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Preferences and Normal Goods: An Easy-to-Check Necessary and Sufficient Condition*

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Abstract

We provide a necessary and sufficient condition for goods to be normal when utility functions are differentiable and strongly quasi-concave. Our condition is equivalent to the condition proposed by Alarie et al. (1990), but it is easier to check: it only requires to compute the minors associated with the border column (or row) of the bordered Hessian matrix of the utility function.

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1 Introduction

The present paper is devoted to the problem of characterizing normality of goods in terms of the properties of the utility function. Leroux (1987) raised the issue providing a sufficient condition for all goods to be normal when preferences can be represented by differentiable strongly quasi-concave utility functions.[1] His condition is based on the Hessian and the bordered Hessian of the utility function. Although the condition is only sufficient, and therefore possibly dependent on the utility representation, Leroux (1987) showed that indeed it is a property of the preference order.

Fisher (1990) provided a necessary and sufficient condition for a good to be normal in terms of the expenditure function. Essentially, his condition says that a good is normal if and only if, holding constant utility, an increase in the price of the good would decrease the marginal utility of income. Since the condition is on the expenditure function, Leroux's problem of characterizing normality only in terms of the utility function was however not solved.

Alarie et al. (1990) provided a necessary and sufficient condition for all goods to be normal in terms of the derivative of shadow prices with respect to quantities. Moreover, they showed that their condition can be restated in terms of a suitable Allais matrix, implying that it can be tested against observed consumption choices. They also provided an interpretation of Leroux's sufficient condition: it requires any two goods to be Allais complements.[2]

We pick up the original Leroux's problem of characterizing normality of goods only in terms of the utility function. More precisely, we provide an easy-to-check condition which only requires to compute the minors associated with the border column (or row) of the bordered Hessian matrix. This is obtained by looking at the problem of the consumer who faces a marginal increase in income as the problem of finding, in the consumption space, the direction along which pairwise marginal rates of substitution do not change.

While our condition cannot be directly tested against observed behavior, it has the advantage of involving only first and second derivatives of the utility function and requiring a few simple computations to be checked. Moreover, our condition independently characterizes the normality of each good so that, if one is only interested in the normality of a subgroup of goods, calculations are even simpler.

The paper is organized as follows. In next section we provide the re-

[1] In an earlier contribution Chipman (1977) has shown that if a strongly concave and twice-differentiable utility function is such that all commodities are complements according to the Auspitz-Lieben-Edgeworth-Pareto definition then all goods must be normal.

[2] The work by Quah (2007) about comparative statics under constrained optimization has provided a condition for normality of goods in a rather general framework. However, such a condition is only sufficient.

quired preliminaries (notation, assumptions and a suitable description of the consumer problem). In section 3 we present our necessary and sufficient condition, we illustrate it by means of a graphical representation with two goods, and we provie a simple example which shows that our condition may be much simpler to check than the equivalent one by Alarie et al. (1990). Section 4 briefly summarizes our contribution.

2 Preliminaries

2.1 Assumptions and Notation

Following Leroux (1987) and Alarie et al. (1990) we conduct our analysis in a neighborhood of x^*, interior point of the consumption set $X \subset \mathbb{R}^n_+$. Moreover, let $U : X \to \mathbb{R}$ be twice continuously differentiable in a neighborhood of x^* and let U represent a preference order on X.

Vectors are column vectors. Denote with ${}^t x$ the transpose of a vector $x \in X$. Let G and H be the gradient vector and the Hessian matrix of U, respectively. The bordered Hessian of U is

$$\tilde{H} \equiv \begin{bmatrix} H & G \\ {}^t G & 0 \end{bmatrix}$$

We assume that U is strongly increasing, i.e. $G > 0$, and strongly quasi-concave, i.e. $\tilde{H}$ is negative definite.

Finally, we indicate with $|A|$ the determinant of a matrix A, with A^{-1} its inverse and with $A_{i,j}$ its (i, j)-minor, i.e. the sub-matrix obtained by deleting the i-th row and the j-th column of A.

2.2 The consumer problem

Let $p \in \mathbb{R}^n$ be the price vector and $R \in \mathbb{R}$ be the individual income. Then, the consumer problem is

$$\max_x U(x) \quad \text{s.t. } {}^t p x = R \tag{1}$$

Let x^* be the solution to problem (1). From first order conditions (FOCs) we get

$$G(x^*) = kp, \quad k \in \mathbb{R}_+ \setminus \{0\} \tag{2}$$

Now, suppose that R is increased by an arbitrarily small amount. Strong quasi-concavity of U grants that the new optimum is unique and fully determined by the FOCs. Moreover, the gradient G of the new optimum lies in the one-dimensional sub-space of $\mathbb{R}^n$ identified by the span of p. Therefore, by differentiating (2) along the direction that goes from x^* to the new optimum, i.e. $\partial x^* / \partial R$, we have that

$$H(x^*)\frac{\partial x^*}{\partial R} = skp = sG(x^*), \quad s \in \mathbb{R} \tag{3}$$

The linear system (3) imposes that a marginal movement along $\partial x^*/\partial R$ produces a proportional change in marginal utilities.

Let us define the $(n+1)$-dimensional vector

$$\tilde{x} \equiv \begin{bmatrix} \dfrac{\partial x^*}{\partial R} \\ \\ -s \end{bmatrix}$$

Then, (3) and the fact that utility must increase in the new optimum imply that

$$\tilde{H}(x^*)\tilde{x} = \begin{bmatrix} \mathbf{0} \\ c \end{bmatrix}, \quad \mathbf{0} \in \mathbb{R}^n, c > 0 \tag{4}$$

3 Result

3.1 Condition

We are now ready to state our necessary and sufficient condition for normality.

Proposition 1 *Let x^* be the maximizer of $U(x)$ subject to ${}^tpx = R$. Then the i-th good is normal at x^*, i.e. $\tilde{x}_i \geq 0$, if and only if*

$$|\tilde{H}_{i,n+1}| = |\tilde{H}_{n+1,i}| \begin{cases} \leq 0 & \text{if } i \text{ is even} \\ \geq 0 & \text{if } i \text{ is odd} \end{cases} \tag{5}$$

Strict normality of the i-th good, i.e. $\tilde{x}_i > 0$, obtains if and only if the inequality holds strictly.

Proof. We know that strong quasi-concavity of U implies that $\tilde{H}$ has full rank (see e.g., Barten and Bohm, 1982, Theorem 11.2). Therefore, system (4) can be solved by inverting $\tilde{H}$. Given the symmetry of $\tilde{H}$ we set

$$\tilde{H}^{-1} = \begin{bmatrix} B & b \\ {}^tb & \beta \end{bmatrix}, \quad b \in \mathbb{R}^n, \beta \in \mathbb{R} \tag{6}$$

From (6) and (4) we get $\tilde{x} = (cb, c\beta)$. Since $c > 0$, we have that the i-th element of $\tilde{x}$, with $1 \leq i \leq n$, has the same sign of the i-th element of b.

The i-th element of b has the same sign of $|\tilde{H}_{i,n+1}|/|\tilde{H}|$ if $(i+n+1)$ is even and has opposite sign if $(i+n+1)$ is odd. Moreover, since $\tilde{H}$ is negative definite, the sign of $|\tilde{H}|$ is negative if n is odd and positive if n is even. Note that, when i is odd, $(i+n+1)$ is odd if and only if n is odd;

instead, when i is even, $(i+n+1)$ is odd if and only if n is even. From these observations and the symmetry of $\tilde{H}^{-1}$ the result follows. □

Of course, the value of $|\tilde{H}_{n+1,i}|$ is not invariant to positive monotone transformations of U. However, Proposition 1 implies that the sign of $|\tilde{H}_{n+1,i}|$ is indeed a characteristic of the preference order.

3.2 A graphical representation with two goods

Admittedly, the statement of condition (5) in Proposition 1 is not easily interpretable. A brief illustration of the case with two goods will help to give the geometric intuition behind our result.

When $n = 2$ Proposition 1 states that good 1 is normal if and only if

$$\frac{\partial^2 U}{\partial x_1 \partial x_2}\frac{\partial U}{\partial x_2} - \frac{\partial U}{\partial x_1}\frac{\partial^2 U}{\partial x_2 \partial x_2} \geq 0 \tag{7}$$

where x_1 and x_2 denote the quantities of, respectively, good 1 and good 2. We note that (7) can be rewritten as an inequality relating the rate of change of the marginal utility of good 1 with the rate of change of the marginal utility of good 2 when we increase the consumption of good 2, i.e.:

$$\frac{\dfrac{\partial^2 U}{\partial x_1 \partial x_2}}{\dfrac{\partial U}{\partial x_1}} \geq \frac{\dfrac{\partial^2 U}{\partial x_2 \partial x_2}}{\dfrac{\partial U}{\partial x_2}} \tag{8}$$

Inequality (8) says that good 1 is normal at x^* if and only if a marginal increase in the consumption of good 2 does not modify the marginal rate of of substitution between the two goods in favor of good 2. In other words, at x^* an increase in the consumption of good 2 should not make the gradient of the utility function rotate counter-clockwise. This is shown in more detail in figure 1.

3.3 Example

Let us illustrate the ease of use of our condition with respect to the one by Alarie et al. (1990) with the following example. Consider $n = 3$ and the following utility

$$U(x) = x_1^{\gamma} x_2^{\delta} + x_3^{\xi} \tag{9}$$

where $\gamma > 0$, $\delta > 0$ and $\xi > 0$ are such that U is strongly quasi-concave.[3] The bordered Hessian of $U(x)$ is

[3]Note that, for every $\gamma > 0$ and $\delta > 0$ there exists $\hat{\xi}$ such that for every $\xi \in (0, \hat{\xi})$ function U is strongly quasi-concave.

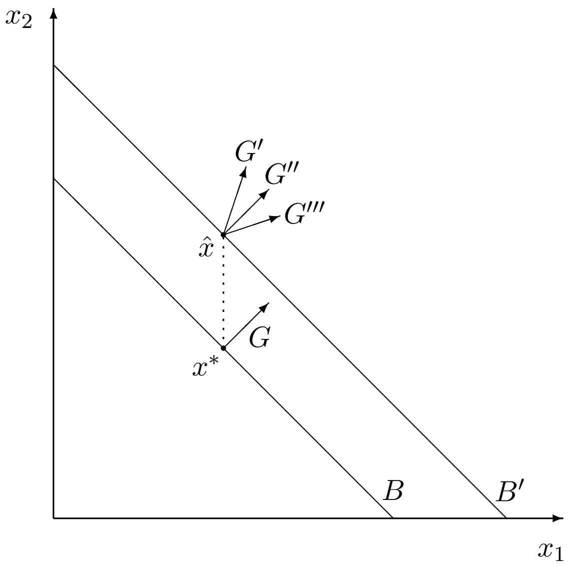

Figure 1: *Normality of good* 1. An increase in income raises the budget line from B to B'. The maximizer in B is x^*, with G its gradient. Suppose that we spend the additional income on good x_2 only, and we reach $\hat{x}$ on B'. If the gradient at $\hat{x}$ is counter-clockwise rotated with respect to G, like in G', then moving further north-west along the budget line B' increases utility and x_1 turns out to be non-normal. Instead, x_1 proves normal if the gradient is like in G'' or G''' (in the latter case x_2 is strictly normal).

$$\tilde{H} = \begin{bmatrix} \gamma(\gamma-1)x_1^{(\gamma-2)}x_2^{\delta} & \gamma\delta x_1^{(\gamma-1)}x_2^{(\delta-1)} & 0 & \gamma x_1^{(\gamma-1)}x_2^{\delta} \\ \gamma\delta x_1^{(\gamma-1)}x_2^{(\delta-1)} & \delta(\delta-1)x_1^{\gamma}x_2^{\delta-2} & 0 & \delta x_1^{\gamma}x_2^{(\delta-1)} \\ 0 & 0 & \xi(\xi-1)x_3^{(\xi-2)} & \xi x_3^{(\xi-1)} \\ \gamma x_1^{(\gamma-1)}x_2^{\delta} & \delta x_1^{\gamma}x_2^{(\delta-1)} & \xi x_3^{(\xi-1)} & 0 \end{bmatrix} \tag{10}$$

Now, if we use the condition by Alarie et al. (1990), in order to check whether goods are normal we have to: (i) calculate the Jacobian of the vector of shadow prices, (ii) calculate the inverse of one of its non-singular square sub-matrices of order 2, (iii) eliminate from the column which has not been used to build the sub-matrix the element whose index coincides with that of the column itself, (iv) use the resulting column vector to post-multiply the previously inverted matrix, (v) check the sign of the resulting values.

Since the vector of shadow prices is of the form $mG/^t wG$, where $m \in \mathbb{R}$ and $w \in \mathbb{R}^n$ identify the chosen normalization, such calculations may be tedious.

On the contrary, if we apply our condition (5) we get

$$\frac{\partial x_1^*}{\partial R} \geq 0 \quad \Leftrightarrow \quad (1-\xi)x_1^{(2\gamma-1)}x_2^{(2\delta-2)}\left[\delta-(\delta-1)\right] \geq 0 \Leftrightarrow \xi \leq 1 \tag{11}$$

$$\frac{\partial x_2^*}{\partial R} \geq 0 \quad \Leftrightarrow \quad (1-\xi)x_1^{(2\gamma-2)}x_2^{\delta}x_2^{(2\delta-1)}\left[(\gamma-1)-\gamma\right] \Leftrightarrow \xi \leq 1 \tag{12}$$

$$\frac{\partial x_3^*}{\partial R} \geq 0 \quad \Leftrightarrow \quad x_1^{(2\gamma-2)}x_2^{(2\delta-2)}\left[(\gamma-1)(\delta-1)-\gamma\delta\right] \geq 0 \Leftrightarrow \gamma+\delta \leq 1 \tag{13}$$

which immediately show that concavity of U in x_3 implies normality of both good 1 and good 2, while normality of good 3 requires concavity of U in (x_1, x_2).

4 Conclusions

In this paper we have provided a necessary and sufficient condition for goods to be normal when utility functions are differentiable and strongly quasi-concave. The interest in our condition stems from the fact that it only requires to compute the minors associated with the border column (or row) of the bordered Hessian matrix which is obtained from the first and second derivatives of the utility function. Equivalent conditions such as the one proposed by Alarie et al. (1990) typically require more cumbersome calculations.

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