex, $\sum_{s \in S} \nu(\{s\}) \leq 1$. Thus $0 < a \leq 1$. Let $P_0(\{s\}) = \frac{\nu(\{s\})}{a}$ for all $s \in S$. We have $\sum_{s \in S} P_0(\{s\}) = 1$, and therefore P_0 is a strictly positive probability measure on (S, Σ). From equation (1) $\nu(A) = aP_0(A)$ for all $A \neq S$ (this equality is obviously true if $A = \emptyset$). Let $a = 1 - \varepsilon$. Since $0 < a \leq 1$, we have $0 \leq \varepsilon < 1$ and $\nu(A) = (1 - \varepsilon)P_0(A)$ for all $A \neq S$, $\nu(S) = 1$. Thus ν is an $\varepsilon-$contamination with $\varepsilon \in [0,1)$. □

Proposition 6 does not hold true if ν is not strictly positive. For instance, the capacity described in Example 2 is not an $(\varepsilon, \delta)-$contamination, and cannot therefore be an $\varepsilon-$contamination. It satisfies, however, the Dempster-Shafer consistency property.

5 Conclusion

In this paper, we interpreted capacities as representing objective information. This interpretation is standard in robust statistics, and had been used in decision theory by Jaffray (1989). However, Schmeidler (1989) provided a subjective interpretation of capacities. Capacities then embody the decision maker's attitude towards uncertainty. A huge literature had grown since in this direction. It is clear that, if one interprets capacities as representing individuals' preferences, then the updating process is related to dynamic behavior, and specifically to dynamic consistency.

Now, assume that one can provide a behavioral interpretation of regular updating (which is indeed the case, as shown e.g. in Pires (2002)). Then, it is clear that our results could be revisited in a behavioral perspective, and pave the way of a characterization of a parametric Choquet Expected Utility model. A related question would be the behavioral signification of the parameters of $(\varepsilon,\delta)-$capacities. We leave these questions for future research.

Appendix

Proof of Proposition 1

Sufficiency.

Let E be such that $\nu(E) > 0$, and assume that condition $(\star)$ holds. By Proposition 6 in Chateauneuf and Jaffray (1995), ν^E is convex, and thus $\text{core}(\nu^E) \neq \emptyset$. Identify $\text{core}(\nu^E)$ to a subset of the $|E|-$dimensional simplex. Let Π^E be the set of the permutations of all elements of E, and $|E| = m$. Given $\pi = \{\theta_{i_1},\ldots,\theta_{i_m}\} \in \Pi^E$ let $B_k(\pi) = \{\theta_{i_1},\ldots,\theta_{i_k}\}$, for all $k \in \{1,\ldots,m\}$. Let $Q_\pi \in \mathbb{P}^E$ be defined as $Q_\pi(B_k(\pi)) = \nu^E(B_k(\pi))$. We know from Proposition 13 in Chateauneuf and Jaffray (1989)) that $\text{core}(\nu^E)$ is a bounded convex polyhedron with extreme points $\{Q_\pi | \pi \in \Pi\}$.

We have by definition $(\text{core}(\nu))^E \subseteq \text{core}(\nu^E)$. We thus have to show that $\text{core}(\nu^E) \subseteq (\text{core}(\nu))^E$. We know that $(\text{core}(\nu))^E$ is convex (see e.g. Kyburg (1987)). It is thus enough (since $\text{core}(\nu^E)$ is a bounded convex polyhedron) to prove that any extreme point of $\text{core}(\nu^E)$ belongs to $(\text{core}(\nu))^E$.

Fix $\pi \in \Pi^E$. We have to show that $Q_\pi \in (\text{core}(\nu))^E$, i.e., that there exists $P \in \text{core}(\nu)$ (and hence satisfying $P(E) > 0$ since $P(E) \geq \nu(E) > 0$) such that:

$$\frac{P(B_i(\pi))}{P(B_i(\pi)) + 1 - P(B_i(\pi) \cup E^c)} = \frac{\nu(B_i(\pi))}{\nu(B_i(\pi)) + 1 - \nu(B_i(\pi) \cup E^c)}$$

or, equivalently, such that:

$$P(B_i(\pi))(1-\nu(B_i(\pi)\cup E^c)) = \nu(B_i(\pi))(1-P(B_i(\pi)\cup E^c)). \qquad (2)$$

By monotonocity of ν, there exist unique $0 \le i_1 \le i_2 \le m$ such that

$$\begin{cases} \nu(B_i(\pi)) = 0, \text{ for all } 1 \le i \le i_1; \\ \nu(B_i(\pi)) > 0 \text{ and } \nu(B_i(\pi)\cup E^c) < 1 \text{ for all } i_1 < i \le i_2; \\ \nu(B_i(\pi)) > 0 \text{ and } \nu(B_i(\pi)\cup E^c) = 1 \text{ for all } i_2 < i \le m. \end{cases}$$

Observe that if $\nu(B_i(\pi)\cup E^c) = 1$ then $P(B_i(\pi)\cup E^c) = 1$ for all $P \in \text{core}(\nu)$, and thus equation (2) is satisfied for all $P \in \text{core}(\nu)$.

Because ν is convex, Proposition 9 in Chateauneuf and Jaffray (1989) implies that there exists $\tilde{P} \in \text{core}(\nu)$ such that $\tilde{P}(B_i(\pi)) = \nu(B_i(\pi))$ for all $1 \le i \le i_2$ and $\tilde{P}(B_{i_2}(\pi)\cup E^c) = \nu(B_{i_2}(\pi)\cup E^c)$. Note that if $1 \le i \le i_1$, $\tilde{P}(B_i(\pi)) = \nu(B_i(\pi)) = 0$ and thus $\tilde{P}$ satisfies equation (2). Assume now $i_1 < i \le i_2$. Let $A_1 = B_{i_2}(\pi)$ and $A_2 = B_i(\pi)\cup E^c$. This implies $A_1 \cap A_2 = B_i$ and $A_1 \cup A_2 = B_{i_2}(\pi)\cup E^c$. Thus $\nu(A_1\cap A_2) = \nu(B_i) > 0$ and $\nu(A_1\cup A_2) = \nu(B_{i_2}(\pi)\cup E^c) < 1$. Therefore condition $(\star)$ implies

$$\nu(A_1\cap A_2) + \nu(A_1\cup A_2) = \nu(A_1) + \nu(A_2),$$

i.e.,

$$\tilde{P}(B_i) + \tilde{P}(B_{i_2}(\pi)) + \tilde{P}(E^c) = \tilde{P}(B_{i_2}(\pi)) + \nu(B_i(\pi)\cup E^c),$$

which simplifies to: $\tilde{P}(B_i(\pi)\cup E^c) = \nu(B_i(\pi))\cup E^c)$. Therefore $\tilde{P}$ satisfies equation (2).

Necessity. Assume condition $(\star)$ is not satisfied, but that the regular updating property holds. Thus, there exist A_1 and A_2 such that $\nu(A_1\cup A_2) < 1$, $\nu(A_1\cap A_2) > 0$ and $\nu(A_1\cup A_2) + \nu(A_1\cap A_2) > \nu(A_1) + \nu(A_2)$. Let $A = (A_1\cap A_2)\cup A_1^c$. Since ν is convex, we have $\nu(A) \ge \nu(A_1\cap A_2) + \nu(A_1^c) > 0$. Let $B = A_2$ and $C = A_1\cap A_2$. By Proposition 6 in Chateauneuf and Jaffray (1995), ν^A is convex. Thus by Proposition 11 in Chateauneuf and Jaffray (1989) there exists a probability measure $Q \in \text{core}(\nu^A)$ such that $Q(B) = \nu^A(B)$ and $Q(C) = \nu^A(C)$. Assume $Q \in (\text{core}(\nu))^A$. Then there exists a probability measure $P \geq \nu$ such that:

$$\frac{P(B)}{P(B)+1-P(B\cup A^c)} = \frac{\nu(B)}{\nu(B)+1-\nu(B\cup A^c)},$$

i.e.,

$$P(B)(1-\nu(B\cup A^c)) = \nu(B)(1-P(B\cup A^c)).$$

But $P(B) \geq \nu(B)$ and $(1-\nu(B\cup A^c)) \geq (1-P(B\cup A^c))$. Furthermore, $B\cup A^c = A_1 \cup A_2$ and therefore $\nu(B\cup A^c) < 1$, which implies $(1-\nu(B\cup A^c)) > 0$. Thus $P(B) = \nu(B) \geq \nu(A_1\cap A_2) > 0$, from which we deduce $\nu(B\cup A^c) = P(B\cup A^c)$. Hence $P(A_2) = \nu(A_2)$ (because $B = A_2$) and $P(A_1\cup A_2) = \nu(A_1\cup A_2)$ (because $B\cup A^c = A_1\cup A_2$). Replacing B by C in the preceeding argument leads to $P(A_1\cap A_2) = \nu(A_1\cap A_2)$ and $P(A_1) = \nu(A_1)$. Therefore $\nu(A_1\cup A_2)+\nu(A_1\cap A_2) = \nu(A_1)+\nu(A_2)$, and thus condition $(\star)$ is satisfied, which yields a contradiction.

Proof of Proposition 2

The equivalence between (i) and (ii) follows from Proposition 1. It is easy to check that $(iii) \Rightarrow (i)$. We will thus only prove that $(i) \Rightarrow (iii)$.

We first show that there exists $\delta \in [0,1)$ such that for all $A, B \in \Sigma$, $A \neq \emptyset$, $B \neq \emptyset$, $A\cap B = \emptyset$ and $A\cup B \neq S$,

$$\nu(A\cup B) - \nu(A) - \nu(B) = \delta.$$

Let us prove that this is true for singletons. For any distincts r, s, t in S (note that this is possible since $|S| \geq 4$), condition $(\star)$ implies:

$$\begin{cases} \nu(\{r,s\}) + \nu(\{s,t\}) = \nu(\{s\}) + \nu(\{r,s,t\}) \\ \nu(\{r,s\}) + \nu(\{r,t\}) = \nu(\{r\}) + \nu(\{r,s,t\}). \end{cases}$$

Thus:

$$\begin{cases} \nu(\{r,s\}) - \nu(\{s\}) = \nu(\{r,s,t\}) - \nu(\{s,t\}) \\ -\nu(\{r\}) = \nu(\{r,s,t\}) - \nu(\{r,s\}) - \nu(\{r,t\}). \end{cases}$$

Summing up these two equations, we obtain:

$$\nu(\{r,s\})-\nu(\{r\})-\nu(\{s\}) = 2\nu(\{r,s,t\})-\nu(\{r,s\})-\nu(\{s,t\})-\nu(\{r,t\}. \quad (3)$$

Permuting r and t in equation (3) leads to:

$$\nu(\{s,t\})-\nu(\{s\})-\nu(\{t\}) = 2\nu(\{r,s,t\})-\nu(\{r,s\})-\nu(\{s,t\})-\nu(\{r,t\}. \quad (4)$$

Let $u \in S$ be such that $u \notin \{r,s,t\}$ (which is possible, since $|S| \geq 4$). Substituting u to r in equation (4) entails:

$$\nu(\{s,t\})-\nu(\{s\})-\nu(\{t\}) = 2\nu(\{s,t,u\})-\nu(\{s,t\})-\nu(\{t,u\})-\nu(\{u,s\}. \quad (5)$$

Permuting s and u in equation (5), we get:

$$\nu(\{t,u\})-\nu(\{t\})-\nu(\{u\}) = 2\nu(\{s,t,u\})-\nu(\{s,t\})-\nu(\{t,u\})-\nu(\{u,s\}. \quad (6)$$

Finally, equations (3) to (6) yield

$$\nu(\{r,s\}) - \nu(\{r\}) - \nu(\{s\}) = \nu(\{t,u\}) - \nu(\{t\}) - \nu(\{u\}),$$

the desired result.

It remains to show that for all $A, B \in \Sigma$, where A or B is not a singleton, $A \neq \emptyset$, $B \neq \emptyset$, $A \cap B = \emptyset$ and $A \cup B \neq S$, there exist distinct $r, s \in S$ such that $\nu(A \cup B) - \nu(A) - \nu(B) = \nu(\{r,s\}) - \nu(\{r\}) - \nu(\{s\})$.

Observe that for all $C, D \in \Sigma$ such that $S \supsetneq C \supseteq D \supseteq \{t,u\}$, condition $(\star)$ implies $\nu(D \cup (C \setminus \{t\})) + \nu(D \cap (C \setminus \{t\})) = \nu(D) + \nu(C \setminus \{t\})$ or equivalently

$$\nu(C) - \nu(D) = \nu(C \setminus \{t\}) - \nu(D \setminus \{t\}). \quad (7)$$

Let $A, B \in \Sigma$, where without loss of generality A is assumed not to be a singleton, be such that $A \neq \emptyset$, $B \neq \emptyset$, $A \cap B = \emptyset$ and $A \cup B \neq S$, and $r \in A$, $s \in B$. By successive applications of equation (7),

$$\begin{aligned} \nu(A \cup B) - \nu(A) &= \nu\left((A \cup B) \setminus (A \setminus \{r\})\right) - \nu\left(A \setminus (A \setminus \{r\})\right) \\ &= \nu(B \cup \{r\}) - \nu(\{r\}). \end{aligned}$$

Thus:

$$\nu(A \cup B) - \nu(A) - \nu(B) = \nu(B \cup \{r\}) - \nu(B) - \nu(\{r\}). \quad (8)$$

Either $B = \{s\}$ and $\nu(A \cup B) - \nu(A) - \nu(B) = \nu(\{r,s\}) - \nu(\{r\}) - \nu(\{s\})$ or applying equation (7) again, we obtain:

$$\begin{aligned}\nu(B\cup\{r\})-\nu(B) &= \nu\left((B\cup\{r\})\setminus(B\setminus\{s\})\right)-\nu\left(B\setminus(B\setminus\{s\})\right)\\ &= \nu(\{rs\})-\nu(\{s\}).\end{aligned}$$

Substituting this last equality in equation (8) yields:

$$\nu(A\cup B)-\nu(A)-\nu(B)=\nu(\{rs\}-\nu(\{r\})-\nu(\{s\}),$$

the desired result. Hence, since ν is a strictly positive convex capacity, there exists $\delta\in[0,1)$ such that for all $A,B\in\Sigma$, $A\neq\emptyset$, $B\neq\emptyset$, $A\cap B=\emptyset$ and $A\cup B\neq S$,

$$\nu(A\cup B)-\nu(A)-\nu(B)=\delta. \tag{9}$$

For any $A\in\Sigma$, $A\neq\emptyset$, $A\neq S$, define: $Q(A)=\nu(A)+\delta$. For all $A,B\in\Sigma$, $A\neq\emptyset$, $B\neq\emptyset$, $A\cap B=\emptyset$ and $A\cup B\neq S$, equation (9) implies $Q(A\cup B)=Q(A)+Q(B)$. Therefore, for all $A\in\Sigma$, $A\neq\emptyset$, $A\neq S$, $Q(A)=\sum_{s\in A}Q(\{s\})$. Therefore, $Q(A)+Q(A^c)=\sum_{s\in S}Q(\{s\})$. Define: $Q(S)=\sum_{s\in S}Q(\{s\})$. We then have, for all $A,B\in\Sigma$, $A\neq\emptyset$, $B\neq\emptyset$, $A\cap B=\emptyset$, $Q(A\cup B)=Q(A)+Q(B)$. Let $a=\sum_{s\in S}Q(\{s\})$. Because ν is strictly positive, $\nu(\{s\})>0$ for all $s\in S$. Since $\delta\in[0,1)$, we have $Q(\{s\})=\nu(\{s\})+\delta>0$ for all $s\in S$, and therefore $a>0$. Define: $P_0(A)=\frac{1}{a}Q(A)$ for all $A\in\Sigma$, $A\neq\emptyset$ and $P_0(\emptyset)=0$. Clearly, P_0 is a probability measure on (S,Σ), and we have, for all $A\in\Sigma$, $A\neq\emptyset$, $A\neq S$,

$$\nu(A)=aP_0(A)-\delta.$$

We already know that $\delta\in[0,1)$. Let $a=1-\varepsilon$. Because S is finite with $|S|\geq 4$, there exists $A,B\in\Sigma$ such that $A\neq\emptyset$, $A\neq S$, $B\neq S$, $A\cup B=S$ and $A\cap B\neq\emptyset$. From the strict positivity of ν, one obtains:

$$\begin{aligned}\nu(A)+\nu(B) &= (1-\varepsilon)P_0(A)-\delta+(1-\varepsilon)P_0(B)-\delta\\ &= (1-\varepsilon)P_0(A\cup B)-\delta+(1-\varepsilon)P_0(A\cap B)-\delta.\end{aligned}$$

Hence $\nu(A)+\nu(B)-\nu(A\cap B)=1-\varepsilon-\delta$. Therefore, convexity of ν implies $1-\varepsilon-\delta\leq 1$, i.e., $\varepsilon\geq-\delta$. Furthermore, since ν is strictly positive, P_0 is also strictly positive and $(1-\varepsilon)P_0(\{s\})-\delta>0$ for all $s\in S$. Therefore $\varepsilon\in\left[-\delta,1-\frac{\delta}{\min_{s\in S}P_0(\{s\})}\right)$. Thus ν is a strictly positive $(\varepsilon,\delta)-$contamination.

Proof of Corollary 1

We first prove that (i) implies (ii). Let ν be a convex capacity, and let $E_1, E_2 \in \Sigma$ be such that $\nu(E_1 \cap E_2) > 0$. By (i), we have $\text{core}(\nu^(E_1)) = \{P^{E_1} | P \in \text{core}(\nu)\}$. On the other hand, (i) also implies:

$$\begin{aligned} \text{core}\left(\left(\nu^{E_1}\right)^{E_2}\right) &= \left\{P^{E_2} \middle| P \in \text{core}(\nu^{E_1})\right\} \\ &= \left\{P^{E_1 \cap E_2} | P \in \text{core}(\nu)\right\} \\ &= \text{core}\left(\nu^{E_1 \cap E_2}\right). \end{aligned}$$

Thus $\nu^{E_1 \cap E_2} = \left(\nu^{E_1}\right)^{E_2}$.

We now prove that (ii) implies (i). Let us consider $A, B \in \Sigma \setminus \{\emptyset, S\}$ such that (i) is not satisfied, i.e., $0 < \nu(A \cap B), \nu(A \cup B) < 1$ and $\nu(A \cap B) + \nu(A \cup B) > \nu(A) + \nu(B)$.

The proof will be completed if we show that there exist $E_1, E_2 \in \Sigma$ with $E_2 \subset E_1$ such that $\left(\nu^{E_1}\right)^{E_2} \neq \nu^{E_2}$. Define $E_1 = B \cup A^c$, $E_2 = (A \cap B) \cup (A^c \cap B^c)$ and $C = A \cap B$ (thus $C \cup E_1^c = A$, $C \cup (E_1 \setminus E_2) = B$ and $C \cup B^c = A \cup B$).

We have:

$$\begin{aligned} \nu^{E_2}(C) &= \frac{\nu(C)}{\nu(C) + 1 - \nu(C \cup E_2^c)} \\ &= \frac{\nu(A \cap B)}{\nu(A \cap B) + 1 - \nu(A \cup B)}. \end{aligned}$$

Similarly, from:

$$\begin{aligned} \left(\nu^{E_1}\right)^{E_2}(C) &= \frac{\nu^{E_1}(C)}{\nu^{E_1}(C) + 1 - \nu^{E_1}(C \cup (E_1 \setminus E_2))} \\ \nu^{E_1}(C) &= \frac{\nu(C)}{\nu(C) + 1 - \nu(C \cup E_1^c)} \\ \nu^{E_1}(C \cup (E_1 \setminus E_2)) &= \frac{\nu(C \cup (E_1 \setminus E_2))}{\nu(C \cup (E_1 \setminus E_2)) + 1 - \nu(C \cup E_2^c)}, \end{aligned}$$

it follows that:

$$\left(\nu^{E_1}\right)^{E_2}(C) = \frac{\nu(A \cap B)}{\nu(A \cap B) + \frac{[1 - \nu(A \cup B)][\nu(A \cap B) + 1 - \nu(A)]}{\nu(B) + 1 - \nu(A \cup B)}}.$$

From $\nu(A \cap B) + \nu(A \cup B) > \nu(A) + \nu(B)$ it follows that $\left(\nu^{E_1}\right)^{E_2}(C) < \nu^{E_2}(C)$, hence $\left(\nu^{E_1}\right)^{E_2} \neq \nu^{E_2}$, which completes the proof.

Proof of Proposition 5

$[(i) \Rightarrow (ii)]$

The arguments used in the proof of Proposition 2 and the weak lower continuity of ν imply that there exists $\delta \geq 0$, $\varepsilon \in [0,1)$ and a strictly positive $\sigma-$additive probability measure P_0 on $(\mathbb{N}, 2^{\mathbb{N}})$ such that $\nu(A) = (1-\varepsilon)P_0(A) - \delta$ for all $A \neq S$. Since $\nu(\{k\}) = (1-\varepsilon)P_0(\{k\}) - \delta$, $\nu(\{k\}) > 0$ and $\lim_{k\to+\infty} P_0(\{k\}) = 0$, we have $\delta = 0$. Therefore, $\nu(A) = (1-\varepsilon)P_0(A)$, for all $A \neq S$.

$[(ii) \Rightarrow (i)]$ Let ν be an $\varepsilon-$contamination with respect to the strictly positive $\sigma-$additive probability measure P_0, and $\varepsilon \in [0,1)$. It is straightforward to check that the core of ν is:

$$\text{core}(\nu) = \left\{ P \in \mathbb{P} \;\middle|\; \begin{array}{c} P(\{s\}) = (1-\varepsilon)P_0(\{s\}) + \lambda(\mathbb{N}, s)\varepsilon \\ \text{with } \lambda(S,s) \geq 0 \text{ and } \sum_{s\in\mathbb{N}} \lambda(\mathbb{N}, s) = 1 \end{array} \right\}.$$

For all $A \in 2^{\mathbb{N}}$, let $\lambda(\mathbb{N}, A) = \sum_{s\in A} \lambda(\mathbb{N}, s)$.

Let $E \in 2^{\mathbb{N}} \setminus \{\mathbb{N}, \emptyset\}$ and $Q \in \text{core}(\nu^E)$. We have to show that there exists $P \in \text{core}(\nu)$ such that $P^E(A) = Q(A)$ for all $A \subset E$, i.e., $P^E(\{s\}) = Q(\{s\})$ for all $s \in E$.

Note that $\nu^E(\{s\}) = (1-\varepsilon')P_0^E(\{s\})$ for all $s \in E$, with $\varepsilon' = \frac{\varepsilon}{\varepsilon + (1-\varepsilon)P_0(E)}$. Let λ^E be the weight function of Q, i.e., for all $s \in E$,

$$Q(\{s\}) = (1-\varepsilon')P_0^E(\{s\}) + \lambda^E(E,s)\varepsilon'.$$

We have to show that there exists $P \in \text{core}(\nu)$ such that $P^E(\{s\}) = Q(\{s\})$, i.e., that there exists a weight function $\lambda(\mathbb{N}, s)$ such that:

$$Q(\{s\}) = (1-\varepsilon')P_0^E(\{s\}) + \lambda^E(E,s)\varepsilon' = \frac{(1-\varepsilon)P_0(\{s\}) + \lambda(\mathbb{N}, s)\varepsilon}{(1-\varepsilon)P_0(E) + \lambda(\mathbb{N}, E)\varepsilon} = P^E(\{s\}),$$

which is equivalent to

$$\lambda(\mathbb{N}, s)\left((1-\varepsilon)P_0(E) + \varepsilon\right)$$

$$= \lambda(\mathbb{N}, E)\left((1-\varepsilon)P_0(\{s\}) + \varepsilon\lambda^E(E,s) + (1-\varepsilon)(P_0(E)\lambda^E(E,s) - P_0(\{s\})\right). \quad (10)$$

Summing equation (10) on $s \in E$ leads to:

$$\lambda(\mathbb{N}, E)\left((1-\varepsilon)P_0(E) + \varepsilon\right) = \lambda(\mathbb{N}, E)\left((1-\varepsilon)P_0(E) + \varepsilon\right). \quad (11)$$

Thus the only constraint on $\lambda(\mathbb{N}, E)$ is that $0 \leq \lambda(\mathbb{N}, E) \leq 1$. It is thus necessary and sufficient to find λ satisfying $\lambda(\mathbb{N}, s) \geq 0$ for all a $s \in E$ and $0 \leq \lambda(\mathbb{N}, E) \leq 1$. From equation (10) we have:

$$\lambda(\mathbb{N}, s) \geq 0 \Leftrightarrow \lambda(\mathbb{N}, E) \geq \frac{(1-\varepsilon)P_0(\{s\}) - (1-\varepsilon)P_0(E)\lambda^E(E,s)}{(1-\varepsilon)P_0(\{s\}) + \varepsilon\lambda^E(E,s)}, \forall s \in E.$$

Therefore,

$$\lambda(\mathbb{N}, E) \geq \sup_{s\in E} \frac{(1-\varepsilon)P_0(\{s\}) - (1-\varepsilon)P_0(E)\lambda^E(E,s)}{(1-\varepsilon)P_0(\{s\}) + \varepsilon\lambda^E(E,s)}. \quad (12)$$

Note that the right-hand side of equation (12) is less or equal than 1. Indeed, if it were greater than 1, there would exist $s \in E$ such that

$$-(1-\varepsilon)P_0(E)\lambda^E(E,s) > \varepsilon\lambda(E,s),$$

or equivalently

$$\left(\varepsilon + (1-\varepsilon)P_0(E)\right)\lambda^E(E,s) < 0,$$

which is impossible. Thus, choosing arbitrarily $\lambda(\mathbb{N}, E)$ such that equation (12) is satisfied and $0 \leq \lambda(\mathbb{N}, E) \leq 1$ leads to $\lambda(\mathbb{N}, s) \geq 0$ for all $s \in \mathbb{N}$, and therefore, by equation (10) $\sum_{s\in E} \lambda(\mathbb{N}, s) = \lambda(\mathbb{N}, E)$, the desired result.

References

Chateauneuf, A., and J. Jaffray (1989): "Some characterizations of lower probabilities and other monotone capacities through the use of Mobius inversion," *Mathematical Social Sciences*, 17, 263–283.

Chateauneuf, A., and J.-Y. Jaffray (1995): "Local Möbius Transforms on Monotone Capacities," in *ECSQARU*, pp. 115–124.

Dempster, A. P. (1967): "Probabilities induced by a multi-valued mapping," *Annals of Mathematical Statistics*, 38, 325–339.

——— (1968): "A generalization of Bayesian inference. (With discussion)," *Journal of the Royal Statistical Society. Series B. Methodological*, 30, 205–247.

Eichberger, J., S. Grant, and D. Kelsey (2007): "Updating Choquet beliefs," *Journal of Mathematical Economics*, 43(7-8), 888–899.

Eichberger, J., S. Grant, and J.-P. Lefort (2009): "Neo-additive Capacities and Updating," *Working Paper*.

Fagin, R., and Y. Halpern (1991): "A new approach to updating beliefs," in *Uncertainty in Artificial Intelligence 6*, pp. 347–374.

Gilboa, I., and D. Schmeidler (1993): "Updating ambiguous beliefs," *Journal of Economic Theory*, 59, 33–49.

Huber, P. (1981): *Robust Statistics*. John Wiley & Sons, New York.

Jaffray, J.-Y. (1989): "Linear utility for belief functions," *Operations Research Letters*, 8, 107–112.

——— (1992): "Bayesian updating and belief functions," *IEEE trans. on systems, man and cybernetics*, 22(5), 1144–1152.

Kyburg, H. (1987): "Bayesian and non-Bayesian evidential updating," *Artificial Intelligence. An International Journal*, 31(3), 271–293.

Pires, C. (2002): "A rule for updating ambiguous beliefs," *Theory and Decision*, 53(2), 137–152.

Schmeidler, D. (1989): "Subjective Probability and Expected Utility Without Additivity," *Econometrica*, 57(3), 571–587.

Shafer, G. (1976): *A Mathematical Theory of Evidence*. Princeton University Press, Princeton, New Jersey.

Walley, P. (1991): *Statistical Reasoning with Imprecise Probabilities*. Chapman and Hall, London.

WANG, T. (2003): "Two classes of multiple priors," Discussion paper, Department of Finance, University British Columbia,.

WASSERMAN, L. A., AND J. B. KADANE (1990): "Bayes' theorem for Choquet capacities," *The Annals of Statistics*, 18, 1328–1339.