March 2004 version

The Role of the Hamiltonian in Dynamic Price Index Theory

By Chuan-Zhong Li and Karl-Gustaf Löfgren[1]

Uppsala University *Umeå University*

Department of Economics *Department of Economics*

Abstract

This paper is an attempt to investigate the cost-of-living index problem in a general equilibrium multi-sector growth model. Instead of using the utility function as a compensation criterion as Konüs' (1924) did in his original contribution, we take advantage of the current-value Hamiltonian in constructing our dynamic price index. Since the Hamiltonian is a constancy-equivalent of future utilities (Weitzman, 1976), the dynamic price index is defined in terms of the minimum expenditure that, under alternative prices, would support the constancy-equivalent-utility level in the future. We show that, when properly deflated by the dynamic price index, the real comprehensive net national product becomes an ideal measure for dynamic welfare comparisons. For some special cases, we show that the dynamic price index reduces to the simple static index.

1. Introduction

This paper attempts to investigate the cost-of-living index problem in a general equilibrium multi-sector growth model, and to explore its implications for dynamic welfare comparisons. One reason for the paper is to exploit the implications for index theory of two results in Weitzman (1976, 2001). The result in the 1976 paper connects the optimal value function of the optimal growth problem to the corresponding current value Hamiltonian. The connection

[1] The authors acknowledge a research grant from Formas, and comments from Jeff La France, Dept. of Agricultural Economics, UCB.

is used to suggest a dynamic subindex based on the Hamiltonian to be used for compensation purposes. The result in the 2001 paper concerns the appropriate deflator to be able to move the utility metrics result presented in the 1976 paper into a money metrics. In our paper it is used to derive an expenditure function based on the money metrics version of the Hamiltonian, where the latter is directly proportional to the present value of future money wealth.

The set up is a perfect foresight economy, where all capital goods that are relevant for production are included. We treat the consumption services of the capital goods as consumption goods, and, hence, we implicitly assume that the economy produces the corresponding rental prices[2]. We also assume a full spectrum of competitive prices for capital goods, which for example means that net additions to pollution stocks are correctly priced[3].

The theory of intertemporal cost-of-living indexes has been surveyed and studied by Pollak (1989)[4]. He works in a perfect foresight discrete time intertemporal model, where capital goods have no visible role in the determination of the optimal consumption plan. He defines the intertemporal cost of living index in a similar way as in the first part of section 3 below. To start with, he uses a general utility function which lacks all separability properties and ends up with a rather sterile expenditure function.[5] In the concluding sections of the chapter, he moves to a subindex idea that was developed in Pollak (1975), where the utility function is weakly separable over time and/or over commodity groups[6]. In this manner, he can define single period subindexes that are "isolated from intertemporal complications". To construct a

[2] There are at least two ways to handle practical CPI- computations of the cost of consumption services from capital goods. One is to use existing rental prices, the other is to compute user costs. A classical, and essentially unsolved, practical problem is how to incorporate the cost of the consumption services of owner occupied housing. This problem is assumed away here, since adding the services derived from capital assets to the consumption vector implies that rental prices are included in the index formula. Moreover, under first best, rental prices and user costs produce the same answer. However, user costs in real world markets do not seem to be appropriately caught by current interest and tax rates, since the resulting index numbers become very volatile over time. Rental prices can, of course, also be volatile over time, but even if they are, they have at least the advantage of being directly observable. Hence, volatility in rental prices implies hopefully that the crossing, indeed, is rough. Rental prices have also "natural" revealed preference upper and lower bounds. The lower bound is the user cost of the owner (a lower rental price would make renting unprofitable), and the upper bound is the user cost of the tenant. This speaks in favor of rental prices, even if the markets that generate them typically are thin.

[3] This is not necessary for the results. It is enough to say that there are no externalities.

[4] See chapter 3.

[5] It looks much like equation (18) below. The latter is, however, based on a utility function which is additively separable over time, implying that demand curves are independent over time.

subindex for period *t,* he holds the levels of consumption in all other periods fixed and calculates the ratio of the expenditure on the goods in period t required to reach a particular indifference curve. The result looks like a static index number. However, as our approach shows in section 3, it is not clear that saving can be neglected in an intertemporal environment, even if the utility function is additively separable over time[7].

2.The static (atemporal) price index theory[8]

The modern static (atemporal) price index number theory was originally developed by Konüs (1924). It is based on the existence of an optimal expenditure function. The idea can be illustrated by solving the static optimization problem of a representative consumer, i.e.

$$\max_{c} u(\mathbf{c}) \tag{1}$$

subject to

$$\mathbf{pc} = y \tag{2}$$

where $\mathbf{c} = (c_1, c_2, ..., c_n)$ is a vector of consumption goods, $\mathbf{p} = (p_1, p_2, ..., p_n)$ is a vector of the corresponding prices, and y is the exogenous income. The utility function $u(\mathbf{c})$ is assumed to be strictly convex and C^2. The resulting optimal value function can be written as

$$u^* = v(\mathbf{p}, y) = \max_{c} u(\mathbf{c}) \tag{3}$$

and the corresponding optimal expenditure function is obtained by solving for y by inverting the optimal value function. Since the optimal utility level u^* is an increasing function of income, the inverse exists and we can write

$$y = v^{-1}(\mathbf{p}, u^*) \tag{4}$$

[6] Pollak prefers a non-symmetric form of separabilty, where one group of commodities is separable from another, but not necessarily the other way around.
[7] The paper by Alchian and Klein (1973) contains, as far as we can understand, a related idea.
[8] An almost up to date version of the theory was presented by Konüs (1924). The paper was reprinted in Econometrica (1939) with introductory remarks by Henry Schultz, who had a translation of the Russian version prepared in 1934/35. He used the translation during his lectures at Chicago University. Other early and important contributions are papers by Allen (1933) and Houthakker (1952). Allen developed the theory independently of Konüs.

A compensating price index now answers the following question: Suppose that the maximum attainable utility level at prices $\mathbf{p}_0$ and income y_0 is $u^0 = u(\mathbf{c}_0)$ with $y_0 = \mathbf{p}_o\mathbf{c}_o$, what is the minimum expenditure (income) level that, under alternative prices $\mathbf{p_1}$ would yield the same utility level, u^0? The answer is obtained by using the expenditure function in equation (4) with prices equal to $\mathbf{p}_1$ and utility level equal to u^0. The solution can be written

$$y_1 = v^{-1}(\mathbf{p}_1, u^0) = \mathbf{p}_1\mathbf{c}^c(\mathbf{p}_1, u^0) \tag{5}$$

and the exact price index number formula becomes

$$\pi(u^{\mathbf{0}}) = \frac{y_1}{y_0} = \frac{\mathbf{p_1 c^c}(\mathbf{p}_1, u^0)}{\mathbf{p_0 c}_0(\mathbf{p_0}, u^0)} \tag{6}$$

where $\mathbf{c}^c(\mathbf{p}_1, u^0)$ is the vector of compensating demand functions, and $\mathbf{c}_0(\mathbf{p}_0, y_0)$ coincides with the Marshallian demand vector. The scalar $\pi(u^0)$ tells us how income at the original consumption vector has to be scaled to preserve the original utility level at the new prices, or how the income at the new prices has to be deflated to be comparable to the original-price-income.

The expenditure function in equation (3) is called the direct expenditure function. The indirect expenditure function is obtained by substituting the demand functions into the utility function in equation (5) to obtain $y = v^{-1}(\mathbf{p}, u(\mathbf{p}_0, y_0)) = \mu(\mathbf{p}, u(\mathbf{p}_0, y_0))$. For $\mathbf{p} = \mathbf{p_0}$, we obtain y_0, and for $\mathbf{p} = \mathbf{p_1}$, $y_1 = \mu(p_1, u(p_0, y_0))$.

The practical applications of the index formula in equation (6) involves the approximation of the expression. This can be done by, for example, keeping the denominator intact and approximating the nominator by the cost of the base year consumption vector, evaluated at the price vector at the time for compensation. This results in Laspeyres' price index. Alternatively, the nominator can be estimated by the cost of the consumption vector at the year of compensation evaluated at current prices, while the denominator is the consumption vector at the time of compensation, evaluated at base year prices. This results in Paasche's index formula. The Laspeyres' formula will overcompensate the utility level at zero, while the Paashe index will under-compensate the cost of buying the utility level, at base year prices, at the time of compensation at base year prices. Both results are a consequence of the convexity of preferences. They were first proved by Konüs (1924) and are sometimes referred to as Konüs inequalities.

3. Index numbers in an intertemporal context

It is relevant to ask how the index number problem changes in an intertemporal world where consumption takes place simultaneously with capital accumulation. Firstly, capital accumulation means that the prices of capital goods enter the picture as well as the prices of the services that are rendered by the capital stocks. For example, how do we price the consumption services that are rendered by private owner occupied housing? Secondly, the expenditure function has to be generalized into an intertemporal framework.

We will, in this section, shed some light on these topics. To start with the first problem, we need a general multi-sector growth model. Following Weitzman (1976, 2001, 2003), let $\mathbf{c}(t) = (c_1(t),...,c_n(t))$ denote the vector of consumption at time t, and $\mathbf{k}(t) = (k_1(t),...,k_m(t))$ the vector of capital stocks at time *t*. The former contains all consumption goods relevant for human welfare. This means, in particular, that the consumption services rendered by the capital stocks are included. The vector of capital goods is comprehensive in the sense that it contains all goods that are relevant for the productive capacity of the economy. It means, for example, that it contains human capital stocks, natural resource stocks as well as stocks of pollution. Moreover, let $\mathbf{p}(t) = (p_1(t),..., p_n(t))$ be the nominal prices of consumption goods, including the rental prices of the consumption services rendered by capital goods at time t, and let $\mathbf{q}(t) = (q_1(t),...,q_m(t))$ be the corresponding prices of capital goods.

For this economy, define comprehensive NNP at time t as $NNP(t) = \mathbf{p}^*(t)c^*(t) + \mathbf{q}^*(t)\dot{k}^*(t)$, where $\dot{\mathbf{k}}(t)$ denotes the vector of net investment at time *t*. The top indexes denote that the economy follows a perfect foresight competitive/ optimal path.

We introduce the technology in terms of a multidimensional production possibility set. Hence, a consumption-investment pair $[\mathbf{c}(t), \dot{\mathbf{k}}(t)]$ is attainable at time t from the capital stock

$\mathbf{k}(t)$ if, and only if $\mathbf{c}(\mathrm{t}), \dot{\mathbf{k}}(\mathrm{t}), \mathbf{k}(\mathrm{t}) \in \mathrm{A}$, where $\mathbf{A}$ is a convex attainable production possibility set.

The general multi-sector growth problem can now be formulated in the following manner

$$Max \int_0^\infty u(\mathbf{c}(t)) e^{-\theta t} dt \tag{7}$$

subject to the constraints

$$\mathbf{c}(\mathrm{t}), \dot{\mathbf{k}}(\mathrm{t}), \mathbf{k}(\mathrm{t}) \in \mathrm{A} \tag{8}$$

and the differential equations

$$\dot{\mathbf{k}}(t) = \mathbf{i}(t) \tag{9}$$

with initial conditions

$$\mathbf{k}(0) = \mathbf{k}_0 \tag{10}$$

The maximum principle is valid and requires that the current value Hamiltonian

$$H(t) = u(\mathbf{c}(t)) + \boldsymbol{\lambda}^{\mathrm{c}}(t)\mathbf{i}(t) \tag{11}$$

is maximized with respect to $[\mathbf{c}(t), \mathbf{i}(t)]$ subject to the restriction (8). Here $\boldsymbol{\lambda}^{\mathbf{c}}$ is an n-dimensional vector of utility shadow prices of capital goods (co-state variables) which satisfies

$$\dot{\lambda}^{c}(t) = \theta\lambda^{\mathbf{c}}(t) - H_{\mathbf{k}}^{*}(t) \tag{12}$$

where $H_{\mathbf{k}}^{*}$ is the gradient of the maximized current-value Hamiltonian with respect to the capital stocks along the optimal path. In the Ramsey growth model, the nominal interest rate is determined by the marginal productivity of capital. In our model things are a bit more complicated, since the technology is very general and there are many capital stocks. A no-arbitrage argument is, however, available. If the consumer, along an optimal path, abstains from one dollar at time *t*, she would abstain from $\lambda_t(t)$ units of utility. At time $t+\Delta t$, she would enjoy $(1+\theta\Delta t)\lambda\ (t)$ units of utility. This is equivalent to consuming, in period $t+\Delta t$, the dollar amount

$$\frac{(1+\theta\Delta t)\lambda\ (t)}{\lambda\ (t+\Delta t)}=1+r(t)\Delta t \qquad (13)$$

where $r(t)$ is the nominal interest rate. Rewriting the equation by multiplying both sides with $\lambda\ (t+\Delta t)/\Delta t$, and taking limits, yields

$$\dot{\lambda}\ (t)=[\theta-r(t)]\lambda\ (t) \qquad (14)$$

which is the differential equation for the marginal utility of income along the optimal path. The solution is

$$\lambda\ (s)e^{-\theta(s-t)}=\lambda_0(t)e^{-\int_t^s r(\tau)d\tau} \qquad (15)$$

Equation (15) can be used to transfer the utility discount factor into the money discount factor. It is also instrumental for decomposing utility along an optimal path into a monetary component and a component measuring marginal utility of income..

). Technically, this is an infinite time multidimensional maximization problem with respect to consumption, which can be solved using the standard Lagrangian method. To derive the intertemporal budget constraint, one has to invoke the so-called Non-Ponzi game condition, which means that the present value of wealth, asymptotically, will remain It turns out that even in this generalized framework, there exists an (intertemporal) expenditure function. The following theorem is useful in order to see this.

Theorem 1. *If a time path* $\{\mathbf{c}^*(s),\mathbf{i}^*(s),\mathbf{k}^*(s)\}$ *for* $s\geq t$ *solves the dynamic optimization problem (1) - (3), with a maximal welfare* $W^*(t)=\int_t^\infty u(\mathbf{c}^*(s))\exp(-\theta(s-t))ds$, *then it also maximizes the present value of the stream of future consumption* $\{\mathbf{c}^*(s)\}$ *,i.e.*

$$M^*(t)\equiv\int_t^\infty \exp(-\int_t^s r(\tau)d\tau)\mathbf{p}^*(s)\mathbf{c}^*(s)ds \qquad (16)$$

evaluated at efficiency prices $\mathbf{p}^*(s) = \nabla U(\mathbf{C}^*(s))/\lambda(s)$ *and discounted at the money interest rate* $r(s)$ *for* $s \geq t$.

This theorem, which is a dynamic version of a well known separating hyper-plane result, is proved in Li and Löfgren (2002). It tells us that the money wealth measure $M^*(t)$ gives locally the same preference ordering of growth paths as the utility welfare measure $W^*(t)$, even though they are not defined in the same units of measurement. Theorem 1 can also be envisioned as a consequence of the representative agent's lifetime consumption allocation problem, i. e., to maximize (dynamic) welfare $W(t)$ subject to the lifetime budget constraint in (16nonnegative[9]. Another possibility is to use the fact, introduced in Dixit et al (1980), that the optimal growth path obeys certain competitive conditions. This was used in Li and Löfgren (2002) to prove Theorem 1.

In other words, given efficiency prices $\{p^*(s)\}_t^\infty$ and money wealth $M^*(t)$, we solve[10]

$$W^*(t) = \max_c \int_t^\infty u(\mathbf{c}(s))e^{-\theta s}ds \tag{7a}$$

subject to

$$M^*(t) = \int_t^\infty \mathbf{p}^*(s)\mathbf{c}^*(s)e^{-\int_t^s r(\iota)d\iota}ds \tag{16a}$$

The resulting solution will depend on the efficiency prices and the present value of money wealth. We can write the optimal value function as

$$W^*(t) = W[\{\mathbf{p}^*(s)\}_t^\infty, M^*(t)] \tag{17}$$

---

9 $M^*(t)$ consists of the present value of the capital stocks.

10 As pointed out by Diewert (2002), a key problem with a temporal equilibrium interpretation for a true intertemporal cost of living index has to do with aging. The expenditure function of an individual who expects to live for *T+1* periods at the beginning of period one, will no longer be relevant when he gets to period two, and (typically) expects to live for *T* periods. This problem is, let be somewhat artificially, circumvented here by introducing an infinite planning horizon[10].

Since utility wealth is monotone in $M^*(t)$ we can invert the value function (17) to obtain the dynamic expenditure function[11]

$$M^*(t) = W^{-1}[\{\mathbf{p}^*(s)\}_t^\infty, W^*] \tag{18}$$

However, to approximate $W^{-1}(\circ)$ for a price path different from $\{p^*(s)\}_t^\infty$ seems to be a rather hopeless task. We do not know prices in the future when we want to compensate a price change in the open interval $(t, t+dt)$, and we are left with an idea we cannot implement.

A less ambitious approach would be to look exclusively at the optimization problem on this interval. To this end, we use an ingenious observation in Weitzman (2001). Conditional on the market prices along the first best path of the economy, one can represent consumer choice at time *t* as the solution to the following optimization problem

$$\max_{[\mathbf{c}(t),\kappa(t)]} H(t) = u(\mathbf{c}(t)) + \lambda(t)\kappa(t) \tag{19}$$

subject to

$$\mathbf{p}(t)\mathbf{c}(t) + \kappa(t) = y(t) \tag{20}$$

where $\kappa(t) = \mathbf{q}(t)\mathbf{i}(t)$ is the total aggregate money value of net investments in the *n* capital stocks. The marginal utility of income is treated as a constant during the period as is money NNP, $y(t)$. Since the objective function in (19) is quasi-linear, the solution for current consumption is $\mathbf{c}(t) = \mathbf{d}(\mathbf{p}(t), \lambda(t))$, where $\mathbf{d}(\cdot)$ is the *m* dimensional vector of demand functions. The corresponding net investment value is $\kappa(t) = y(t) - \mathbf{p}(t)\mathbf{c}(t)$.

We now define an expenditure function

$$E(\mathbf{p}(t), \lambda(t), H) = \min_{c(t),\kappa(t)} [\mathbf{p(t)c(t)} + \kappa(t)] \tag{21a}$$

subject to

$$u(\mathbf{c}(t)) + \lambda(t)\kappa(t) \geq H \tag{21b}$$

[11] For an excellent analysis of intertemporal household theory, and the properties of the value and expenditure

Why use a money metrics measure of utility based on the current value Hamiltonian? The reason is near at hand. The Hamiltonian measures current utility that is obtained from current consumption plus future utility that is obtained from net investment today. Weitzman (1976) shows that the current value Hamiltonian is directly proportional to the current value of future utility along the first best path of the economy. The factor of proportionality being the utility discount rate, i.e.

$$H^*(t) = \theta\int_t^\infty u(\mathbf{c}^*(s)e^{-\theta(s-t)}ds = \theta W(t) \tag{22}$$

In other words, keeping the present purchasing power (including that arising from capital formation) constant means that future consumption possibilities (ceteris paribus) are kept intact. Using equation (22), we can write the expenditure function in equation (21a) in terms of the intertemporal value function

$$E(\mathbf{p}(t), \lambda(t), \theta W(t)) = \min_{c(t),i(t)} [\mathbf{p(t)c(t)} + \kappa(t)] \tag{21c}$$

A remaining problem seems to be that the marginal utility of income in (21b) moves over time, meaning that the utility function (Hamiltonian) changes over time, which, in turn, may make intertemporal comparisons of expenditure functions irrelevant for compensation purposes. However, over the short interval a compensatory index based on the expenditure function in equation (21a) can now be written

$$\pi(H) = \frac{\mathbf{p}'(t)\mathbf{c}^{c'} + \kappa^{c'}}{\mathbf{p}(t)\mathbf{c}(t) + \kappa} \tag{23}$$

where $\mathbf{c}^c(t) = D(\mathbf{p}'(t))$ is the compensating (Hicksian) as well as the (Marshallian) demand vector, and $\kappa^c(t) = H(t) - u(\mathbf{c}^c(t))$. Note that the nominator in (23) is the minimum expenditure required to, at the new prices $\mathbf{p}'(t)$, attain the same welfare level (H) obtained at the status quo prices $\mathbf{p}(t)$ and income $y(t)$. When $\mathbf{p}'(t) = \mathbf{p}(t)$, this expenditure is equal to the status quo income $\mathbf{p}(t)\mathbf{c}(t) + \kappa$.

functions the reader is referred to LaFrance (2001)

In other words, keeping the present purchasing power (including that arising from capital formation) constant means that future consumption possibilities (ceteris paribus) are kept intact.

One might claim that a price index based on only consumption would work, provided that income compensation is conditioned on the magnitude of net investment. However, in order to make it conditional on capital formation, net investment must be known. Moreover, the prices of investment goods may change as well as the prices of consumption goods. In other words, sub-indexes based on consumption can be used to simplify a dynamic approach to the cost-of-living-index. However, in order to fully reflect that we both consume today and, at the same time, invest to consume in the future, a compensation function should also reflect the saving decision. In this manner, we also connect the sub-index to the optimal value function.

4. An ideal deflator based on normalized prices

Although the dynamic compensatory index (23) is defined in a similar way as the static index in (6), it is not obvious that it can be used for deflating income from one point in time to another. The reason is that the current-value Hamiltonian function $H(t) = u(\mathbf{c}(t)) + \lambda(t)\kappa(t)$ is not stationary. This is due to the non-constant marginal utility of income $\lambda(t)$. In the best case, the index in (22) can be used to deflate counter-factual prices within the same time interval $[t, t+dt]$ when the marginal utility of income $\lambda(t)$ remains constant. Since our main interest is to construct a compensatory index to facilitate welfare comparisons over time, it is necessary to first normalize the utility price of investment $\kappa(t)$**,** i.e. the marginal utility of income $\lambda(t)$. For this purpose, we define the first deflator (the Ideal Weitzman Index)

$$\pi^0(t) = \lambda(t_0)/\lambda(t) = \frac{\tilde{\mathbf{p}}(t)\mathbf{c}(t_0)}{\mathbf{p}(t_0)\mathbf{c}(t_0)} \tag{24}$$

where $\tilde{\mathbf{p}}(t)$ denotes the "imputed" market clearing price that would be observed at time t if the market basket of goods consumed in the economy is $\mathbf{c}(t_0)$. The index is independent of the consumption vector, the benchmark. The term "ideal measure" is chosen by Weitzman (2001) to denote the ideal towards which the makers of a CPI - PPP- type index strive when they try to select a representative market basket straddling two economies, or two points in time in the same economy. Let be that the practical imputation problems are difficult to solve. In the latter case of two points in time in the same economy, the scalar $\pi^0(t)$ measures the price level at time t relative that of time t_0. If we choose a consumption vector in the base period, we have a market revealed measure of what it costs in the base year. The estimation of the nominator is much more difficult. One approach could be to try and construct intertemporal purchasing parity numbers, the function $\pi^0(t)$, possibly from historical data[12].

Benchmark independence is a property which makes the index consistent with, or perhaps better than , a mix of the two approaches presented in Francis Ysidro Edgeworth's pioneer work[13] on index numbers. These are known as *the aggregative approach*[14] *and the stochastic approach,* respectively . The latter has a broad focus, typically the general level of prices or value of money, without any specific reference to a particular group or application to a particular set of circumstances. The former can, like here, refer to the aggregate expenditure of all consumers with the object of saying something about the standard of living of the group. Under the stochastic approach, which is Edgeworth's first choice, the objective is the "determination of an index irrespective of quantities of commodities; based upon the hypothesis that there is a numerous group of articles whose prices vary after the manner of a perfect market, with changes affecting the supply of money"[15].

With help of the first price index above, the static-like problem in (19-20) can be rewritten as

$$\max_{[\mathbf{c}(t),\bar{\kappa}(t)]} H(t) = u(\mathbf{c}(t)) + \lambda(t_0)\bar{\kappa}(t) \tag{25}$$

subject to

$$\bar{\mathbf{p}}(t)\mathbf{c}(t) + \bar{\kappa}(t) = \bar{y}(t) \tag{26}$$

where $\bar{\kappa}(t) = \kappa(t)/\pi^0(t)$ denote the normalized value of investment, $\bar{\mathbf{p}}(t) = \mathbf{p}(t)/\pi^0(t)$ are the normalized consumption prices, and $\bar{y}(t)$ the normalized income (comprehensive NNP) at time t. With such a normalization, the price of investment $\bar{\kappa}(t)$ is made constant at the reference level $\lambda(t_0)$, and, thus, the current-value Hamiltonian functional form (26) becomes stationary over time. It now becomes possible to define an intertemporal indifference map over the n+1 dimensional space $(\mathbf{c}(t),\bar{\kappa}(t))$ by $H(\mathbf{c}(t),\bar{\kappa}(t)) = H^0$.

[12] Remember that the index depends only on time.
[13] The work was done in the 1880s, see Edgeworth (1925a,b).
[14] The name is due to Frisch(1936).
[15] Edgeworth (1925), p 233.

First, let us consider the base year problem at time t_0, i.e. maximizing the current-value Hamiltonian $H(t_0) = u(\mathbf{c}(t_0)) + \lambda(t_0)\overline{\kappa}(t_0)$ under the static-like budget constraint $\overline{\mathbf{p}}(t_0)\mathbf{c}(t_0) + \overline{\kappa}(t_0) = \overline{y}_0$. Let $(\mathbf{c}^0, \overline{\kappa}^0)$ denote the optimal solution, then the maximized current-value Hamiltonian can be expressed by $\hat{H}^0 = u(\mathbf{c}^0) + \lambda^0\overline{\kappa}^0$ and the expenditure by $\overline{y}_0 = \overline{\mathbf{p}}(t_0)\mathbf{c}^0 + \overline{\kappa}^0$. Now, our question is this: given a price vector $(\overline{\mathbf{p}}(t),1)$ for consumption and normalized investment at any time t, what is the minimum expenditure $\overline{y}_t$ which can support a current-value Hamiltonian at the same level as $\hat{H}^0$? Following Konûs (1924), we can express this expenditure by

$$\overline{y}_t = e(\overline{\mathbf{p}}(t),1,\hat{H}^0(\overline{\mathbf{p}}(t_0),1,\overline{y}_0) = \overline{\mathbf{p}}(t)\mathbf{c}^c + \overline{\kappa}^c \qquad (27)$$

where $\mathbf{c}^c$ and $\overline{\kappa}^c$ denote the compensating demand for consumption and investment such that $H(c^c, \overline{\kappa}^c) = \hat{H}^0$. With these devices, we can now define Hamilton-Konüs-dynamic price index by

$$\pi(t) = \frac{\overline{y}_t}{\overline{y}_0} = \frac{\overline{\mathbf{p}}(t)\mathbf{c}^c + \overline{\kappa}^c}{\overline{\mathbf{p}}(t_0)\mathbf{c}^0 + \overline{\kappa}^0} \qquad (28)$$

which can be written as

$$\pi(t) = \alpha\pi_c(t) + (1-\alpha)\pi_i(t) \qquad (28')$$

where $\alpha = \frac{\overline{\mathbf{p}}(t_0)\mathbf{c}^0}{\overline{\mathbf{p}}(t_0)\mathbf{c}^0 + \overline{\kappa}^0}$ and $1-\alpha = \frac{\overline{\kappa}^0}{\overline{\mathbf{p}}(t_0)\mathbf{c}^0 + \overline{\kappa}^0}$ are the weights for the consumer price index $\pi_c = \frac{\overline{\mathbf{p}}(t)\mathbf{c}^c}{\overline{\mathbf{p}}(t_0)\mathbf{c}^0}$ and the investment price index $\pi_i = \frac{\overline{\kappa}^c}{\overline{\kappa}^0} = \frac{\overline{\mathbf{q}}(t)\mathbf{i}^c}{\overline{\mathbf{q}}(t_0)\mathbf{i}^0}$, respectively.

It is seen that the dynamic price index is a weighted average of two static-like indexes, one for the current consumption and the other for investment related to the value of future consumption.

The dynamic price index defined in (28) will prove valuable for welfare comparisons over time. Consider the following two situations, one with (normalized) prices $\overline{\mathbf{p}}(t_0)$, $\overline{\mathbf{q}}(t_0)$ and national income (or comprehensive NNP) $\overline{y}_0$ at time t_0, and the other with (normalized) prices $\overline{\mathbf{p}}(t)$, $\overline{\mathbf{q}}(t)$ and national income (or comprehensive NNP) $\overline{y}(t)$ at any other time t. To compare the intertemporal welfare between the two situations, we can now make use of the dynamic price index defined in (28) to arrive at a very real national income (real NNP) measure. Let $\overline{y}_r(t) = \overline{y}(t)/\pi(t)$ be the deflated real income at time t, then the following claim is true

Proposition 1. *Intertemporal welfare at time t is higher than at time t_0 if $\overline{y}_r > \overline{y}_0$.*

The reason is simple: $\overline{y}_r(t) > \overline{y}_0$ implies that $\overline{y}(t) > \pi(t)\overline{y}_0 = \overline{y}_t$, i.e. the normalized income at time t is greater than the minimum expenditure required to reach the reference welfare level $\hat{H}^0$. Since marginal utility of income $\lambda\ (t_0) = \lambda(t)\pi(t) > 0$, the excess income $\overline{y}(t) - \overline{y}_t > 0$ also implies a higher welfare level at time t than at time t_0.

Note that our dynamic price index in (28) was defined in terms of the normalized prices rather than the original nominal prices. This means that to arrive at real income in Proposition 1, we have used the Hamilton-Konûs-Weitzman - chain index

$$\Pi(t) = \pi^0(t)\pi(t) \tag{29}$$

such that

$$\overline{\mathbf{p}}_r(t) = \frac{\mathbf{p}(t)}{\pi^0(t)\pi(t)} = \frac{\overline{\mathbf{p}}(t)}{\pi(t)} \text{ and } \overline{y}_r(t) = \frac{y(t)}{\pi^0(t)\pi(t)} = \frac{\overline{y}(t)}{\pi(t)} \tag{30}$$

However, it is also possible to arrive at a welfare criterion by staying with only one index, the ideal Weitzman – index, i.e., using normalized prices.

5. Compensating income in normalized prices

By the property of the static-like formulation in (25) and (26), it is possible to derive an exact expression of the compensating income and thereby the dynamic price index defined in (28). For this purpose, we consider the following two cases: given $\overline{\mathbf{p}}(t_0)\mathbf{c}(t_0) + \overline{\kappa}(t_0) = \overline{y}_0$, the maximum attainable current-value Hamiltonian is $\hat{H}(t_0) = u(\mathbf{c}^0(t_0)) + \lambda(t_0)\bar{\kappa}^0(t_0)$; given another counter-factual static-like budget constraint $\overline{\mathbf{p}}(t)\mathbf{c}(t) + \bar{\kappa}(t) = \bar{y}_0$ with the same income, the maximum attainable current-value Hamiltonian is $\hat{H}(t) = u(\mathbf{c}^*(t)) + \lambda(t_0)\bar{\kappa}^*(t)$. The difference in welfare levels between the two points in time is

$$\hat{H}(t) - \hat{H}(t_0) = u(\mathbf{c}^*(t)) - u(\mathbf{c}^0(t)) + \lambda(t_0)(\bar{\kappa}^*(t) - \bar{\kappa}^0(t))$$

which can be written as[16]

$$\hat{H}(t) - \hat{H}(t_0) = \lambda(t_0)[\overline{\mathbf{p}}(t)\mathbf{c}^*(t) - \overline{\mathbf{p}}(t_0)\mathbf{c}^0(t)) - \lambda(t_0)\int_{\overline{\mathbf{p}}(t_0)}^{\overline{\mathbf{p}}(t)} D(\overline{\mathbf{p}})d\overline{\mathbf{p}} + \lambda(t_0)(\bar{\kappa}^*(t) - \bar{\kappa}^0(t)] \quad (31)$$

Since the income levels are the same for both situations, we have

$$(\overline{\mathbf{p}}(t)\mathbf{c}^*(t) - \overline{\mathbf{p}}(t_0)\mathbf{c}^0(t)) + (\bar{\kappa}^*(t) - \bar{\kappa}^0(t)) = \bar{y}_0 - \bar{y}_0 = 0$$

and thus we can simplify (31) to

$$\hat{H}(t) - \hat{H}(t_0) = -\lambda(t_0)\int_{\overline{\mathbf{p}}(t_0)}^{\overline{\mathbf{p}}(t)} D(\overline{\mathbf{p}})d\overline{\mathbf{p}} \quad (32)$$

or, in a money metrics $\Delta = -\int_{\overline{\mathbf{p}}(t_0)}^{\overline{\mathbf{p}}(t)} D(\overline{\mathbf{p}})d\overline{\mathbf{p}}$

This implies that maximizing $H(t) = u(\mathbf{c}(t)) + \lambda(t_0)\bar{\kappa}(t)$ subject to a "new" budget constraint including the compensating income $\Delta = -\int_{\overline{\mathbf{p}}(t_0)}^{\overline{\mathbf{p}}(t)} D(\overline{\mathbf{p}})d\overline{\mathbf{p}}$, would yield the exact maximum utility level as $\hat{H}(t_0)$. This means that the Hamilton-Konûs - dynamic price index defined in equation (28) can be rewritten as

$$\pi(t) = \frac{\bar{y}_0 + \Delta}{\bar{y}_0} \qquad (33)$$

Now, consider the following two situations, one with (normalized) prices $\bar{\mathbf{p}}(t_0)$, $\bar{\mathbf{q}}(t_0)$ and national income (or comprehensive NNP) $\bar{y}_0$ at time t_0, and the other with (normalized) prices $\bar{\mathbf{p}}(t)$, $\bar{\mathbf{q}}(t)$ and national income (or comprehensive NNP) $\bar{y}(t)$ at any other time t. If $\bar{y}_r(t) = \bar{y}(t)/\pi(t) > \bar{y}_0$, or $\bar{y}(t) > \pi(t)\bar{y}_0 = \bar{y}_0 + \Delta$, then we can say that welfare at t is higher than at t_0. This is summed up in Proposition 2.

Proposition 2. *If the sum of income change and consumer surplus, both measured in normalized prices, is positive, i.e.,*

$\bar{y}(t) - \bar{y}_0 + CS(t) > 0$

or

$\Delta\bar{y}(t) + CS(t) > 0$

then welfare is higher at t than at t_0.

This result is closely related to a welfare comparison between two economies at the same point in time in Weitzman (2001).

The main theorem

The index formula in Propositions 1 is rather involved, and , since it involves two deflators, is far from practical. How wrong is our first approximation in equation (23) ? The answer is given in Theorem 1.

Theorem 1. *The index formulas in equations (28) and (23) are equivalent, i.e.,* $\pi(t) = \pi(H)$..

[16] The consumer surplus is obtained from rewriting the Hamiltonian into a money metrics, starting from a utility metrics, i.e. $u(\mathbf{c}) = \int_0^{c^*} \mathbf{u_c}(\mathbf{c})\mathbf{dc} = \lambda[\mathbf{p}^*\mathbf{c}^* + \int_{\mathbf{p}^*}^{\bar{\mathbf{p}}^c} \mathbf{d}(\mathbf{p})\mathbf{dp}] = \lambda_0[\mathbf{p}_\mathbf{r}^*\mathbf{c}^* + \int_{\mathbf{p}_\mathbf{r}^*}^{\bar{p}^c{}_r} \mathbf{d}(\mathbf{p}_\mathbf{r})\mathbf{dp}_\mathbf{r}]$, where $\bar{\mathbf{p}}^c$ is a vector of choke off prices.

Proof: $\pi(t) = \frac{\bar{y}_t}{\bar{y}_0} = \frac{\bar{\mathbf{p}}(t)\mathbf{c}^c + \bar{\kappa}^c}{\bar{\mathbf{p}}(t_0)\mathbf{c}^0 + \bar{\kappa}^0} = \frac{\mathbf{p(t)c^c} + \kappa^c}{\mathbf{p(t_0)c^0} + \kappa^0} \underset{*}{=} \pi(H)$

The first two equalities follow from definitions, the third after multiplying the two members by $\pi^0(t)$, and $\underset{*}{=}$ follows since the two implied minimization problems, under $[[\mathbf{p(t)}, \lambda(t)]\ and\ [\bar{\mathbf{p}}\mathbf{(t)}, \lambda(t_0)]$, yield the same answers, i. e., $\mathbf{c^c} = \mathbf{c^{c'}}$ and $\kappa^c = \kappa^{c'}$, q.e.d.

5. Special cases

It is worth mentioning that, for a few special cases, it is possible to simplify the composite index formula. Firstly, when the utility function $u(\mathbf{c})$ is homogenous of degree 1, then the consumer surplus term becomes zero so that the dynamic price index in (33) $\pi(t) = 1$. The reason is that, by Euler's theorem, we have $u(\mathbf{c}) = \nabla u(\mathbf{c})\mathbf{c} = \lambda^0 \bar{\mathbf{p}}\mathbf{c} = \lambda^0 \bar{y}$, i.e. a linear-in-income utility function, and thus no extra consumer term is left. In this case, the first price index becomes $\pi^0(t) = \frac{\bar{\mathbf{p}}(t)\mathbf{c}(t)}{\bar{\mathbf{p}}(t_0)\mathbf{c}(t_0)} = \frac{\bar{y}(t)}{\bar{y}(t_0)} = \frac{\lambda^0}{\lambda(t)}$, which is indeed observable. Secondly, when the current-value Hamiltonian is constant over time, i.e. $\dot{H}(t) = 0$, then we know that $\kappa(t) = \mathbf{q}(t)\mathbf{i}(t) = 0$ since $\dot{H}(t) = \theta\kappa(t)$. In this case, $H(t) = u(\mathbf{c}(t))$ is a constant. In the case of heterogeneous consumption goods, in contrast to a case with an aggregated homogenous consumption good, this does not imply that consumption is constant. The n-dimensional consumption vector is allowed to move along an indifference map. However, with a zero value of net investment, the dynamic price index $\pi(t) = \alpha\pi_c(t) + (1-\alpha)\pi_i(t)$ reduces to the simple static case $\pi_c(t)$ as $\alpha = 1$ in this case. At this stage, nothing more can be said about the first index $\pi^0(t)$ introduced by Weitzman. As a third special case, when both $\dot{H}(t) = 0$ and the nominal prices $\mathbf{p}(t)$ are constants, then we have $\pi^0(t) = 1$, $\pi(t) = 1$ and the HKW – chain index reduces to $\Pi(t) = \pi^0(t)\pi(t) = 1$.

6. Concluding comments

The main conclusion that follows from this paper is that any intertemporal cost-of-living. subindex should contain, not only the value of consumption, but also the value of investment (saving). The reason is that a consumer at each instant of time makes an optimal choice between consumption, resulting in instantaneous utility, and investment, resulting in future utility. The sum of these two components are, at each instant of time, proportional to the intertemporal optimal value function showing that a subindex based on the Hamiltonian does, indeed, reflect the underlying value function in an appropriate manner. The only way to avoid investment expenditures (saving) is to use a fully fledged intertemporal cost-of-living index, but as Pollak(1989) puts it, such an index involves "some morbid assumptions". These stem from the ambition to keep the present value of future utility constant. The subindex expenditure function, conditioned on normalized prices, keeps an annuity of the present value of future utility constant.

In this paper, we have defined the composite price index in two steps. In the first step, we use the Weitzman ideal price index to transform the nominal prices to (ideal) normalized prices, which makes the marginal utility of normalized income constant. Secondly, based on the quasi-linear current-value Hamiltonian functional form and the normalized prices, we define our dynamic price index as a cost-of-living index measure. The final composite index is the product of the two sub-indices. We have also shown that the results derived are consistent with the conventional wisdom that income difference plus a consumer surplus term reflect welfare changes even for the dynamic economy. Finally, the main theorem shows that a straight forward, but seemingly incomplete approach to the index number problem gives the appropriate result.

The beauty is that all data inputs required are the current entities that are, in principle, observable today, and the welfare conclusions, conditional on the future price path, are drawn for the entire future.

7. References

Alchian, A.A., and Klein B. On a Correct Measure of Inflation, *Journal of Money Credit and Banking 5, 173-191.*

Allen, R.G.D. (1933) On the Marginal Utility of Money and Its Applications, *Economica 12, 200-209.*

Allen, R.G.D. (1975) *Index Numbers in Theory and Practice*, London: Mac Millan

Diewert, E. (2002) Harmonized Indexes of Consumer Prices: Their Conceptual Foundation, Swiss Journal of Economics and Statistics 138, 547-637.

Edgeworth, F.Y. (1925a) *Paper Relating to Political Economy*, Volume 1, London.

Edgeworth, F.Y. (1925b) The Plurality of Index Numbers, *Economic Journal 35, 379-388.*

Gillingham, R.F. (1983) Measuring the Cost of Shelter of Homeowners: Theoretical and Empirical Considerations, *Review of Economics and Statistics 65, 254-265.*

Goodhart, C. (2001) What Weights should be Given to Asset Prices in the Measurement of Inflation, *The Economic Journal 111, 335-356.*

Houthakker, H.S. (1952) Compensated Changes in Quantities and Qualities Consumed, *Review of Economic Studies 19, 155-164.*

Konüs, A.A. (1924) The problem of the True Index of the Cost of Living, The Economic Bulletin of the Institute of Economic Conjuncture, Moscow, No 9-10, 64-71. Reprinted in Econometrica, January, 1939, 10-29.

LaFrance, J. (2001) Duality for the Household: Theory and Applications, in *Handbook of Agricultural Economics, Volume 1.* Edited by B.Gardner and G. Rausser

Li, C. Z. and Löfgren, K. G. (2002), On the Choice of Metrics in Dynamic Welfare Analysis: Utility versus Money Measures, *Umeå Economic Studies* No. 590.

Pollak, R.A. (1975) Subindexes of the Cost of Living Index, *International Economic Review, 16, 135-150.*

Pollak, R.A. (1989) *The Theory of the Cost of Living Indexes*, Oxford: Oxford University Press

Weitzman, M L (1976) On the Welfare Significance of National Product in a Dynamic Economy, *Quarterly Journal of Economics* 90, 156-162.

Weitzman, M.L (2001) A Contribution to the Theory of Welfare Accounting, *Scandinavian Journal of Economics* 103, 1-24.

Weitzman, M.L. (2003) *Income, Capital, and the Maximum Principle,* Cambridge: Harvard University Press.