Umeå University, March 2002

Stochastic Cost Benefit Rules: A Back-of-a-Lottery-Ticket Calculation Method*

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Abstract

In this paper, we introduce cost benefit rules for projects embedded in a stochastic optimal growth framework. We model uncertainty in terms of Brownian motion and Ito integrals. Taking the mathematical expectation of the project means that the Ito integrals vanish, and we end up with a cost benefit rule that closely resembles its deterministic counterpart.

JEL Classification: C60, D61, D81

Keywords: Cost benefit analysis, stochastic optimal control theory, Brownian motion

1. Introduction

A result in optimal control theory presented more than two decades ago[1] has had a considerable influence on the study of cost benefit analysis in deterministic dynamic continuous time models. The basic result – sometimes referred to as the 'dynamic envelope theorem' – greatly simplifies the calculation of the value of a project. In its most basic form, this result means that the value of a small project can be measured by differentiating the present value Hamiltonian partially with respect to the relevant parameter and then integrating over the planning horizon along the optimal path. This result follows because the indirect effects of the parameter via control, state and costate variables vanish as a consequence of

* A research grant from FORMAS is gratefully acknowledged.
[1] See Seierstad (1981). See also Seierstad and Sydsaeter (1987), Léonard (1987), Caputo (1990) and LaFrance and Barney (1991).

optimization. More recently, a number of studies have tried to extend the analysis to apply in imperfect market economies[2], where the maximized Hamiltonian is not as well defined as it is in the context of the planner models where the dynamic envelope theorem was originally applied.

In this paper, we introduce cost benefit rules for projects embedded in a stochastic optimal growth framework. Such an extension of the literature is important from a theoretical point of view in the sense that the methods used to solve stochastic optimal control problems differ from their deterministic counterparts. It is also relevant because many aspects of behavior in intertemporal economies are related to uncertainty in a fundamental way. We will show how optimization will add envelope properties that greatly reduce the measurement problem. More specifically, since we model uncertainty in terms of Brownian motion and Ito integrals, taking the mathematical expectation of the project means that the stochastic integrals vanish. Therefore, except for the expectations operator, we are left with the corresponding deterministic cost benefit rule.

The outline of the study is as follows. In Section 2, we present the model. Section 3 contains the main results, whereas Section 4 exemplifies the back-of-a-lottery-ticket calculation method in the context of a simple numerical framework.

2. The Model

In this section, we introduce a Ramsey model with a stochastic pollution equation and a pollution externality. The corresponding deterministic model is due to Brock (1977). The stochastic components are population growth, which will influence the stochastic differential equation for capital accumulation, and the assimilative capacity of the environment. The latter means introducing a stochastic differential equation for the accumulation of pollution. Although our model is specific in the sense of focusing on environmental aspects of optimal growth, the results are easy to generalize to any stochastic optimization problem.

The value function reads

[2] Cost benefit analysis of public projects in imperfectly competitive market economies are often applied to environmental policy problems; see e.g. Aronsson et al. (1997) and Aronsson (1999).

$$U(0) = E_0\{\int_0^\infty u(c(t), x(t))e^{-\theta t}dt\} \tag{1}$$

where $c(t)$ is consumption per capita and $x(t)$ is the stock of pollution. In other words, in comparison with the stochastic Ramsey problem analyzed by Merton (1975), we insert the stock of pollution as an additional argument in the utility function.

Let $F(L, K, G)$ be a linear homogeneous net production function (i.e. depreciation has been accounted for), where *L* denotes the units of labor input, *K* the units of capital input and *G* the units of energy input. The capital stock is assumed to evolve according to

$$\dot{K}(t) = F(L(t), K(t), G(t)) - C(t) \tag{2}$$

where *C* is the aggregate consumption. Let $k = K/L$ and $g=G/L$ and then differentiate totally with respect to time. By using linear homogeneity of the production function, it follows that

$$\dot{k}(t) = f(k(t), g(t)) - nk(t) - c(t) \tag{3}$$

in which $f(k, g)$ is net output per capita and n the rate of population growth. It is assumed that $L(t) = L(0)\exp(nt)$ with $L(0) > 0$ and $0 < n < 1$. Equation (3) is a variation of the Solow neoclassical differential equation for the capital stock per capita under certainty. Note that $dL/dt = nL$ or $dL = nLdt$.

Suppose that the growth of the labor force is described by the geometric Brownian motion

$$dL = nLdt + \sigma_1 L dw_1 \tag{4}$$

Equation (4) should be interpreted in the sense of Ito, and we assume that the Brownian motion, $w_1 = w_1(t)$, is defined on some probability space. Intuitively, the increments, dw_1, should be thought of as normally distributed variables with mean zero and variance dt. An important property of Brownian motion is that $w_1(s) - w_1(t)$ is independent of $w_1(t)$ for $s \geq t$. The drift of the process in equation (4) is governed by the expected rate of labor growth per unit of time, *n*. In other words, over a short interval of time, *dt*, the proportionate change of the labor force, *dL*/*L*, is normally distributed with mean *ndt* and variance $\sigma_1^2 dt$.

We are now ready to transform the uncertainty about the growth of the labor force into uncertainty about the growth of the capital stock per capita, $k = K/L$. Define

$$k(t) = \frac{K(t)}{L} = \Phi(L,t) \tag{5}$$

and apply Ito's lemma to obtain

$$\begin{aligned} dk &= [\, f\,(k(t), g(t)) - c(t) - (n - \sigma_1^2)k(t)]dt - \sigma_1 k(t) dw_1 \\ k(0) &= k_0 \end{aligned} \tag{6}$$

In the same spirit, we assume that the stock of pollution evolves over time. The emissions at each instant are related to the use of energy in production and possibly also dependent on the stock itself. By simplifying and writing the emission production function as $e(t) = g(t)x(t)$, it follows that $g(t)$ is also interpretable as the emission rate at time *t*. The stock of pollution accumulates according to the stochastic differential equation

$$\begin{aligned} dx &= g(t)x(t)dt - \sigma_2 \gamma x(t) dw_2 \\ x(0) &= x_0 \end{aligned} \tag{7}$$

where γ is the rate of depreciation, and w_2 is another stochastic variable that follows a Brownian motion. Note also that of a process evolving according to a geometric Brownian motion remains positive over time. This means that $x(t)$ is positive.

To shorten the notations, let us define

$$y(t) = \begin{bmatrix} c(t) \\ g(t) \end{bmatrix} \qquad z(t) = \begin{bmatrix} k(t) \\ x(t) \end{bmatrix} \tag{8}$$

As permissible controls, we choose the feedback controls $y(t) = y(t, z(t))$, where $y(t)$ is a deterministic control function. By substituting the control functions into the stochastic differential equations (6) and (7), we obtain

$$dz(t) = \begin{bmatrix} dk \\ dx \end{bmatrix} = \begin{bmatrix} h_1(y(t,k,x),k;\sigma_1,n) \\ h_2(t,k,x) \end{bmatrix} dt - \begin{bmatrix} \sigma_1 k \\ \sigma_2 \gamma x \end{bmatrix} dw \tag{9}$$

$$z(0) = \begin{bmatrix} k_o \\ x_0 \end{bmatrix}$$

with self explanatory notation. We assume that an admissible control implies that the system of stochastic differential equations has a unique solution as well as require that $y(t) \geq 0$. Moreover, to avoid a nonessential solution, we introduce the time horizon condition $T = \inf[t > 0 | k(t) = 0] \Lambda \infty$, and write the optimal value function as follows;

$$V(0, z_0) = \sup_y \int_0^T u(c(t), x(t)) e^{-\theta t} dt \tag{10}$$

subject to equation system (9). From stochastic optimal control theory, we know that an optimal control must satisfy a partial differential equation, which is called the Hamilton-Jacobi-Bellman (HJB) equation. The generalized HJB-equation can be written

$$\frac{\partial V(t,z)}{\partial t} + \sup_y \{u(c(t), x(t)) e^{-\theta t} + \Delta^y V(t,z)\} = 0 \quad \forall (t,z) \in D \tag{11}$$

with the transversality conditions $V(T, 0, x) = 0$. The term Δ^y is a differential operator. To define the differential operator, let us write equation system (9) in compact vector notation as

$$dz_t^y = a^y(t,z)dt - \sigma^y(t,z)dw_t \tag{12}$$

where the top index denotes that the process is driven by the control function, $y(t)$, or a fixed vector, y. Define the matrix[3]

$$\mathrm{M}^{y} = \sigma^{y}(t,z,y)\sigma^{y}(t,z,y)' \tag{13}$$

in which the prime symbol denotes transpose. The partial differential operator can now be written

$$\Delta^{y} = \sum_{i=1}^{2} a^{y}(t,z)\frac{\partial}{\partial z_i} + \frac{1}{2}\sum_{i=1}^{2}\sum_{j=1}^{2} M_{ij}^{y}\frac{\partial^2}{\partial z_i \partial z_j} \tag{14}$$

with an obvious modification for a case when we are dealing with N stochastic differential equations.

For the present case with two SDE’s, we can apply the differential operator to obtain

$$\begin{aligned} -\frac{\partial V(t,z)}{\partial t} &= \sup_{y}\{u(c(t)x(t))e^{-\theta t} + \frac{\partial V(t,z)}{\partial k}h_1(t,z) + \frac{\partial V(t,z)}{\partial x}h_2(t,z) \\ &\quad + \frac{1}{2}[\frac{\partial^2 V(t,z)}{\partial k^2}\sigma_1^2 k^2 + 2\frac{\partial^2 V(t,z)}{\partial k \partial x}\sigma_1\sigma_2 kx + \frac{\partial^2 V(t,z)}{\partial x^2}\sigma_2^2\gamma^2 x^2]\} \end{aligned} \tag{15}$$

By introducing explicit co-state variables for notational convenience, equation (15) can be rewritten as

$$\begin{aligned} -\frac{\partial V(t,s)}{\partial t} &= \sup_{y} H(t,z,y,p,\frac{\partial p_k}{\partial k},\frac{\partial p_x}{\partial x},\frac{\partial p_k}{\partial x}) \\ &= H^{*}(t,k,x,p,\frac{\partial p_k}{\partial k},\frac{\partial p_x}{\partial x},\frac{\partial p_k}{\partial x}) \end{aligned} \tag{15a}$$

where H^{*} is interpretable as a maximized generalized present value ‘Hamiltonian’, and

$$p(t) = (p_k, p_X) = (\frac{\partial V}{\partial k}, \frac{\partial V}{\partial x})$$

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[3] Here σ^{y} is a 2×1 vector.

is a vector of co-state variables in present value terms. Note that the co-state variables are defined as partial derivatives of the optimal value function. In case the increments dw_1 and dw_2 are independent, the multidimensional analogues of the co-state stochastic differential equations have the following shape

$$dp_k = -H_k^* dt - \frac{\partial^2 V}{\partial k^2}\sigma_1 k dw_1 - \frac{\partial^2 V}{\partial k \partial x}\sigma_2 \gamma x dw_2$$

$$dp_x = -H_x^* dt - \frac{\partial^2 V}{\partial x^2}\sigma_2 \gamma x dw_2 - \frac{\partial^2 V}{\partial x \partial k}\sigma_1 k dw_1 \tag{16}$$

The derivation of the general form of the stochastic co-state differential equations can be carried out in the following way. Each co-state variable is the derivative of the optimal value function with respect to the corresponding state variable. By applying Ito's Lemma, the resulting expression will contain a term representing the cross derivative of the optimal value function with respect to time and the state variable. We can then derive an expression for this derivative by taking the first derivative of the HJB-equation with respect to the state variable and again using Ito calculus. Substituting the resulting expression for the cross derivative into the original co-state differential equation, one arrives at the result in equations (16) after some cancellations. We supply the details in the Appendix. Note finally that the more general case with *N* state variables is a straight forward extension[4].

3. Cost Benefit Rules

The form of the co-state equation contains the key to the shape of a cost benefit rule under Brownian motion. Since a co-state variable measures the contribution to the value function of a marginal increase in a state variable, we can use the concept of co-state variable to derive a cost benefit rule. The trick is to introduce an artificial state variable in terms of the parameter that describes the project. In the model set out above, the parameter γ will be used as a project that improves the assimilate capacity of the environment. Since γ is time independent, we can write its differential equation as $d\gamma = 0,\ \gamma(0) = \gamma$. This gives us three stochastic differential equations, one of them being deterministic. We can, nevertheless, elicit a co-state

[4] The *N*-dimensional case can be easily guessed by the reader.

variable, which is defined as a partial derivative of the optimal value function, i.e. $p_\gamma = \partial V / \partial \gamma$. We can then use the general form of the co-state equations (16) to write

$$dp_\gamma = -H_\gamma^*(\cdot)dt + \frac{\partial^2 V}{\partial \gamma^2}\sigma_3 dw_3 - \frac{\partial^2 V}{\partial \gamma \partial k}\sigma_1 k dw_1 - \frac{\partial^2 V}{\partial \gamma \partial x}\sigma_2 \gamma x dw_2 \tag{17}$$

However, $\sigma_3 \equiv 0$ by assumption, and we can integrate equation (17) over the interval (t, t_1) to obtain

$$p_\gamma(t_1) = p_\gamma(t) - \int_t^{t_1} H_\gamma^*(\cdot)ds - \int_t^{t_1} \frac{\partial^2 V}{\partial \gamma \partial k}\sigma_1 k dw_1 - \int_t^{t_1} \frac{\partial^2 V}{\partial \gamma \partial x}\sigma_2 \gamma x dw_2 \tag{18}$$

Since $p_\gamma(T) = 0$ according to the tranversality condition, we can write the cost benefit rule as

$$p_\gamma(t) = \int_t^T H_\gamma^*(\tau)d\tau + \int_t^T \frac{\partial^2 V}{\partial \gamma \partial k}\sigma_1 k dw_1 + \int_t^T \frac{\partial^2 V}{\partial \gamma \partial x}\sigma_2 \gamma x dw_2 \tag{19}$$

Finally, taking expectations and using the fact that the second and third terms on the right hand side of equation (19) are Ito-integrals, we have[5]

$$E_t(p_\gamma) = E_t\{\int_t^T H_\gamma^*(\tau)d\tau\} \tag{20}$$

which is a close analogue to the deterministic dynamic envelope theorem.

Project uncertainty can be introduced by specifying the differential equation for the project state variable, i.e.

$$d\gamma = \sigma_3 dw_3, \quad \gamma(0) = \gamma \tag{21}$$

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[5] See e.g. Björk (2000), pp 31-32. The reason is that the process is adapted, and that the increments are independent.

With this extension, equation (19) will contain one more Ito-integral. In expectations, the answer will be the same.

4. Exemplification: Back-of-a-lottery-ticket calculations

In this section, we want to exemplify the method developed in the previous section, i.e., that the calculations of $E_t(p_\gamma)$ can be reduced to a back-of-a-lottery-ticket calculation through the relation

$$E_t(p_\gamma) = E_t\{\int_t^T H_\gamma^*(\tau)d\tau\}$$

We consider the following, to some extent oversimplified, stochastic control problem:

$$V(t,x) = \inf_c E_t[\int_t^\infty (X_s^2 + c_s^2)e^{-\rho s}ds]$$

where the underlying process is given by

$$dX_t = c(X_t)dt + \sigma\gamma dw_t,\ X_t = x$$

Defining $p_\gamma = \partial V / \partial \gamma$, we want to calculate $E_t(p_\gamma)$ where the subindex t indicates that the underlying process starts at time t. One can approach the problem in two ways; either explicitly solve the stochastic optimal control problem and develop all expression explicitly before carrying out the calculation, or use the method described above. The second approach will imply less hard work.

We start with the first approach. The HJB-equation becomes

$$-\frac{\partial V(t,x)}{\partial t} = \inf_c\{e^{-\rho t}(x^2 + c^2) + c\frac{\partial V(t,x)}{\partial x} + \frac{1}{2}\sigma^2\gamma^2\frac{\partial^2 V(t,x)}{\partial x^2}\}$$

Maximizing with respect to the control variable gives

$$c = -\frac{1}{2}e^{\rho t}\,\frac{\partial V(t,x)}{\partial x}.$$

Inserting the expression for the control variable into the HJB-equation, we obtain

$$0 = e^{-\rho t}x^2 - \frac{1}{4}e^{\rho t}(\frac{\partial V(t,x)}{\partial x})^2 + \frac{\partial V(t,x)}{\partial t} + \frac{1}{2}\sigma^2\gamma^2\frac{\partial^2 V(t,x)}{\partial x^2}\}$$

By using separation of variables, so $V(t,x) = e^{-\rho t}\phi(x)$ and $\phi(x) = ax^2 + b$, one may solve for the parameters. If the control is only allowed to assume negative values we may conclude, by referring to Theorem 11.2.2 in Oksendahl (2000), that we have found the unique solution to the simplified stochastic control problem under consideration. In fact, the parameter a does not depend on γ. The optimal value function is given by

$$V(t,x) = e^{-\rho t}(x^2 + \frac{\sigma^2\gamma^2}{\rho})a$$

and the maximized present value 'Hamiltonian' becomes

$$H^* = e^{-\rho t}(x^2(1-a^2) + \sigma^2\gamma^2 a)$$

Therefore,

$$p_\gamma = 2e^{-\rho t}\frac{\sigma^2}{\rho}\gamma a = \int_t^T H_\gamma^*(\tau)d\tau$$

In this particular case, taking expectations makes no difference.

Let us now consider the back-of-a-lottery-ticket calculation. By definition,

$$-\frac{\partial V(t,x)}{\partial t} = H^*(t,x,\gamma,V_x,V_{xx})$$

Differentiation of $H^*(t, x, \gamma, V_x, V_{xx})$ with respect to γ gives

$$H_\gamma^* = \sigma^2 \gamma V_{xx} = 2e^{-\rho t}\sigma^2 \gamma a$$

in which we have made use of the explicit solution. Therefore, by our result,

$$E_t(p_\gamma) = E_t\{\int_t^T H_\gamma^*(\tau)d\tau\} = 2e^{-\rho t}\frac{\sigma^2}{\rho}\gamma a$$

Appendix: The shape of the SDE:s for the co-state variables

We have defined

$$p = (p_k, p_x) = (V_k, V_x)$$

Using Ito's formula on p_k implies

$$\begin{aligned} dp_k &= \{V_{kt} + V_{kk}h_1 + V_{kx}h_2 + \frac{1}{2}[V_{kkk}\sigma_1^2 k^2 + V_{xxx}\sigma_2^2\gamma^2 x^2 + 2V_{kxk}\sigma_1\sigma_2 kx]\}dt \\ &\quad - V_{kk}\sigma_1 k_1 dw_1 - V_{kx}\sigma_2 \gamma x dw_2 \end{aligned}$$

Since $V_{kt} = V_{tk}$, it follows from equation (15a) that[6]

$$-V_{tk} = H_k^* + V_{kk}h_1 + V_{kx}h_2 + \frac{1}{2}[V_{kkk}\sigma_1^2 k^2 + V_{xxx}\sigma_2^2\gamma^2 x^2 + 2V_{kxk}\sigma_1\sigma_2 kx]$$

which, if inserted into the SDE-equation for the co-state, yields the first part of equation system (16). The co-state equation for x follows analogously.

The same procedure as above can be used in the *N+1* state variable case, where the project is the first state variable, to show that

$$p_1(t) = \int_t^T H_1^*(s)ds + \sum_{i=2}^{N+1} \int_t^T V_{1i}(x,s)\sigma_1(x,s)dw_1$$

where x is the vector of state variables, and dw_i i.e. the Wiener increment of the i:th stochastic process. In expectations taken at time *t*, this reduces to equation (20).

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[6] We have omitted the topindex $*$ on *k* and *x*.