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Effects of Explanatory Variables in Count Data Moving Average Models

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Abstract

This note gives dynamic effects of discrete and continuous explanatory variables for count data or integer-valued moving average models. An illustration based on a model for the number of transactions in a stock is included.

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Key Words: INMA model, Marginal effect, Intra-day, Financial data.

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1. Introduction

This note gives dynamic effects of explanatory variables for count data or integer-valued moving average (INMA) models with time dependent parameters.

The INMA model of order q was introduced by Al-Osh and Alzaid (1988) and McKenzie (1988). Brännäs and Hall (2001) summarize previous model characterizations and add two new ones. In essence, the model can be interpreted in alternative ways, but importantly the first both conditional and unconditional moments remain the same. Higher order moments such as autocorrelations differ for the different interpretations.

When it comes to introducing explanatory variables our focus is on the first order moment functions, and obviously we wish to adhere to an integer-valued data generating process. This can be accomplished by letting parameters become functions of explanatory variables and by adopting functional forms that are consistent with an INMA process. Brännäs et al. (2002) and Brännäs

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and Quoreshi (2006) give empirical results for INMA model with explanatory variables in tourism and financial research settings. Given such specifications, obtaining marginal and other dynamic effects on the conditional mean over time is feasible and of great practical interest.

2. Model

An INMA(q) can be written as

$$y_t = \theta_{0,t} \circ \varepsilon_t + \theta_{1,t-1} \circ \varepsilon_{t-1} + \theta_{2,t-2} \circ \varepsilon_{t-2} + ... + \theta_{q,t-q} \circ \varepsilon_{t-q},$$

where $\circ$ indicates binomial thinning and the INMA parameters, $\theta_{0,t} = 1, \theta_{i,t} \in [0,1], i = 1, \dots, q-1$, and $\theta_{q,t} \in (0,1]$, are time dependent thinning probabilities.[1] In addition, $E(\varepsilon_t) = \lambda_t$. This representation is a generalized model in that both the $\theta_{i,t}$s and the λ_t are time dependent. It is through these parameters that we most simply can include exogenously determined explanatory variables contained in a vector $\mathbf{x}_t$.

For INAR models it appears reasonable to include the effect of explanatory variables through the AR-parameters, and possibly in terms of λ_t (Brännäs, 1995). When it comes to including explanatory in INMA models, Brännäs et al. (2002) in an INMA(1) model lets both $\theta_{1,t}$ and λ_t vary, while Brännäs and Quoreshi (2006) lets only λ_t vary with explanatory variables.

For the INMA(q) and conditional on $\mathbf{x}_t$ it holds that

$$E(y_t) = \theta_{0,t}\lambda_t + \theta_{1,t-1}\lambda_{t-1} + \theta_{2,t-2}\lambda_{t-2} + ... + \theta_{q,t-q}\lambda_{t-q}.$$

For $\theta_{i,t}$ a logistic specification is chosen, i.e. $\theta_{i,t} = 1/[1 + \exp(\mathbf{x}_t\boldsymbol{\beta}_i)]$, while for λ_t we consider a static as well as a dynamic specification. Obviously, other specifications are also possible.

3. Dynamic Effects

3.1 Time Dependent λ_t

Consider first the conventional count data specification $\lambda_t = \exp(\mathbf{x}_t\boldsymbol{\beta})$ together with time invariant $\theta_i, i = 1, \dots, q$. Then $\partial\lambda_t/\partial x_{k,t} = \beta_k\lambda_t$, with k indicating the continuous x-variable of interest. Since, $\partial E(y_t)/\partial\lambda_{t-s} = \theta_s$, for $s \leq q$, and equal to zero otherwise, a marginal change in variable k at time $t-s$ gives the marginal effect

$$m_{t,s}^k = \frac{\partial E(y_t)}{\partial x_{k,t-s}} = \frac{\partial E(y_t)}{\partial \lambda_{t-s}} \frac{\partial \lambda_{t-s}}{\partial x_{k,t-s}} = \begin{cases} \theta_s\beta_k\lambda_{t-s}, & s \leq q \\ 0, & s > q \end{cases}.$$

[1] The binomial thinning operator $\alpha \circ x = \sum_{i=1}^{x} z_i$, where $\{z_i\}$ is a 0-1 iid random sequence, with $\Pr(z_i = 1) = \alpha$ as the thinning probability. Conditionally on x, $\alpha \circ x$ is binomially distributed with mean αx and variance $\alpha(1-\alpha)x$. Given independence between the z_i and x, $E(\alpha \circ x) = \alpha E(x)$ and $V(\alpha \circ x) = \alpha^2 V(x) + \alpha(1-\alpha)E(x)$. Also, $\alpha \circ x \in [0, x]$.

We may determine the variance for the effect by the delta method, i.e. $V(m_{k,s}) = V\left(\hat{\theta}_s\hat{\beta}_k\hat{\lambda}_{t-s}\right) \approx \mathbf{h}'V(\boldsymbol{\psi})\,\mathbf{h}$, where $\boldsymbol{\psi}' = (\theta_1, ..., \theta_q; \beta_1, ..., \beta_m)$, and both the covariance matrix of the parameter estimator, i.e. $V(\boldsymbol{\psi})$, and $\mathbf{h} = \partial m_{k,s}/\partial\boldsymbol{\psi}$ are evaluated at the estimates.

If x_k is a discrete-valued variable, say a dummy variable, it appears more reasonable to give results of a unit change in x_k in terms of absolute or relative changes in the mean. Let $\mathbf{x}_t$ denote the base level and the new level by $\mathbf{x}_t^k$ for a unit change in variable k. Then

$$\begin{aligned}\nabla_{t,s}^k &= E\left(y_t \mid \mathbf{x}_{t-s}^k\right) - E\left(y_t \mid \mathbf{x}_{t-s}\right) = \theta_s \exp\left(\mathbf{x}_{t-s}\boldsymbol{\beta}\right)\left[\exp(\beta_k) - 1\right] \\ &\approx \theta_s\beta_k\lambda_{t-s}\end{aligned}$$

and the percentage change

$$\begin{aligned}D_{t,s}^k &= 100\,\frac{E\left(y_t \mid \mathbf{x}_{t-s}^k\right) - E\left(y_t \mid \mathbf{x}_{t-s}\right)}{E\left(y_t \mid \mathbf{x}_{t-s}\right)} \\ &= 100\,\left(e^{\beta_k} - 1\right)\left(\theta_s\lambda_{t-s}/\sum_{i=0}^{q}\theta_i\lambda_{t-i}\right) \approx 100\beta_k\left(\theta_s\lambda_{t-s}/\sum_{i=0}^{q}\theta_i\lambda_{t-i}\right).\end{aligned}$$

The approximations are for small β_k.

Consider next a case of a dynamic specification $\lambda_t = \lambda_{t-1}^{\alpha}\exp\left(\mathbf{x}_t\boldsymbol{\beta}\right)$. Since $\ln\lambda_t = \alpha\ln\lambda_{t-1} + \mathbf{x}_t\boldsymbol{\beta}$ and since $\partial\ln\lambda_t/\partial x_{k,t-s} = \left(\partial\ln\lambda_t/\partial\lambda_t\right)\left(\partial\lambda_t/\partial x_{k,t-s}\right)$ it follows that $\partial\lambda_t/\partial x_{k,t-s} = \lambda_t\left(\partial\ln\lambda_t/\partial x_{k,t-s}\right)$. By recursive substitution

$$\ln\lambda_t = \alpha^t\ln\lambda_0 + \sum_{i=1}^{t}\alpha^{t-i}\mathbf{x}_i\boldsymbol{\beta}.$$

Therefore, $\partial\ln\lambda_t/\partial x_{k,t-s} = \alpha^s\beta_k$ and $\partial\lambda_t/\partial x_{k,t-s} = \alpha^s\beta_k\lambda_t$. In the INMA($q$) model we then get the marginal effect

$$m_{t,s}^k = \sum_{i=0}^{s}\theta_i\frac{\partial\lambda_{t-i}}{\partial x_{k,t-s}} = \beta_k\sum_{i=0}^{s}\alpha^{s-i}\theta_i\lambda_{t-i}$$

with here and in the sequel $\theta_0 = 1$ and $\theta_i = 0, i > q$. The effects of a discrete change of a unit size in a variable x_k are given by

$$\begin{aligned}\nabla_{t,s}^k &= \sum_{i=0}^{s}\theta_i\lambda_{t-i}\left[\exp(\beta_k\alpha^{s-i}) - 1\right] \\ D_{t,s}^k &= 100\,\nabla_{t,s}^k/\sum_{i=0}^{q}\theta_i\lambda_{t-i}.\end{aligned}$$

For the more general case of

$$\lambda_t = \lambda_{t-1}^{\alpha_1}...\lambda_{t-p}^{\alpha_p}\exp(\mathbf{x}_t\boldsymbol{\beta})$$

we employ the companion form

$$\begin{pmatrix} \ln \lambda_t \\ \ln \lambda_{t-1} \\ \vdots \\ \ln \lambda_{t-p+1} \end{pmatrix} = \begin{pmatrix} \alpha_1 & \alpha_2 & ... & \alpha_p \\ 1 & 0 & ... & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & ... & 1 & 0 \end{pmatrix} \begin{pmatrix} \ln \lambda_{t-1} \\ \ln \lambda_{t-2} \\ \vdots \\ \ln \lambda_{t-p} \end{pmatrix} + \begin{pmatrix} \mathbf{x}_t \boldsymbol{\beta} \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

or

$$\mathbf{\Lambda}_t = \mathbf{A}\mathbf{\Lambda}_{t-1} + \mathbf{z}_t = \mathbf{A}^t \mathbf{\Lambda}_0 + \sum_{i=1}^{t} \mathbf{A}^{t-i} \mathbf{z}_i.$$

We assume that the largest eigenvalue of $\mathbf{A}$ is smaller than one, so that $\mathbf{A}^t \to 0$ with increasing t. Here, t is assumed large. Since $\partial \mathbf{z}_i / \partial x_{k,i} = (\beta_k, 0, ..., 0)'$ and $\mathbf{A}^k = \mathbf{R}\boldsymbol{\theta}^k \mathbf{S}$, with $\mathbf{R}$ the right and $\mathbf{S}$ the left matrix of eigenvectors, respectively, and $\mathbf{\Theta}$ the diagonal matrix of eigenvalues, we have

$$\frac{\partial \ln \lambda_t}{\partial x_{k,t-s}} = \mathbf{1}' \frac{\partial \mathbf{\Lambda}_t}{\partial x_{k,t-s}} = \beta_k \mathbf{1}' \mathbf{R} \boldsymbol{\theta}^s \mathbf{S} \mathbf{1},$$

where $\mathbf{1} = (1, 0, ..., 0)'$. We use this result and $\partial \lambda_t / \partial x_{k,t-s} = \lambda_t \partial \ln \lambda_t / \partial x_{k,t-s}$ to obtain

$$m_{t,s}^k = \beta_k \sum_{i=0}^{s} \theta_i \lambda_{t-i} \mathbf{1}' \mathbf{R} \boldsymbol{\theta}^{s-i} \mathbf{S} \mathbf{1}.$$

The effects of a discrete, unit change in variable x_k are in this case given by

$$\begin{aligned} \nabla_{t,s}^k &= \sum_{i=0}^{s} \lambda_{t-i} \theta_i \left[\exp\left(\beta_k \mathbf{1}' \mathbf{A}^{s-i} \mathbf{1} \right) - 1 \right] \\ D_{t,s}^k &= 100 \frac{\sum_{i=0}^{s} \lambda_{t-i} \theta_i \left[\exp\left(\beta_k \mathbf{1}' \mathbf{A}^{s-i} \mathbf{1} \right) - 1 \right]}{\sum_{i=0}^{q} \theta_i \lambda_{t-i}}. \end{aligned}$$

3.2 Time Dependent $\theta_{i,t}$

In this case $\lambda_t = \lambda$ and $\partial \theta_{i,t} / \partial x_{k,t} = -\beta_{i,k} \theta_{i,t} (1 - \theta_{i,t})$, for $s \leq q$, so that

$$m_{t,s}^k = -\lambda \beta_{s,k} \theta_{i,t-s} (1 - \theta_{s,t-s}).$$

The effects of a discrete change in x_k are

$$\begin{aligned} \nabla_{t,s}^k &= \lambda \left(\theta_{i,s}^k - \theta_{i,s} \right) \\ D_{t,s}^k &= 100 \left(\theta_{i,s}^k - \theta_{i,s} \right) / \sum_{i=0}^{q} \theta_{i,t-i}, \end{aligned}$$

where superscript k denotes the new level.

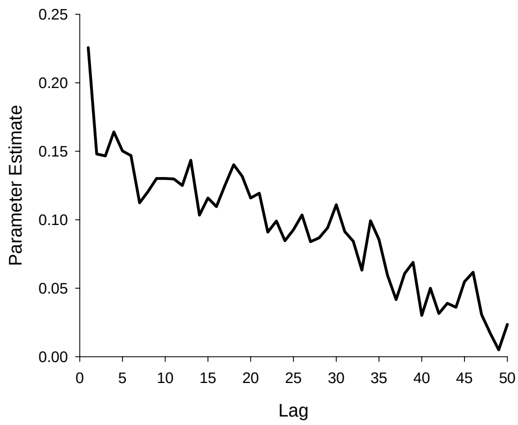

Figure 1: Estimates of moving average parameters.

3.3 Time Dependent $\theta_{i,t}$ and λ_t

Setting $\theta_{i,t} = 1/\left[1 + \exp\left(\mathbf{x}_t\boldsymbol{\beta}_i\right)\right]$ and $\lambda_t = \exp\left(\mathbf{w}_t\boldsymbol{\gamma}\right)$ and letting $x_{k,t} = w_{l,t}$ be the variables of which we wish to obtain effects we get the marginal effect on the form

$$m_{t,s}^{kl} = \theta_{s,t-s}\lambda_{t-s}\left[\gamma_l - \beta_{s,k}\left(1 - \theta_{s,t-s}\right)\right]$$

and

$$\begin{aligned} \nabla_{t,s}^{kl} &= \lambda_{t-s}\left[e^{\beta_k}\theta_{s,t-s}^{l} - \theta_{s,t-s}\right] \\ D_{t,s}^{kl} &= 100\ \nabla_{t,s}^{kl}/\sum_{i=0}^{q}\theta_{i,t}\lambda_{t-i}. \end{aligned}$$

4. Empirical Illustration

The empirical illustration is based on the INMA(50) model for intra-day data of Brännäs and Quoreshi (2006). The model explains the number of traded stocks in Ericsson B, with 50 INMA-lags, see Figure 1 for the estimates, and a time dependent λ_t function. The λ_t is of the form $\lambda_{t-1}^{\alpha}\exp(\mathbf{x}_t\boldsymbol{\beta}) = \exp(0.283 + 0.328\ln\lambda_{t-1} - 3.892\nabla p_t - 4.154\nabla p_t^{+} + 11.128\nabla s_t + 0.242\cdot 1_t)$, where $\nabla p_t^{+} = 0$ for $\nabla p_t \leq 0$ and $\nabla p_t^{+} = \nabla p_t$ for positive price changes, ∇s_t is a spread change, and $1_t = 1(t \leq 1100)$. The explanatory variables are discrete and we set $\lambda_0 = 8$, $\nabla s_t = 0$ and $1_t = 1$.

Figure 2 reports effects ($\nabla_{t,s}^{k}$) over time of price changes at time $t = 50$. To calculate the effects we use an extension of the $\nabla_{t,s}^{k}$ of Section 3.1, i.e. $\nabla_{t,s}^{k} = \sum_{i=0}^{s}\theta_i\lambda_{t-i}\left[\exp\left(\alpha^{s-i}\delta_{ks}\beta_k\right) - 1\right]$, with $\delta_{ks} = \pm 2$ and ± 5. Two features

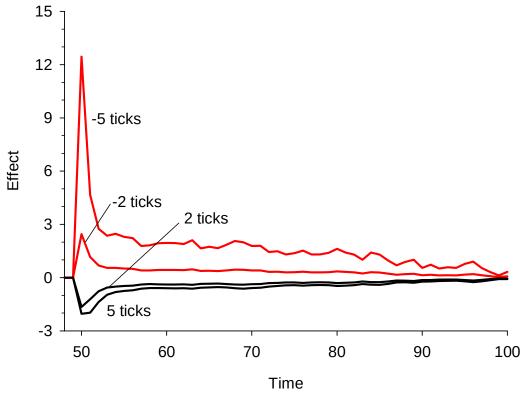

Figure 2: Effects of discrete price changes at $t = 50$.

are noteworthy. First, effects converge the further we move from the period of change. Smaller changes gives faster convergence. Second, the effects of positive and negative changes in price are not symmetrical. The initial effect of a negative change increases trading frequency by much more than the trade frequency decrease due to a positive change. Over all lags the effects of a -5-ticks change is about 2.5 times larger, while there is hardly any size difference for the ± 2-ticks changes.

References

Al-Osh M. and Alzaid A. (1988). Integer-Valued Moving Average (INMA) Process. *Statistical Papers* **29**, 281-300.

Brännäs, K. (1995). Explanatory Variables in the AR(1) Count Data Model. *Umeå Economic Studies* **381**.

Brännäs, K. and Hall, A. (2001). Estimation in Integer-Valued Moving Average Models. *Applied Stochastic Models in Business and Industry* **17**, 277-291.

Brännäs, K., Hellström, J. and Nordström, J. (2002). A New Approach to Modelling and Forecasting Monthly Guest Nights in Hotels. *International Journal of Forecasting* **18**, 19-30.

Brännäs, K. and Quoreshi, A.M.M.S. (2006). Integer-Valued Moving Average Modelling of the Number of Transactions in Stocks. *Umeå Economic Studies* **637**.

McKenzie, E. (1988). Some ARMA models for Dependent Sequences of Poisson Counts. *Advances in Applied Probability* **20**, 822-835.