THE DUAL OF A COST MINIMIZING LINEAR TRANSSHIPMENT MODEL: AN ECONOMIC INTERPRETATION OF AN ASSEMBLY-PLANT PROCESSING-DISTRIBUTION NETWORK FOR A FIRM

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The general economic meanings and mathematical structure of the dual of a primal mathematical programming model have been discussed for many years [1, 2, 3, 4]. However, within the mathematical programming realm, many interesting formulization variations have developed partly in response to variations in particulars of problems.

A number of authors have discussed the economic meaning and mathematical structure of the primal of a linear cost minimizing transportation model [1, 2, 3, 4, 5]. Some authors discussed the economic meaning and mathematical structure of the dual as well as the primal of the transportation model [10]. Several authors discussed cost minimizing transshipment models [6, 7, 8, 9]. Recently, greater interest has been shown in specific economic meanings of the dual of the cost minimizing transshipment model [11].

PURPOSE

The purpose of this article is to provide some interpretive insights into economic meanings of the dual of a cost minimizing linear transshipment primal model as formulated for an assembly-processing-distribution network of a firm.

ASSUMPTIONS

For the sake of illustrative convenience the following simple conditions are assumed in the model:

1. Fixed and given supply and demand quantities,
2. Fixed and given unit costs of transportation and plant processing,
3. One product,
4. No retaliatory actions by rivals, and
5. Aggregate supply equals aggregate demand.

Though an assembly-processing-distribution network of a firm can cover any number of regions, two-region primal and dual transshipment models will be presented as examples.

PRIMAL TRANSSHIPMENT FORMULATION

The transshipment model is a generalization of the basic linear programming transportation model allowing shipments of goods to go through any sequence of points rather than just from m surplus regions to n deficit regions [9]. Table 1 shows a primal transshipment formulation.

The objective is to allocate quantities to and from plants in such a way as to minimize the sum of assembly, processing and product shipment costs (row 1). The constraints are that: The amount shipped from a plant must equal the amount processed at the plant (rows 2 and 3); the amount of raw product available in a region[1] is equal to or less than the sum of the amount of processing in the region less the amount exported from the region plus the amount imported into the region (rows 4 and 5);[2] the amount of product received at a destination must equal the amount shipped from all plants to that region (rows 6 and 7), and only non-negative amounts may be shipped and processed.

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[1] The absence of $t_{1,1}R_{1,1}$ and $t_{2,2}R_{2,2}$ is to avoid duplication of intraregion supply provided by S_1 and S_2, respectively, to P_1 and P_2, respectively.

[2] Rows 4 and 5 have been divided through by -1 to get all inequality senses in the same direction.

Table 1. A PRIMAL TRANSSHIPMENT FORMULATION

Row

1. $t_{1,2}R_{1,2}+t_{2,1}R_{2,1}+C_1P_1+C_2P_2+T_{1,1}X_{1,1}+T_{1,2}X_{1,2}+T_{2,1}X_{2,1}+T_{2,2}X_{2,2} = \text{Min } Z$

subject to:

Row									
2.			P_1		$-X_{1,1}$	$-X_{1,2}$			$= 0$
3.				P_2			$-X_{2,1}$	$-X_{2,2}$	$= 0$
4.	$-R_{1,2}$	$+R_{2,1}$	$-P_1$						$\geqslant -S_1$
5.	$R_{1,2}$	$-R_{2,1}$		$-P_2$					$\geqslant -S_2$
6.					$X_{1,1}$		$+X_{2,1}$		$\geqslant D_1$
7.						$X_{1,2}$		$+X_{2,2}$	$\geqslant D_2$

where:

Xij = processed product shipment from region i to region j,
Rij = raw product shipment from region i to region j,
Pi = processed amount of product in region i,
Tij = processed product outbound transportation unit cost from region i to region j,
tij = assembly unit cost (raw product plus inbound transportation) from region i to region j,
Ci = processing unit cost in region i,
Si = supply of raw product in region i, and
Di = demand for processed product in region i.

THE DUAL TRANSSHIPMENT FORMULATION

Table 2 shows a formulation of the dual of the primal formulation shown above in Table 1.

The known or given supply (S_i) and demand (D_i) constraints of the primal have become the known or given objective function coefficients of the dual. The given transportation and processing cost coefficients of the primal objective function (t_{ij}, C_i, T_{ij}) have become the dual constraints. The transformation of the primal to a dual has in part involved a trading of the structural positions of given values.

ECONOMIC MEANING OF THE DUAL

The particular objective of this dual formulation (row 8) is to find prices, v_i, and w_i, for supplies, S_i, and demands, D_i, respectively, so that these prices correspond to the most economical allocation of the assembly, processing and distribution functions from the viewpoint of the minimum sum of costs of these functions (as found by the primal). The prices that satisfy this objective will maximize an imputed or "shadow" evaluation of the scarce or fixed supply and demand availabilities.

The maximization objective represents an imputed total revenue (sum of wi D_i) less an imputed total cost of supplies (sum v_i S_i). The values of the prices v_i and w_i are such that under- and over-utilization of resources are avoided throughout assembly, processing, and distribution.

The maximum value W of the dual will, of course, equal the minimum cost value, Z, of the primal. Thus, the dual of the transshipment primal finds those prices of supply and demand which provide a net total revenue that exactly rewards the inputs of assembly, processing, and transportation.

The presence of u_i0 in the objective function is an algebraically logical necessity but is of no economic significance.

The attainment of the dual objective must satisfy particular constraint relationships. Specifically, the constraints are:

1. Row 9 could be restated to read $v_2 \leqslant v_1 + t_{1,2}$. This says that the delivered price of raw product at plant 2 (*before* processing occurs) must not be more than delivered price of raw product at plant 1 (*before* processing occurs) plus the unit transport cost of raw product from origin 1 to plant 2. Row 10 has an analogous interpretation.

Table 2. A DUAL TRANSSHIPMENT FORMULATION

Row

8. $u_1 \cdot O + u_2 \cdot O - v_1 S_1 - v_2 S_2 + w_1 D_1 + w_2 D_2 = \text{Max } W$

subject to:

Row							
9.			$-v_1$	$+v_2$			$\leqslant t_{1,2}$
10.			$+v_1$	$-v_2$			$\leqslant t_{2,1}$
11.	u_1		$-v_1$				$= C_1$
12.		u_2		$-v_2$			$= C_2$
13.	$-u_1$				$+w_1$		$\leqslant T_{1,1}$
14.	$-u_1$					$+w_2$	$\leqslant T_{1,2}$
15.		$-u_2$			$+w_1$		$\leqslant T_{2,1}$
16.		$-u_2$				$+w_2$	$\leqslant T_{2,2}$

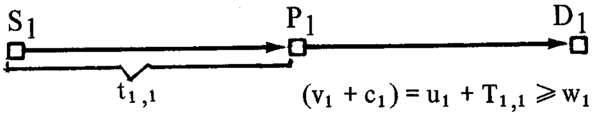
where:

ui = imputed unit value (shadow price) of processed product at plant i,
vi = imputed unit value (shadow price) of raw product at plant i but before plant processing begins, and
wi = imputed unit value (shadow delivered price) of processed product in market destination region i.

The remaining symbols were defined earlier in the primal.

2. Row 11 could be restated as $u_1 = v_1 + c_1$, that is, the unit value of *processed* product at plant 1 is the sum of delivered raw product unit value at plant 1, v_1, plus the unit processing cost at plant 1, c_1. Similar reasoning applies to row 12.
3. Row 13 could be restated as $w_1 \leqslant u_1 + T_{1,1}$ which requires that the delivered price of processed product at market destination 1, w_1, will not exceed u_1 (as defined above) plus the unit transport cost for processed product from plant 1 to market region 1, $T_{1,1}$. Rows 14, 15, and 16 can be treated similarly.

The economic interpretation of the above constraint parts might be seen more clearly in terms of an illustrative route. Suppose the route started in raw material region, S_1, and transshipped through processing plant, P_1, with processed product delivered from plant, P_1, to market region, D_1. Then a simple graph would appear as follows:



$T_{i,j}$, c_i and $t_{i,j}$ are given values from the primal, whereas v_i, u_i, and w_i are solved unknowns of the dual.

In effect, the dual solution prices fill in value-added accumulations along the route. Given unit costs of assembly, processing and delivery, the dual specifies ultimately that the optimally delivered price in each market should not exceed the sum of the optimally combined unit costs of assembly, processing, and delivery. Thus, the dual solution specifies values which assure that market prices neither undervalue nor overvalue the input values required to optimally satisfy the market.

The dual values of v_i, u_i, and w_i are marginal values. If the manager of a firm wishes to buy more supplies, say from S_1, then he knows not to pay more than v_i per supply unit. If the manager wished to process additional volume at, say, plant 1, then he knows from the dual that an optimizing value-added cost would be u_1. If the manager wants to sell additional quantities in, say, market 2, then he knows to charge a price w_2.

The dual prices as reward values may be applied to suppliers, transporters, plant processors, and consumers regardless of whether any combinations of these components are exogenous or endogenous to a particular system.

EXTENSIONS OF LINEAR MODEL

Though specific multiproduct and non-linear models are beyond the scope of this paper, it should be emphasized that the economic meanings of the dual of the linear transshipment model carry forward into expanded models.

Essentially, the meanings are the same except that additional refinements are attached to non-linear models in which functions replace points and in which "givens" of linear models become unknowns to be sought by quadratic or more general convex programming techniques [11].

CONCLUSIONS

The dual linear transshipment formulation and solution for an assembly-processing-distribution network of a firm has separate but closely related economic meanings to a primal cost minimizing linear transshipment model of the same network.

The primal cost minimizing model specifies an optimal combination of tonnage to be assembled, processed, and distributed subject to supply and demand constraints. In contrast, the dual solution finds imputed prices for supply and demand regions resulting in an imputed net total revenue which exactly rewards and exhausts the optimal input costs of assembly, processing, and transportation.

In terms of any single route from a raw material origin through a processing plant to a final market destination, the dual specifies that the delivered price in each final market should not exceed the sum of the optimally combined unit costs of assembly, processing, and distribution along the route.

A manager may use dual solution values as prices to assign to extra units of input or output so that over- and under-utilization of resources will be avoided.

Economic meanings of the dual of a linear transshipment model may be transferred and expanded in more developed non-linear transshipment models.

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