GENERALIZED STOCHASTIC DOMINANCE: AN EMPIRICAL EXAMINATION

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Abstract

Use of generalized stochastic dominance (GSD) requires one to place lower and upper bounds on the risk aversion coefficient. This study showed that breakeven risk aversion coefficients found assuming the exponential utility function delineate the places where GSD preferences switch between prospects. However, between these break points, multiple, overlapping GSD intervals can be found. Consequently, when one does not have risk aversion coefficient information, discovery of breakeven coefficients instead of GSD use is recommended. The investigation also showed GSD results are insensitive to wealth and data scaling but are sensitive to rounding.

Key words: risk, generalized stochastic dominance, risk aversion coefficients

Stochastic dominance has become a popular method for the analysis of agricultural data. It provides a way of ranking risky alternatives without detailed knowledge of decision-maker preferences. However, in many cases, first and second degree stochastic dominance cannot fully rank the alternatives. Consequently, some analysts have had to turn to stronger assumptions. In particular, Meyer's (1977) generalized stochastic dominance (GSD) with its accompanying computer program (Meyer 1975) has become a common technique. Raskin and Cochran cite 17 studies using risk aversion coefficients (RAC), the majority of which use some form or another of GSD. Furthermore, articles employing GSD have appeared in each of the last four volumes of this journal including Goh *et al.*'s recent GSD software article. The theoretical background regarding GSD is presented in Appendix A.

GSD supports preference rankings when the decision-maker's RAC is assumed to fall within a predetermined, numerically specified interval. Such numerical specification can be difficult since RAC's are, in general, individualistic. However, the GSD analyses in the literature generally involve an *a priori* numerical specification of the RAC bounds. Alternatively, one could use a numerical search technique to discover RAC intervals wherein GSD differentiates among the risky prospects. Hammond, in a non-GSD context, developed theory and methods for the discovery of numerical RACs that differentiate between two prospects (hereafter called breakeven risk aversion coefficients——BRACs) under a constant absolute risk aversion assumption. McCarl (1988) reviewed agricultural applications of Hammond's procedure and developed a computer implementation.

The ultimate purpose of this study was to see whether numerical techniques can be used to find GSD risk aversion bounds. The interrelationships between the BRAC and GSD techniques were examined. In addition this study examined: (a) intervals where dominance holds, (b) cases under which GSD preference switches between prospects, and (c) the numerical properties of GSD.

GSD Definition Used in this Paper

Meyer (1977) originally developed GSD as a technique that could guarantee dominance under the assumption that the decision-maker's RAC fell between a lower [$r_1(x)$] and an upper [$r_2(x)$] bound, where x represents wealth. In Meyer's (1975) computer program and in the empirical literature, a special case of GSD is used where the RAC bounds are independent of wealth, thus being constants (r_1, r_2). Herein, GSD preference is defined to occur when:

Distribution F dominates distribution G as long as the decision-maker's RAC (r) is in the interval $r_1 \leq r \leq r_2$.

FINDING RAC BOUNDS

The investigation began with the objective of finding values for r_1 and r_2 such that the interval between r_1 and r_2 is as large as possible with GSD preference maintained. However, empirical experience quickly revealed two things:

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1. The maximum upper bound (r_2) for the RAC interval, such that GSD preference exists, depends, in general, on the numerical value of the lower bound (r_1), or the minimum lower bound depends on the upper bound.
2. The RAC bound interval could not span a breakeven risk-aversion coefficient in the Hammond and McCarl (1988) sense.

The evidence and concepts supporting these statements is given below.

First, the interrelationship between r_1 and the maximum r_2 is an empirically observed relation. This can be demonstrated using Klemme's data on corn tillage practices (p. 552). RAC intervals were found over which GSD preference held using Klemme's distributions for chisel tillage and till plant. These were derived using a procedure that finds the "largest permissible" r_2 given an r_1 such that GSD preference is maintained. This is done by first finding an r_1, r_2 pair where GSD preference holds (such a pair can be found by setting r_2 to a value very close to r_1) and then successively making r_2 larger [1] until $\hat{r}_2$ is found where GSD preference exists and where $\bar{r}_2$ GSD preference fails. Subsequently, a numerical search is done between $\hat{r}_2$ and $\bar{r}_2$ until the "largest permissible" value of r_2 is found for which GSD preference occurs (see McCarl (1990) for a computer program implementing this procedure).

The consequent results for selected r_1 values are given in Table 1. Note, when starting from r_1 equals -.02 that GSD preference can be found for $r_1 = -0.02 \leq r \leq -0.00485 = r_2$. However, when starting from an r_1 value within this interval of -0.005, the GSD results hold all the way up to $r_2 = 0.01259$.[2] The Table 1 data support the first statement, showing in general the maximum r_2 depends on r_1 and vice versa. Evidence presented later shows such dependency does not always occur.

Table 1. Dependency of Maximum r_2 on r_1 for Klemme Data at Selected Values of r_1

Intervals where GSD can be found[a]	
r_1	Maximum r_2
-0.020	-0.00485
-0.019	-0.00452
-0.018	-0.00415
-0.017	-0.00373
-0.016	-0.00325
-0.015	-0.00270
-0.014	-0.00207
-0.013	-0.00132
-0.012	-0.00043
-0.011	+0.00066
-0.010	+0.00203
-0.009	+0.00379
-0.008	+0.00524
-0.007	+0.00708
-0.006	+0.00946
-0.005	+0.01259
-0.004	+0.01684
-0.003	+0.02283
-0.002	+0.03183
-0.001	+0.04668
+0.001	+0.17962
+0.002	>0.50000[b]

[a]In all cases, the "till plant" distribution is preferred.

[b]This range exceeds .5, but above .5, numerical difficulties occur.

RACs where preferences shift are also important pieces of information. Hammond introduced the concept of breakeven risk-aversion coefficients (BRACs) to define places where preferences shift under constant absolute risk aversion.[3]

The constant absolute risk aversion BRACs are important determinants of GSD preference shifts as can be shown using the data given in King and

[1] Small steps are used in this process to avoid the researcher's being fooled by cases where there are multiple shifts in an interval.

[2] These results were verified for accuracy using the Meyer (1975) program with identical results achieved for the indicator function (Appendix A, equation 2) but numerical tolerances are used in deriving the results herein. Therefore, dominance conclusions can differ (*i.e.* Meyer required the function to be ≤ 0, whereas this study required that the function be 10^{-24}).

[3] BRACs are those risk aversion coefficients where the expected utility difference between two prospects f and g equals zero. For example, given the utility function $u(w) = -e^{-rw}$ and the wealth distributions of two prospects f(w) and g(w), then $\int_0^\infty - e^{-rw}$ $(f(w)-g(w))\,dw=0$ is solved for all r. The resultant set of r's are the BRACs mentioned in the text. For decision makers with a risk-aversion coefficient slightly larger than a BRAC, one distribution will dominate, while slightly below the BRAC, the other will dominate. Hammond and McCarl elaborate on this concept

Robison (KR). There is a single BRAC that differentiates among KR's distributions 1 and 2 (p. 514) which equals 0.0003634. Employing the "largest permissible" r_2 finding algorithm and searching upward from just above the BRAC, GSD preference is found in the interval (0.00036341 , ∞). Similarly, if r_2 is fixed just below the BRAC and a downward search is conducted for the smallest permissible r_1, the GSD preference interval ($-\infty$, 0.00036339) is found.[4] Inconclusive GSD preference results arise from intervals with r_1 and r_2 spanning over the BRAC (*i.e.*, $r_1 < 0.0003634 < r_2$). Thus, in this case, the BRAC delineates the types of GSD preferences that can be found.

It is not surprising that BRACs delineate places where GSD preferences switch. Constant absolute risk-aversion is a special case of the utility functions falling in the GSD risk-aversion bounds. However, in general, BRACs do not fully define GSD preference intervals as Meyer's analytical results do not extend to the entire range between the BRACs. In the Table 1 case, there are no BRACs in the interval $-0.0848 \leq RAC \leq 0.0848$ with till plant being dominant everywhere. However, many "largest permissible" GSD intervals arise in this interval, all exhibiting identical preference to those found using the BRAC assumptions. A number of additional investigations were done on the various data sets, with the universal finding being that the BRACs denoted preference shifts but not necessarily the limits of "largest permissible" GSD intervals.

Apparently, GSD intervals can be found continuously and overlapping between BRACs. For example, using Klemme's data for conventional and till plant corn tillage, McCarl reported BRACs of -0.0077755 and -0.0042593, stating that till plant dominates for RACs below -0.0077755 or above -0.0042593 while conventional tillage dominates between these BRACs. A substantial number of runs were done exploring intervals around these BRACs. Some are reported in Table 2. There, the interval between BRACs is completely covered with overlapping "largest permissible" GSD preference intervals exhibiting the same preference as under constant absolute risk aversion. This leads to the case-specific finding that overlapping GSD preference intervals can appear between the BRACs, but that the BRACs delineate GSD preference shifts. However, this cannot be proved using numerical methods, although it appears eminently reasonable given that the constant absolute risk-aversion utility function is a special case of those considered by GSD.

Table 2. Intervals Found Yielding GSD Preferences for Klemme Conventional Tillage Versus Till Plant Data

r_1	r_2	Dominant Distribution
-0.0200000	-0.0113730	Till Plant
-0.0114000	-0.0095605	Till Plant
-0.0096000	-0.0088256	Till Plant
-0.0088300	-0.0084267	Till Plant
-0.0084300	-0.0081952	Till Plant
-0.0082000	-0.0080537	Till Plant
-0.0080600	-0.0079645	Till Plant
-0.0079650	-0.0079025	Till Plant
-0.0079026	-0.0078612	Till Plant
-0.0078613	-0.0078336	Till Plant
-0.0078337	-0.0078150	Till Plant
-0.0078151	-0.0078024	Till Plant
-0.0077800	-0.0077786[a]	Till Plant
-0.0077600	-0.0077529	Conventional
-0.0075300	-0.0074281	Conventional
-0.0074300	-0.0072932	Conventional
-0.0073000	-0.0071230	Conventional
-0.0071300	-0.0069089	Conventional
-0.0069100	-0.0066448	Conventional
-0.0066450	-0.0063444	Conventional
-0.0063445	-0.0060248	Conventional
-0.0060249	-0.0057068	Conventional
-0.0057068	-0.0054101	Conventional
-0.0054102	-0.0051494	Conventional
-0.0042595	-0.0042594	Conventional
-0.0042590	-0.0042589[b]	Till Plant
-0.0030000	-0.0021699	Till Plant

[a]McCarl (1988) indentified a breakeven risk aversion coefficient at -0.0077755. Thus, preferences switched around this point.

[b]McCarl (1988) identified another breakeven risk aversion coefficient at -0.0042593.

The results also show GSD preference results can oscillate. Several cases arose where prospect 1 was found to be preferred over a range (RAC < a), then prospect 2 over a higher range (a < RAC < b), followed by a return to prospect 1 over a yet higher range (RAC > b). Table 2 provides such an example.

One other question involves intervals where indifference occurs. Authors such as King and Robison (p. 515) discussed intervals where GSD indifference exits. Examination with both the original Meyer (1975) program and the program developed based

[4] Obviously, $\pm\infty$ could not be tried. However, the distributions crossed only once and dominance persisted for large positive and large negative RACs.

on the "largest permissible" bound interval procedure found such intervals do not exist unless either the RAC bound spread is too large or a BRAC is spanned.

OTHER EXPERIMENTATION

In addition to the results presented above, a number of other findings were generated.

Level of Wealth and Addition of Constants

Experiments were done on the effect of adding/subtracting a constant from all the data. These experiments were motivated by curiosity as to what happens when including or excluding wealth. The hypothesis was that changes in wealth would alter the GSD preference intervals. This hypothesis was not verified. All "largest permissible" GSD preference results were unchanged by the addition of positive constants, with the interval identical to all reported significant digits (6). Intuitively this result can be explained as follows. The key result in preference determination involves the final sign of the recursive relation given in Appendix A equation 2. The final value of this equation potentially consists of a number of terms each involving r_1 or r_2. However, the switch from r_1 to r_2 or vice versa occurs at the point where an intermediate term (Q_{n+1}) in the recursive relationship equals zero. Addition of wealth under constant r_1, r_2 would not alter the F and G terms but would only shift their location. However, the first derivative of the utility function would be multiplied by a positive constant (equaling the utility of initial wealth). This would not change the RAC roots, because the place where a function equals zero is not affected by multiplication by a positive constant (*i.e.* if $f(r)^*$) equals zero, then $Kf(r^*)$ also equals zero where K is a positive constant). Also, the final function result would alter in magnitude but not sign, because it is also multiplied by a positive coefficient. Thus, insensitivity to wealth is not unexpected. In addition, insensitivity to wealth is fortuitous given the difficulty of dealing with wealth and the lack of attention to it inherent in the literature.

Scaling

Yet another possible question involves the impact of multiplicative data transformations on GSD interval results (*i.e.* changes in units from dollars to thousands of dollars). Raskin and Cochran show in a different context that, when the data are divided by N, identical results arise if the RACs from before the multiplication are multiplied by N. These results also apply to GSD preference intervals. GSD interval results for scaled data with RAC bounds appropriately multiplied are identical to those found for unscaled data with the original RAC bounds.

Sensitivity to Rounding

McCarl found BRACs to be sensitive to data precision and rounding. Given the interrelationships between BRACs and GSD preference intervals as discussed above, it appears obvious that GSD would exhibit similar sensitivity. A limited amount of experimentation verified this. Rounding or otherwise altering the data alters the results. Therefore, GSD results are sensitive to data format and certainly sampling error (for investigations of stochastic dominance sensitivity to scaling, see the literature cited in Tolley and Pope). Consequently, it is important to do external work on the distribution before using GSD or finding BRACs. Perhaps density function estimation, smoothing, bootstrapping, or some other procedure should be employed.

Numerical Stability

The final issue worthy of brief mention regards numerical stability. The GSD approach using fixed r_1 and r_2 requires evaluation of an exponential utility function of the form $-e^{-rx}$ where r is the RAC and x is the level of wealth. Authors such as Danok, McCarl and White, and Kramer and Pope have used risk aversion parameters with values as large as 0.1 with corresponding x values in the 100,000s. This leads to an exponentiation with about -10,000 as the power. The original Meyer (1975) GSD program as well as the program used herein are not accurate on virtually any computer when dealing with sums of such numbers. GSD users need to be careful to size properly the potential RAC intervals using procedures such as given in McCarl and Bessler (where a bound on the maximum RAC of 20 divided by the standard error is suggested). For example, the Danok, McCarl and White data need a maximum RAC of 0.000011 to rank, while Kramer and Pope's data do not require a RAC above 0.000017.

CONCLUDING COMMENTS AND IMPLICATIONS

The recent release of Goh *et al.*'s GSD software, coupled with the above results, seems to call for some guidance to potential GSD users. Three types of implications can be drawn: (1) guidance regarding when to use GSD and how to select RAC intervals, (2) guidance regarding data preparation before using GSD and GSD result sensitivity, and (3) guidance on specifying GSD intervals.

GSD Use and Interval Selection

Probably the most difficult decision when considering using GSD is the selection of the upper and lower bounds for the RAC intervals. The recommendations arising from this study are conditional on whether or not decision-maker risk-aversion coefficient information is available.

If risk-aversion coefficient information is not available, GSD should not be used. A BRAC-finding procedure such as that given in McCarl (1988) is preferred. Use of arbitrarily chosen non-overlapping intervals, as is common in the literature, appears to be little more than shooting in the dark. BRACs show where preferences switch and do not require any assumption on RAC magnitude.

On the other hand, if clientele RAC information is available, BRACs should be found to identify the places where preferences shift. Subsequently, the clientele RAC information should be examined relative to closeness to the BRACs. In turn, GSD may be used to derive preference information given the clientele RAC information. If so, a modified GSD procedure should be used to find the "largest permissible" GSD preference interval. On the other hand, if working with a decision-maker, one may directly present the choices, or elicit and evaluate actual utility.

Data Preparation for GSD

During the experimentation in this study when the absolute value of the interaction of r_1 or r_2 times wealth was large (say greater than 100), the GSD programs had substantial numerical difficulties. Users of GSD should scale r_1 and r_2 or the data so that this limit is not exceeded. Also, a constant may be freely subtracted from all of the probability distributions (*i.e.* changing from wealth to current income) without altering the GSD preference results. Thus, the researchers may wish to subtract a constant uniformly from all distributions such that the minimum observation over all the data has a value of one. Simultaneously, the values of r_1 and r_2 should be sized appropriately. Limits such as 20 divided by the standard error of the risky prospect could be used as discussed in McCarl and Bessler.

The experiments indicate that GSD results are sensitive to data presentation, manipulation, and sampling error. Precision is also an issue. During manuscript review, one reviewer found that small changes in the risk aversion coefficient bounds drastically altered the nature of the output from Meyer's (1975) program which does not control for numerical stability (*i.e.* changing a result from dominance to non-dominance). GSD users should properly size data as discussed above. Also, it would be a good idea for GSD program developers, such as Goh *et al.* to control for numerical errors.

GSD Result Interpretation

The numerical results arising above indicate four things that should be considered when interpreting GSD preference results. First, regions of non-dominance are composed of smaller regions of dominance. Apparently, only breakeven points and no true regions of GSD indifference exist. Second, between Hammond and McCarl's breakeven risk aversion coefficients, there is a continuous, but potentially overlapping, set of GSD intervals where the same prospect is always preferred. Third, preferences may switch more than once. Thus, if for an interval, prospect f is found to be preferred to prospect g, similar preferences may also be found in non-adjacent intervals. Fourth and finally, incorporation of wealth does not affect GSD preference interval results.

APPENDIX A.

Theoretical Development of Meyer's GSD Technique

Given two continuous probability distributions, f and g, the theory of expected utility (von Neumann and Morganstern) asserts that in order for distribution f to dominate distribution g, from an economic agent's viewpoint, the expected utility of distribution f must be greater than the expected utility of distribution g. Mathematically, this has been shown to be equivalent to

$$(1) \quad \int_0^1 [\, F(x) - G(x) \,]\, u'(x)\, dx < 0 \,,$$

where F and G are the cumulative probability density functions of f and g, and u'(x) is the first derivative of the utility function with respect to x.

Meyer (1977) utilized this condition in developing GSD preference conditions. However, rather than dealing with a known utility function, he dealt with all utility functions bounded by constraints upon the RAC. Thus, the conditions Meyer found were conditions where GSD preference occurs for all decision makers regardless of their RAC as long as their Pratt RAC measure falls between $r_1(x)$ and

$r_2(x)$. $r_1(x)$ and $r_2(x)$ are upper and lower bounds on RAC that potentially depend on x. To derive GSD conditions, Meyer set up an optimal control problem with the RAC as the control variable:

Problem A

$$\text{maximize} \quad \int_0^1 [\, F(x) - G(x) \,]\, u'(x)\, dx$$

Subject to

$$(u'(x))' = \left(\frac{u''(x)}{u'(x)}\right) u'(x)$$

$$r_1(x) \leq \frac{-u''(x)}{u'(x)} \leq r_2(x)\,.$$

If the solution to this problem yields a negative objective function value, the maximum value of the expected utility is negative for all possible RAC function choices [r(x)] between the RAC bounds $r_1(x)$ and $r_2(x)$ and, therefore, f must dominate g for all such r(x) choices. The Meyer GSD preference proposition may be summarized as:

> If the maximand of the optimal control problem stated in problem A is negative, then f dominates g by GSD for all r(x) falling between $r_1(x)$ and $r_2(x)$.

Meyer (1977) analytically derived the solution to the control problem relying on the linearity of the state variable. Such a problem has a discontinuous solution arising from the theory of the bang bang control problem (See Kamien and Schwartz, p. 186-192). The optimal control involves setting the risk aversion parameter as follows:

$$r(x) = \frac{-u''(x)}{u'(x)} = \begin{cases} r_1(x) \text{ if } \int_x^1 [F(z) - G(z)]u'(z)dz > 0 \\ r_2(x) \text{ if } \int_x^1 [F(z) - G(z)]u'(z)dz \leq 0\,. \end{cases}$$

Meyer indicated that calculation of the Problem A maximand must be done through a numerical, backward recursive calculation. Namely, given a discrete set of points x, the backward recursion involves integration from x_n to x_{n+1} plus the maximand value from x_{n+1} forward (Q_{n+1}) as follows:

$$(2) \qquad Q_n = \int_{x_n}^{x_{n+1}} [F(z) - G(z)]\, u'(z)\, dz + Q_{n+1}\,.$$

Where $r_2(x)$ is used if Q_{n+1} is nonpositive, and $r_1(x)$ is used if Q_{n+1} is negative. Subsequently, one proceeds to the next point (x_{n-1}) and continues. After treating the last point, the overall maximand ($Q^* = Q_1$) is obtained. If Q^* is negative, there is dominance. However, if Q^* is positive, one cannot guarantee GSD preference. It is important to note that even if, as in Meyer's (1975) program, $r_1(x)$ and $r_2(x)$ are constants, constant absolute risk aversion utility functions are not assumed. Rather r(x) is constrained to fall in between the constants r_1 and r_2, but r(x) can exhibit any pattern whether it be increasing, decreasing, constant, or oscillating within these bounds. An algorithmic step by step overview of Meyer's (1975) computer program for GSD is given in McCarl (1990).

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