PREFERENCE AMONG RISKY PROSPECTS UNDER CONSTANT RISK AVERSION

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Abstract

Risk analyses often require a measure of individual risk aversion. Here a procedure is presented to calculate risk aversion parameter ranges wherein individuals would exhibit preference among a set of risky prospects.

Key words: expected utility, risk aversion coefficient.

The comparison of risky prospects usually requires an assumption about individual risk preference. Sometimes risk preference assumptions can be relatively simple, such as indifference to risk (profit maximization) or risk aversion (second degree stochastic dominance). However, more complex assumptions are often required for conclusive dominance results. For example, one might define a range for the risk aversion coefficient (RAC) as commonly done with stochastic dominance with respect to a function (Meyer) or with mean variance programming models (Apland et al.). Specifying such a range can be difficult and often requires complex deduction or wholesale adoption of the results of other researchers (Raskin and Cochran).

In this manuscript, an alternative approach is presented wherein risk aversion coefficients are found which differentiate among the prospects. However, to do this, assumptions are needed regarding utility function form and data availability. In particular, a constant absolute risk aversion utility function is assumed as well as the availability of a discrete set of data on a finite number of mutually exclusive risky prospects. Furthermore, these data are assumed to be free of sampling error. Thus, the primary purpose of this paper is to present a method (hereafter called RISKROOT) which finds those RACs at which preferences among prospects change (hereafter called breakeven risk aversion coefficients—BRACs) given known distributions and a constant risk aversion utility function. The secondary purpose of this paper is to present some information on how RISKROOT might be used. Discussion is also presented about the similarities of this procedure to the Meyer procedure.

JUSTIFICATION OF ASSUMPTIONS

Four principle assumptions were identified in the previous section:

1. constant absolute risk aversion utility functions,
2. finite number of mutually exclusive prospects,
3. discrete distributions, and
4. data free of sampling error.

The justification for these assumptions and the effects of relaxing them, where known, are presented in this section.

The first and most basic assumption is that of constant absolute risk aversion. This assumption has been used or dealt with by many previous researchers. For example, Freund showed that this assumption, coupled with normality, justifies use of the E-V model. Pratt presented functional forms exhibiting such characteristics (which will be used herein). Hammond assumed such utility functions and derived results indicating when decision makers with nonconstant risk aversion could make decisions using constant RAC functions as proxies. Yassour et al. used the assumption in conjunction with continuous distributions to derive their EUMGF approach which was later used by Collander and Zilberman. Kramer and Pope (1986) argue that constant absolute risk aversion is assumed "in most applications. . . largely because wealth

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data is [sic] frequently unavailable or of questionable accuracy" (p. 189). Finally, Tauer, following Hammond, utilized the assumption to study alternative RACs to find intervals in which the BRACs lie. In the studies assuming such a utility form, there are three rationales used for the constant RAC assumption: 1) analytical convenience, 2) empirical inability to specify the wealth dependency of the RAC, and 3) the implication of Hammond's (p. 1061) derivation which shows that when decision makers have decreasing absolute risk aversion but their RAC, at their current wealth level, is at or above a BRAC, the preference orderings will be consistent with the decision makers' preferences, given that the distributions cross only once. One would also speculate that the same result would hold for the largest BRAC above a decision maker's current RAC when the decision maker has increasing absolute risk aversion. These justifications will be used herein, although their implications will be discussed when multiple distribution crossings and BRACs are present.

The finite number of mutually exclusive prospects assumption is adopted to allow pairwise comparison of a finite set of alternatives. A continuous set of alternatives cannot be handled herein and is more conveniently done using methods such as E-V analysis (McCarl et al.).

The discrete distribution assumption is used herein to differentiate from the EUMGF approach of Yaussour et al., which is the continuous distribution analogue of the method developed here. The assumption of a distributional form and a constant absolute risk aversion utility function would allow one to set the moment-generating functions for the prospects equal and solve for the BRAC. If the distribution form is of a known continuous nature and a moment-generating function is known or derivable, then the moment-generating function approach should be used and, following Hammond, the BRAC calculated. The procedure here should only be used with discrete distributions or when moment-generating functions applicable to the distributions at hand are not available.

The fourth assumption, that the data are free of sampling error, is subject to further research. Pope and Ziemer have explored related issues showing the sensitivity of stochastic dominance analysis to sampling errors. The RISKROOT method is undoubtably sensitive to sampling error.

THE PROBLEM—BEHAVIOR OF BRACS

Hammond proved that given two risky prospects whose cumulative distributions crossed once, there would be a BRAC such that below the BRAC one prospect dominated, while above it the other dominated. Hammond then suggested that the BRAC could be simply computed. In particular, he states that if one makes distributional assumptions, this would require "little more than a table of moment-generating functions and a few pencil calculations" (p. 1059).[1] Hammond's requirement of one crossing is potentially restrictive. The underlying basis for this requirement is Karlin's result which implies that there are no more BRACs than there are distribution crossings. Thus, one crossing means a maximum of one BRAC, but 10 crossings means there could be as many as 10 BRACs. Moving away from Hammond's continuous distribution and single crossing assumptions constitutes the essence of this paper.

First let us investigate the number of crossings. Two questions arise:

1) Do cases exist where there are multiple crossings? and
2) If so, what are the implications on the behavior of the BRACs within those cases?

To investigate these questions, data were drawn from a number of previously published agricultural economic studies. Table 1 presents the results, by study, summarizing: a) the number of observations in the study (there could be as many crossings as one less than the number of observations); b) the number of distributions considered; c) the number of possible pairwise comparisons of distributions; d) the number of distribution comparisons with zero crossings, one crossing, and more than one crossing; and e) the maximum number of crossings observed. Note that multiple crossings were observed somewhere in all data sets except that of Kramer and Pope (1981).[2] Thus, it is not unreasonable to expect multiple crossings. Consequently, we now turn to the implica-

[1]The Yassour et al. EUMGF approach is an implementation of this procedure.

[2]Kramer and Pope argue that "in all of the empirical studies we are familiar with, distributions have been found to cross no more than once" (1986, p.189), but this does not appear to be generally true.

tions of multiple crossings for BRACs.

One way of gaining insight into the effects of multiple crossings involves consideration of a graph of the RAC versus the utility difference for a case set of data. Klemme presents returns to land and management data for four corn tillage options. Considering the alternatives Conventional Tillage (CT) and Till Plant (TP), the data cross five times. A graph of the utility difference between CT and TP from two constant risk aversion curves as the RAC changes are given in Figure 1. In this case, TP dominates CT for risk aversion coefficients smaller than −0.00778 and larger than −0.00426, while CT is dominant for RACs between −0.00778 and −0.00426.[3] Here, there are two BRACs, occurring at −0.00778 and −0.00426. This shows that multiple crossings can mean multiple BRACs.

A few words of interpretation are in order to indicate why one might expect two distributions to exhibit multiple BRACs. Consider the following hypothetical data for two prospects.

Observation	Distribution 1	Distribution 2
1	10	11
2	20	12
3	30	13
4	40	14
5	50	15
6	60	16
7	70	17
8	80	81
Mean	45.0	25.6
Standard Deviation	22.9	23.7

Note that distribution 1 has a much higher mean and a slightly smaller variance than distribution 2. Thus, one would expect distribution 1 to dominate in most cases. However, note that distribution 2 has a higher minimum and a higher maximum. Thus, following the arguments in Grube, for appropriate values of the RAC, distribution 2 will be preferred to distribution 1 for both highly risk preferring and highly risk averse individuals.

The above graph and data show several notable things about the problem of finding BRACs:

1) the difference between utilities forms a function which can oscillate (i.e., no particular nice properties such as concavity or convexity can be counted on);
2) multiple BRACs can occur; and
3) the utility difference always approaches zero as the RAC approaches zero or positive infinity. This can be seen by plugging these values for the RAC into equation (1) below.

We now turn our attention to the problem of finding BRACs.

FINDING BRACS

The problem of finding BRACs between two

TABLE 1: SUMMARY OF CROSSINGS OF CUMULATIVE DISTRIBUTION FUNCTIONS IN PREVIOUS STUDIES

Study	Number of Data Points Per Distribution	Number of Distributions in Study	Possible Number of Pairwise Comparisons	Number of Pairwise Comparisons: Zero Crossings	One Crossing	Multiple Crossings	Signs for any Pair
Danok et al.[a]	15	15	105	14	71	20	5
Lee et al.[b]							
−377 acres	31	2	1	0	0	1	2
−720 acres	31	2	1	0	0	1	10
−930 acres	31	2	1	0	0	1	3
Klemme[c]	8	4	6	0	0	6	5
Kramer and Pope[d] (1981)	10	8	28	21	7	0	1

[a] Drawn from the table on p. 706.

[b] Drawn from Lee (pp. 184–94).

[c] Drawn from the corn data, p. 552.

[d] Drawn from p. 125.

[3] In addition there is a root at about 0.5727, but at this point utility is on the order of 10^{-44} and finding the root is numerically difficult. In addition, this result implies that a very high risk premium (more than $1000/acre) will be paid to insure a slightly greater minimum return per acre ($1.25).

distributions with n_1 observations in the first and n_2 in the second is to find a RAC such that the utility difference (UD) is effectively zero. Algebraically, the problem is to find those RACs such that:

$$(1)\ UD = \left[\sum_{i=1}^{n_1} P_{i1}[-\Delta e^{-rx_{i1}}]\right] - \left[\sum_{i=1}^{n_2} P_{i2}[-\Delta e^{-rx_{i2}}]\right] = 0,^4$$

where r is a RAC; Δ is positive one if r > 0, negative one if r < 0; i denotes observation; P_{ij} denotes the probability of observation i for distribution j; and x_{ij} denotes the ith observation on the jth distribution compared.

Given that the UD function is not convex or concave, a general grid search is used to discover the BRACs. The algorithm used, hereafter called RISKROOT, has been computerized in FORTRAN for PC and other computers (McCarl, 1987) and can be obtained by contacting the author. The basic procedure used in this algorithm is: a) initially develop a grid of possible RACs; b) evaluate whether the utility difference changes signs (has a root) between any two of the grid points; and c) if sign changes are found, then find the final BRAC using a binary search. Steps b and c of this procedure are repeated until all RACs intervals have been examined.

Example #1

The results arising using the RISKROOT procedure are possibly best demonstrated using an example. Klemme's four corn tillage data depicts distributions for Conventional Tillage (CT), Chisel Tillage (CP), Till Plant (TP), and No Till (NT). In this data, six unique pairwise comparisons are possible. The results are as follows:

1) For the pair CT/CP, the distributions cross four times, but CT is dominant for all risk aversion coefficients.
2) For the pair CT/TP, the distributions cross five times, and BRACs are found at −0.00778 and −0.00426 with CT dominant between them and TP outside of them. Furthermore, the maximin rule indicates that CT should dominate as risk aversion increases. Thus, a search above the upper search limit (set following McCarl and Bessler) was conducted. As the RAC gets large there is a crossing at 0.57265, but at this point the risk premium is $1030/acre, which far overwhelms the means of CT and TP (around $230/acre). Simultaneously utility is of the order 10^{-44} and the utility difference is of the order ≅ 10^{-50}. Thus, this point could well be disregarded.
3) For the pair CT/NT, the distributions cross twice, but CT is always dominant.
4) For the pair CP/TP, TP is always dominant.
5) For the pair TP/NT, TP is always dominant.
6) For the pair CP/NT, three crossings were found, but CP was dominant everywhere. However, dominance everywhere by CP is inconsistent with the maximin rule since the maximum returns under NT exceed those for CP. Exploring large RACs yields a root at 0.11838. Again, this root is suspect as it corresponds to a risk premium of $185/acre.

Summarizing BRACs for Multiple Comparisons

When the algorithm is applied to a set of data containing more than two distributions, results are generated for each pairwise comparison. These can be summarized into an overall set of results. The summarization procedure basically places all the BRACs on one scale and examines all pairwise results between the BRACs to identify the dominant set in each interval. Redundant information is deleted (see McCarl, 1987).

Example #2

The RISKROOT procedure when applied to the Klemme data set yielded multi-distributional results identical to the results above (since the CT and TP alternatives dominated the other alternatives). Thus, we present summary results using data from Danok et al., regarding the returns to machinery complements on a midwest crop farm under stochastic weather events. The results involve 105 pairwise comparisons. These comparisons may be summarized as follows:

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[4]This equation is not defined at r = 0. The RISKROOT algorithm and the computer program written to implement RISKROOT use a comparison of the means to insure consistency at r=0.

Dominant Machinery Complement	RAC Range
2	RAC ≤ −0.0000557
11	−0.0000557 ≤ RAC ≤ −0.0000073
3	−0.0000073 ≤ RAC ≤ 0.0000121
9	0.0000121 ≤ RAC

These results show, for example, that for risk averters with a constant RAC exceeding .0000121, machinery complement 9 is the best while complement 3 is dominant for those below this value and above −.0000073. These results are consistent with the Danok et al. results which identify complements 3 and 9 as those meriting consideration when assuming risk aversion.

HOW BRACS CAN BE USED

Now that the RISKROOT method for finding BRACs and the basic nature of the results has been introduced, it is worthwhile to address the issue of how the resultant BRACs can be used. The author foresees four ways BRACs can be used:

1) presenting choices to decision makers,
2) developing order of magnitude estimates on BRACs for use with methods such as the Meyer and/or E-V approaches,
3) studying how BRACs affect choices and drawing implications for other approaches, and
4) studying how data manipulations affect BRACs and distribution choices.

The use of RISKROOT in each of these settings is discussed below.

Presenting Choices

As partially illustrated in the above examples, one of the possible usages of RISKROOT involves sorting through a set of data to identify which prospects are preferred for which RAC range. The RISKROOT procedure would give unequivocal results in this setting regardless of utility function: a) if a single alternative was found to dominate everywhere, b) for all risk averters if all roots found were in the negative risk aversion range, c) for all risk preferers if the roots were only in the positive range, d) for preference results above the largest BRAC for those with decreasing absolute risk aversion with a RAC at current wealth smaller than that BRAC, e) for preference results below the smallest BRAC for those with increasing absolute risk aversion below the smallest (including most negative) BRAC, and f) for those with constant risk aversion utility functions or utility functions closely approximated by such (Tsaing).

Developing Magnitude Estimates on BRACs

Many researchers have difficulty establishing the appropriate RAC range in an applied study. Cases have arisen where values too large have been used (e.g., Grube) or where values have been simply (and possibly inappropriately) adopted from other studies (see Raskin and Cochran). Furthermore, the Meyer program is notorious for numerical overflow errors when the maximum RAC is too large, while E-V analysis frequently reports alternatives which do not use the full land endowment for a RAC which is too large. Thus, information on the appropriate order of magnitude for the RAC would be helpful. Such data could be developed in the E-V case by using RISKROOT on the probability distributions under alternatives constituted by a minimax, a maximax, and a maximum expected value plan (all of which could easily be generated using the programming model). Turning to Meyer's program, one can find unanimous intervals anywhere between the interior BRACs or anywhere outside the extreme values but not crossing the BRACs (McCarl, 1988). Thus, the program results provide a guide for selecting intervals when using the Meyer program.

Studying How BRACs Affect Choices

Under this application, RISKROOT can be used to see how sensitive the choice among prospects is to variations in the risk aversion parameter. As such, one would be able to investigate the type of behavior expected from an E-V or EUMGF approach when comparing a number of selected alternatives. Results from the analysis in the development of RISKROOT show that the results can be quite sensitive with flipflops in preferences where multiple crossings are present as in the example above.

Studying Consequences of Data Manipulations on BRACs

RISKROOT provides an interesting way of studying the consequences of changes in the

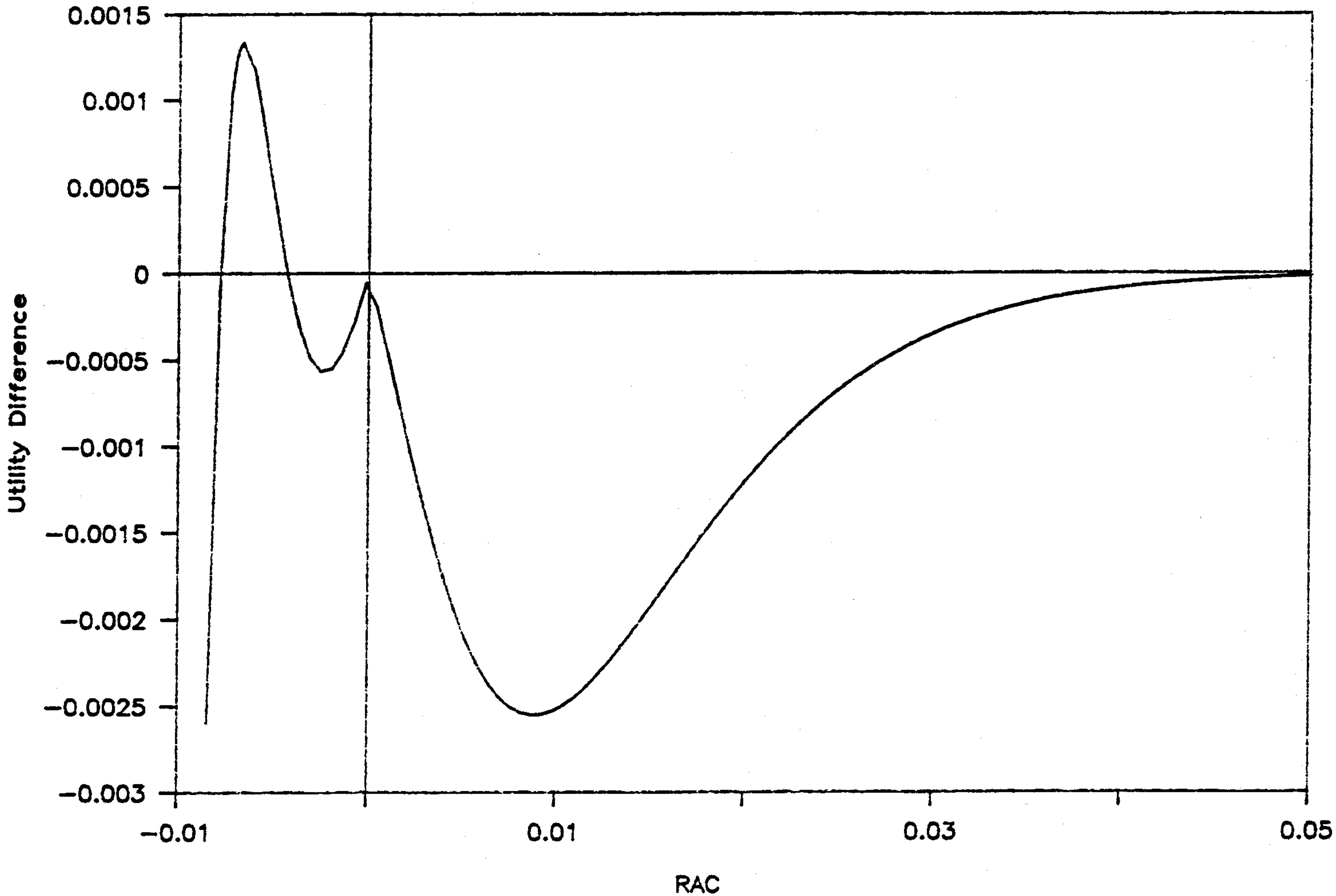

Figure 1: Graph of Utility Difference
Under Varying Risk Aversion.

basic data assumptions. For example, in experiments with rounded, smoothed, and manipulated data, it was found that:

a) rounding Klemme's data to the nearest \$ (dropping pennies with the mean on the order of \$250/acre) led to the elimination of a crossing and a BRAC, plus a 400% change in the largest BRAC;
b) comparing results using Day's raw data for corn nitrogen fertilization versus a Pearson I distribution fitted to the data altered the BRAC from .1226 to .0883, while applying Anderson's recommendation to Klemme's data in example #1 above altered the BRACs found from two to none and changed the number of crossings from 5 to 3; and
c) adding a constant (as in wealth) to the data did not alter the risk aversion results.[5] However, multiplying the data by a constant led to a new BRAC equal to the old one divided by the constant (as proved in Raskin and Cochran).

General Results with RISKROOT

There also were general results that were revealed when developing and using RISKROOT.

First, no fixed relationship was found between the number of crossings and the number of BRACs, other than conforming to Karlin's result that the number of crossings provide a bound on the maximum number of BRACs. Cases were found where there were ten crossings but no BRACs, while simultaneously cases were found with four crossings and three roots.

Second, cases were found in the risk preferring range where an item may initially dominate, then be dominated, then dominate again. This was the case in the Klemme data above. However, multiple roots among a pair were not found for risk averse RACs (i.e., those greater than 0). But this would probably occur if Klemme's data were all changed in sign.

Third, multiple BRACs were frequently found to be the case.

Fourth, one distribution is always dominant for each RAC value except at the exact BRAC crossing points where one is indifferent between the prospects. This result carries through to the multi-distribution case. Consequently, more definitive dominance results can be expressed than, say, under other stochastic dominance forms. However, stronger assumptions are being made relative to the utility function.

COMPARISON WITH MEYER'S PROCEDURE

Readers may be interested in some comparison with the Meyer procedure. First, we must note there is a fundamental difference. Meyer's results, while derived using a computer program containing an exponential utility function (as noted in Kramer and Pope, 1986), are developed based on a theorem which holds for any shape of the risk aversion parameter, r(x), such that the numerical values of the r(x) are between the two constants. RISKROOT identifies BRACs, but under the constant r(x) assumption. Experimentation with Meyer's program shows that if, for example, RISKROOT identifies a

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[5]This is probably best seen by investigating the effects of adding wealth in equation (1). For simplicity here we assume $n_1=n_2=n$. The result is

$$UD = \sum_{i=1}^{n} P_i[-\Delta e^{-rx_{i1}} - (-\Delta e^{-rx_{i2}})] = 0.$$

Now, assuming that each of the x_{ik}'s are really wealth (w) plus some observed specific income level (Y_{ik}), the equation becomes

$$\sum_{i=1}^{n} P_i[-\Delta e^{-r(w+Y_{i1})} - (-\Delta e^{-r(w+Y_{i2})})]=0$$

$$= \sum_{i=1}^{n} P_i[\Delta e^{-rw} e^{-rY_{i1}} - (-\Delta e^{-rw} e^{-rY_{i2}})]=0$$

$$= e^{-rw}\left[\sum_{i=1}^{n} P_i [\Delta e^{-rY_{i1}} - (-\Delta e^{-rY_{i2}})]\right]=0$$

and e^{-rw} can be divided out not affecting the root.

pair of BRACs, that anywhere between the BRACs intervals for the Meyer program can be found exhibiting the same preference (McCarl, 1988). Results of no dominance from Meyer are only found when the interval crosses a BRAC or when too large of an interval is used. For example, when applied to the example #1 data, a set of overlapping preference intervals could be found between −0.00778 and −0.00426.

CONCLUDING COMMENTS

This paper outlined the RISKROOT procedure which finds breakeven risk aversion coefficients between pairs of distributions under the assumption of an exponential utility function. RISKROOT finds the RAC values where preferences change. However, these RACs, while more discriminating, are based on more restrictive underlying assumptions. RISKROOT should be useful for sorting out preferences if the assumptions are met, developing RAC estimates for use in other studies, studying the relationship between RACs and dominance, and studying the consequences of distributional smoothing and/or data manipulation.

The FORTRAN program underlying this procedure is available and documented in McCarl (1987). The procedure is available for the PC or any machine with a FORTRAN compiler and costs $5.00 plus the price of a 360K floppy disk.

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