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Existence of equilibrium in financial markets: Hart's securities exchange model with consumption in the first period*

Jean Pietro Bonaldi Varón†

Abstract

Hart has established necessary and sufficient conditions for the existence of equilibrium in an economy consisting of two time periods in which agents trade assets whose returns depend on an uncertain state of nature. Hammond has enounced an equivalent condition from an alternative approach to Hart's model. In both cases, it is assumed that agents maximize the expected value of their utility in the second period, when the asset returns are paid. In this paper, Hart's model is modified in such a way that agents also value consumption in the first period and the implications of this modification on the conditions proposed by these authors are analyzed.

JEL Classification: D53

Keywords: General Equilibrium, financial markets, securities model.

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Existencia del equilibrio en los mercados financieros: el modelo de intercambio de activos de Hart con consumo en el periodo inicial.*

Jean Pietro Bonaldi Varón†

Abstract

Hart ha establecido condiciones necesarias y suficientes para la existencia del equilibrio, en una economía de dos periodos en la cual los agentes intercambian activos cuyos retornos dependen de un estado de la naturaleza incierto. Hammond ha enunciado una condición equivalente a partir de una aproximación alternativa al modelo de Hart. En ambos casos, se supone que los agentes maximizan el valor esperado de su utilidad en el segundo periodo, cuando se pagan los retornos de los activos. En este artículo, el modelo de Hart se modifica de tal forma que los agentes también valoran el consumo en el primer periodo y se analizan las implicaciones de esta modificación sobre las condiciones propuestas por los autores mencionados.

Clasificación JEL: D53

Palabras clave: Equilibrio General, mercado financiero, modelo de activos.

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1 Introduction

Assets or securities whose returns are subject to almost unpredictable contingencies are usually exchanged in financial markets. This is why uncertainty plays a central role in the explanation of agents' behavior when they enter the market. Although it is true that individuals can not easily foresee the future value or dividends of those assets, it is also clear that they can gather information from multiple sources in virtue of which they can form expectations about these returns. Without having to consider the true probability distribution of future events determining asset returns, the subjective probabilities that agents assign to these events, based on some signals that markets may send them and on their previous knowledge of the relevant surrounding circumstances, let us model their behavior under the uncertainty conditions previously described and, therefore, increase our understanding of the market.

In this paper, special emphasis is made on the degree of divergence that agents' expectations may exhibit. The question of whether or not these expectations correctly predict future events is left aside. Following a line of research originally developed by Green (1973) and Hart (1974), among other authors that will be mentioned later, some conditions concerning agents' expectations about asset returns will be analyzed here in order to determine their implications for the existence of equilibrium in financial markets.

The model proposed here is a modification of Hart's (1974) securities exchange model. To describe the agents' trading decisions, Hart assumes that they are only interested in maximizing the expected value of their future utility from wealth, which directly depends on asset returns, but deliberately ignores the utility they can obtain in the period in which transactions take place, if they spend part of their income on something different than buying assets. Hart argues that this omission lets him derive necessary and sufficient conditions for the existence of equilibrium that are weaker than Green's (1973), who actually incorporates in his model the fact that agents value present consumption and not only care about expected utility derived from future wealth. It is convenient to observe, however, that Green's model differs from Hart's , and also from the one to be proposed here, in other aspects. The main difference being that, in his own terms, Green's model includes markets for both currently deliverable commodities and contracts for future delivery through which borrowing and lending is possible.

Hammond (1983) supports Hart's (1974) position sustaining that Green's conditions are unnecessarily strong; instead he proposes the overlapping expectations assumption and proves that it is equivalent to Hart's (1974) condition. Furthermore, Page (1987) makes use of Hart's model to express the conditions for the existence of equilibrium in terms of the non-existence of arbitrage prices. On the other hand, Carvajal and Riascos (2006) include a term on the utility function that depends on the wealth remaining after exchanging assets but before their returns are paid. In this aspect, the model proposed in this paper follows Carvajal and Riascos (2006).

Broadly speaking, the purpose of this paper is to show that the addition of this term in the agents' utility functions is not innocuous. More precisely, it will be proved in section 5, by means of an example, that in an economy identical to Hammond's (1983), except for the fact that agents value present consumption, the overlapping expectations condition is not sufficient for the existence of equilibrium.

Specifically, this work focuses mainly in the following aspects. To begin with, a similar model to Hart's (1974) is proposed, with an additional term in the utility functions used to incorporate in the model the fact that agents can derive utility from wealth in the first period. Then, generalizing a condition initially proposed by Carvajal and Riascos (2006) for an economy consisting of two agents, a necessary condition for the existence of equilibrium is established, and it is proved that it is equivalent to Hart (1974) and Hammond's (1983) conditions. To emphasize why the model with utility derived from consumption in the first period has to be considered, it is shown that the overlapping expectations condition is not sufficient for the existence of equilibrium in an economy in which utility in the first period matters. Finally, the existence of equilibrium in the proposed economy is proved, under some additional assumptions.

2 The model

As it was mentioned in the introduction, the economy considered in this paper is identical to Hart´s (1974) securities exchange model, except for the fact that consumers are also considered here to be interested in present consumption and not just in the utility that they expect to obtain from the returns of the assets they exchange in present markets. In fact, the economy described here consists of two time periods. In the first one, traders can exchange assets whose returns are paid in the second period. An asset return includes both the total value of one unit of the asset and the dividends that it could have paid and is contingent to the state of nature that takes place in the second period. Although traders ignore future asset returns in the first period, they are assumed to form expectations about the returns based on which they decide how to exchange assets.

Let there be a finite number of traders and assets in the economy, represented by the sets $\mathcal{I} = \{1, ..., I\}$ and $\mathcal{A} = \{1, ..., A\}$, respectively. In the second period, one of infinitely many possible states of nature occurs, each of which is identified with a unique vector of returns of the A assets in the economy. Additionally, it is assumed that those returns are non-negative, so each state of nature may be formally represented by a point in $\mathbb{R}^A_+$.

A bundle of assets is called a *portfolio* and is represented by a vector $z \in \mathbb{R}^A$. Any component of this vector can be positive, negative or zero, since all traders are allowed to buy or sell every asset. To be more specific, $z^i_a < 0$ means that trader i is short selling asset a, assuming no trader receives a positive endowment of any asset.

Traders are completely characterized by their beliefs or expectations on asset returns, utility functions describing their preferences for wealth and initial endowments of wealth. For each $i \in \mathcal{I}$, $u^i : \mathbb{R} \longrightarrow \mathbb{R}$ is a utility function satisfying the following assumptions:[1]

(A.1) *u^i is concave*

(A.2) *u^i is strictly increasing in $\mathbb{R}$.*

It is also assumed that consumers maximize the sum of the utility of present consumption and the expected utility of future wealth. Clearly, for each trader, his expected utility depends on his beliefs concerning future asset returns, and it seems reasonable to suppose those beliefs can be based on previous knowledge, such as someone else´s beliefs or another kind of information gathered from the market that may be reflected in asset prices. Following Hart (1974), it is accordingly supposed that trader´s expectations on future returns depend on asset prices.

Formally, let $p \in \mathbb{R}_+^A$ be a price vector for assets traded in the first period, $r \in \mathbb{R}_+^A$, a vector of returns or a state of nature, $\mathcal{B}\left(\mathbb{R}_+^A\right)$, the Borel $\sigma-$algebra in $\mathbb{R}_+^A$ and $M\left(\mathbb{R}_+^A\right)$, the set containing all probability measures in the measurable space $\left(\mathbb{R}_+^A, \mathcal{B}\left(\mathbb{R}_+^A\right)\right)$.[2] For any trader $i \in \mathcal{I}$, his beliefs about the asset returns are defined by a function $\mu^i : \mathbb{R}_+^A \rightarrow M\left(\mathbb{R}_+^A\right)$, where $\mu^i(p, X)$ denotes the probability that i assigns to the fact that the returns vector r belongs to the set X, given asset prices p. Two regularity assumptions are made initially about the functions representing traders' beliefs:

(A.3)* *For all $p \in \mathbb{R}_+^A$, there is a bounded subset C of $\mathbb{R}_+^A$ such that $\mu^i(p, C) = 1$.*

(A.4) *For all $i \in I$, the function $\mu^i : \mathbb{R}_+^A \rightarrow M\left(\mathbb{R}_+^A\right)$ is continuous in the topology of weak convergence of probability measures.*[3]

Although the economy concerned involves two time periods, Hart assumes the trader´s utility is completely determined by his second period expected utility of wealth. Here, an additional term is added to the utility function, representing the utility that traders gain from the remaining wealth they keep after trading assets in the first period. Additionally, it is assumed in this model that each trader i receives positive initial endowments of wealth, w_0^i and w_1^i, in the first and second periods, respectively.[4] So, given a price vector $p \in \mathbb{R}^A$ for the first period, trader i chooses a portfolio $z = (z_1, ..., z_m) \in \mathbb{R}^A$ to maximize the following function describing his preferences on the set of feasible portfolios at prices p:[5]

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[1] The same notation is used here for Hart (1974) and Hammond´s (1983) assumptions.

[2] For a formal definition of the measurable space $\left(\mathbb{R}_+^A, \mathcal{B}\left(\mathbb{R}_+^A\right)\right)$, see Cohn (1980).

[3] Hart's condition (A.3) has been replaced here temporarily by (A.3)*, which is weaker. Latter in this paper, the consequences of the assumption (A.3) on the existence of equilibrium in the financial market will be made clear.

[4] It would be broader to assume the second period initial endowment is contingent on the corresponding state of nature, as it is done by Carvajal and Riascos (2006). However, the proof of the existence of equilibrium proposed in section 6 would have to be modified.

[5] When defining function $V^i(p, z)$, pz and rz denote the dot product of the respective vectors and $\int f d\mu^i(p)$ is the Lebesgue integral of function f with respect to the measure $\mu^i(p)$. It should be noted that, if $f(r) = u^i\left(w_2^i + rz\right)$, since u^i is a coninuous function, by (A.2), then f is also continuous (in r) and Borel-measurable.

$$V^i(z,p) = u^i(w_0 - pz) + \int u^i(w_1 + rz)\, d\mu^i(p) \quad (1)$$

Note that, by assumption (A.3)*, $\int u^i(w_1 + rz)\, d\mu^i(p) = \int_C u^i(w_1 + rz)\, d\mu^i(p)$, then the function $f(r) = u^i(w_1 + rz)$ is $\mu-$integrable, for all $p \in \mathbb{R}^A_+$ and $z \in \mathbb{R}^A$, so V^i is well defined. Moreover, since u is concave and the dot product is linear, V^i is concave and therefore continuous in z, for a given p.

As it should be expected, traders are subject to certain restrictions. Particularly, at prices p, trader´s i budget restriction can be defined by the set:

$$B^i(p) = \left\{z \in \mathbb{R}^A : w_0 - pz \geq 0\right\} \quad (2)$$

Furthermore, an additional restriction is imposed here to the agents, concerning the impossibility to plan bankruptcy for the second period. Intuitively, if traders derive utility from wealth, it seems reasonable to suppose they do not consider viable to make transactions that could imply a negative level of wealth in the future. That is, if a trader assigns a positive probability to a state of nature, he will not be willing to acquire a portfolio that would lead him to receive a negative income if such a state occurs. Correspondingly, at prices p, trader i´s feasible portfolio set is defined as:

$$X^i(p) = \left\{z \in \mathbb{R}^A : \forall r \in S^i(p) : rz + w_1 \geq 0\right\} \quad (3)$$

where $S^i(p)$ is the support of the probability measure $\mu^i(p)$.[6]

Specifically in this aspect, the model proposed here is less general than Hart´s (1974), because his feasible portfolio sets are assumed just to satisfy some conditions, but they are not assigned any explicit form. In fact, the feasible sets $X^i(p)$ defined in 3 satisfy Hart's assumptions (A.5) - (A.7) and (A.9), and it is enough to assume further on that $S^i(p) \neq \{0\}$, for all $i \in \mathcal{I}$ and $p \in \mathbb{R}^A_+$, for condition (A.8) to be satisfied.[7] As it will be made clear, the fact that the supports of the probability measures contain more than one point is an immediate consequence of Hammond's assumption (A.12). It is convenient to add that Green (1973) imposes a restriction on the set of actions, equivalent to

[6] Let μ be a measure in $(X, \mathcal{B}(X))$. The support of μ, denoted by suppμ, is defined as the set $\{x \in X : \mu(U) > 0$, for all open set U such that $x \in U\}$

Note further that assumption (A.3)* holds if and only if, for all p, supp$\mu^i(p)$ is a bounded set.

[7] Hart make the following assumptions on the feasible sets X^i:

(A.5) X^i is closed and convex for all $i \in \mathcal{I}$.

(A.6) For each $i = 1, ..., n$ there exists $\widehat{z}^i \in X^i$ satisfying $\widehat{z}^i \ll \overline{z}^i$.

(A.7) Given $a \in \mathcal{A}$, there exists some $i \in \mathcal{I}$ with the following property: if $z'_a > z_a$ and $z'_k = z_k$ for all $k \in \mathcal{A}$, $k \neq a$, then $z \in X^i$ implies that $z' \in X^i$.

(A.8) Given any $i \in \mathcal{I}$, $p \in \mathbb{R}^A_+$ and $z \in X^i$, there exists $\widehat{z}^i \in X^i$ such that $V^i(\widehat{z}^i, p) > V^i(z^i, p)$

It should be noted that (A.6) does not apply strictly to our model since we have no initial endowments of assets $\overline{z}$. However, our representative trader i receives an initial amount of wealth $\overline{w}^i > 0$ and, at any price $p \in \mathbb{R}^A_+$, there are infinite portfolios $\overline{z}^i \in \mathbb{R}^A_+$, such that $p\overline{z}^i = \overline{w}^i$, for each of which there exists a $\widehat{z}^i \in X^i$ satisfying $\widehat{z}^i \ll \overline{z}^i$.

the previous one defining the feasible sets. Moreover, Hammond (1983) considers this a plausible restriction and shows how it satisfies his assumptions (A.10) and (A.11), an important fact that will be used later on in this paper.

Briefly speaking, any trader i in this economy faces the problem of choosing a z in $B^i(p) \cap X^i(p)$ that maximizes the function $V^i(\cdot, p)$, taken prices p as given. Therefore, trader i's individual demand correspondence, $Z^i : \mathbb{R}^A \rightrightarrows \mathbb{R}^A$, can be defined as follows:

$$Z^i(p) = \underset{z \in B^i(p) \cap X^i(p)}{\arg\max} V^i(z, p)$$

For the sake of briefness, the economy described in this section will be denoted $\mathcal{E}$ hereafter.

3 Individual Demands

It could be the case that the individual demand correspondence takes empty values. In other words, the utility maximization problem, as it was stated previously, could have no solution for some prices and, in that case, it is obvious that equilibrium would not exist. This is why it is important to establish necessary and sufficient conditions for the existence of such a solution. Carvajal and Riascos (2006), for example, prove (lemma 1) that $Z^i(p) \neq \varnothing$ if and only if the following conditions are satisfied:

1. There does not exist a $z \in \mathbb{R}^A$ such that $pz < 0$ and $\mu^i(p, \{r : rz < 0\}) = 0$, and
2. There does not exist a $z \in \mathbb{R}^A$ such that $pz = 0$, $\mu^i(p, \{r : rz < 0\}) = 0$ and $\mu^i(p, \{r : rz > 0\}) > 0$

Intuitively, it is clear that if there were a portfolio that does not fulfill condition 1 or 2, trader i would have incentives to demand an infinite amount of it, since, in either case, he could obtain more utility by augmenting the quantity hold of such portfolio, without assuming the risk of loosing with the transaction, according to his beliefs. This argument can be easily translated into formal terms to demonstrate that if any of those conditions does not hold, trader i´s demand for assets is not determined. The proof of that result, concerning economy $\mathcal{E}$, is very similar to the one proposed by Carvajal and Riascos (2006). However, their proof of the other direction of lemma 1 supposes an economy with a finite number of possible states of nature in the second period and does not apply to our case of interest.

Green (1973) also states necessary and sufficient conditions for the existence of a solution to the utility maximization problem faced by the agents in his model, described exclusively in terms of their expectations concerning future commodity prices. As it will be proved, under an additional assumption about trader´s beliefs, conditions 1 and 2 in Carvajal and Riascos' (2006) lemma 1 are equivalent to a natural adaptation of Green's condition, appropriate for the asset market, and also necessary and sufficient for the individual demand correspondences to be non empty.

In what follows, some concepts and results belonging to the field of Convex Analysis will be needed, so an appendix has been added including respective definitions and references. Most of this material was based on, or taken directly from Rockafellar (1970) and the corresponding notation is the same as in Hammond (1983). This said, $K^i(p)$ denotes the convex cone generated by $S^i(p)$, the support of the probability measure $\mu^i(p)$, and for any cone C, $C^+ = \{z : \forall r \in C, rz \geq 0\}$ is the polar cone of C.

Following Hammond (1983), and using the notation just introduced, a new assumption on trader´s beliefs is introduced now:

(A.12) *For all $i \in I$ and $p \in \mathbb{R}^A_+$, $intK^i(p) \neq \varnothing$, where $intC$ denotes the interior of C (in the usual topology of $\mathbb{R}^A$).*

As it will be shown later, this assumption is necessary for the existence of equilibrium in the proposed economy. This is why, despite the fact that it implies some restrictions on the functions $\mu^i : \mathbb{R}^A_+ \to M\left(\mathbb{R}^A_+\right)$ describing trader's beliefs, its introduction does not make the existence proof less general. Moreover, as it was mentioned previously, under this assumption it is possible to establish necessary and sufficient conditions for the existence of a solution to the utility maximization problem of the traders, just in terms of their beliefs or expectations concerning asset returns, as it is shown in theorem 1.

Before stating a proof of theorem 1, it is convenient to enounce a proposition that will be useful in its demonstration.

Proposition 1 *Let $K \subseteq \mathbb{R}^A$ be a closed convex cone and $k \in K^+$, then there exists $z \in K$ such that $z \neq 0$ and $kz = 0$ if and only if $k \in \partial K^+$, where $\partial K^+ = clK^+ \backslash intK^+$ is the boundary of K^+ and clK^+ its closure.*

Proof See the Appendix. ■

Theorem 1 *Let $p \in \mathbb{R}^A_+$ be a asset prices vector, if (A.12) holds then the following are equivalent:*

1. *$Z^i(p) \neq \varnothing$.*
2. *Conditions 1 and 2 in Carvajal and Riascos' (2006) lemma 1 are satisfied at prices p*
3. *$p \in intK^i(p)$*

Proof As it was mentioned before, Carvajal and Riascos (2006) proved that 1 implies 2. Now, suppose that 2 holds and let $z \in K^i(p)^+$, which is a closed convex cone (see Rockafellar, 1970. Section 14). If $pz < 0$, condition 1 of lemma 1 implies that $\mu^i(p, \{r : rz < 0\}) > 0$ or, equivalently, that there exists a $r \in S^i$ such that $rz < 0$, then $z \notin K^i(p)^+$, therefore $pz \geq 0$ and $p \in K^i(p)^{++} = \text{cl}K^i(p)$ (see Rockafellar, 1970. Section 14). If, on the other hand, $pz = 0$ and $z \in K^i(p)^+$, by condition 2 of lemma 1, $\mu^i(p, \{r : rz > 0\}) = 0$ or, equivalently, for all $r \in S^i$, $rz \leq 0$, and then, for all $k \in K^i(p)$, $kz = 0$. However, by assumption (A.12), $\mathrm{int}K^i(p) \neq \varnothing$, and it is easy to verify that $\mathrm{int}K^i(p) = \mathrm{int}\left(K^i(p)^{++}\right)$ (see Rockafellar, 1970. Sections 6 and 14), therefore there exists a $\widehat{k} \in \mathrm{int}\left(K^i(p)^{++}\right)$ and proposition **1** implies that for all $z \in K^i(p)^+$, if $z \neq 0$, then $kz > 0$, which contradicts a previous result. It follows that $z = 0$ or $pz > 0$, in other words, that there does not exist a $z \in K^i(p)^+$ such that $z \neq 0$ and $pz = 0$. Then, again by proposition **1**, $p \notin \partial K^i(p)^{++}$ or, equivalently, $p \notin \partial K^i(p)$ and given that $p \in \mathrm{cl}K^i(p)$ it follows that $p \in \mathrm{int}K^i(p)$.

To complete the proof, let's suppose that $p \in \mathrm{int}K^i(p)$. It will be shown now that in such case the set $X^i(p) \cap B^i(p)$ is compact. It is clear that $X^i(p)$ and $B^i(p)$ are closed and convex, in fact, $X^i(p)$ is the intersection of the closed upper half-spaces determined by all the hyperplanes $rz = -w$, with $r \in S^i(p)$, and $B^i(p)$ is the closed lower half-space determined by the hyperplane $zp = -w_o$, therefore $X^i(p) \cap B^i(p)$ is also closed and convex. Moreover, it is evident that $0 \in X^i(p) \cap B^i(p)$, and since this set is closed and convex, it is also bounded if and only if its unique direction of recession is 0 (Rockafellar, 1970. Theorem 8.4). To obtain a contradiction, let's suppose that $e \neq 0$ is a direction of recession of $X^i(p) \cap B^i(p)$, then, it is evident that e is a direction of recession of $X^i(p)$ and $B^i(p)$, so $zp + \lambda ep \leq -w_o$, for all $z \in B^i(p)$ and $\lambda > 0$, and then $ep \leq 0$. If e is a direction of recession of $X^i(p)$ it is easy to verify that e is also a direction of recession of $K^i(p)^+$, that is, if $zk \geq 0$ for all $k \in K^i(p)$ then $(z + \lambda e)k \geq 0$ for all $\lambda > 0$, hence, $ek \geq 0$ for all $k \in K^i(p)$ and then $e \in K^i(p)^+$, particularly, $ep \geq 0$, because $p \in \mathrm{int}K^i(p)$. Consequently $ep = 0$ and, since $\mathrm{int}K^i(p) = \mathrm{int}\left(K^i(p)^{++}\right)$, proposition **1** implies that $p \in \partial K^i(p)^{++}$, which is impossible. In conclusion, $X^i(p) \cap B^i(p)$ has no directions of recession different from zero, so it is bounded.

Finally, as the utility function V^i is continuous on the compact set $X^i(p) \cap B^i(p)$, it reaches a maximum in that set, so at prices p there exists a solution to trader i's maximization problem, and then $Z^i(p) \neq \varnothing$. ■

4 Necessary conditions for the existence of equilibrium

Definition 1 $\langle z, p \rangle \in \mathbb{R}^{AI} \times \mathbb{R}^A_+$ *denotes an equilibrium for the described economy if:*

1. $z^i \in Z^i(p)$ *for all* $i \in \mathcal{I}$
2. $\sum_{i=1}^{I} z^i = 0.$

As it can be observed, these are the standard properties defining an equilibrium. The first one requires that, at prices p, there exists a vector of demands, one for each trader in the economy, such that everyone maximizes his utility, and the second one establishes that the corresponding resource allocation should be feasible, furthermore, that the aggregate demand for assets equals the aggregate supply, which is assumed to be constant and equal to zero.

For a less general economy than the one previously described, consisting of two agents whose beliefs do not depend on the asset prices, with a finite number of possible states of nature and incomplete asset markets, Carvajal and Riascos (2006) proposed a necessary and sufficient condition for the existence of equilibrium called constrained-compatibility. Their objective is to show that when markets are incomplete, belief equivalence, which is a necessary condition for the existence of equilibrium in an economy with infinite periods and complete financial markets, as Araujo and Sandroni (1999) prove, is so strong that it is no longer necessary. In fact, Carvajal and Riascos (2006) prove that, if the market is sufficiently incomplete, the constrained-compatibility condition holds generically, and thus equilibrium exists generically, although agents' beliefs are not equivalent, i.e., they disagree on the null events.

Even though the question about the completeness of the market is not going to be dealt with explicitly in this paper, it is convenient to point out that the proposed model includes the two relevant cases. Note that every point in $\mathbb{R}_+^A$ represents a possible return for each of the A assets in the economy, therefore, there could be as many returns for each asset as there are (positive) real numbers. If this is the case, asset a can be described as a function $\alpha : \mathbb{R}_+^A \to \mathbb{R}_+$, where $\alpha(s)$ denotes its return if the state of nature $s \in \mathbb{R}_+^A$ occurs; in fact, $\alpha(s)$ is just the projection of s to its a-th coordinate. According to this representation, the asset market is incomplete since it is not possible to express every function from $\mathbb{R}_+^A$ to $\mathbb{R}_+$ as a linear combination of a given finite set of such functions. However, an additional restriction can be included to confine the model to the case where the states of nature belong to a finite subset of $\mathbb{R}_+^A$, as in Carvajal and Riascos (2006). This is why it can be said that Hart's general representation of the financial structure and the corresponding traders' expectations contains, as particular cases, both complete and incomplete markets. This said, we can turn our attention back on the constrained-compatibility condition proposed by Carvajal and Riascos.

Definition 2 *j's beliefs are constrained compatible with i's beliefs if there does not exist a $z \in \mathbb{R}$ such that $\mu^i(\{r : rz < 0\}) = 0$, $\mu^i(\{r : rz > 0\}) > 0$ and $\mu^j(\{r : r(-z) < 0\}) = 0$.*

It is clear that if such a $z \in \mathbb{R}$ exists, agents i and j could make a transaction, consisting of buying and selling portfolio z respectively, from which non of them would expect negative returns. Moreover, in such case agent i would assign a positive probability to the possibility of deriving some gain from that transaction, and therefore he could be interested in acquiring portfolio z in unlimited amounts which, on the other hand, agent j could sell him, being, at least, indifferent between making or not making the corresponding transaction.

For an economy with more than two traders, to have constrained-compatibility of beliefs between pairs of agents is also necessary for the existence of equilibrium, but no longer guarantees it. It is possible that a trader can not find an arbitrage opportunity as the one previously described, if he just considers making bilateral transactions with some other trader, but that he can instead take advantage from all the market, if he satisfies his demand for assets with the corresponding offers of various traders simultaneously. This is why it is necessary to generalize the notion of belief compatibility, for an economy with a finite, but arbitrary large, number of agents. [8]

Definition 3 *At prices $p \in \mathbb{R}_+^A$, traders' beliefs are constrained compatible if there does not exist a $z \in \mathbb{R}^{AI}$ such that:*

1. $\sum_{j \in \mathcal{I}} z^j = 0$
2. $\mu^j \left(p, \left\{r : rz^j < 0\right\}\right) = 0$*, for all* $j \in \mathcal{I}$.
3. $\mu^i \left(p, \left\{r : rz^i > 0\right\}\right) > 0$*, for some* $i \in \mathcal{I}$.

It is easy to verify that this condition on traders' beliefs is necessary for the existence of equilibrium in economy $\mathcal{E}$.

Proposition 2 *If $\langle z, p\rangle \in \mathbb{R}^{AI} \times \mathbb{R}_+^A$ is an equilibrium of economy $\mathcal{E}$ (definition 1), then traders' beliefs are constrained compatible at prices p.*

Proof This result is an immediate consequence of Carvajal and Riascos's (2006) lemma 1 and its proof, for economy $\mathcal{E}$, is very similar to the one proposed there. ■

The constrained-compatibility condition (definition 3) is equivalent, and similar in its formulation, to others founded in the literature. Hart (1974), for example, establishes necessary conditions for the existence of equilibrium very similar to those in the definition 3. Hammond (1983), expresses these conditions in terms of the supports of the probability measures defining traders' belief. Page (1987) and Werner (1987), on the other hand, enounce necessary and sufficient conditions based mainly on the non- existence of arbitrage prices and show that, under some assumptions, such conditions are equivalent to Hart and Hammond's. A relevant question is why this other conditions also apply to the case of an economy in which traders are interested in consumption in the first period.

Definition 4 *At prices $p \in \mathbb{R}_+^A$,[9] Hart's (1974) necessary condition is satisfied if there does not exists a $z \in \mathbb{R}^{AI}$ such that:*

[8] Alvaro Riascos suggested the compatibility condition of definition 3 in a conversation we had.

[9] Hart only considers prices in the simplex $\Delta = \left\{p \in \mathbb{R}_+^A : \sum_{a \in \mathcal{A}} p_a = 1\right\}$. This normalization is valid and convinient, given that agents only are interested in their (second period) expected utility.

Our model, however, can be described as if it contains a single consumption good called wealth, whose price is numeraire, in terms of which asset prices are expressed. In other words, asset prices in economy $\mathcal{E}$ can not be further normalized, so it is necessary to consider $\mathbb{R}_+^A$, and not just a proper subset, as the set containing all possible vector prices.

(H.1) $\sum_{j\in\mathcal{I}} z^j = 0$

(H.2) z^j *is a direction of recession of* $X^j(p)$ *and* $\widehat{S}_j^+ E_z^{+j}(p) + \widehat{S}_j^- E_z^{-j}(p) \geq 0$, *for all* $j \in \mathcal{I}$.

(H.3) $\mu^i\left(p, \left\{r : rz^j = 0\right\}\right) < 1$ *for some* $i \in \mathcal{I}$.

Where $\widehat{S}_j^+ = \lim_{w\to\infty} \frac{du^j}{dw}$, $\widehat{S}_j^- = \lim_{w\to-\infty} \frac{du^j}{dw}$, $E_z^{+j}(p) = \int_{\{r:rz\geqslant 0\}} rz\ d\mu^j(p)$ *and* $E_z^{-j}(p) = \int_{\{r:rz<0\}} rz\ d\mu^j(p)$[10]

Remark 1 *Note that, according to the definition of the feasible portfolio sets (3), the second part of condition (H.2) follows directly from the first one.*

In fact, if z^j is a direction of recession of $X^j(p)$, then $z^j \in K^j(p)^+$, in other words, $rz^j \geq 0$ for all r in the support of $\mu^j(p)$, hence $\mu^j(p, \{r : rz < 0\}) = 0$ and, consequently, $E_z^{-j}(p) = 0$. Since $\widehat{S}_j^+ \geq 0$, because u^j is increasing, and clearly $E_z^{+j}(p) \geq 0$, then $\widehat{S}_j^+ E_z^{+j}(p) + \widehat{S}_j^- E_z^{-j}(p) \geq 0$.

Proposition 3 *For a fixed* $p \in \mathbb{R}_+^A$, $z \in \mathbb{R}^{AI}$ *satisfies (H.1) - (H.3) if and only if it also satisfies conditions 1, 2 and 3 of definition 3. In other words, Hart's necessary condition and the constrained-compatibility condition are equivalent.*

Proof Let $z \in \mathbb{R}^{AI}$ satisfy (H.1) - (H.3), since z^j is a direction of recession of X^j, $z^j \in K^{j+}$, thus , for all $r \in S^j$, $rz \geq 0$, which implies that $\mu^j\left(\left\{r : re^j < 0\right\}\right) = 0$. By (H.3) there exists some i such that $\mu^i\left(\left\{r : re^i = 0\right\}\right) < 1$ or, equivalently, there exists a $r \in S^i$ such that $rz^i \neq 0$; since $z^i \in K^{j+}$, then $rz^i > 0$ and it follows that $\mu^i\left(\left\{r : rz^i > 0\right\}\right) > 0$.

Now let's suppose that $z \in \mathbb{R}^{AI}$ satisfies 1 - 3 (of definition 3). By condition 2, we know that $\mu^j\left(p, \left\{r : rz^j < 0\right\}\right) = 0$, for all $j \in \mathcal{I}$, then, for all $r \in S^j$, $re^j \geq 0$, i.e., $e^j \in K_j^+$. Therefore, if $x \in X_j$, $r \in S_j$ and $\lambda \geq 0$, it is clear that $r\left(x + \lambda e^j\right) + w_1 \geq 0$, thus e^j is a direction of recession of X^j. Finally, by condition 3, there exists an $i \in \mathcal{I}$ for which $\mu^i\left(\left\{r : rz^i > 0\right\}\right) > 0$, particularly, $\mu_i\left(\left\{r : rz^i = 0\right\}\right) < 1$. ■

Hammond (1982) introduces another condition on the supports of the probability measures representing traders' expectations, and then Werner (1987) interprets it as a non arbitrage condition in a more general model that includes markets for goods and assets. Furthermore, Hammond proves that, under assumption (A.12), his condition is equivalent to Hart's necessary condition (definition 4).

Definition 5 *Traders in the economy have overlapping expectations at* p *if* $\bigcap_{j\in\mathcal{I}} intK^j(p) \neq \varnothing$.

Proposition 4 *Hammond's (1982) theorem 1: Under assumption (A.12), traders have overlapping expectations at* p *if and only if the economy satisfies Hart´s necessary condition.*

[10] Hart admits the possibility that $\widehat{S}_j^- = \infty$, and assumes that $0 \times \infty = 0$.

Proof See Hammond (1982) ■

Proposition 5 *Under (A.12), traders have overlapping expectations at* p *if and only if their beliefs are constrained-compatible in the sense of definition 3.*

Proof Such equivalence is an immediate consequence of propositions 3 and 4. ■

Proposition 6 *The overlapping expectations condition is necessary for the existence of equilibrium in economy* $\mathcal{E}$*.*

Proof This follows directly from propositions 5 and 2. ■

Note that, as Hammond (1983) points out following Green (1973), a stronger necessary condition is that $p \in \bigcap_{j \in \mathcal{I}} \text{int}K^j(p)$. However, Hammond's assertion applies to an economy in which traders do not derive utility from their wealth in the first period, since this is precisely his case of interest. Despite this fact, as it was mentioned before, theorem 1 implies that this condition is also necessary for the existence of equilibrium in economy $\mathcal{E}$.

Corollary 1 *Given assumption (A.12), if* $p \in \mathbb{R}_+^A$ *is an equilibrium price vector for economy* $\mathcal{E}$*, then* $p \in \bigcap_{j \in \mathcal{I}} intK^j(p)$*.*

Proof If $p \in \mathbb{R}_+^A$ is an equilibrium price vector for economy $\mathcal{E}$ then, by the definition of equilibrium, $Z^j(p) \neq \varnothing$ for all $j \in \mathcal{I}$ and, in consequence, theorem 1 implies that $p \in \bigcap_{j \in \mathcal{I}} \text{int}K^j(p)$. ■

Hammond also proves that the overlapping expectations condition is sufficient for the existence of equilibrium in Hart's securities exchange model. In fact, by Hammond's (1984) theorem 2, under Hart's assumptions (A.1), (A.2), (A.4) and (A.6) besides Hammond's (A.10) and (A.11), if $\bigcap_{j \in \mathcal{I}} \text{int}K^j(p) \neq \varnothing$ for all p, then the equilibrium exists. Nevertheless, in the next section it will be shown, by means of an example, that in an economy that differs from Hart's (1974) because traders assign some value, in terms of utility, to consumption in the first period, the hyphothesis of Hammond's (1984) theorem 2, specifically the overlapping expectations condition, does not guarantee the existence of equilibrium. It should be noted that Page (1987) also proposes an example showing that the overlapping expectations conditions is not sufficient for the existence of equilibrium (in Hart´s (1974) model!), however, his example does not fulfill assumption (A.10) which is part of the hypothesis of Hammond's theorem.

5 Example

Let's suppose an economy consisting of two traders, $\mathcal{I} = \{1, 2\}$, and markets for two assets. In the first period both traders receive the same positive initial endowment of wealth, w_0, and the markets for the two assets are open. Asset

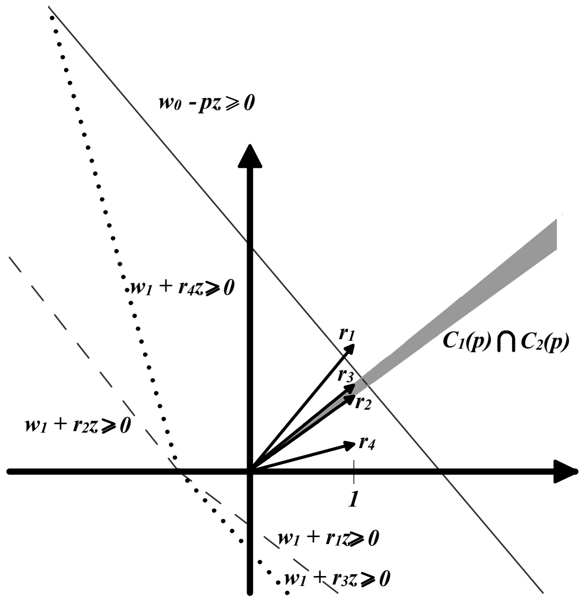

Figure 1:

returns are paid in the second period, and no more exchange is possible. $1 \in \mathcal{A}$ is a risk free asset; it pays one unit of wealth with certainty, $2 \in \mathcal{A}$, instead, is a risky asset; its return depends on the state of nature, and traders form expectations on this return based on its price. More precisely, if p_2 is the price of asset 2, trader 1 assigns the same probability to the fact that its return will be $\frac{1}{2}p_2 + \varepsilon$, or $\frac{1}{2}p_2 + 0.5\varepsilon$, for some $\varepsilon > 0$. On the other hand, trader 2 believes that the return of the risky asset could be $\frac{1}{2}p_2 + 0.6\varepsilon$ or $\frac{1}{2}p_2$, and assigns the same probability to both events. As it can be noted, the previous description defines two functions μ^1 and μ^2 from $\mathbb{R}^2_+$ to $M\left(\mathbb{R}^2_+\right)$. Moreover, the beliefs represented by these functions satisfy the overlapping expectations condition for all $p \in \mathbb{R}^2_+$, as it can clearly be seen in figure 1.[11]

Traders budget sets and portfolio feasible sets are defined as before, that is; $B^1(p) = B^2(p) = \left\{z \in \mathbb{R}^2 : w_0 - pz \geq 0\right\}$ and $X^i(p) = \left\{z \in \mathbb{R}^2 : \forall r \in S^i(p), w_1 + rz \geq 0\right\}$, for all $i \in \mathcal{I}$, and it is supposed further that $w_0 > w_1 > 0$.

The utility function of wealth for each trader is linear, moreover, $u^1(w) = u^2(w) = w$, and their total utility is the sum of their utility of wealth in the first period and the expected value of their utility in the second period, then:

$$V^1(p,z) = w_0 - p_1 z_1 - p_2 z_2 + \frac{1}{2}\left(z_1 + \left(\frac{1}{2}p_2 + \varepsilon\right) z_2\right) + \frac{1}{2}\left(z_1 + \left(\frac{1}{2}p_2 + 0.5\varepsilon\right) z_2\right) \text{ and}$$

[11] In figure 1 and further on, $C^i(p)$ denotes $\text{int}K^i(p)$.

$$V^2(p,z) = w_0 - p_1 z_1 - p_2 z_2 + \frac{1}{2}\left(z_1 + \left(\frac{1}{2}p_2 + 0.6\varepsilon\right) z_2\right) + \frac{1}{2}\left(z_1 + \frac{1}{2}p_2 z_2\right)$$

or, in a more concise fashion,

$$V^1(p,z) = w_0 + (1-p_1) z_1 + \left(0.75\varepsilon - \frac{1}{2}p_2\right) z_2 \text{ and}$$

$$V^2(p,z) = w_0 + (1-p_1) z_1 + \left(0.3\varepsilon - \frac{1}{2}p_2\right) z_2$$

Note that this is a particular case of economy $\mathcal{E}$, and that it satisfies all the assumptions in the hypothesis of Hammond's (1984) theorem 2 also. In fact, the utility functions satisfy (A.1) and (A.2) because they are linear. Besides, it is clear, intuitively, that functions μ^1 and μ^2 satisfy (A.4), since the probability measures $\mu^1(p)$ and $\mu^2(p)$ change continuously with changes in p.[12] Since the portfolio feasible sets have the general form in (3), they satisfy Hart's (1974) (A.6) and Hammond's (A.10) and (A.11), as Hammond asserts. Finally, it is clear that traders in this economy have overlapping expectations as it is shown in figure 1. Briefly speaking, the only difference between the economy just described and Hammond's (1983) is that in the latter, consumption in the first period matters.

Now it will be proved, by contradiction, that there does not exist an equilibrium for this economy. Lets suppose then that $p \in \mathbb{R}^2_+$ is an equilibrium price vector, by corollary 1, it follows that $p \in C_1(p) \cap C_2(p)$ and, for the economy concerned, this condition is equivalent to:

$$\frac{p_2}{2} + 0.5\varepsilon < \frac{p_2}{p_1} < \frac{p_2}{2} + 0.6\varepsilon \tag{4}$$

Two different cases will be considered now separately. First, if $0 < p_1 < 1$, it is clear that V^1 and V^2 are strictly increasing in z_1 and then, the corresponding equilibrium allocation must satisfy the budget constraint of both traders, $w_0 - pz \geq 0$, with equality. In fact, since the points in the supports of the probability measures $\mu^1(p)$ and $\mu^2(p)$ have non-zero coordinates, $(1,0)$ is a direction of recession of $X^i(p)$, hence, if $z \in X^i(p)$, but $w_0 - pz > 0$, there exists a $\lambda > 0$ such that $\widehat{z} = z + (1,0)\lambda \in X_i(p)$ and $w_0 - p\widehat{z} = w_0 - pz - \lambda p_1 = 0$. Clearly $V^1(p,\widehat{z}) > V^1(p,z)$, so z can not be an equilibrium allocation. Summarizing,

[12]Formally, let (p^n) be a sequence in $\mathbb{R}^2_+$ such that $p^n \to p$ and $B \in \mathcal{B}(\mathbb{R}^2_+)$, then $\mu^1(p^n, B) = \frac{1}{2}\left(\chi_B\left(1, \frac{1}{2}p^n + \varepsilon\right) + \chi_B\left(1, \frac{1}{2}p^n + 0.5\varepsilon\right)\right)$, where $\chi_B : \mathbb{R}^2_+ \to \{0,1\}$ is the indicator function of $B \in \mathcal{B}(\mathbb{R}^m_+)$. Now, if B is such that $\mu^1(p, \partial B) = 0$, then $\left(1, \frac{1}{2}p + \varepsilon\right) \in \text{int}B$ or $\left(1, \frac{1}{2}p + \varepsilon\right) \notin \text{cl}B$, and the same is true for $\left(1, \frac{1}{2}p + 0.5\varepsilon\right)$. In any of the four possible cases, those two points belong to open sets contained in B or in B^c, therefore $\mu^1(p^n, B) \to \mu^1(p, B)$ and, by Pormateau´s theorem (See Aliprantis and Border, 1994. Theorem 12.3) $\mu^1(p^n)$ converges weakly to $\mu^1(p)$. In a very similar fashion, it can be shown that μ^2 is continuous in the topology of weak convergence of probability measures.

if z^1 and z^2 were the equilibrium allocations corresponding to $p_1 < 1$, then $w_0 - pz^1 = w_0 - pz^2 = 0$, but $w_0 > 0$, so it follows that $z^1 + z^2 \neq 0$,i.e., z^1 and z^2 can not be equilibrium allocations, therefore $p_1 \geq 1$.

In the second case, where $p_1 \geq 1$, it will be shown initially that the demands for assets that maximizes traders' utility are on the boundary of their portfolio feasible sets. As it can be seen in figure 1, each of these sets is determined by two linear inequalities:

$$X_1(p) = \left\{ z \in \mathbb{R}^2 : z_1 + \left(\frac{p_2}{2} + 0.5\varepsilon\right) z_2 \geq -w_1 \text{ and } z_1 + \left(\frac{p_2}{2} + \varepsilon\right) z_2 \geq -w_1 \right\} \text{ and}$$

$$X_2(p) = \left\{ z \in \mathbb{R}^2 : z_1 + \frac{p_2}{2} z_2 \geq -w_1 \text{ and } z_1 + \left(\frac{p_2}{2} + 0.6\varepsilon\right) z_2 \geq -w_1 \right\}.$$

If $p_1 \geq 1$, one of these inequalities is satisfied with equality for both traders, more specifically, if $z^1 \in Z^1(p)$ and $z^2 \in Z^2(p)$ then, as it will be shown now:

$$z_1^1 + \left(\frac{p_2}{2} + 0.5\varepsilon\right) z_2^1 = -w_1 \text{ and} \tag{5a}$$

$$z_1^2 + \left(\frac{p_2}{2} + 0.6\varepsilon\right) z_2^2 = -w_1 \tag{5b}$$

So let's suppose that $z^1 \in X^1(p) \cap B^1(p)$, but $z_1^1 + \left(\frac{p_2}{2} + 0.5\varepsilon\right) z_2^1 > -w_1$. Clearly, there exists a $\delta > 0$ such that $z_1^1 - \left(\frac{1}{2}p_2 + \varepsilon\right)\delta + \left(\frac{p_2}{2} + 0.5\varepsilon\right)\left(z_2^1 + \delta\right) = z_1^1 + \left(\frac{p_2}{2} + 0.5\varepsilon\right) z_2^1 - 0.5\varepsilon\delta = -w_1$. Let $\left(\widehat{z}_1^1, \widehat{z}_2^1\right) = \left(z_1^1 - \left(\frac{1}{2}p_2 + \varepsilon\right)\delta, z_2^1 + \delta\right)$, then:

$$V^1\left(p, \widehat{z}^1\right) = w_0 + (1 - p_1)\left(z_1^1 - \left(\frac{1}{2}p_2 + \varepsilon\right)\delta\right) + \left(0.75\varepsilon - \frac{1}{2}p_2\right)\left(z_2^1 + \delta\right)$$

$$= V^1\left(p, z^1\right) + \left(p_1 p_2 + \left(2p_1 - \frac{1}{2}\right)\varepsilon - 2p_2\right)\frac{1}{2}\delta$$

By (4), in equilibrium $2p_2 < p_1 p_2 + 1.2p_1\varepsilon$ which, together with $p_1 \geq 1$, implies that $p_1 p_2 + \left(2p_1 - \frac{1}{2}\right)\varepsilon \geq p_1 p_2 + \left(2p_1 - \frac{1}{2}p_1\right)\varepsilon = p_1 p_2 + 1.5p_1\varepsilon > 2p_2$. Therefore, $p_1 p_2 + \left(2p_1 - \frac{1}{2}\right)\varepsilon - 2p_2 > 0$ and then $V^1\left(p, \widehat{z}^1\right) > V^1\left(p, z^1\right)$. Besides, it can be easily verified that $\widehat{z}_1^1 + \left(\frac{p_2}{2} + \varepsilon\right)\widehat{z}_2^1 \geq -w_1$ and $w_0 - p\widehat{z}^1 \geq 0$, so $\widehat{z}^1 \in X^1(p) \cap B^1(p)$ and it can be concluded that $z^1 \notin Z^1(p)$.

With a very similar argument, it can be shown that, if $z^2 \in X^2(p) \cap B^2(p)$ but $z_1^2 + \left(\frac{p_2}{2} + 0.6\varepsilon\right) z_2^2 > -w_1$, there exists a $\delta > 0$ such that $\widehat{z}^2 = \left(z_1^2 + \delta\frac{p_2}{2}, z_2^2 - \delta\right) \in X^2(p) \cap B^2(p)$ and $V^2\left(p, \widehat{z}^2\right) > V^2\left(p, z^2\right)$, then $z^2 \notin Z^2(p)$.

Summarizing, if p is an equilibrium price vector with $p_1 \geq 1$ and z^1, z^2 are the corresponding demands for assets, then they satisfy equations (5a) and (5b), respectively. Furthermore, it must be the case that $z_2^1 \neq 0$ (and hence $z_2^2 \neq 0$ also), otherwise $z_1^1 = -w_1$ and $z_1^2 = -w_1$ contradicting the fact that, in equilibrium, $z_1^1 + z_1^2 = 0$.

Therefore, since the utility functions V^1 and V^2 are linear and their gradients are the vectors $\left(1-p_1, 0.75\varepsilon - \frac{1}{2}p_2\right)$ and $\left(1-p_1, 0.3\varepsilon - \frac{1}{2}p_2\right)$, respectively, z^1 and z^2 satisfy the conditions stated above only if $p_1 > 1$, $-\frac{0.75\varepsilon - \frac{1}{2}p_2}{1-p_1} \leq \frac{1}{2}p_2 + 0.5\varepsilon$ and $-\frac{0.3\varepsilon - \frac{1}{2}p_2}{1-p_1} \geq \frac{1}{2}p_2 + 0.6\varepsilon$, but this is impossible since $\frac{1}{2}p_2 + 0.5\varepsilon < \frac{1}{2}p_2 + 0.6\varepsilon$ and $\frac{0.3\varepsilon - \frac{1}{2}p_2}{p_1 - 1} < \frac{0.75\varepsilon - \frac{1}{2}p_2}{p_1 - 1}$, for $p_1 > 1$.

After considering the two cases, $0 < p_1 < 1$ and $p_1 \geq 1$, it is concluded that there does not exist an equilibrium price vector for this economy[13], despite the fact that, for all $p \in \mathbb{R}^2_+$, the overlapping expectations condition holds.

It must be noted that Hart's (1974) proof of the existence of equilibrium is based on more assumptions than Hammond's. Particularly, Hart's assumption (A.3), claiming the existence of a bounded set $C \subseteq \mathbb{R}^A_+$ such that $\mu^i(p, C) = 1$ for all $p \in \mathbb{R}^A_+$ and $i \in \mathcal{I}$, is not included in Hammond's hypothesis. Furthermore, this assumption is not satisfied in the previous example, in fact, the sets $\bigcup_{p \in \mathbb{R}^A_+} S^i(p)$ are not bounded. So, it is interesting to ask what would be the implications for the existence of equilibrium if we replace assumption (A.3)* with (A.3) in economy $\mathcal{E}$. In section 6 it will be proved that under an even stronger assumption, that is, that traders' beliefs do not depend on asset prices (together with (A.3)*), which trivially implies condition (A.3), the equilibrium actually exists in the asset markets.

6 Sufficient conditions for the existence of equilibrium

As Hart (1974) points out, if traders' beliefs do not depend on asset prices, the necessary conditions of definition 4 are also sufficient for the existence of equilibrium. Such claim is also true for economy $\mathcal{E}$, despite of the emphasized difference between the two models. However, Hart's proof of the existence of equilibrium, using assumption (A.3) which is weaker than assuming beliefs that are constant in asset prices, may not be applied to economy $\mathcal{E}$, since it is based on a result that does not hold when traders derive utility from first period consumption.

This result is Hart's (1974) lemma 1, and it states that, under assumptions (A.1) - (A.3), if $p \in \mathbb{R}^A_+$ and $z, e \in \mathbb{R}^A$, then $\widehat{V}^k(p, z + \lambda e) \geq \widehat{V}^k(p, z)$, for all $\lambda \geq 0$, if and only if

$$\widehat{S}^+_k E^{+k}_e(p) + \widehat{S}^-_k E^{-k}_e(p) \geq 0. \text{(See definition 4)} \qquad (6)$$

Utility function $\widehat{V}^k$ differs from the one proposed here (1) precisely because of the fact that it does not include the term $u^k(w - pz)$ representing the utility derived from consumption in the first period. If such term is added and the

[13] If $p_1 = 0$, the non-existence of equilibrium follows trivialy, in fact, $Z_1(0, p_2) = Z_2(0, p_2) = \varnothing$.

feasible portfolio sets have the form of expression (3), it could easily happen that (6) is satisfied and, nevertheless, there exists $\lambda > 0$ such that $V^k(p, x + \lambda e) < V^k(p, x)$.

In fact, as it was mentioned previously, in economy $\mathcal{E}$ inequality 6 holds trivially if e is a direction of recession of $X^k(p)$ or, equivalently, if $e \in K^k(p)^+$, and, in such case, it is does not depend on the values taken by the limits $\lim_{w \to -\infty} \frac{du^k}{dw}$ and $\lim_{w \to \infty} \frac{du^k}{dw}$[14]. In general terms, what is being pointed out here is that the concavity of function u^k can play no role in determining if (6) is satisfied or not.

For example, suppose that:

$$V^k(p, z) = -\exp(-(w - pz)) - \exp(-(w + 0.25z_1 + 0.25z_2))$$

and let $e = (1, 1)$. It is obvious that e is a direction of recession of X^k, however, $V^k(p, 0) = -2\exp(-w)$ and $V^k(p, 0 + \lambda e) = -\exp(-(w - p_1\lambda - p_2\lambda)) - \exp(-(w + 0.5\lambda))$. Clearly, there exists some $\widehat{\lambda} > 0$ and a price vector p such that $V^k(p, 0 + \lambda e) < V^k(p, 0)$.

This example shows that a proof of the existence of equilibrium in economy $\mathcal{E}$, as the one included in the next section, must be proposed, even though the assumptions under which it is established here are stronger than the conditions supposed by Hart (1974).

Finally, it could be noted further that the equivalence claimed by Page (1986) between his non-arbitrage condition, Hart's (1974) necessary condition and Hammond's (1982) overlapping expectations condition, holds for Hart's (1974) securities exchange model, but not necessarily for economy $\mathcal{E}$. In fact, Page's proofs of these equivalences are based on Hart's lemma 1.

7 Proof of the existence of equilibrium

Definition 6 *It is said that trader i's beliefs do not include zero if* $0 \notin S^i(p)$, *for all* $p \in \mathbb{R}^A_+$.

Remark 2 *Given that* $S^i(p)$ *is a closed set, it follows immediately from the previous definition that trader i's beliefs do not include zero if and only if there does not exist a sequence in* $S^i(p)$ *that converges to zero, for all* $p \in \mathbb{R}^A_+$.

Lemma 1 *Let* $S \subseteq \mathbb{R}^A_+$, $w \in \mathbb{R}_{++}$ *and* $X = \left\{x \in \mathbb{R}^A : \forall r \in S, rx + w \geq 0\right\}$. *If there does not exist a sequence in* S *that converges to zero then there exists a* $v \in \mathbb{R}^A_+$ *such that* $X + v \subseteq K^+$, *where* K^+ *is the polar of the convex cone generated by* S.

Proof As it was mentioned previously, $K^+ = \left\{x \in \mathbb{R}^A : \forall r \in S, rx \geq 0\right\}$. To obtain a contradiction, let's suppose that there does not exist a $v \in \mathbb{R}^A_+$ such

[14] The limits $\lim_{w \to -\infty} \frac{du^k}{dw}$ and $\lim_{w \to \infty} \frac{du^k}{dw}$ can be considered as a measure of the concavity of the function u^k and, hence, of agent k's risk aversion degree.

that $X+v \subseteq K^+$,i.e., that for all $v \in \mathbb{R}_+^A$ there are $x^v \in X$ and $r^v \in S$ such that $r^v(x^v+v) < 0$, and thus $x^v + v \notin K^+$. Particularly, for all $n \in \mathbb{N}$ positive, there are x^n and r^n such that $r^n(x^n + \mathbf{1}n) < 0$ or, equivalently, $r^n x^n < -n \sum_{i=1}^{A} r_i^n$, where $\mathbf{1}$ is a vector in $\mathbb{R}^A$ such that all its coordinates are equal to 1. Since $x_n \in X$ and $r_n \in S$, $r_n x_n \geq -w$, therefore $n \sum_{i=1}^{A} r_i^n < w$. This inequality holds for all $n \in \mathbb{N}$ positive, so it follows that $\sum_{i=1}^{A} r_i^n \to 0$ as $n \to \infty$, and then $r^n \to 0$, because $r^n \in \mathbb{R}_+^A$, which contradicts the fact that there are no sequences in S converging to zero. ■

The proof of theorem 2, is a reconstruction of the corresponding proof in Carvajal and Riascos (2006). However, the one proposed here is more general in various aspects. In fact, economy $\mathcal{E}$ consists of more than two agents, the number of states of nature is infinite and traders can assign positive probabilities to states in which some assets have zero returns. The strategy of the proof is standard. First of all a sequence of bounded economies for which equilibrium exists is built. Then, to prove the existence of those equilibria, a correspondence to which a fixed point theorem can be applied is conveniently defined; in this particular case Kakutani's theorem is used. Finally, it is shown that from some point in the sequence of bounded economies the equilibrium lies in their interior and then it is also an equilibrium of the unbounded economy. Hart (1974) and Page (1987), among many others, use similar strategies in their corresponding proofs.

Theorem 2 *Suppose that all traders' beliefs are constant in asset prices, do not include zero and are constrained-compatible, then an equilibrium exists for economy $\mathcal{E}$.*

Proof Let $\mathbf{P} = \left\{ \mathbf{p} = (p_0, p) \in \mathbb{R}_+ \times \mathbb{R}_+^A : \sum_{a=0}^{A} p_a = 1 \right\}$ and $n \in \mathbb{N}$ fixed.

For each $k \in \mathcal{I}$, the truncated budget correspondence $\mathbf{B}_n^k : \mathbf{P} \rightrightarrows \mathbb{R}^{A+1}$, can be defined by:

$$\mathbf{B}_n^k(\mathbf{p}) = \left\{ \mathbf{z} = (z_0, z) \in \mathbb{R} \times \mathbb{R}^A : \begin{array}{c} p_0\left(w_0^k - z_0\right) - pz \geq 0 \\ \forall r \in S^k, w^k + rz \geq 0 \\ 0 \leq z_0 \leq (n+1) w_0^k \\ \forall a \in \mathcal{A}, -n \leq z_a \leq n \end{array} \right\}$$

It is easy to verify that this correspondence has non-empty, compact and convex values and is upper hemicontinuous. Let $n \in \mathbb{N}$, $k \in \mathcal{I}$ and $\mathbf{p} \in \mathbf{P}$ be fixed, it is clear that, $\mathbf{z} = (0, \mathbf{0}) \in \mathbf{B}_n^k(\mathbf{p})$, so $\mathbf{B}_n^k(\mathbf{p}) \neq \varnothing$. Now, let $\mathbf{B}_1 = \left\{ (z_0, z) \in \mathbb{R} \times \mathbb{R}^A : p_0\left(w_0^k - z_0\right) - pz \geq 0 \right\}$, $\mathbf{B}_r = \left\{ (z_0, z) \in \mathbb{R} \times \mathbb{R}^A : w^k + rz \geq 0 \right\}$ and $\mathbf{B}_2 = \left\{ (z_0, z) \in \mathbb{R} \times \mathbb{R}^A : 0 \leq z_0 \leq (n+1) w_0^k \ \wedge \ \forall a \in \mathcal{A}, -n \leq z_a \leq n \right\}$. $\mathbf{B}_1$ is closed because it is the inverse image of a closed interval in $\mathbb{R}$ under the continuous function $f : \mathbb{R} \times \mathbb{R}^A \longrightarrow \mathbb{R}$ defined by $f(z_0, z) = p_0 z_0 + pz$. Every

$\mathbf{B}_r$ is closed because it is the product of a closed set in $\mathbb{R}^A$ (the inverse image of a closed set under a continuous function) and all $\mathbb{R}$. Moreover, it is evident that $\mathbf{B}_2$ is compact, because it is the product of compact sets. Since $\mathbf{B}_n^k(\mathbf{p}) = \mathbf{B}_1 \cap \bigcap_{r \in \mathcal{S}^k} \mathbf{B}_r \cap \mathbf{B}_2$ is the intersection of closed sets with a compact set, it follows that it is also compact. Note further that $\mathbf{B}_1$ is the closed lower halfspace determined by a hyperplane in $\mathbb{R} \times \mathbb{R}^A$, each $\mathbf{B}_r$ is the product of a closed lower halfspace in $\mathbb{R}^A$ and $\mathbb{R}$, and $\mathbf{B}_2$ is a box in $\mathbb{R} \times \mathbb{R}^A$, that is, the product of $A+1$ closed intervals in $\mathbb{R}$. Each of these sets are convex and so is their intersection, $\mathbf{B}_n^k(\mathbf{p})$.

To prove upper hemicontinuity, take fixed n and k and define the correspondences $\widetilde{\mathbf{B}} : \mathbf{P} \rightrightarrows \mathbb{R}^{A+1}$ and $\overline{\mathbf{B}} : \mathbf{P} \rightrightarrows \mathbb{R}^{A+1}$as:

$$\widetilde{\mathbf{B}}_n^k(\mathbf{p}) = \left\{(z_0, z) \in \mathbb{R} \times \mathbb{R}^A : p_0\left(w_0^k - z_0\right) - pz \geq 0\right\} \text{ and}$$

$$\overline{\mathbf{B}}_n^k(\mathbf{p}) = \left\{ \mathbf{z} = (z_0, z) \in \mathbb{R} \times \mathbb{R}^A : \begin{array}{c} \forall r \in S^k, w^k + rz \geq 0 \\ 0 \leq z_0 \leq (n+1)\, w_0^k \\ \forall a \in \mathcal{A}, -n \leq z_a \leq n \end{array} \right\}$$

The graph of $\widetilde{\mathbf{B}}$ is the set $\left\{(\mathbf{p}, \mathbf{z}) = (p_0, p, z_0, z) \in \mathbf{P} \times \mathbb{R}^{A+1} \mid p_0\left(w_0^k - z_0\right) - pz \geq 0\right\}$. Since the function $g : \mathbf{P} \times \mathbb{R}^{A+1} \longrightarrow \mathbb{R}$ defined by $g(\mathbf{p}, \mathbf{z}) = p_0\left(w_0^k - z_0\right) - pz$ is continuous, it follows that the graph of $\widetilde{\mathbf{B}}$, which is the inverse image of the closed set $\mathbb{R}_+$ under function g, is closed. In other words, the correspondence $\widetilde{\mathbf{B}}$ is closed.

Note that $\overline{\mathbf{B}}$ is constant in $\mathbf{P}$, in fact, for all $\mathbf{p} \in \mathbf{P}$, $\overline{B}(\mathbf{p}) = \bigcap_{r \in \mathcal{S}^k} \mathbf{B}_r \cap \mathbf{B}_2$. Additionally, the lower inverse image of a set $A \subset \mathbb{R}^{A+1}$ under correspondence $\overline{\mathbf{B}}$ is, by definition, the set $\overline{B}^l(A) = \left\{\mathbf{p} \in \mathbf{P} \mid \overline{\mathbf{B}}(\mathbf{p}) \cap A \neq \varnothing\right\}$, therefore, $\overline{B}^l(A) = \left\{\mathbf{p} \in \mathbf{P} \mid \bigcap_{r \in \mathcal{S}^k} \mathbf{B}_r \cap \mathbf{B}_2 \cap A \neq \varnothing\right\}$. Let F be a closed subset of $\mathbb{R}^{A+1}$, then it follows that $\overline{B}^l(F) = \mathbf{P}$ or $\overline{B}^l(F) = \varnothing$, in any case $\overline{B}^l(F)$ is closed in $\mathbf{P}$. It is concluded that the correspondence $\overline{\mathbf{B}}$ is upper hemicontinuous (see Aliprantis and Border (1994), lemma 14.4) and it is clear that it has compact values. Finally, $\mathbf{B}_n^k = \overline{\mathbf{B}} \cap \widetilde{\mathbf{B}}$ is upper hemicontinuous (see Aliprantis and Border's (1994) theorem 14.24).

$\mathbf{B}_n^k$ is also lower hemicontinuous, the proof is the same as in Carvajal and Riascos (2006).

Applying Berges' Maximum Theorem to the correspondence $\mathbf{B}_n^k$ and the continuous function $v^k(z_0, z) = u^k(z_0) + \int u^k(w_1 + rz)\, d\mu^k(p)$, it can be deduced straightforward that the bounded individual demand correspondence, $\mathbf{Z}_n^k : \mathbf{P} \rightrightarrows \mathbb{R}^{A+1}$, defined by:

$$\mathbf{Z}_n^k(\mathbf{p}) = \arg\max_{\mathbf{z} \in \mathbf{B}_n^k(\mathbf{p})} u^k(z_0) + \int u^k\left(w^k + rz\right) d\mu^k(p),$$

is upper hemicontinuous with non empty and compact values. Besides, this correspondence has convex values, as it will be shown now. In fact, the function u^k is monotone and concave, and $h_r(z) = w_1^k + rz$ is linear in z, then the function $u^k \circ h_r$ is concave and, since $v^k(z_0, z) = u^k(z_0) + \int u^k(w_1 + rz)\, d\mu^k(p)$ is the sum of concave functions, it also is concave and thus, quasiconcave.

Let $\mathbf{z}^1 \in \mathbf{Z}_n^k(\mathbf{p})$ and $\mathbf{z}^2 \in \mathbf{Z}_n^k(\mathbf{p})$, it follows that, for all $\lambda \in [0,1]$:

$$u^k\left(\lambda z_0^1 + (1-\lambda) z_0^2\right) + \int u^k\left(w^k + r\left(\lambda z^1 + (1-\lambda) z^2\right)\right) d\mu^k(p)$$

$$\geq u^k\left(z_0^1\right) + \int u^k\left(w^k + rz^1\right) d\mu^k(p)$$

$$= u^k\left(z_0^1\right) + \int u^k\left(w^k + rz^2\right) d\mu^k(p)$$

therefore, $\lambda \mathbf{z}^1 + (1-\lambda)\mathbf{z}^2 \in \mathbf{Z}_n^k(\mathbf{p})$.

Let $\mathbf{N}_n = \left[0, (n+1)\sum_{k=1}^{I} w_0^k\right] \times [-In, In]^A$ and $\mathbf{\Phi} : \mathbf{P} \times \mathbf{N}_n \rightrightarrows \mathbf{P} \times \mathbf{N}_n$ defined by:

$$\mathbf{\Phi}(\mathbf{p}, \mathbf{z}) = \left(\arg\max_{\mathbf{p}\in\mathbf{P}} p_0 \left(z_0 - \sum_{k=1}^{I} w_0^k\right) + pz\right) \times \left(\sum_{k=1}^{I} \mathbf{Z}_n^k(\mathbf{p})\right)$$

This correspondence can be represented as the product $\mathbf{\Phi}_z \times \mathbf{\Phi}_p$, where $\mathbf{\Phi}_z : \mathbf{N}_n \rightrightarrows \mathbf{P}$ and $\mathbf{\Phi}_p$: $\mathbf{P} \rightrightarrows \mathbf{N}_n$ are defined by:

$$\mathbf{\Phi}_z(\mathbf{z}) = \arg\max_{\mathbf{p}\in\mathbf{P}} p_0 \left(z_0 - \sum_{k=1}^{I} w_0^k\right) + pz \text{ and } \mathbf{\Phi}_p(\mathbf{p}) = \sum_{k=1}^{I} \mathbf{Z}_n^k(\mathbf{p})$$

Given that $\mathbf{P}$ is compact and the function $\widehat{g}(\mathbf{p}, \mathbf{z}) = p_0 \left(z_0 - \sum_{k=1}^{I} w_0^k\right) - pz$ is continuous, Berges' Maximum Theorem implies that $\mathbf{\Phi}_z$ is upper hemicontinuous and with non-empty and compact values. On the other hand, $\mathbf{\Phi}_p$ is the finite sum of upper hemicontinuous correspondences with compact values and then, (see Aliprantis and Border (1994), theorem 14.31) it is also upper hemicontinuous and with compact values. Since the product of upper hemicontinuous correspondences with compact values is upper hemicontinuous with compact values too (see Aliprantis and Border (1994), theorem 14.27), it follows that $\mathbf{\Phi}$ is an upper hemicontinuous correspondence with non-empty and compact values.

Moreover, for all $(\mathbf{p}, \mathbf{z}) \in \mathbf{P} \times \mathbf{N}_n$, $\mathbf{\Phi}(\mathbf{p}, \mathbf{z})$ is convex. In fact, for a fixed $\mathbf{z} \in \mathbf{N}_n$, the function $\widehat{g}_{\mathbf{z}}(\mathbf{p}) = p_0 \left(z_0 - \sum_{k=1}^{I} w_0^k\right) - pz$ is concave in $\mathbf{p}$, so it follows that the set $\mathbf{\Phi}_z(\mathbf{z})$ is convex, by very similar argument than the one used to prove the convexity of the set $\mathbf{Z}_n^k(\mathbf{p})$. Since the sum and the product of convex sets are also convex, it is concluded that $\mathbf{\Phi}(\mathbf{p},\mathbf{z})$ is convex.

Note that $\mathbf{\Phi}$ satisfies the hypothesis of Kakutani's fixed point theorem, and then, for each $n \in \mathbb{N}$, there exists a $(\mathbf{p}_n, \mathbf{z}_n) \in \mathbf{P} \times \mathbf{N}_n$ such that $(\mathbf{p}_n, \mathbf{z}_n) \in \mathbf{\Phi}(\mathbf{p}_n, \mathbf{z}_n)$.

Since $\mathbf{z}_n \in \sum_{k=1}^{I} \mathbf{Z}_n^k(\mathbf{p}_n)$, then $\mathbf{z}_n = \sum_{k=1}^{I} \mathbf{z}_n^k$, where $\mathbf{z}_n^k = \left(z_{n,0}^k, z_n^k\right) \in \mathbf{B}_n^k(\mathbf{p}_n)$, hence, by the monotonicity of preferences, $p_{n,0}\left(z_{n,0} - \sum_{k=1}^{I} w_0^k\right) + p_n z_n = 0$.

From $\mathbf{p}_n \in \arg\max_{\mathbf{p}\in\mathbf{P}} p_0\left(z_{n,0} - \sum_{k=1}^{I} w_0^k\right) + pz_n$, it follows that $z_{n,0} - \sum_{k=1}^{I} w_0^k \leq 0$ and $z_n \leq 0$. Besides, since $p_n > 0$, if $z_{n,a} < 0$ for some $a \in \mathcal{A}$, then $p_{n,a} = 0$, otherwise $p_{n,0}\left(z_{n,0} - \sum_{k=1}^{I} w_0^k\right) + p_n z_n < 0$. Let $e^a \in \mathbb{R}^A$ be defined as follows: $e_i^a = 0$, if $i \neq a$ and $e_i^a = 1$, if $i = a$; then $v^k(z_0, z) < v^k(z_0, z + \lambda e^a)$ for all $\lambda > 0$ because, by hypothesis, $\mathrm{int}K_k$ is non-empty, and thus there is some $r \in S^k$ such that $r_a > 0$. Now let's suppose that $z_{n,a}^k < n$, in this case there would exist a sufficiently small $\lambda > 0$ such that $\left(z_{n,0}^k, z_n^k + \lambda e^a\right) \in \mathbf{B}_n^k(\mathbf{p}_n)$ which is not possible because $\left(z_{n,0}^k, z_n^k\right) \in \mathbf{Z}_n^k(\mathbf{p}_n)$. Therefore $z_{n,a}^k = n$, for all $k \in \mathcal{I}$, hence $z_{n,a} = In > 0$ which, by contradiction, implies $z_n = 0$.

In a similar fashion, if $p_{n,0} = 0$, for all $\mathbf{z}_n^k \in \mathbf{Z}_n^k(\mathbf{p})$, $z_0^k = (n+1)\, w_0^k$, then $z_{n,0} - \sum_{k=1}^{I} w_0^k = n \sum_{k=1}^{I} w_0^k > 0$, because it was assumed that $w_0^k > 0$, which contradicts a previous assertion. It follows that $p_{n,0} > 0$ and thus $z_{n,0} - \sum_{k=1}^{I} w_0^k = 0$, given that $p_{n,0}\left(z_{n,0} - \sum_{k=1}^{I} w_0^k\right) + p_n z_n = 0$ and $z_n = 0$.

Therefore, $\frac{1}{p_{n,0}} p_n z_n^k = \frac{1}{p_{n,0}} p_n \left(-\sum_{j\neq k} z_n^j\right)$ is bounded in n, because $\frac{1}{p_{n,0}} p_n z_n^k = w_0^k - z_{n,0}^k$ and $z_{n,0}^k \geq 0$, then $-\sum_{j\neq k} w_0^j \leq \frac{1}{p_{n,0}} p_n z_n^k \leq w_0^k$, so $z_{n,0}^k = w_0^k - \frac{1}{p_{n,0}} p_n z_n^k$ and, consequently, the sequence $\left(z_{n,0}^k\right)$ is bounded.

Now it must be proved that z_n^k is also bounded. Let's suppose, on the contrary, that there exists a $j \in \mathcal{I}$ such that the sequence $\left(z_n^j\right)$ is not bounded. It is clear that $z_n^j \in X^j = \left\{z \in \mathbb{R}^A : \forall r \in S^j, rz + w \geq 0\right\}$ for all n and, by lemma **1,** there exists a $v^j \in \mathbb{R}_+^A$ such that $X_j + v^j \subseteq K^{j+}$, therefore, the sequence whose terms are $\widehat{z}_n^j = z_n^j + v^j$ is completely contained in K^{j+} and it is not bounded. Similarly, there is a $v^k \in \mathbb{R}_+^m$ for all $k \in \mathcal{I}$ such that

$\widehat{z}_n^k = z_n^k + v^k \in K^{k+}$, for all n.

By hypothesis, $\bigcap_{k\in\mathcal{I}} \text{int}K^k$ is non-empty and, by remark **3** (see the Appendix), it is clear that $K^{k+} \subseteq \left(\bigcap_{k\in\mathcal{I}} \text{int}K^k\right)^+$, for all $k \in \mathcal{I}$, and hence $\bigcap_{k\in\mathcal{I}} \text{int}K^k = \text{int} \bigcap_{k\in\mathcal{I}} K^k$ (see Rockafellar (1970), theorem 6.5). Then, proposition **1** implies that for all $p \in \text{int}\left(\bigcap_{k\in\mathcal{I}} K^k\right)^{++} = \bigcap_{k\in\mathcal{I}} \text{int}K^k$ and for all $z \in \left(\bigcap_{k\in\mathcal{I}} \text{int}K^k\right)^+$ different from zero, $pz > 0$. Now let $\widehat{p} \in \bigcap_{k\in\mathcal{I}} \text{int}K^k$ then, since $\widehat{z}_n^k \in \left(\bigcap_{k\in\mathcal{I}} \text{int}K^k\right)^+$ for all n and $k \in \mathcal{I}$, $\widehat{z}_n^j \in \left(\bigcap_{k\in\mathcal{I}} \text{int}K^k\right)^+$, hence $\widehat{p}\widehat{z}_n^j \to \infty$ and $\widehat{p}\sum_{k\in\mathcal{I}} \widehat{z}_n^k \to \infty$. It follows that $\widehat{p}\sum_{k\in\mathcal{I}} \widehat{z}_n^k = \widehat{p}\sum_{k\in\mathcal{I}} z_n^k + \widehat{p}\sum_{k\in\mathcal{I}} v^k \to \infty$, thus $\widehat{p}\sum_{k\in\mathcal{I}} z_n^k \to \infty$, which is impossible because, as it was previously shown, $\sum_{k\in\mathcal{I}} z_n^k = z_n = 0$. In conclusion, the sequence $\left(z_n^k\right)$ is bounded for all $k \in \mathcal{I}$.

This implies that there exists a bounded set M such that $\mathbf{z}_n^k = \left(z_{n,0}^k, z_n^k\right) \in M$ for all n and $k \in \mathcal{I}$. The sequence of sets $(\text{int}\mathbf{N}_n)$ is increasing and unbounded, so it is clear that there exists a $L \in \mathbb{N}$ such that $M \subseteq \text{int}\mathbf{N}_l$, for all $l \geq L$. Therefore, for all $l \geq L$ the sequence $\left(\mathbf{z}_n^k\right)$ is contained in $\text{int}\mathbf{N}_l$. Now it will be shown that $\left\langle z_L, \frac{p_L}{p_{L,0}}\right\rangle$ is an equilibrium of the unbounded economy. We already know that $p_{L,0} > 0$, $z_{L,0}^k = w_0^k - \frac{p_L}{p_{L,0}} z_L^k$ and $\sum_{k\in\mathcal{I}} z_L^k = 0$, moreover, if $\widehat{p}_L = \frac{p_L}{p_{L,0}}$, then $\mathbf{Z}_L^k(\mathbf{p}) = \arg\max_{\mathbf{z}\in\mathbf{B}_L^k(\mathbf{p})} u^k(z_0) + \int u^k\left(w_1^k + rz\right) d\mu^k(p) = \arg\max_{\mathbf{z}\in\mathbf{B}_L^k(\mathbf{p})} V^k(\widehat{p}_L, z)$. Now let's suppose that $z_L^j \notin Z^j(\widehat{p}_L)$ for some $j \in \mathcal{I}$, hence there exists a $z^* \in Z^j(\widehat{p}_L)$ such that $V^j(\widehat{p}_L, z^*) > V^j\left(\widehat{p}_L, z_L^j\right)$. Since V^j is concave in its second argument and $\mathbf{z}_L^k \in \text{int}N_L$, there is a $\lambda \in (0,1)$ close enough to 1 such that, for $z^\lambda = \lambda z_L^j + (1-\lambda) z^*$ and $z_{L,0}^\lambda = w_0^k - \widehat{p}_L z^\lambda$, $V^j(\widehat{p}_L, z^\lambda) > V^j\left(\widehat{p}_L, z_L^j\right)$ and $\left(z_{L,0}^\lambda, z^\lambda\right) \in \mathbf{B}_n^j(\mathbf{p})$, which is impossible because $\mathbf{z}_n^j \in \mathbf{Z}_n^j(\mathbf{p})$. Finally, $z_L^k \in Z^k(\widehat{p}_L)$ for all $k \in \mathcal{I}$ and then $\langle z_L, \widehat{p}_L\rangle$ is an equilibrium for the unbounded economy. ■

8 Concluding Remarks

The main results found in this paper highlight the importance of an aspect of the assets exchange model that has not been explored exhaustively so far in the literature. Considering the value agents assign to consumption in the first period introduces a significant difference in the model because it changes the conditions for the existence of equilibrium. Even when those conditions coincide under an additional set of assumptions, their application to the case concerned here is not trivial and requires modifying the corresponding proofs substantially, or proposing entirely new ones. Moreover, the example presented questions the forcefulness of previous results highly widespread in the literature on financial markets, particularly, the overlapping expectations condition.

9 Appendix

This appendix includes the definitions of some fundamental concepts from convex analysis that are useful in the paper. Some elementary properties of this concepts are also stated here without demonstration, since they follow easily from the definitions or were taken directly from Rockafellar (1970). Finally, the proof of proposition 1 is established.

A subset of $\mathbb{R}^A$ is a *convex cone* if it is closed under addition and multiplication by a positive scalar.

If S is an arbitrary subset of $\mathbb{R}^A$, the *convex cone generated by* S, denoted co$S$, is the set whose elements are all the non negative linear combinations of the elements of S.

Note that co$S$ is the union of the smallest convex cone containing S and the set $\{0\} \subseteq \mathbb{R}^A$. In general, a convex cone does not necessarily includes the origin.

Definition 7 *Let K be a non-empty convex cone in $\mathbb{R}^A$. The polar of K is the set $K^+ = \left\{x \in \mathbb{R}^A : \forall k \in K, kx \geq 0\right\}$, where kx denotes the dot product of these two vectors.*[15]

It follows immediately from the definition of K^+ that it is a closed convex cone. Moreover, as it can be easily verified, K^{++}, the polar of K^+, is equal to cl$K$, the closure of K.

Remark 3 *Let K_1 and K_2 be two convex cones, with K_1^+ and K_2^+ their respective polar cones. If $K_1 \subseteq K_2$ then $K_2^+ \subseteq K_1^+$.*

Remark 4 *If $K = coS$ then $K^+ = \left\{x \in \mathbb{R}^A : \forall s \in S, sx \geq 0\right\}$*

Another important notion from convex analysis is that of a direction of recession. If C is a non-empty convex cone in $\mathbb{R}^A$ and $e \in \mathbb{R}^A$, then e is a *direction of recession* of C if and only if $x + \lambda e \in C$, for all $\lambda > 0$ and $x \in C$. The set consisting of all the directions of recession of a non-empty convex set C, is called *recession cone of* C and it is denoted 0^+C.

Proposition 7 *Let C be a non-empty, closed convex set in $\mathbb{R}^A$ and $e \in \mathbb{R}^A$. If there exists a $x \in C$ such that $x + \lambda e \in C$, for all $\lambda > 0$ then $z + \lambda e \in C$, for all $\lambda > 0$ and all $z \in C$, that is, $e \in 0^+C$.*

Proof See Rockafellar (1970), theorem 8.3. ■

Proposition 8 *Let C be the set of all solutions to an arbitrary system (not necessarily finite or countable) of linear weak inequalities, i.e., $C = \left\{x \in \mathbb{R}^A : \forall j \in J, r_j x \geq \alpha_j\right\}$,*

[15] The definition of the polar of a convex cone proposed here differs from Rockafellar's. In terms of his definition, our set K^+ is the negative of the polar of K, that is, the polar of K multiplied by -1. However, all the properties attributed by Rockafellar to the polar still holds for K^+, as previously defined, making the respective changes in the signs and the inequalities direction.

where J is an arbitrary index set, $r_j \in \mathbb{R}^A$ and $\alpha_j \in \mathbb{R}$, for all $j \in J$. The recession cone of C is determined by the corresponding system of homogeneous inequalities, that is, $0^+C = \left\{x \in \mathbb{R}^A : \forall j \in J, r_j x \geq 0\right\}$.

Proof See Rockafellar (1970), section 8. ■

Proposition 1 *Let $K \subseteq \mathbb{R}^A$ be a closed convex cone and $k \in K^+$, then there exists a $z \in K$ such that $z \neq 0$ and $kz = 0$ if and only if $k \in \partial K^+$, where $\partial K^+ = clK^+ \backslash intK^+$.*

Proof Let's suppose there exists a $z \in K$ such that $z \neq 0$ and $kz = 0$, then for some n, $z_n \neq 0$. Let $\widehat{k}^\delta$ be defined as follows: $\widehat{k}_i^\delta = k_i - \delta$, if $i = n$ and $z_i > 0$, $\widehat{k}_i^\delta = k_i + \delta$, if $i = n$ and $z_i < 0$ and finally $\widehat{k}_i^\delta = k_i$, if $i \neq n$. Clearly, $\widehat{k}^\delta z = kz \pm \delta z_i < 0$, thus $\widehat{k}^\delta \notin K^+$, for all $\delta > 0$. It follows that $k \notin \text{int}K^+$, i.e., $k \in \partial K^+$.

To prove the other direction, let's suppose now that $k \in \partial K^+$ and for all $z \in K$, if $z \neq 0$ then $zk > 0$. Hence, since $k \notin \text{int}K^+$, for all $n \in \mathbb{N}$, with $n \geq 1$, there exists $\widehat{k}^n \in B_{\frac{1}{n}}(k) = \left\{x \in \mathbb{R}^A : \|x - k\| < \frac{1}{n}\right\}$, such that $\widehat{k}^n \notin K^+$, where $\|\cdot\|$ denotes the euclidean norm in $\mathbb{R}^A$. Since $\widehat{k}^n \notin K^+$, there is a $z^n \in K$ such that $\widehat{k}^n z^n < 0$, moreover, for all $\lambda > 0$, $\widehat{k}^n(\lambda z^n) < 0$, therefore z^n can be chosen conveniently such that $\varepsilon \leq \|z^n\| \leq \varepsilon + 1$, where ε is a positive arbitrary number. Is clear that $\widehat{k}^n \to k$ and, by construction, the sequence (z^n) is contained in a compact set, so it has a convergent subsequence, (z^{n_i}), such that $z^{n_i} \to z$ for some $z \in K$, given that K is closed. It follows that $\widehat{k}^{n_i} z^{n_i} \to kz$, which is impossible because $zk > 0$ and $\widehat{k}^{n_i} z^{n_i} < 0$ for all i. In conclusion, if $k \in \partial K^+$, there exists a $z \neq 0$ in K such that $kz = 0$. ■

References

[1] Aliprantis, C. D. and K. C. Border (1994) Infinite Dimensional Analysis: A Hitchhiker's Guide, Springer-Verlag, Berlin.

[2] Araujo, A. and A. Sandroni (1999), On the convergence to homogeneous beliefs when markets are complete, *Econometrica*, 67: 663-672.

[3] Carvajal, A. and A. Riascos (2006), Belief Non-equivalence and Financial Trade: A Comment on a Result by Araujo and Sandroni. Documento CEDE No. 2006-27. Facultad de Economía, Universidad de los Andes.

[4] Cohn, D. L. (1980), Measure Theory, Birkhäuser, Boston.

[5] Green, J.R. (1973) Temporary general equilibrium in a sequential trading model with spot and futures transactions, *Econometrica* 41: 1103-24.

[6] Hammond, P. J.(1983) Overlapping expectations and Hart's condition for equilibrium in a securities model, *Journal of Economic Theory* 31: 170–175.

[7] Hart, O. (1974). On the Existence of Equilibrium in a Securities Model, *Journal of Economic Theory*, 9: 293-311.

[8] Page, F. H. (1987) On Equilibrium in Hart's Securities Exchange Model, *Journal of Economic Theory*, 9: 293–311.

[9] Rockafellar, T.(1970) Convex Analysis. Princeton University Press.

[10] Werner, J.(1987) Arbitrage and the existence of competitive equilibrium, *Econometrica*, 55: 1403–1418.