Temporal Aggregation of the Returns of a Stock Index Series*

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Abstract

The effects of temporal aggregation on asymmetry properties and the kurtosis of returns based on the NYSE composite index are studied. There is less asymmetry in responses to shocks for weekly and monthly frequencies than for the daily frequency. Kurtosis is not smaller for the lower frequencies.

Key Words: Asymmetric moving average, QGARCH, estimation, kurtosis, Pearson IV, NYSE.
JEL Classification: C13, C22, C51, C53, G12, G14.

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1. Introduction

This study focuses on temporal aggregation of the daily returns of the composite index of the New York stock exchange (NYSE) to weekly and monthly frequencies. More than 37 years of daily returns are used for the empirical study. The study is based on the nonlinear asymmetric moving average (asMA) model (Wecker, 1981; Brännäs and De Gooijer, 1994) with an asymmetric quadratic GARCH specification (Brännäs and De Gooijer, 2003) for the conditional variance. The density is specified as Pearson type IV (Pearson, 1894, 1895; Premaratne and Bera, 2000; Brännäs and Nordman, 2003b).

For the conditional mean specification, the temporal aggregation results of Brewer (1973) have previously been employed by Brännäs and Ohlsson (1999) for the asMA model when extended by an autoregressive component. Drost and Nijman (1993) gave temporal aggregation results for weak GARCH models in combination with ARMA models. They also indicated that the density will approach the Gaussian as the aggregation horizon becomes longer. Empirical results appear to support this tendency in

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distributional convergence (Brännäs and Ohlsson, 1999). It will be an empirical issue to see whether this remains true in the current setup. Meddahi and Renault (2003) recently generalized the results of Drost and Nijman (2003) by, e.g., dropping the symmetry assumption inherent in the weak GARCH model and using a wider class of volatility models. This class contains our conditional variance model as a special case. The current model specification is then richer than, e.g., the GARCH(1,1) of Jacobsen and Dannenburg (2003).

The temporal aggregation in this study is performed by averaging from the daily index series and then forming the returns for different frequencies.

2. Model Class and Temporal Aggregation

The present modelling exercise builds on Brännäs and De Gooijer (2003) and Brännäs and Nordman (2003ab), who used the normal and the log-generalized gamma as well as the Pearson type IV densities, respectively. Their models have as conditional mean the asMA specification

$$e_t = \mathrm{E}(y_t|Y_{t-1}) = \theta_0 + \sum_{i=1}^{q_1} \theta_i^+ u_{t-i}^+ + \sum_{i=1}^{q_2} \theta_i^- u_{t-i}^-,$$

where $u_t^+ = \max(0, u_t)$, $u_t^- = \min(u_t, 0)$, and $Y_t = (y_1, \ldots, y_t)$ is the information set (Wecker, 1981; Brännäs and De Gooijer, 1994). The zero mean prediction error is $u_t = y_t - \mathrm{E}(y_t|Y_{t-1})$ and $u_t = \varepsilon_t h_t$, where $h_t > 0$ is a conditional standard deviation. The random variable ε_t is treated as having zero mean and unit variance, and being conditionally independent of h_t. The conditional variance of Brännäs and De Gooijer (2003) is an asQGARCH specification

$$h_t^2 = \mathrm{V}(y_t|Y_{t-1}) = \alpha_0 + \sum_{i=1}^{p_1} (\alpha_i^+ u_{t-i}^+ + \alpha_i^- u_{t-i}^-) + \sum_{i=1}^{p_2} \beta_i u_{t-i}^2 + \sum_{i=1}^{p_3} \gamma_i h_{t-i}^2.$$

The e_t and h_t^2 conditional moments both catch shocks u_t asymmetrically around zero. One may view the conditional mean as containing not a full risk measure h_t^2, as in the CAPM-M model, but an unrestricted reduced form of such a measure.

The density specifications target ε_t, but knowing more about the implied marginal or conditional density of the y_t variable is obviously also of interest. Brännäs and De Gooijer (1994) demonstrated that the marginal density for the ARasMA$(1,1)$ model (no conditional heteroskedasticity) under a Gaussian assumption on the $\{u_t\}$ sequence can be skewed. Brännäs and De Gooijer (2003) obtained and used some partial moment results for an asMA-asQGARCH model under a Gaussian assumption on ε_t.

We use standardization of the Pearson IV density to zero mean and unit variance, i.e. we use $\varepsilon = (\eta - \mu)/\sigma$ where μ is the mean and σ the standard deviation of the Pearson IV distributed variable η. The density for $u = \varepsilon h = (\eta - \mu)\sigma^{-1} h$ is obtained from the density of η as $f(u) = f_\eta(u)\partial\eta/\partial u$. Additional details are given in the Appendix. The Pearson IV density contains, e.g., widely used densities such as t and noncentral t as special cases and the Gaussian density as a limiting case.

Brewer (1973), Brännäs and Ohlsson (1999) and others used $q_i^* = [(k-1+q_i)/k]$ to relate the order q_i of the daily level to the order q_i^* of the aggregate level for MA or asMA models. The $[x]$ denotes the integer value of x and k is the time span, e.g., $k = 5$ for weekly data. For small q_i we expect the order of a MA or asMA model to remain unaltered by temporal aggregation. The presence of conditional variance has no bearing on this result.

Drost and Nijman (1993) in a slightly different context found that temporal aggregation of a flow variable yields a weak GARCH model also at the aggregate level. They also gave expressions relating the parameters of two aggregation levels. Empirically they found the persistence parameter to become smaller. Jacobsen and Dannenburg (2003) reported similar results in a multi-country study.

Meddahi and Renault (2003) extended the weak GARCH framework of Drost and Nijman (1993) by their square-root stochastic autoregressive (SR-SARV) model. This is closed under temporal aggregation. It can be demonstrated that the asQGARCH belongs to the SR-SARV class so that asQGARCH will remain the specification also at lower frequencies. Meddahi and Renault (2003) among other things also showed that skewness at a low frequency may be due to genuine skewness or to leverage effects at a higher frequency. The presence of a leverage effect at a lower frequency is due to leverage at a higher frequency.

3. Results

The estimation results are based on the New York Stock Exchange composite index for the period December 31, 1965 – May 2, 2003 ($T = 9741$ observations, source: Datastream). Daily returns are formed as $y_t = 100[\ln(I_t) - \ln(I_{t-1})]$, where I_t is the index. For lower frequencies I_t corresponds to the average over the appropriate period. The daily series has mean 0.023, variance 0.801, skewness -1.51, and kurtosis 37.75. Nonparametrically (kernel) estimated densities for the daily, weekly and monthly frequencies are displayed in Figure 1. There appears to be less peaked shape for lower frequencies and average returns and standard deviations are larger for the lower frequencies. Figure 2 exhibits the autocorrelation functions for the series and the squared series. At lag one autocorrelations increase as frequency decreases, while for larger lags there are small and no systematic differences.

We present results according to the following plan. Using daily data we search for the best (in terms of the AIC and SBIC information criteria) conditional mean function using conditional ML under normality. Both information criteria vary little across specifications and we choose the most parsimonious representation among those with identical criterion values (two decimals). Next, we add the asQGARCH(1,1,1) to the conditional mean function and employ the Pearson IV density $f(u)$ to estimate the best fitting daily model along with the a, r, δ parameters by conditional ML. This model is employed for the other frequencies as well. Finally, we estimate restricted specifications, focusing on the asymmetry in the conditional mean and variance functions. Table 1 summarizes the estimation results for models estimated this way.

For the daily model the two asMA(1) estimates are rather close, but the log-likelihood values suggest that equality must be rejected by a likelihood ratio test. In

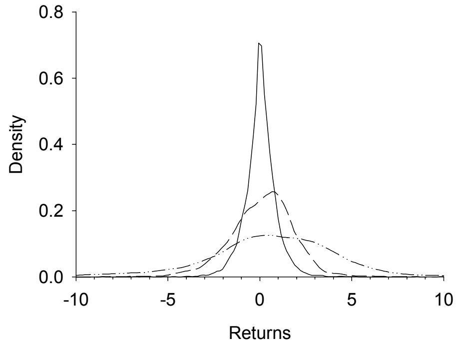

Figure 1: Kernel estimated densities for daily (solid line), weekly (dashed line) and monthly (dot-dashed line) return series.

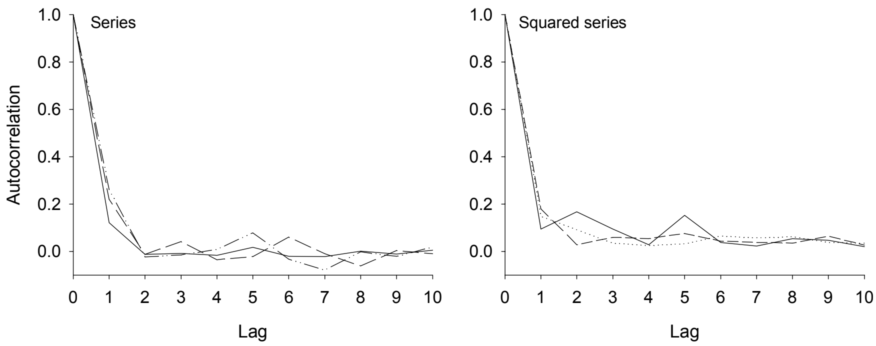

Figure 2: Autocorrelations for daily (solid line), weekly (dashed line) and monthly (dot-dashed line) return series.

Table 1: Parameter estimates (standard errors in parentheses).

Variable	Day		Week			Month		
			Conditional mean function					
u_{t-1}^{+}	0.142	–	0.280	–	–	0.285	–	–
	(0.018)		(0.044)			(0.072)		
u_{t-1}^{-}	0.152	–	0.213	–	–	0.293	–	–
	(0.018)		(0.045)			(0.091)		
u_{t-1}	–	0.147	–	0.248	0.245	–	0.289	0.273
		(0.010)		(0.023)	(0.023)		(0.045)	(0.047)
Constant	0.024	0.028	0.048	0.081	0.075	0.460	0.444	0.461
	(0.010)	(0.008)	(0.058)	(0.042)	(0.042)	(0.237)	(0.190)	(0.187)
			Conditional variance function					
Constant	0.008	0.008	0.164	0.159	0.079	0.833	0.753	0.974
	(0.002)	(0.002)	(0.074)	(0.070)	(0.029)	(0.688)	(0.640)	(0.489)
u_{t-1}^{+}	-0.045	-0.048	-0.151	-0.140	–	0.055	0.100	–
	(0.011)	(0.010)	(0.108)	(0.103)		(0.462)	(0.446)	
u_{t-1}^{-}	-0.062	-0.061	-0.324	-0.336	-0.438	-1.578	-1.609	-1.348
	(0.012)	(0.011)	(0.113)	(0.110)	(0.070)	(0.604)	(0.605)	(0.430)
u_{t-1}^{2}	0.054	0.055	0.065	0.064	0.032	-0.058	-0.063	–
	(0.007)	(0.007)	(0.030)	(0.029)	(0.016)	(0.058)	(0.057)	
h_{t-1}^{2}	0.928	0.928	0.842	0.843	0.846	0.822	0.826	0.786
	(0.005)	(0.005)	(0.027)	(0.026)	(0.024)	(0.058)	(0.056)	(0.058)
			Pearson IV parameters					
a	0.734	0.418	0.031	0.141	0.175	0.059	0.021	1.009
	(0.929)	(0.528)	(0.078)	(0.603)	(0.653)	(0.174)	(0.076)	(8.856)
r	6.034	5.915	10.766	10.931	10.602	6.369	6.375	5.682
	(0.441)	(0.186)	(2.019)	(1.928)	(2.061)	(3.121)	(3.126)	(2.560)
δ	0.359	0.424	3.016	3.157	3.037	2.244	2.251	1.919
	(0.159)	(0.157)	(1.000)	(0.941)	(1.035)	(1.613)	(1.563)	(1.248)
T	9740	9740	1942	1942	1942	449	449	449
$\ln L$	-10949.5	-10953.7	-3629.7	-3630.2	-3631.6	-1146.6	-1146.6	-1147.7
LB_{10}(sr)	15.34	14.99	11.14	11.14	11.84	5.05	4.70	7.43
LB_{10}(sq-sr)	11.27	10.61	9.36	9.51	8.41	6.68	9.69	6.75
Variance (u_t)	0.790	0.789	3.059	3.057	3.057	12.245	12.203	11.900
Skewness	-0.47	-0.47	-0.57	-0.56	-0.57	-0.54	-0.56	-0.54
Kurtosis	4.42	4.37	3.05	2.97	3.15	1.11	1.18	1.30

Notes: LB_{10}(sr) and LB_{10}(sq-sr) is the Ljung-Box test against serial correlation in standardized residuals and their squares, respectively. $\ln L$ is the log-likelihood function value.

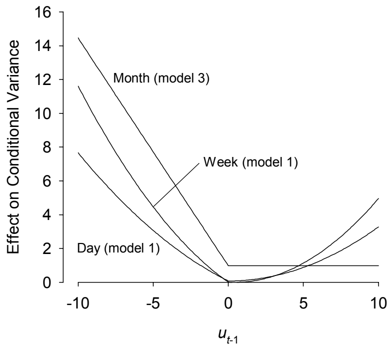

Figure 3: Effects of changes in u_{t-1} on the conditional variance h_t^2 (h_{t-1}^2 set at zero).

the asQGARCH model there is a negative effect of u_{t-1}^+, while a larger and positive effect of u_{t-1}^- (as u_{t-1}^- takes only 0 or negative values). Figure 3 demonstrates that u_{t-1} cannot give rise to negative h_t^2. Note also that neither of the daily models (nor any other model) have excess serial correlation in standardized (i.e. $\hat{\varepsilon}_t = \hat{u}_t/\hat{h}_t$) nor squared standardized residuals.

For models estimated at weekly and monthly frequencies the sizes of parameters are larger than those of the daily model, with the exception of the effect (the persistence) of h_{t-1}^2 which becomes smaller with lower frequency. In neither case can a MA(1) specification of the conditional mean be rejected against the more general asMA(1) specification. For weekly data the effect of u_{t-1}^+ is not significant in the asQGARCH, and at the monthly frequency neither of the effects of u_{t-1}^+ nor u_{t-1}^2 are significant. The response to shocks in u_{t-1} is asymmetric at all frequencies, and the asymmetry appears to be stronger at lower frequencies, cf. Figure 3.

Hence, the nonlinearity in the conditional mean function disappears as the sampling frequency becomes lower. For the asQGARCH, persistence becomes weaker with lower sampling frequency, while the asymmetric response to shocks through u_{t-1} appears to become stronger.

For the localization parameter of Pearson IV, the null hypothesis of $a = 0$ can not be rejected for any of the models. The r parameter has estimates around 6 for daily and monthly frequencies, but around 11 for the weekly frequency. This suggests that the unstandardized $f_\eta(\eta)$ density is most leptokurtic for the weekly data.[1] For the $f(u)$ density the shape is, however, influenced by the other parameters of the model. The estimates of δ are throughout positive (negative skewness) and significantly so for daily and weekly frequencies.

In summary, the Pearson IV parameters suggests that a non-central t-density may be a good approximation for daily and weekly data, while a t-density appears sufficient

[1] cf. the Appendix for the density definitions.

for monthly data.[2]

4. Conclusions

The results support the finding of Brännäs and Ohlsson (1999) that nonlinearity or asymmetry in the conditional mean asMA specification disappears as the sampling frequency becomes lower.

The persistence of shocks in the conditional variance becomes smaller, while the short-term response to shocks becomes larger but remains asymmetric. Except for the asymmetry in response, similar results were also found by Jacobsen and Dannenburg (2003) in their multi-country study. The persistent and strengthened asymmetry in the response to shocks as frequencies become lower is explained by the results of Meddahi and Renault (2003). The estimated Pearson IV densities become more symmetric with lower sampling frequency.

Appendix

The Pearson type IV density was introduced by Pearson (1894, 1895). Following Kendall and Stuart (1969, ch. 6) the density arises as the solution to the derivative

$$\frac{df_\eta(\eta)}{d\eta} = \frac{(\eta-\beta)f_\eta(\eta)}{b_0+b_1\eta+b_2\eta^2}, \quad b_1 \neq 0, b_2 \neq 0,$$

where the roots of $b_0+b_1\eta+b_2\eta^2=0$ are complex. The solution is a density function that we may write

$$f_\eta(\eta) = c_{r\delta}^{-1}\left(1+\frac{\eta^2}{a^2}\right)^{-(1+r/2)} \exp\left[-\delta \arctan(\frac{\eta}{a})\right],$$

where $c_{r\delta} = a\int_{-\pi/2}^{\pi/2}\cos^r(\omega)\exp(-\delta\omega)d\omega$ is a constant that depends on a, r and δ but not on the observations. The resulting density has skewness (using Kendall and Stuart, 1969, p. 153) $s_\eta = \mathrm{E}(\eta-\mu)^3/[\mathrm{E}(\eta-\mu)^2]^{3/2} = -4\delta(r-2)^{-1}[(r-1)/(r^2+\delta^2)]^{1/2}$. Obviously, the skewness measure is closely linked to the δ parameter. The kurtosis is $k_\eta = \mathrm{E}(\eta-\mu)^4/[\mathrm{E}(\eta-\mu)^2]^2 = 3(r-1)[(r+6)(r^2+\delta^2)-8r^2]/(r-2)(r-3)(r^2+\delta^2)$.

Premaratne and Bera (2000) used a δ definition of opposite sign and do not employ division by σ in their standardization. For that reason their h^2 is normalized to have a constant term equal to one. Their parameter μ does not correspond to the expected value as here, but is treated as a parameter reflecting the mode. We prefer the conventional standardization as it eases direct comparison of models. From the standardized density we obtain the density to be used for estimation on the form

$$f(u) = c_{r\delta}^{-1}\sigma h^{-1}\left(1+\frac{(\sigma u/h+\mu)^2}{a^2}\right)^{-(1+r/2)} \exp\left[-\delta \arctan(\frac{\sigma u/h+\mu}{a})\right].$$

Numerical integration of $c_{r\delta}$ is fast in practise.

[2] One gets the kurtosis of the normal distribution for $\delta = 0$ with $r \to \infty$.

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