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Electoral Competition in 2-Dimensional Ideology Space with Unidimensional Commitment

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Abstract

We study a model of political competition between two candidates with two orthogonal issues, where candidates are office motivated and committed to a particular position in one of the dimensions, while having the freedom to select (credibly) any position on the other dimension. We analyse two settings: a homogeneous one, where both candidates are committed to the same dimension and a heterogeneous one, where each candidate is committed to a different dimension. We characterise and give necessary and sufficient conditions for existence of convergent and divergent Nash equilibria for distributions with a non-empty and an empty core. We identify a special point in the ideology space which we call a strict median, existence of which is strictly related to existence of divergent Nash equilibria. A central conclusion of our analysis is that for divergent equilibria, strong extremism (or differentiation) seems to be an important equilibrium feature.

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1. Introduction

The seminal Hotelling-Downs model (Hotelling (1929) and Downs (1957)) of electoral competition and its well known result of policy convergence to the median voter have remained central to the literature on formal political economics. The crucial assumption of this model is that candidates care only about winning the elections and are able to commit to any pre-election announcement of policies (c.f. Duggan (2005)). This assumption leads to problems with existence of Nash equilibria even with two competing politicians in multi-dimensional ideology spaces. A multi-dimensional generalization of the median voter theorem states that a Nash equilibrium choice of policies with two competing candidates must be the point which is weakly majority preferred by all voters (a *core point*) (Duggan (2005)). The formal requirements for the existence of such a point in multi-dimensional ideology spaces is so restrictive (Plott (1967)) that for almost all specifications of voter preferences, the core point does not exist.[1]

In this paper we study the Hotelling-Downs setting with two competing

[1]Duggan and Jackson (2005) attempt to overcome this problem by studying a model with mixed strategies of candidates and assuming that candidates are unable to predict each others policy positions. It is shown that if indifferent voters are allowed to randomize with any probability ability between zero and one (instead of voting for each candidate with equal probability), then it is possible to have existence of mixed strategy Nash equilibria.

candidates on a two-dimensional ideology space where the set of policies each candidate can propose is restricted by his commitment to one of the issues/dimensions. We take the stand that while a candidate's pre-electoral position may have more than one orthogonal issues, a candidate may not be able to make credible commitments on all of these issues due to history or his own identity. For example, while L. K. Advani (the right-wing BJP leader) of India can credibly commit to a large extent to any position on issues such as budget deficits, voters in India will never believe that he could take an anti-Hindu stand if voted to power. Similarly, it would be hard for Barack Obama to credibly commit to any policy that can directly hurt African Americans. Also, a candidate may represent a particular political party, which by history has adopted a certain stance towards some of the issues which cannot be relaxed by the candidate (that is the current leader). Similar phenomenon may occur when new issues are introduced during the political campaign (e.g. see Hinich and Munger (2008)). Appearance of new issue might give a candidate a new degree of freedom in selecting his political position, while he has to 'stick' his position towards the older ones. In other words, by virtue of popularity, a candidate (or a political party) is at times identified by ideologies in certain issues on which they establish irrefutable reputation, while such a label does not appear for them in other issues. We also take the stand that in a mature democracy with well-established leaders and parties, it is almost inevitable that each party or leader faces such reputations. Our framework is a simplification of this stand and results in each candidate having to choose a policy from a unidimensional ideology space to win the election which is otherwise contested in a 2-dimensional ideology space. Additionally we allow each candidate to stay out of the elections, if winning is impossible for him. We restrict our attention to pure strategies only and we are interested in the existence of Nash equilibria where both candidates contest. Our focus is to address this in situations where the core is empty, and where in equilibrium the two parties take very different stands.

The idea of restricting ideology spaces of candidates is new. A similar idea is studied very recently by Beeler Asay (2008) where 'feasible' policies of candidates are restricted by linear constraints in a setting with two-dimensional ideology space, a finite set of voters and two office-motivated candidates. The central objective of that study is this: given that an incumbent faces competition from a potential entrant, what policy must he choose to minimize the probability of his opponent's victory. New solution concepts called a *constrained strong point* and a *constrained core* are developed to add to the stream of literature in political competition where new solution concepts are sought to overcome the problem of emptiness of the core (c.f. Ferejohn et al. (1984), Owen and Shapley (1989), Wuffle et al. (1989)). Since our focus is on sustaining equilibria where both parties contest and there is no gain from votes unless it leads to victory with positive probability, the notions of the strong point or the constrained strong point are not helpful. We identify another 'special' point in the ideology space that has strict relation (that is either they both exist or they both do not) with the notion of the constrained core and in high dimensions plays the role of a 'median voter'. We call this point the *strict median* and we show that this point is the projection of the constrained cores (which we show, if it exists, is always unique with a continuum of voters) of each candidate in their respective feasible strategy sets. Our analysis suggests that in electoral games with two competing candidates and with restricted strategies, either the core is non-empty in which case the core coincides with the strict median and an equilibrium with both candidates contesting exists, or that the core is empty while the strict median may still exist and the existence of the strict median is necessary for existence of equilibrium in pure strategies where both candidates contest. The more interesting scenario for us is where the two contesting parties announce different platforms. We show that in every such equilibrium, the parties must be far apart from each other. In particular when they share a common committed issue, the parties must hold *strongly extremist and opposite* positions in that issue while their announced policy converges in the other issue in which they are free to choose. In the case where the committed issues of the two parties are different, their overall positions must remain significantly distant from each other.

The rest of the paper is structured as follow. We introduce the model formally in Section 2. Then we provide some preliminary notions and results in Section 3 and the main results in Section 4 and comment on one-party equilibrium. We discuss our findings and the notion of the strict median in Section 5 and conclude in Section 6.

2. The Model

Two candidates 1 and 2 compete for office in an election governed by the majority rule. There is a continuum of voters, denoted by the set $\mathcal{C}$ and each voter in $\mathcal{C}$ has a single-peaked preference over the set of *policy positions*, which is assumed to be the real plane $\mathbb{R}^2$ (that is each policy consists of two independent (orthogonal) issues, the set of issues is denoted by $I = \{1, 2\}$). We shall use δ to denote the Euclidean distance on $\mathbb{R}^2$. The ideal policies of the voters are distributed on $\mathbb{R}^2$ with a distribution function given by density function f (throughout the text we will identify the respective distribution with f as well). We assume that f is non-atomic, that is the support X of f is such that there exists a connected open set $Z \subseteq X$ such that $X \subseteq \bar{Z}$, where $\bar{Z}$ denotes the closure of Z.[2]

Each of the candidates $d \in \{1, 2\}$ is committed to a particular value of only one of the issues, while he has freedom of credible choice of his political position over the remaining one. The *committed issue* of candidate d is denoted by $\mathrm{c}_d \in I$ and the *free issue* is denoted by n_d. The unique ideal value of the committed issue of candidate d is denoted by v_d. We will also use a (instead of v_1) to denote this ideal value of candidate 1 and use b (instead of v_2) for candidate 2. Points a and b are common knowledge amongst all players so that these are the only credible pre-electoral announcements for the committed issues. However each candidate $d \in \{1, 2\}$ is free to choose his policy from his *feasible set of policies* $L_d = \{\bar{x} \in \mathbb{R}^2 : \bar{x} \downarrow \mathrm{c}_d = v_d\}$, which is a line in $\mathbb{R}^2$. Thus, each candidate is committed to one of the issues and to announce his proposed policy $\bar{x}_d \in \mathbb{R}^2$, candidate d has to choose a value of the issue n_d.

Elections are conducted as follows: each candidate makes a decision (simultaneously and independently) of whether (or not) to contest the elections

[2]We see this as a natural generalization of the notion of non-atomicity of unidimensional distributions.

and, if he decides to contest, what policy to propose. These announcements become common knowledge amongst voters and all voters cast votes, voting for their most preferred policies (in case of a tie, they vote for each candidate with equal probability). A candidate who receives the maximal mass of support is selected as the winner of the elections (while a tie is broken by an equiprobable draw). The winner implements the policy he announced. We assume the following preferences of candidates: they prefer (a) not contesting to losing ; (b) a tie to not contesting; (c) winning to not contesting; (d) to be the unique winner to any other outcome; and (e) a tie to losing.

In summary, the strategic game $\Gamma_{a,b}^{\mathrm{c}_1,\mathrm{c}_2}$ is studied in which the set of candidates is $\{1,2\}$, the committed issue of candidate 1 is c_1 and his preferred value of this issue is a and the committed issue of candidate 2 is c_2 and his preferred value of this issue is b. For each candidate $d \in \{1,2\}$ the set of pure strategies is $\{N\} \cup \mathbb{R}$, where N stays for staying out of the elections, and the preferences are as described above. In what follows we shall focus our attention on issues concerning existence and characteristics of pure strategy Nash equilibria of $\Gamma_{a,b}^{\mathrm{c}_1,\mathrm{c}_2}$ where both candidates contest. We call such equilibria *Full-Participation Equilibria.*

3. Preliminaries

Before presenting the results we need to define some notions. Given a measurable set $X \subseteq \mathbb{R}^2$ we use

$$\mu(X) = \int_X f(\bar{x})\mathrm{d}\bar{x}$$

to denote the mass of voters with ideal policies in X.

Any line $l \subseteq \mathbb{R}^2$ gives a rise to a distribution f_l, which will be called a *distribution f projected on l*, where

$$f_l(\bar{z}) = \int_{l^\perp(\bar{z})} f(\bar{x})\mathrm{d}\bar{x},$$

defined for $\bar{z} \in l$, where $l^\perp(\bar{z})$ denotes the line perpendicular to l intersecting it at $\bar{z}$. Notice that if f is non-atomic, then f_l is non-atomic as well.

A line $m \subseteq \mathbb{R}^2$ such that the masses of both half planes defined by the line are equal is called a *median line*. The following fact is crucial for our results [see appendix for a proof].

Fact 1. *Let f be a non-atomic distribution. Then for any vector $\vec{v} \in \mathbb{R}^2$ there exists a unique median line $m^f_{\vec{v}}$ perpendicular to $\vec{v}$.*

Given an issue $\mathrm{i} \in I$ we will use $\mathbf{e}_\mathrm{i}$ to denote the unit vector associated with the respective dimension of the issue i. Then $m_{\mathbf{e}_\mathrm{i}}$ denotes median line associated with vector $\mathbf{e}_\mathrm{i}$. We will also use a shorter notation m^i to denote the median line $m_{\mathbf{e}_\mathrm{i}}$.

Given two policies $\{\bar{x}_1, \bar{x}_2\} \subseteq \mathbb{R}^2$, let $B(\bar{x}_1, \bar{x}_2) = \{\bar{x} \in \mathbb{R} : \delta(\bar{x}_1, \bar{x}) = \delta(\bar{x}_2, \bar{x})\}$ be the *bisector* of $\bar{x}_1$ and $\bar{x}_2$ (c.f. Aurenhammer and Klein (2000)), that is a line perpendicular to the interval connecting policies $\bar{x}_1$ and $\bar{x}_2$, going through its middle. The bisector of $\bar{x}_1$ and $\bar{x}_2$ separates the half space $D(\bar{x}_1, \bar{x}_2) = \{\bar{x} \in \mathbb{R}^2 : d(\bar{x}_1, \bar{x}) < d(\bar{x}_2, \bar{x})\}$ containing policies that are closer to $\bar{x}_1$ from the half space $D(\bar{x}_2, \bar{x}_1)$ containing the policies that are closer to $\bar{x}_2$. Then $D(\bar{x}_1, \bar{x}_2)$ contains ideal points of voters strictly preferring $\bar{x}_1$ to $\bar{x}_2$.

Similarly, we define a bisector line of two lines $l_1 \subseteq \mathbb{R}^2$ and $l_2 \subseteq \mathbb{R}^2$ to be a line consisting of points equidistant from both lines.[3] Two lines may have one (if their intersection is empty) or two (otherwise) bisector lines. The set of bisector lines of lines l_1 and l_2 is denoted by $B(l_1, l_2)$.

We say that $\bar{x}_1$ is *weakly majority preferred* to $\bar{x}_2$, denoted by $\bar{x}_1 \bar{M} \bar{x}_2$, if $\mu(D(\bar{x}_1, \bar{x}_2)) \geq \mu(D(\bar{x}_2, \bar{x}_1))$, that is the mass of support of $\bar{x}_1$ is not smaller than the mass of support of $\bar{x}_2$. The set of policies $\mathcal{C}_f \subseteq \mathbb{R}^2$ that are weakly majority-preferred to all other policies is called the core policies or simply the *core*. The relation of *strict majority preference* is defined as usual and will be denoted by M.

Given a distribution f we will use $\mathcal{M}_f = \{m^f_{\vec{v}} : \vec{v} \in \mathbb{R}\}$ to denote the set of all median lines associated with f. The following fact is well known, but for completeness we provide its proof in the appendix.

Fact 2. *Given a non-atomic distribution function f, the core of $\mathcal{C}_f \neq \varnothing$ if and only if there exists $\bar{c}_f \in \mathbb{R}^2$ such that for any $\{m_1, m_2\} \in \mathcal{M}_f$, if $m_1 \neq m_2$, then the intersection point of them is $\bar{c}_f$ and $\mathcal{C}_f = \{\bar{c}_f\}$.*

That is the core $\mathcal{C}_f$, if it exists, must consist of (the unique) intersection point $\bar{c}_f$ of all median lines. We call $\bar{c}_f$ the *core point* and every voter whose ideal position is $\bar{c}_f$ is called a *core voter*. Facts 1 and 2 together clearly indicate that non-emptiness of the core is hard to achieve in higher dimensions while its existence is guaranteed in one-dimension (the median).

[3]By 'distance' between a point and a line we mean the standard notion of the shortest distance between the two.

4. The Results

In our analysis we will be interested in Nash equilibria as strategy profiles in $\mathbb{R}^2$, that is in Nash equilibria in which both candidates contest. We call such equilibria *full participation Nash equilibria.* Among full participation Nash equilibria we will also distinguish between two kinds: those where the policies proposed by the candidates are the same, which we call *Convergent Full Participation Equilibria* (**CFPE**), and those where the policies proposed by the candidates are different, which we call *Divergent Full Participation Equilibria* (**DFPE**). *We are mainly interested in the existence and characteristics of a DFPE and particularly so when the core is empty.*

In the subsections to follow we study conditions for existence and properties of a CFPE and a DFPE, analysing two cases separately: one where both candidates are committed to the same issue (*Homogeneous Commitment*) and the other where each candidate is committed to a different issue (*Heterogeneous Commitment*).

4.1. Convergent Full Participation Equilibria

The analysis of a CFPE is quite straightforward and we first state our results in this case.

4.1.1. Homogeneous Commitment

In the case of homogeneous commitment, where the committed issues of both candidates are the same it either holds that $L_1 \cap L_2 = \varnothing$, that is feasible sets of policies of both candidates are disjoint, which is the case when $a \neq b$, or it holds that $L_1 = L_2$, that is both candidates have the same feasible set of policies, which is the case when $a = b$. The necessary condition for the existence of a CFPE is that $a = b$ in which case we have the classical Hotelling-Downsian competition in one dimension and the following well known result holds (thus left without a proof).

Theorem 1. *Consider a game $\Gamma^{\mathrm{i,i}}_{a,b}$. Then a CFPE of $\Gamma^{\mathrm{i,i}}_{a,b}$ exists if and only if $a = b$. Moreover if (x_1, x_2) is a CFPE of $\Gamma^{\mathrm{i,i}}_{a,a}$, then $x_1 = x_2 = m$, where m is the median of the distribution f projected on $L = L_1 = L_2$.*

4.1.2. Heterogeneous Commitment

In the case of heterogeneous commitment it always holds that $L_1 \cap L_2 \neq \varnothing$ and the intersection of the feasible sets of policies of both candidates contains exactly one policy. It turns out that a CFPE exists in this case if and only if L_1 and L_2 are median lines. We show that the policy proposed by the two candidates in a CFPE must be this unique intersection point of the two feasible sets of policies. The central message of the following theorem is that such equilibrium exists if and only if the feasible sets of strategies of the two candidates are the two vertical and horizontal median lines and in any such equilibrium, if a is the value of the committed issue for candidate 1, then it is also the announced value of the free issue for candidate 2 and vice versa (with respect to b).

Theorem 2. *Consider a game $\Gamma^{\mathrm{i,j}}_{a,b}$ with $\mathrm{i} \neq \mathrm{j}$. Then a CFPE of $\Gamma^{\mathrm{i,j}}_{a,b}$ exists if and only if $L_1 = m^{\mathrm{i}}$ and $L_2 = m^{\mathrm{j}}$. Moreover, if this condition is satisfied then a strategy profile (y_1, x_2) is a Nash equilibrium if and only if $y_1 = b$ and $x_2 = a$.*

Proof. If a strategy profile (y_1, x_2) is a CFPE then it must be that $L_1 \cap L_2 = \{(y_1, x_2)\}$ and so it must be that $y_1 = b$ and $x_2 = a$.

For the left to right implication of the existence part of the theorem we will show that if (y_1, x_2) is a CFPE, then it must be that $(y_1, x_2) \in m^{\mathrm{i}} \cap m^{\mathrm{j}}$. Assume the opposite. Then either $(y_1, x_2) \notin m^{\mathrm{i}}$ or $(y_1, x_2) \notin m^{\mathrm{j}}$. Suppose that the first case holds. Let $\bar{x} \in L_1 \cap m^{\mathrm{j}}$. Then $\bar{x} M (y_1, x_2)$ and so candidate 1 can win outright by proposing $\bar{x} \downarrow \mathrm{j}$ instead of y_1. Analogous argument can be used for the case where $(y_1, x_2) \notin m^{\mathrm{j}}$. Thus if (y_1, x_2) is a Nash equilibrium, then it must be that $(y_1, x_2) \in m^{\mathrm{i}} \cap m^{\mathrm{j}}$ and so it must be that $L_1 = m^{\mathrm{i}}$ and $L_2 = m^{\mathrm{j}}$.

For the right to left implication of the existence and characterisation part of the theorem assume that $L_1 = m^{\mathrm{i}}$ and $L_2 = m^{\mathrm{j}}$ and consider a strategy profile (y_1, x_2) such that $y_1 = b$ and $x_2 = a$. Let $\bar{x} \in L_1 \cap L_2$ be the policy proposed by both candidates under this strategy profile. Then for any $\bar{x}' \in L_1$ it holds that $\bar{x} M \bar{x}'$, so it is not profitable for candidate 1 to deviate to any other position. Similarly, it is not profitable for candidate 2 to deviate to any other position as well. Hence (y_1, x_2) is a Nash equilibrium. $\square$

Thus we have provided the necessary and sufficient conditions for existence of a CFPE together with full characterisation of such equilibria in both homogeneous and heterogeneous commitments.

4.2. Divergent Full Participation Equilibria

Before we study conditions for existence and characteristics of a DFPE in both homogeneous and heterogeneous commitment settings, let us investigate the necessary and sufficient condition for the existence of a *winning strategy* for one of the candidates given an arbitrary strategy of its competitor.

Let $\bar{x}_2$ be the policy proposed by candidate 2 and assume that $\bar{x}_2 \notin$

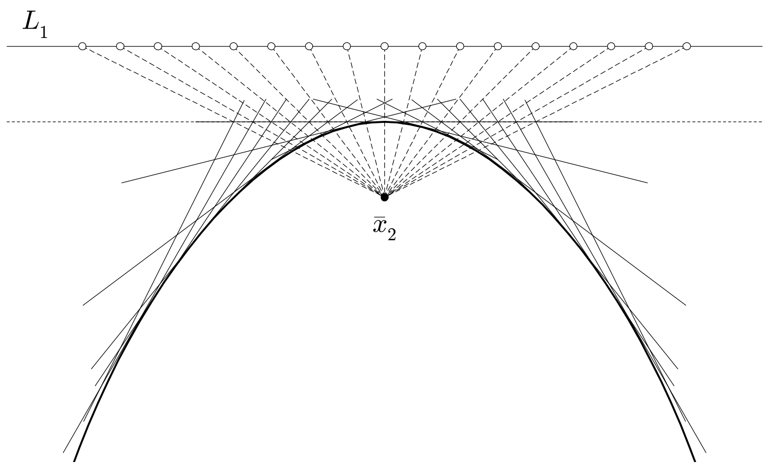

Figure 1: Bisector lines from $\mathcal{B}(\bar{x}_2, L_1)$ and the tangent parabola.

L_1, that is the policy $\bar{x}_2$ is not feasible for candidate 1. Consider the set $\mathcal{B}(\bar{x}_2, L_1) = \{B(\bar{x}_1, \bar{x}_2) : \bar{x}_1 \in L_1\}$ of bisector lines between $\bar{x}_2$ and all feasible policies of candidate 1. Notice that all the lines in $\mathcal{B}(\bar{x}_2, L_1)$ are tangent to the parabola with focus $\bar{x}_2$ and directrix L_1 (see Figure 1).

Let $\mathcal{P}(\bar{x}_2, L_1)$ denote the subset of the plane separated by that parabola together with the parabola and containing policy $\bar{x}_2$. Notice that the interior of region $\mathcal{P}(\bar{x}_2, L_1)$ could be interpreted as the set of ideal positions of the electorate which is inaccessible to candidate 1 (that is, loyal voters of candidate 2). The following lemma gives the necessary and sufficient conditions for candidate 1 to possess a policy that would win over $\bar{x}_2$.

Lemma 1. *Let $\bar{x}_2$ be the policy proposed by candidate 2 and assume that $\bar{x}_2 \notin L_1$. Then there exists $\bar{x} \in L_1$ such that $\bar{x} M \bar{x}_2$ if and only if there exists a median line $m \in \mathcal{M}_f$ such that $m \cap \mathcal{P}(\bar{x}_2, L_1) = \varnothing$.*

Proof. For the right to left implication suppose that $m \in \mathcal{M}_f$ is a median line such that $m \cap \mathcal{P}(\bar{x}_2, L_1) = \varnothing$. Let l be a line parallel to m and tangent to $\mathcal{P}(\bar{x}_2, L_1)$. Then policy $\bar{x}$ which is the reflection of $\bar{x}_2$ in l strictly dominates $\bar{x}_2$ as $m \in D(\bar{x}, \bar{x}_2)$.

For the left to right implication suppose that there exists $\bar{x} \in L_1$ such that $\bar{x} M \bar{x}_2$. Consider the bisector of $\bar{x}_2$ and $\bar{x}$, $B(\bar{x}_2, \bar{x})$ (which is tangent to $\mathcal{P}(\bar{x}_2, L_1)$) and let m be the median line parallel to $B(\bar{x}_2, \bar{x})$. Since $\bar{x} M \bar{x}_2$, so it must be that $m \in D(\bar{x}, \bar{x}_2)$ and since $\mathcal{P}(\bar{x}_2, L_1) \cap D(\bar{x}, \bar{x}_2) = \varnothing$, so it must be that $m \cap \mathcal{P}(\bar{x}_2, L_1) = \varnothing$. □

Lemma 1 provides us with a necessary and sufficient condition for existence of winning best responses for a candidate and shows that they exist if and only if there exists a median line that does not intersect a parabolic region associated with one of the policies proposed under the 'current' strategy profile.

Before we proceed to introducing the results on a DFPE, let us introduce yet another notion that will be crucial. Let $m \in \mathcal{M}_f$ be a median line and assume that f is non-atomic. By Fact 1, every median line in $\mathcal{M}_f$ can be uniquely identified by its gradient r with respect to the median line m (that is $r = \tan(\alpha)$, where α is the angle between m and the median line). We will use $[m]_r$ to denote the median line identified by m and $r \in \mathbb{R}$. The proof of the following fact is moved to the Appendix.

Consider function $\gamma^m : \mathbb{R} \setminus \{0\} \to m$ such that $\{\gamma^m(r)\} = m \cap [m]_r$. That is $\gamma^m(r)$ returns the intersection point of m and $[m]_r$.

Fact 3. *If f is non-atomic, then for any $m \in \mathcal{M}_f$, γ^m is a continuous function on $\mathbb{R} \setminus \{0\}$.*

Finally, Let $\bar{p}_0^+(m) = \lim_{r\to 0^+} \gamma^m(r)$ and $\bar{p}_0^-(m) = \lim_{r\to 0^-} \gamma^m(r)$.

4.2.1. Homogeneous commitment

Suppose that $\mathrm{c}_1 = \mathrm{c}_2 = \mathrm{i}$ (and consequently $\mathrm{n}_1 = \mathrm{n}_2 = \mathrm{j} \neq \mathrm{i}$) and $a \neq b$. To simplify the presentation we will assume that for any $\bar{x} = (x, y) \in \mathbb{R}$, that the first coordinate is the value of the free issue j while the second coordinate is the value of the committed issue i. We will also use x_1 and x_1' to denote values of free issue chosen by candidate 1 and x_2 and x_2' to denote values of the free issue chosen by candidate 2.

It turns out that in any DFPE with homogeneous commitment, both candidates have to propose the same value of their free issues. Thus, divergence here is entirely driven by history of the party. For such an equilibrium to exist, the distribution f must be such that both $\bar{p}_0^+(m^{\mathrm{i}})$ and $\bar{p}_0^-(m^{\mathrm{i}})$ exist and $\bar{p}_0^+(m^{\mathrm{i}}) = \bar{p}_0^-(m^{\mathrm{i}}) = \bar{p}_0(m^{\mathrm{i}})$. However, the interesting aspect of existence is that the feasible sets of policies of both candidates must be distant enough from each other and the median line m^{i} must be the bisector of them. *This implies that strong extremism in the committed dimension must prevail, while these two parties who are known to take two opposing extremist stands in a common issue must take a common stand in the free issue.* The equilibrium strategy of each candidate is the projection of $\bar{p}_0(m^{\mathrm{i}})$ on his feasible set of policies. Finally, the theorem also indicates that stronger divergence in the common committed issue cannot reduce the chance of existence of such equilibria.

Theorem 3. *Consider a game $\Gamma^{\mathrm{i},\mathrm{i}}_{a,b}$ with $a \neq b$. If a DFPE of the game $\Gamma^{\mathrm{i},\mathrm{i}}_{a,b}$ exists, then* (i). $B(L_1, L_2) = \{m^{\mathrm{i}}\}$,

(ii). $\bar{p}_0^{+}(m^{\mathrm{i}}) = \bar{p}_0^{-}(m^{\mathrm{i}}) = \bar{p}_0(m^{\mathrm{i}}) \in \mathbb{R}^2$,

(iii). $|a - b| \geq D_f$, *where* $D_f \in \mathbb{R}_{\geq 0}$ *is a threshold value that depends on the distribution* f *only,*

(iv). *for any* $\varepsilon > 0$ *a DFPE of game* $\Gamma^{\mathrm{i,i}}_{a+\varepsilon, b-\varepsilon}$ *(if* $a > b$*) or game* $\Gamma^{\mathrm{i,i}}_{a-\varepsilon, b+\varepsilon}$ *(if* $a < b$*) exists.*

Moreover, if a DFPE $\Gamma^{\mathrm{i,i}}_{a,b}$ *exists, then* (x_1, x_2) *is a DFPE if and only if* $x_1 = x_2 = \bar{p}_0(m^{\mathrm{i}}) \downarrow \mathrm{j}$.

Proof. We start by showing that if a DFPE of $\Gamma^{\mathrm{i,i}}_{a,b}$ exists, then $B(L_1, L_2) = \{m^{\mathrm{i}}\}$. Assume that the opposite holds, that is there exists a DFPE (x_1, x_2) and $B(L_1, L_2) \neq \{m^{\mathrm{i}}\}$. Notice that since $L_1 \cap L_2 = \varnothing$, so they have exactly one bisector line. Then it holds that $B((x_1, a), (x_1, b)) \neq m^{\mathrm{i}}$ and $B((x_2, a), (x_2, b)) \neq m^{\mathrm{i}}$. Since f is non-atomic, so either $\mu(D((x_2, a), (x_2, b))) > \mu(D((x_2, b), (x_2, a)))$ (if L_1 is closer to m^{i} than L_2) or $\mu(D((x_1, b), (x_1, a))) > \mu(D((x_1, a), (x_1, b)))$ (if L_2 is closer to m^{i} than L_1). Thus either candidate 1 can propose x_2 instead of x_1 and win outright (in the first case) or candidate 2 can propose x_1 instead of x_2 and win outright (in the second case). Hence (x_1, x_2) cannot be a Nash equilibrium and it must be that $B((x, a), (x, b)) = m^{\mathrm{i}}$, for all $x \in \mathbb{R}$.

We show next that if (x_1, x_2) is a DFPE, then it must be that $x_1 = x_2$. Again assume that the opposite holds, that is $x_1 \neq x_2$. Let $\bar{x}_1$ and $\bar{x}_2$ denote the policies proposed by candidates 1 and 2 in equilibrium, respectively. As we have shown above, it must be that $B(L_1, L_2) = m^{\mathrm{i}}$. Then $\mathcal{P}(\bar{x}_1, L_2) \cap \mathcal{P}(\bar{x}_2, L_1) = \varnothing$ and there exist two different parallel lines l_1 and l_2, one tangent to $\mathcal{P}(\bar{x}_1, L_2)$ and another one tangent to $\mathcal{P}(\bar{x}_2, L_1)$. Hence the median

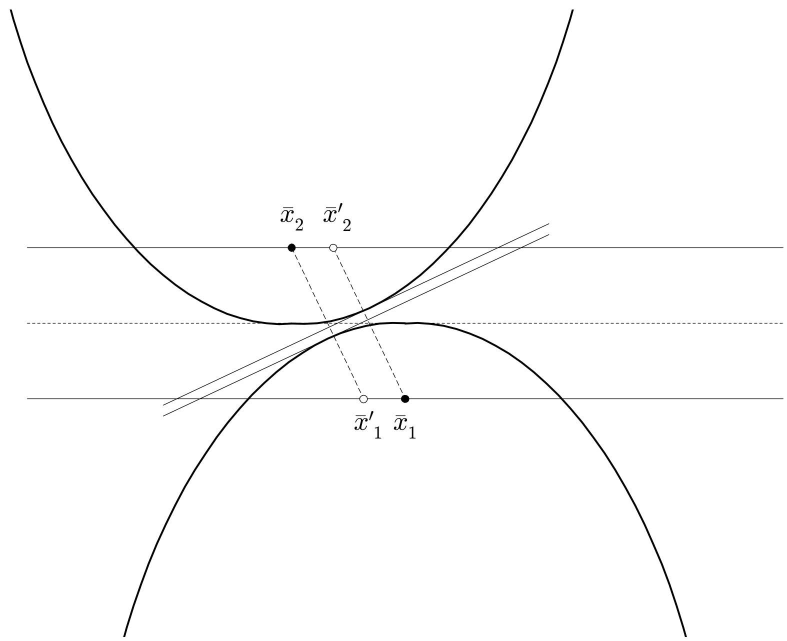

Figure 2: Configuration with $x_1 \neq x_2$ and possible improving deviations (at least one of them improves).

line m parallel to l_1 and l_2 satisfies $m \cap \mathcal{P}(\bar{x}_1, L_2) = \varnothing$ or $m \cap \mathcal{P}(\bar{x}_2, L_1) = \varnothing$ (see Figure 2).

But then, by Lemma 1, there exists an improving deviation for one of the candidates and so (x_1, x_2) cannot be a Nash equilibrium. Hence it must be that if (x_1, x_2) is a Nash equilibrium, then $x_1 = x_2$.

Thirdly, we show that if (x, x) is a Nash equilibrium, then it must be that $x = \bar{p}_0^+(m^\mathrm{i}) \downarrow \mathrm{j}$ and $x = \bar{p}_0^-(m^\mathrm{i}) \downarrow \mathrm{j}$. We will show the first equality, the second one can be shown by analogical arguments. Assume the opposite, that is $x \neq x_0$ where $x_0 = \bar{p}_0^+(m^\mathrm{i}) \downarrow \mathrm{j}$. Let $\bar{x} \in m^\mathrm{i}$ denote the projection of policy proposed by candidate 1 in equilibrium on median line m^i. Then $\bar{x} \neq \bar{p}_0^+(m^\mathrm{i})$. Since $\bar{p}_0^+(m^\mathrm{i}) = \lim_{r \to 0^+} \gamma^{m^\mathrm{i}}(r)$ and, by Fact 3, γ^{m^i} is continuous on $\mathbb{R} \setminus \{0\}$, so for any $\sigma > 0$ there exists $r_\sigma > 0$ such that for all $r' \in [0, r_\sigma)$, $|\gamma^{m^{\mathrm{i}}}(r') - \gamma^{m^{\mathrm{i}}}(0)| < \sigma$. Take $\sigma = |x - x_0|/2$ and let l be the line tangent to $\mathcal{P}(\bar{x}_2, L_1)$ at projection of point $((a+b)/2, (2x-x_0)/2)$ on $\mathcal{P}(\bar{x}_2, L_1)$ (if $x_0 > x$) or the line tangent to $\mathcal{P}(\bar{x}_1, L_2)$ at projection of point $((a+b)/2, (2x_0-x)/2)$ on $\mathcal{P}(\bar{x}_1, L_2)$ (if $x_0 < x$). Let s be the gradient of l with respect to median line m^{i}. Then there exists $r' \in [0, r_\sigma)$ such that $r' < s$. For any such r' the associated median line $[m^{\mathrm{i}}]_{r'}$ does not intersect $\mathcal{P}(\bar{x}_2, L_1)$ (if $x_0 > x$) or $\mathcal{P}(\bar{x}_1, L_2)$ (if $x > x_0$) (see Figure 3). Hence, by Lemma 1, (x, x) cannot be a Nash equilibrium of $\Gamma^{\mathrm{i,i}}_{a,b}$, which contradicts our assumptions. Thus it must be that $x = \bar{p}_0^+(m^{\mathrm{i}}) \downarrow \mathrm{j}$. It can shown, using analogical arguments, that if (x, x) is a divergent full participation Nash equilibrium, then it must be that $x = \bar{p}_0^-(m^{\mathrm{i}}) \downarrow \mathrm{j}$. It follows that if a divergent full participation Nash equilibrium of $\Gamma^{\mathrm{i,i}}_{a,b}$ exists, then it must be that $\bar{p}_0^+(m^{\mathrm{i}}) = \bar{p}_0^-(m^{\mathrm{i}})$.

Before we continue with the proof, we make the following remark.

Remark 1. *Notice that the analysis above implies that if a divergent full participation Nash equilibrium of game $\Gamma^{\mathrm{i,i}}_{a,b}$ exists, then it is unique and has a form (x, x), where $x = \bar{p}_0(m^{\mathrm{i}}) \downarrow \mathrm{j}$. Hence if such equilibrium exists, then a strategy profile (x, x), where $x = \bar{p}_0(m^{\mathrm{i}}) \downarrow \mathrm{j}$, must be a Nash equilibrium.*

In the next part of the proof we show that if a divergent full participation Nash equilibrium of game $\Gamma^{\mathrm{i,i}}_{a,b}$ exists, then there exists a divergent Nash equilibrium of game $\Gamma^{\mathrm{i,i}}_{a+\varepsilon,b-\varepsilon}$ (if $a > b$) or game $\Gamma^{\mathrm{i,i}}_{a-\varepsilon,b+\varepsilon}$ (if $a < b$), for any $\varepsilon > 0$.

Suppose that $a > b$ (proof for $b < a$ is analogical) and let (x, x) be a divergent full participation Nash equilibrium of $\Gamma^{\mathrm{i,i}}_{a,b}$. Let $L_1^\varepsilon = \{(x, a+\varepsilon) : x \in \mathbb{R}\}$ and $L_2^\varepsilon = \{(x, b-\varepsilon) : x \in \mathbb{R}\}$. Notice that for $\varepsilon > 0$ it holds that

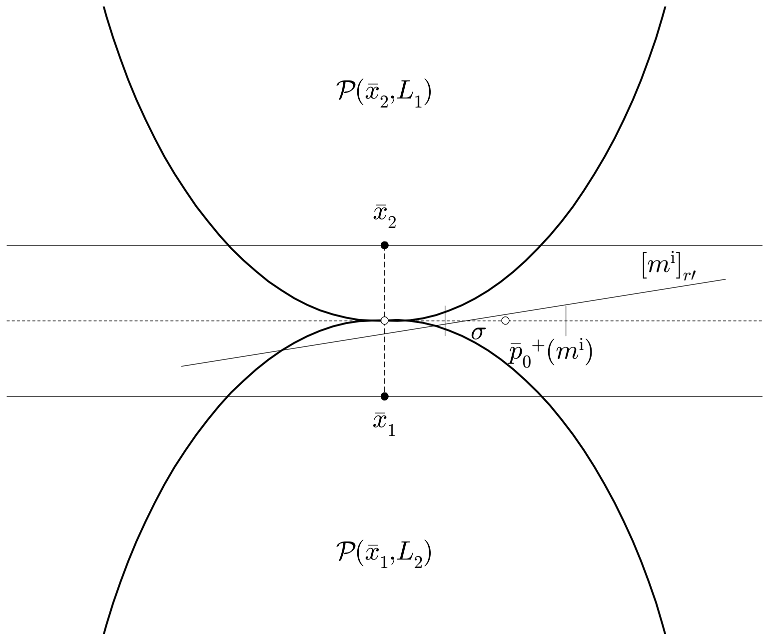

Figure 3: Configuration with $x < x_0$ and a median line lying outside region $\mathcal{P}(\bar{x}_2, L_1)$.

$\mathcal{P}((x,a),L_2) \subseteq \mathcal{P}((x,a+\varepsilon),L_2^\varepsilon)$ and $\mathcal{P}((x,b),L_1) \subseteq \mathcal{P}((x,b+\varepsilon),L_1^\varepsilon)$. Thus any median line intersecting both $\mathcal{P}((x,a),L_2)$ and $\mathcal{P}((x,b),L_1)$ intersects both $\mathcal{P}((x,a+\varepsilon)),L_2^\varepsilon)$ and $\mathcal{P}((x,b-\varepsilon),L_1^\varepsilon)$ as well.

Hence, by Lemma 1, (x,x) is a Nash equilibrium of game $\Gamma^{\mathrm{i,i}}_{a+\varepsilon,b-\varepsilon}$. Moreover, there exists a minimal distance $D_f \geq 0$ between a and b such that $\mathcal{P}((x,a),L_2) \cup \mathcal{P}((x,b),L_1)$ intersects all the median lines. Value of D_f depends on the distribution f only as $x = \bar{p}_0(m^{\mathrm{i}}) \downarrow$ j depends on the distribution f. $\square$

Equilibria with a non-empty core. If the distribution function f has a non-empty core, then we are able to extend Theorem 3 to provide the necessary and sufficient conditions for existence of a DFPE and its full characterisation. The theorem shows that when the core is non-empty, such an equilibrium exists if and only if the two values of the committed issues lie equidistantly on either side of the median line that is vertical to the committed issue and that both candidates choose identical policies on their common free issue. In that sense, there is convergence of policies in the common free issue.

Theorem 4. *Consider a game* $\Gamma^{\mathrm{i,i}}_{a,b}$ *with* $a \neq b$*. If distribution function* f *has non-empty core* $\mathcal{C}_f = \{\bar{c}_f\}$*, then a DFPE of* $\Gamma^{\mathrm{i,i}}_{a,b}$ *exists if and only if* $B(L_1, L_2) = m^{\mathrm{i}}$*. Moreover,* (x_1, x_2) *is a DFPE if and only if* $x_1 = x_2 = \bar{c}_f \downarrow \mathrm{j}$.

Proof. Notice that if $\bar{c}_f$ is a core point, then $\bar{p}_0^+(m^{\mathrm{i}}) = \bar{p}_0^-(m^{\mathrm{i}}) = \bar{c}_f$ and $D_f = 0$. Hence necessary conditions for existence of full participation Nash equilibrium follow immediately from Theorem 3.

Similarly, if (x_1, x_2) is a divergent full participation Nash equilibrium then $x_1 = x_2 = \bar{c}_f \downarrow \mathrm{j}$ follows immediately from Theorem 3.

On the other hand we will show that if $B(L_1, L_2) = m^{\mathrm{i}}$, then a strategy profile (x, x) such that $x = \bar{c}_f \downarrow \mathrm{j}$ is a Nash equilibrium of game $\Gamma^{\mathrm{i,i}}_{a,b}$. Let $\bar{x}_1$ and $\bar{x}_2$ denote the policies proposed by candidates 1 and 2, respectively, under the strategy profile (x, x). Notice that $\mathcal{P}(\bar{x}_1, L_2)$ and $\mathcal{P}(\bar{x}_2, L_1)$ intersect all the median lines. Hence, by Lemma 1, no profitable deviation from (x, x) is possible by any candidate and so (x, x) is a Nash equilibrium. □

Equilibria with an empty core. We are unable to give a full set of conditions for the existence of a DFPE and provide complete characterisation of such equilibria if the core of f is empty. However we provide the following examples with empty core distributions, one where a DFPE exists and another where it fails to exist.

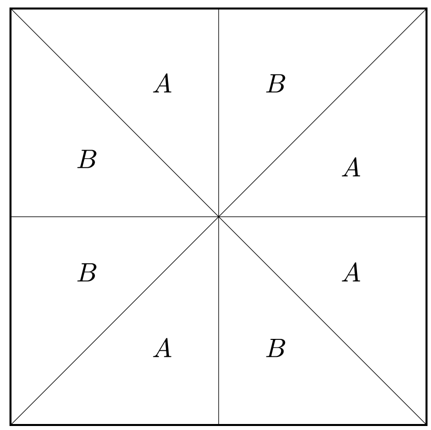

Figure 4: Distribution function for Example 1.

Example 1 (Existence of a DFPE). *Consider the distribution function f which supports a (closed) square with sides of length 2 as presented in Figure 4. The square is divided into 8 parts of equal area. The density function is constant on each of these areas (including the clockwise border, and excluding the centre) and at the centre its value is the same as in areas labelled with B. The mass of each area labelled with A is μ_A and the mass of each area labelled with B is μ_B. We assume that $\mu_B > \mu_A > 0$ and $\mu_A + \mu_B = 1/4$. We will also refer to a difference $\theta = \mu_B - \mu_A$ between the two masses (notice that $0 < \theta < 1/4$).*

Observe that the horizontal median line is the horizontal axis of symmetry of the square and vertical median line is the vertical axis of symmetry of the square. Also observe that the core is indeed empty.

Let the coordinate system be orientated so that the horizontal median line is the horizontal axis (x) and the vertical median line is the vertical axis (y). Consider a game $\Gamma^{\mathrm{i},\mathrm{i}}_{c,-c}$ with committed issue i of both candidates being horizontal dimension and the values to which candidate 1 and 2 are committed *being c and $-c$, respectively. We prove the following claim:*

Claim 1. *If $c \geq 4\theta$, then $(-2\theta, -2\theta)$ is a Nash equilibrium of game $\Gamma^{\mathrm{i,i}}_{c,-c}$.*

Proof. Let $x(r)$ be the distance from the origin to the intersection point $(-x(r), 0)$ of a median line given by the equation $y = r(x + x(r))$ with the horizontal median line as a function of the gradient r. We will show some properties of $x(r)$ that will be useful to prove our claim.

Notice that since the distribution is symmetric about the horizontal axis, so $x(r) = x(-r)$. Hence we will restrict our attention to the case where $r > 0$. Observe that if $r > 0$, then the intersection point cannot be on the right hand side of the origin, as the mass on the left hand side of the line $y = rx$ is larger than the mass on its right hand side. Thus it must be that $x(r) \geq 0$. Moreover, it must be that $x(r) < 1$ (that is the intersection point lies within the bounds of the square), since otherwise the mass on the right hand side of the line would be larger than the mass on the left hand side of the line. Similarly, if $r > 0$, then the associated median line cannot intersect the upper bound of the square to the right of the vertical axis as this axis it a median line. Hence there are three configuration with median line intersecting the square possible as depicted in Figures 5, 6 and 7 respectively. Notice that since $x(r) < 1$, so for $r \in (1, 1/2)$ it must be that median line $y = r(x + x(r))$ intersects the square according to configuration (1).

We will now compute the value of $x(r)$ for configuration (1). Consider a line $y = r(x + d)$ intersecting the square according to configuration (1). To compute the mass on the left hand side of a line $y = r(x + d)$, where $d \in [-1, 0]$, we will need to compute the masses of the regions R_1, R_2, R_3, R_4 and R_5, as presented in Figure 5.

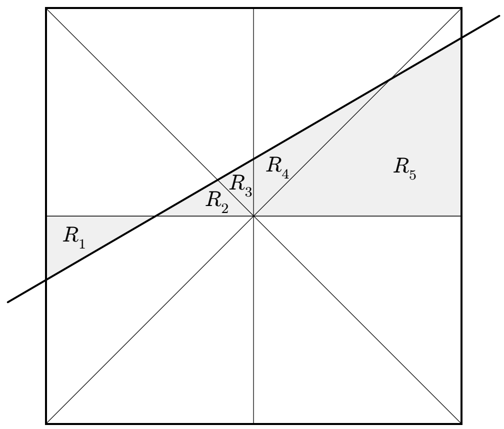

Figure 5: Configuration (1) with median line $y = r(x + x(r))$ intersecting the square.

It is easy to check that the areas of the respective regions are as follows:

$$\begin{aligned}
v_1(r,d) &= \frac{r(1-d)^2}{2}, \\
v_2(r,d) &= \frac{rd^2}{2(r+1)}, \\
v_3(r,d) &= \frac{r^2d^2}{2(r+1)}, \\
v_4(r,d) &= \frac{r^2d^2}{2(1-r)}, \\
v_5(r,d) &= \frac{r((r-1)(d+1)^2+d^2)}{2(r-1)}.
\end{aligned} \tag{1}$$

Now the mass of the left hand side of the line $y = r(x + d)$ is

$$2\mu_B + 2\mu_A - 2\mu_B(v_2(r,d) + v_4(r,d)) - 2\mu_A(v_3(r,d) + v_5(r,d)) + 2\mu_B v_1(r,d)$$

which, after substituting $\mu_B - \mu_A$ by θ is equal to

$$\frac{4r^3\theta d^2 + r(r^2-1)d + (r^2-1)(2r\theta - 1)}{2(r^2-1)}$$

Making this equal to 1/2 and solving for d we find $x(r)$, which is the intersection point of the median line $y = r(x + x(r))$ with the horizontal axis

$$x(r) = \frac{\sqrt{(1-r^2)(32r^2\theta^2 + 1 - r^2)} - (1-r^2)}{8r^2\theta} \tag{2}$$

Notice that

$$x_0 = \lim_{r \to 0^+} x(r) = 2\theta < \frac{1}{2},$$

so $\bar{p}_0^+ = (-2\theta, 0)$. Moreover, by symmetry of the distribution about horizontal median line, $\bar{p}_0^- = \bar{p}_0^+$, and so $\bar{p}_0 = (-2\theta, 0)$.

We will also show that $x(r)$ is decreasing in r for $0 < r \leq 1$. Differentiating $x(r)$ we get

$$x'(r) = \frac{\sqrt{(1-r^2)(r^2(32\theta^2 - 1) + 1)} + r^2(1 - 16\theta^2) - 1}{4r^3\theta\sqrt{(1-r^2)(r^2(32\theta^2-1)+1)}}.$$

The denominator of $x'(r)$ is > 0 for $0 \leq r \leq 1$ and $0 < \theta < 1/4$ and the nominator can be rewritten as

$$\frac{256r^4\theta^4}{r^2(1-16\theta^2) - 1 - \sqrt{(1-r^2)(r^2(32\theta^2-1)+1)}}. \tag{3}$$

Since

$$(1-r^2)(r^2(32\theta^2-1)+1) = r^2(1-16\theta^2) - 1 + 16r^2\theta^2(3-2r^2) + r^4 - 3r^2 + 2$$

and

$$16r^2\theta^2(3-2r^2) + r^4 - 3r^2 + 2 > 0$$

for $0 \leq r \leq 1$ and $0 < \theta < 1/4$, so (3) is < 0 and so $x'(r) < 0$. Thus $x(r)$ is decreasing on $(0, 1]$ for $0 < \theta < 1/4$. This means in particular that $x(r) < 2\theta$ for $0 \leq r \leq 1$ and $0 < \theta < 1/4$. Consider a line $y = r(x + 2\theta)$ and suppose that it intersects the square according to configuration (1). Then it holds

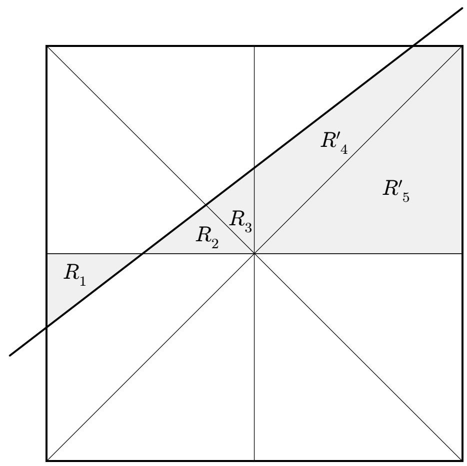

Figure 6: Configuration (2) with median line $y = r(x + x(r))$ intersecting the square.

that $0 \leq r \leq 1/(1+2\theta)$. Notice that for any $2\theta < d \leq 0$, the line $y = r(x+d)$ intersects the square according to configuration (1) as well. Hence, by the fact that $2\theta < x(r) \leq 0$ for configuration (1) and $0 < \theta < 1/4$, it holds that $x(r) < 2\theta$ for $0 \leq r \leq 1/(1+2\theta)$ and $0 < \theta < 1/4$.

For the remaining two configurations, we will show that in each of them it must be that $x(r) < 2\theta$. Consider a line $y = r(x + 2\theta)$ and suppose that it intersects the square according to configuration (2) (see Figure 6). Then it holds that $1/(1 + 2\theta) \leq r \leq 1/(1 - 2\theta)$. We will show that the sum of masses of the regions R_2, R_3, R'_4 and R'_5 is greater than the mass of region R_1, which implies that the median line $y = r(x + x(r))$ must be to the right of the line $y = r(x + 2\theta)$, that is $x(r) < 2\theta$.

The area of the new region R'_4 is

$$v'_4(r, d) = \frac{r - (rd - 1)^2}{2r},$$

and the mass of the area R'_5 is μ_A. Hence the difference between the sum of the masses of regions R_2, R_3, R'_4 and R'_5 and the mass of region R_1 is

$$2\mu_B v_2(r, 2\theta) + 2\mu_A v_3(r, 2\theta) + 2\mu_B v'_4(r, 2\theta) + \mu_A - 2\mu_B v_1(r, 2\theta)$$

which, after substituting $\mu_B - \mu_A$ by θ is equal to

$$D_2(r, \theta) = \frac{r^3(12\theta^2 - 48\theta^3 - 1) + r^2(28\theta^2 + 4\theta + 1 - 16\theta^3) + r(16\theta^2 + 1) - 4\theta - 1}{8r(r+1)}.$$

Differentiating $D_2(r, \theta)$ over r we can easily check that it is increasing with r increasing on the interval $[1/(1+2\theta), 1/(1-2\theta)]$, for $0 < \theta < 1/4$. Thus

$$D_2(r, \theta) \geq D_2\left(\frac{1}{1+2\theta}, \theta\right) = \frac{32\theta^2}{(2\theta+1)^3} > 0$$

for $1/(1+2\theta) \leq r \leq 1/(1-2\theta)$ and $0 < \theta < 1/4$. Hence $x(r) \leq 2\theta$ in this case.

Lastly, consider a line $y = r(x + 2\theta)$ and suppose that it intersects the square according to configuration (3) (see Figure 7). Then it holds that $1/(1-2\theta) \leq r \leq 1/(2\theta)$. We will show that the sum of the masses of regions R_2, R_3, R'_4 and R'_5 is greater than the sum of the masses of regions R'_1 and R_6, which implies that the median line $y = r(x + x(r))$ must be to the right of the line $y = r(x + 2\theta)$, that is $x(r) < 2\theta$.

The areas of the new regions R'_1 and R_6 are

$$\begin{aligned} v'_1(r, d) &= \frac{r - 1 - rd^2}{2(r-1)}, \\ v_6(r, d) &= \frac{(r(d-1)+1)^2}{2r(r-1)}. \end{aligned}$$

Hence the difference between the sum of the masses of regions R_2, R_3, R'_4 and R'_5 and the masses of regions region R'_1 and R_6 is

$$2\mu_B v_2(r, 2\theta) + 2\mu_A v_3(r, 2\theta) + 2\mu_B v'_4(r, 2\theta) + \mu_A - 2\mu_B v'_1(r, 2\theta) - 2\mu_A v_6(r, 2\theta)$$

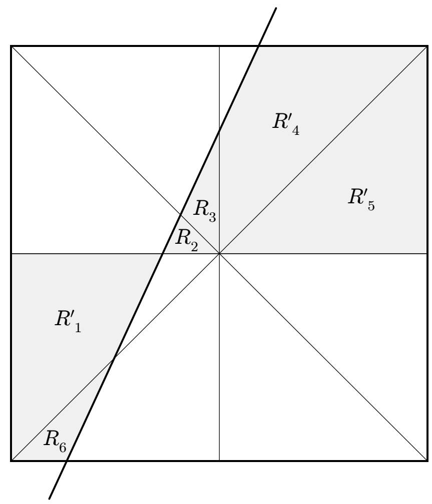

Figure 7: Configuration (3) with median line $y = r(x + x(r))$ intersecting the square.

which, after substituting $\mu_B - \mu_A$ by θ is equal to

$$D_3(r, \theta) = \frac{\theta(4r^2\theta^2(2r + 1 - r^2) + r^3 - r^2 - r + 1)}{r(r^2 - 1)}.$$

It is easy to check that

$$r^3 - r^2 - r + 1 \geq 0$$

for $r \geq 0$ and

$$4r^2\theta^2(2r + 1 - r^2) \geq 0$$

for $0 \leq r \leq 1 + \sqrt{2}$ and $\theta \geq 0$. If $r > 1 + \sqrt{2}$, then

$$4r^2\theta^2(2r + 1 - r^2) + r^3 - r^2 - r + 1 \geq r(r - 1)^2 + 2 > 0,$$

as, $\theta \geq 1/(2r)$. Thus we have shown that $D_3(r, \theta) \geq 0$ for $0 < \theta < 1/4$ and $1/(1 - 2\theta) \leq r \leq 1/(2\theta)$ and so $x(r) \leq 2\theta$ in this case.

Now we are ready to show that $(-2\theta, -2\theta)$ is a Nash equilibrium of game Γ. Notice that the region

$$\mathcal{P}((-c,-2\theta), L_1) = \left\{(x,y) \in \mathbb{R}^2 : y \geq \frac{(x+2\theta)^2}{4c}\right\}$$

and the region

$$\mathcal{P}((c,-2\theta), L_2) = \left\{(x,y) \in \mathbb{R}^2 : y \leq -\frac{(x+2\theta)^2}{4c}\right\}.$$

Observe that both the regions intersect the vertical median line. We will show that for all $r \in \mathbb{R} \setminus \{0\}$ both regions intersect the median line $y = r(x + x(r))$. Consider the region $\mathcal{P}((-c,-2\theta), L_1)$. Since all median lines intersect the horizontal median line line within the interval $(-2\theta, 0]$, so for all $r < 0$ the median line $y = r(x+x(r))$ intersects this region. For $r > 0$ consider the line $y = r(x+x_p(r))$ tangent to this region. Then $x_p(r) = 2\theta - rc$ and if we show that $x(r) \geq x_p(r)$, for $r > 0$, then this will imply that the associated median lines intersect the region. As we observed above, it must be that $x(r) \geq 0$, for $r > 0$. Thus if $x_p(r) \leq 0$, that is $r \geq 2\theta/c$, then the inequality is satisfied. Suppose that $r < 2\theta/c$. Then $r \leq 1/2$, as $c \geq 4\theta$, and we have to show that the inequality holds for configuration (1), that is

$$x_p(r) \leq \frac{\sqrt{(1-r^2)(32r^2\theta^2+1-r^2)} - (1-r^2)}{8r^2\theta}$$

which is equivalent to

$$-r\frac{c}{4\theta} \leq \frac{\sqrt{(1-r^2)(32r^2\theta^2+1-r^2)} - (1-r^2)}{32r^2\theta^2} - \frac{1}{2}. \tag{4}$$

Consider the function

$$\begin{aligned}\varphi(r,\theta) &= \frac{\sqrt{(1-r^2)(32r^2\theta^2+1-r^2)} - (1-r^2+16r^2\theta^2)}{32r^2\theta^2} \\ &= \frac{\sqrt{(1-r^2+16r^2\theta^2)^2 - 256r^4\theta^4} - (1-r^2+16r^2\theta^2)}{32r^2\theta^2}\end{aligned}$$

obtained from the right hand side of the inequality. Let

$$\psi(r,\theta) = \sqrt{(1-r^2+16r^2\theta^2)^2 - 256r^4\theta^4} - (1-r^2+16r^2\theta^2).$$

Notice that $\psi(r,\theta) < 0$ for $0 < r \leq 1/2$ and $0 < \theta \leq 1/4$. Differentiating $\varphi(r,\theta)$ over θ we get

$$\frac{\partial\varphi(r,\theta)}{\partial\theta} = \frac{\psi(r,\theta)\sqrt{(1-r^2)(r^2(32\theta^2-1)+1)}}{16r^2\theta^3(r^2(32\theta^2-1)+1)}.$$

and since $\psi(r,\theta) < 0$ for $0 < r \leq 1/2$ and $0 < \theta \leq 1/4$ so

$$\frac{\partial\varphi(r,\theta)}{\partial\theta} < 0,$$

for $0 < r < 1/2$ and $0 < \theta \leq 1/4$. Thus $\varphi(r,\theta)$ is decreasing in θ for $\theta \in (0, 1/4]$, for any $0 < r \leq 1/2$, and to show inequality (4), we need to show that

$$-r\frac{c}{4\theta} \leq \varphi(r,1/4) = \frac{\sqrt{1-r^4}-1}{2r^2}, \tag{5}$$

for $0 < r \leq 1/2$ and $c \geq 4\theta$. Since

$$\frac{\sqrt{1-r^4}-1}{2r^2} = -\frac{r^2}{2(1+\sqrt{1-r^4})} \geq -\frac{r^2}{2} > -r \geq -r\frac{c}{4\theta}$$

for $0 < r \leq 1/2$ and $c \geq 4\theta$, so the inequality (4) holds. Thus we have shown that all median lines intersect the region $\mathcal{P}((-c,-2\theta),L_1)$. Showing that all median lines intersect the region $\mathcal{P}((c,-2\theta),L_2)$ can be done by symmetrical arguments, due to the fact that f is symmetrical about the horizontal median line. Hence if $c \geq 4\theta$, then $(-2\theta,-2\theta)$ is a Nash equilibrium of $\Gamma^{\mathrm{i,i}}_{c,-c}$. □

To be complete, the next example demonstrates that an equilibrium may of course not exist with an empty core.

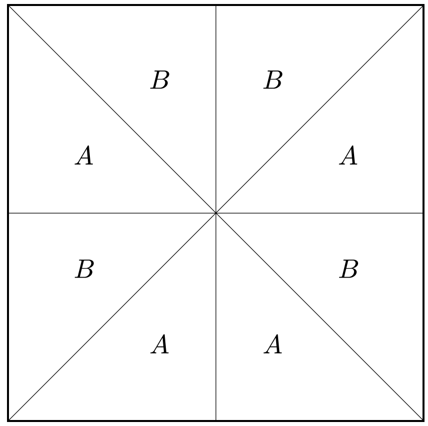

Figure 8: Distribution function for Example 2.

Example 2 (Non-existence of a DFPE)**.** *Consider the distribution function f which is a rotated distribution from Example 1 with support presented in Figure 8.*

Let the coordinates system be orientated so that the horizontal median line is the horizontal axis (x) and the vertical median line is the vertical axis (y). Consider a game $\Gamma^{\mathrm{i},\mathrm{i}}_{c,-c}$ with committed issue i *of both candidates being horizontal dimension and the values to which candidates* 1 *and* 2 *are committed being c and $-c$, respectively. We prove the following claim:*

Claim 2. *If $c \neq 0$, then a Nash equilibrium of game $\Gamma^{\mathrm{i},\mathrm{i}}_{c,-c}$ does not exist, that is there is no DFPE.*

Proof. Like in Example 2, if $0 < r < 1/2$, the median line $r(x + x(r))$ intersects the square according to configuration (1) (see Figure 5). Consider a line $r(x + d)$ intersecting the square according to configuration (1). Then the mass on the left hand side of the line is

$$2\mu_B + 2\mu_A - 2\mu_A(v_2(r,d) + v_5(r,d)) + 2\mu_B(v_3(r,d) + v_4(r,d)) + 2\mu_B v_1(r,d)),$$

where $v_1(r,d)$, $v_2(r,d)$, $v_3(r,d)$, $v_4(r,d)$ and $v_5(r,d)$ are areas of regions R_1, R_2, R_3, R_4 and R_5, respectively, as computed in Example 3 (see Equation (1)). After substituting $\mu_B - \mu_A$ by θ this mass is equal to

$$\frac{2d^2r\theta(1-2r-r^2) - dr(1-r^2) + (1-r^2)(2r\theta+1)}{2(1-r^2)}$$

Making this equal to 1/2 and solving for d we find $x(r)$, which is the intersection point of the median line $y = r(x + x(r))$ with the horizontal axis

$$x(r) = \frac{1 - r^2 - \sqrt{(1-r^2)((1-r^2)(1-16\theta^2) + 32r\theta^2)}}{4\theta(1-2r-r^2)}$$

Notice that

$$x_0 = \lim_{r \to 0^+} x(r) = \frac{1-\sqrt{1-16\theta^2}}{4\theta} > 0$$

for $0 < \theta < 1/4$. Hence it holds that $\bar{p}_0^+ = (-x_0, 0)$. Moreover, by symmetry of the distribution about vertical median line, $\bar{p}_0^- = (x_0, 0)$, and since $x_0 > 0$ it holds that $\bar{p}_0^+ \neq \bar{p}_0^-$. Thus, by Theorem 3, there is no full participation Nash equilibrium of game $\Gamma^{\mathrm{i,i}}_{c,-c}$ if $c > 0$. □

4.2.2. Heterogeneous commitment

Suppose that $\mathrm{c}_1 \neq \mathrm{c}_2$ (and consequently $\mathrm{n}_1 \neq \mathrm{n}_2$). To simplify the presentation, for any $\bar{x} = (x, y) \in \mathbb{R}$, we will assume that the first coordinate is the value of the committed issue candidate 1 or the free issue of candidate 2, while the second coordinate is the value of the free issue of candidate 1 or the committed issue of candidate 2. We will also use y_1 and y_1' to denote values of free issue chosen by candidate 1 and x_2 and x_2' to denote values of the free issue chosen by candidate 2.

In what follows we will refer to median lines $m_{\mathbf{e}_{\mathrm{c}_1} - \mathbf{e}_{\mathrm{c}_2}}$ and $m_{\mathbf{e}_{\mathrm{c}_1} + \mathbf{e}_{\mathrm{c}_2}}$, which will be denoted by m^+ and m^-, respectively, to simplify the notation. The following lemma is itself interesting. It shows that for such an equilibrium to exist, any one of exactly two symmetric cases must hold as far as the committed values of each candidate is concerned:

Lemma 2. *Consider the game* $\Gamma^{\mathrm{i,j}}_{a,b}$ *with* $\mathrm{i} \neq \mathrm{j}$*. If a DFPE exists, then either* $(a, b) \in m^+$ *or* $(a, b) \in m^-$*.*

Proof. Assume the opposite and let (y_1, x_2) be a DFPE of game $\Gamma^{\mathrm{i,j}}_{a,b}$. Let $\bar{x}_1$ and $\bar{x}_2$ denote the policies proposed by candidates 1 and 2 in equilibrium, respectively. Since the equilibrium is divergent, so it must be that $\bar{x}_1 \neq \bar{x}_2$.

Notice first that it must be that $y_1 \neq b$ and $x_2 \neq a$. For suppose the opposite and assume that $y_1 = b$. Since the equilibrium is divergent, so it must be that $x_2 \neq a$. Moreover it must be that $\bar{x}_1$ and $\bar{x}_2$ are symmetric about the median line m^{i}. But then candidate 2 can reposition himself to x_2' such that $(x_2', b) \in L_2 \cap m^{\mathrm{i}}$ and win outright, which contradicts the assumption that (y_1, x_2) is a Nash equilibrium. Hence it must be that $y_1 \neq b$. The case of $x_2 \neq a$ can be shown by analogical arguments.

For the remaining part let l^+ denote the line parallel to m^+ and such that $(a, b) \in l^+$ and let l^- denote the line parallel to m^- such that $(a, b) \in l^-$. Then either $\bar{x}_1$ and $\bar{x}_2$ lie on the opposite sides of l^+ or l^-. Suppose that the first case holds and suppose that $\delta(\bar{x}_1, m^+) < \delta(\bar{x}_2, m^+)$ (where δ denotes the distance between a point and a line). Then candidate 2 can reposition himself to y_1' such that $B((a, y_1'), \bar{x}_2) = l^+$ and win outright. Similarly there would be an improving deviation for candidate 2 if it was that $\delta(\bar{x}_2, m^+) < \delta(\bar{x}_1, m^+)$. Hence it must be that $l^+ = m^+$ and so $(a, b) \in m^+$ in this case. It can be shown, by analogical arguments, that it must be that $l^- = m^-$ in the case where $\bar{x}_1$ and $\bar{x}_2$ lie on the opposite sides of l^-. $\square$

From now on we will restrict our attention to the case where $(a, b) \in m^+$. Analogous results hold for the symmetric case where $(a, b) \in m^-$.

It turns out that in any DFPE with heterogeneous commitment (and with $(a, b) \in m^+$), both candidates must propose the projection of the point $\bar{p}_0(m^+)$ on their respective free issues. Hence, for such equilibrium to exist, the distribution f must be such that both $\bar{p}_0^+(m^+)$ and $\bar{p}_0^-(m^+)$ exist and $\bar{p}_0^+(m^+) = \bar{p}_0^-(m^+) = \bar{p}_0(m^+)$. Moreover the feasible sets of policies of both candidates must be distant enough from the point $\bar{p}_0(m^+)$. From Lemma 2 and points (ii) and (iii) of Theorem 5 it will follow that in equilibrium, *the distance between the policies announced by the two parties must be significantly large.*

Theorem 5. *Consider a game $\Gamma^{\mathrm{i,j}}_{a,b}$ with $\mathrm{i} \neq \mathrm{j}$ and suppose that $(a, b) \in m^+$. If a DFPE exists then*

(i). $\bar{p}_0^+(m^+) = \bar{p}_0^-(m^+) = \bar{p}_0(m^+)$,

(ii). *if $a < \bar{p}_0(m^+) \downarrow \mathrm{i}$, then $\bar{p}_0(m^+) \downarrow \mathrm{i} - a \geq D^l_f$, where $D^l_f \in \mathbb{R}_{\geq 0}$ is a threshold value that depends on the distribution f only,*

(iii). *if $a > \bar{p}_0(m^+) \downarrow \mathrm{i}$, then $a - \bar{p}_0(m^+) \downarrow \mathrm{i} \geq D^r_f$, where $D^r_f \in \mathbb{R}_{\geq 0}$ is a threshold value that depends on the distribution f only,*

(iv). *for any $\varepsilon > 0$ a DFPE of game $\Gamma^{\mathrm{i,j}}_{a-\varepsilon,b-\varepsilon}$ (if $a < \bar{p}_0(m^+) \downarrow \mathrm{i}$) or game $\Gamma^{\mathrm{i,j}}_{a+\varepsilon,b+\varepsilon}$ (if $a > \bar{p}_0(m^+) \downarrow \mathrm{i}$) exists.*

Moreover, if a DFPE of game $\Gamma^{\mathrm{i,j}}_{a,b}$ exists, then a strategy profile (y_1, x_2) is a DFPE if and only if $y_1 = \bar{p}_0(m^+) \downarrow \mathrm{j}$ and $x_2 = \bar{p}_0(m^+) \downarrow \mathrm{i}$.

Proof. We start by showing that if (y_1, x_2) is a DFPE, then it must be that the policies $\bar{x}_1$ and $\bar{x}_2$ proposed by candidates 1 and 2 in equilibrium are

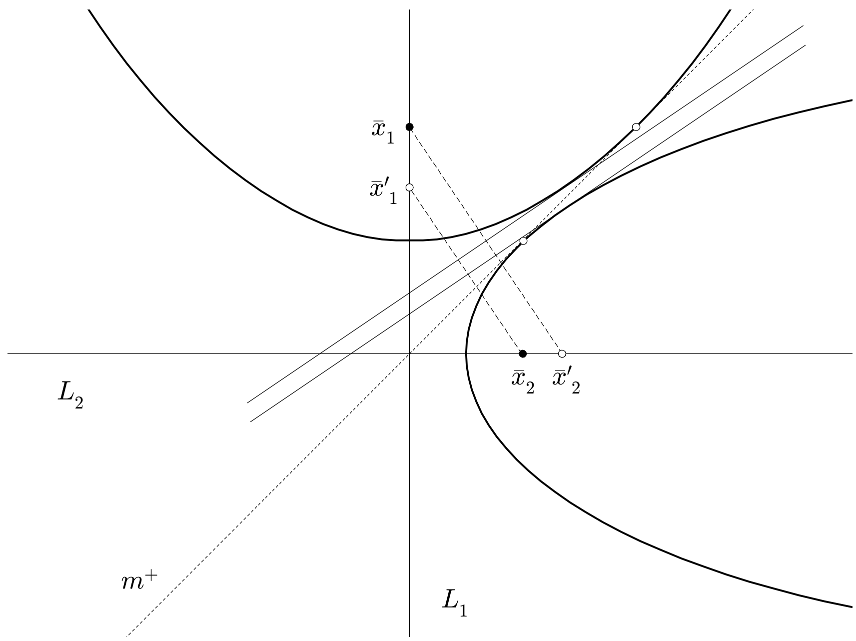

Figure 9: Configuration with $\bar{x}_1$ and $\bar{x}_2$ not symmetric about median line m^+ and possible improving deviations (at least one of them improves).

symmetric about the median line m^+. Assume that the opposite holds. Then $\mathcal{P}(\bar{x}_1, L_2) \cap \mathcal{P}(\bar{x}_2, L_1) = \varnothing$ and there exist two parallel lines l_1 and l_2, one tangent to $\mathcal{P}(\bar{x}_1, L_2)$ and another one tangent to $\mathcal{P}(\bar{x}_2, L_1)$. Hence the median line m parallel to l_1 and l_2 satisfies $m \cap \mathcal{P}(\bar{x}_1, L_2) = \varnothing$ or $m \cap \mathcal{P}(\bar{x}_2, L_1) = \varnothing$. (see Figure 9).

But then, by Lemma 1, there exists an improving deviation for at least one of the candidates, so (y_1, x_2) cannot be a Nash equilibrium. Hence it must be that $\bar{x}_1$ and $\bar{x}_2$ are symmetric about the median line m^+.

Next we show that if (y_1, x_2) is a DFPE, then it must be that $y_2 = \bar{p}_0(m^+) \downarrow \mathrm{j}$ and $x_2 = \bar{p}_0(m^+) \downarrow \mathrm{i}$. Notice that if $\bar{x}_1$ and $\bar{x}_2$ are symmetric about median line m^+, then the regions $\mathcal{P}(\bar{x}_1, L_2)$ and $\mathcal{P}(\bar{x}_2, L_1)$ are tangent

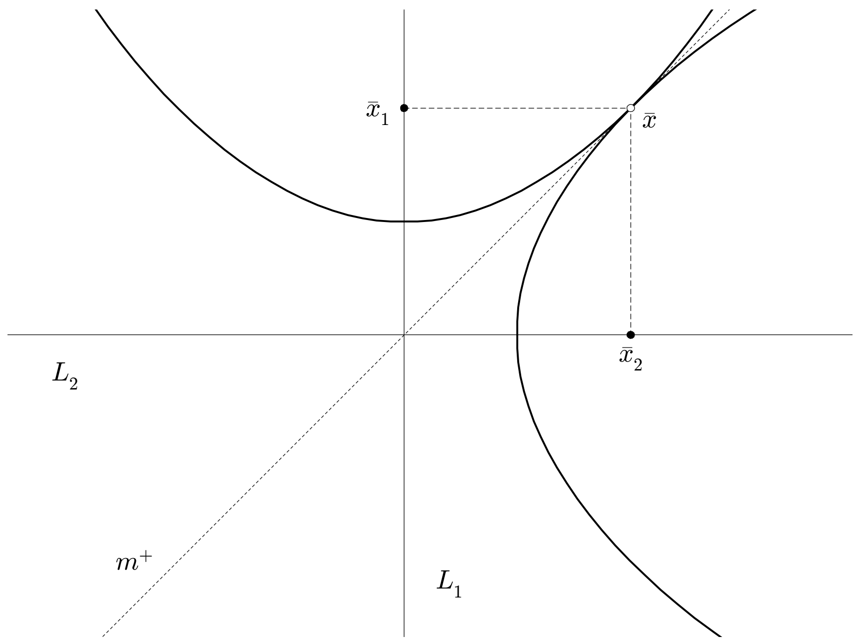

Figure 10: Tangency of regions $\mathcal{P}(\bar{x}_1, L_2)$ and $\mathcal{P}(\bar{x}_2, L_1)$ when $\bar{x}_1$ and $\bar{x}_2$ are symmetric about median line m^+.

to each other and the tangency point $\bar{x} \in m^+$ (see Figure 10).

We will show that it must be that $\bar{x} = \bar{p}_0^+(m^+)$ and $\bar{x} = \bar{p}_0^-(m^+)$. We will show the first equality, the second one can be shown analogically. Arguments here are similar to those used in the proof of Theorem 3. Suppose that $\bar{x} \neq \bar{p}_0^+(m^+)$. Since $\bar{p}_0^+(m^+) = \lim_{r \to 0^+} \gamma^{m^+}(r)$ and γ^{m^+} is continuous on $\mathbb{R} \setminus \{0\}$, so for any $\sigma > 0$ there exists $r_\sigma > 0$ such that for all $r' \in [0, r_\sigma)$, $|\gamma^{m^+}(r') - \gamma^{m^+}(0)| < \sigma$. Take $\sigma = \|\bar{x} - \bar{p}_0^+(m^+)\|/2$ and let s be the gradient of the line tangent to $\mathcal{P}(\bar{x}_2, L_1)$ at the projection of point $(\bar{x} + \bar{p}_0^+(m^+))/2$ on $\mathcal{P}(\bar{x}_2, L_1)$ with respect to median line m^+.

Then there exists $r' \in [0, r_\sigma)$ such that $r' < s$. For any such r' the associated median line $[m]_{r'}$ does not intersect $\mathcal{P}(\bar{x}_2, L_1)$ (if $\bar{p}_0^+(m^+) > \bar{x}$)

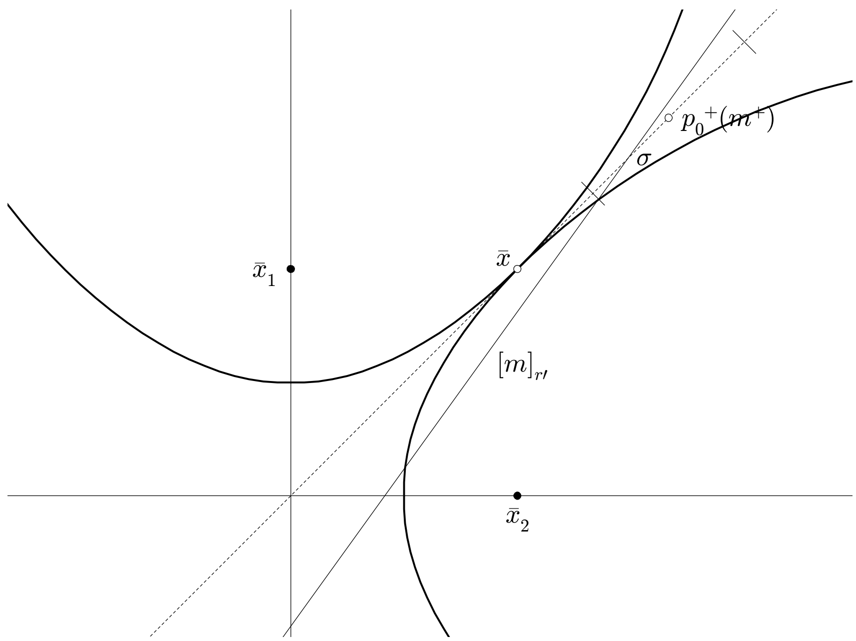

Figure 11: Configuration with $\bar{p}_0^+(m^+) > \bar{x}$ and a median line lying outside region $\mathcal{P}(\bar{x}_2, L_1)$.

or $\mathcal{P}(\bar{x}_1, L_2)$ (if $\bar{p}_0^+(m^+) < \bar{x}$) (see Figure 11). Hence, by Lemma 1, (y_1, x_2) cannot be a Nash equilibrium of $\Gamma^{\mathrm{i,j}}_{a,b}$, which contradicts our assumptions. Thus it must be that $\bar{x} = \bar{p}_0^+(m^+)$. We can show, using analogical arguments, that it must also be that $\bar{x} = \bar{p}_0^-(m^+)$. It follows that if a DFPE of $\Gamma^{\mathrm{i,j}}_{a,b}$ exists, then it must be that $\bar{p}_0^+(m^+) = \bar{p}_0^-(m^+) = \bar{p}_0(m^+)$. Moreover it must be that $\bar{p}_0(m^+) = \bar{x}$, that is $y_1 = \bar{p}_0(m^+) \downarrow \mathrm{j}$ and $x_2 = \bar{p}_0(m^+) \downarrow \mathrm{i}$.

Before we continue with the proof, we make the following remark:

Remark 2. *The analysis above implies that if a DFPE of* $\Gamma^{\mathrm{i,j}}_{a,b}$ *exists,* then it is unique *and has the form* (y_1, x_2), *where* $y_1 = \bar{p}_0(m^+) \downarrow \mathrm{j}$ *and* $x_2 = \bar{p}_0(m^+) \downarrow \mathrm{i}$. *Hence if such equilibrium exists, then a strategy profile* (y_1, x_2), *where* $y_1 = \bar{p}_0(m^+) \downarrow \mathrm{j}$ *and* $x_2 = \bar{p}_0(m^+) \downarrow \mathrm{i}$, *must be a Nash equilibrium of*

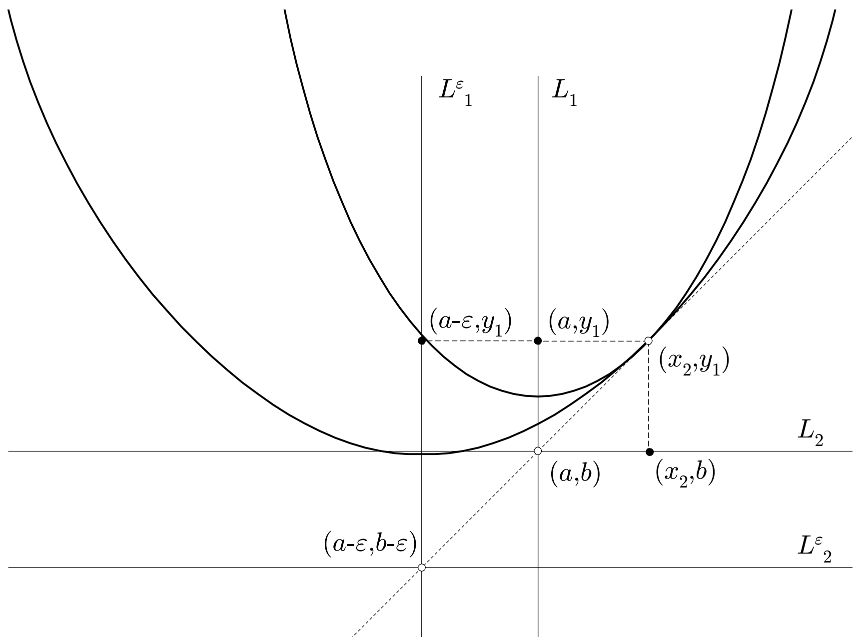

Figure 12: Configuration with $a < \bar{p}_0(m^+) \downarrow \text{i}$, $\varepsilon > 0$ and regions $\mathcal{P}((a, y_1), L_2) \subseteq \mathcal{P}((a - \varepsilon, y_1), L_2^{\varepsilon})$.

game $\Gamma^{\text{i,j}}_{a,b}$.

In the next part of the proof we show that if a DFPE of game $\Gamma^{\text{i,j}}_{a,b}$ exists, then, for any $\varepsilon > 0$, a DFPE of game $\Gamma^{\text{i,j}}_{a-\varepsilon,b-\varepsilon}$ (if $a < \bar{p}_0(m^+) \downarrow \text{i}$) or game $\Gamma^{\text{i,j}}_{a+\varepsilon,b+\varepsilon}$ (if $a > \bar{p}_0(m^+) \downarrow \text{i}$) exists.

Suppose that $a < \bar{p}_0(m^+) \downarrow \text{i}$ (proof for $a > \bar{p}_0(m^+) \downarrow \text{i}$ is analogical) and let (y_1, x_2) be a DFPE of $\Gamma^{\text{i,j}}_{a,b}$. Let $L_1^{\varepsilon} = \{(a - \varepsilon, y) : y \in \mathbb{R}\}$ and $L_2^{\varepsilon} = \{(x, b - \varepsilon) : x \in \mathbb{R}\}$. Notice that for $\varepsilon > 0$ it holds that $\mathcal{P}((a, y_1), L_2) \subseteq \mathcal{P}((a - \varepsilon, y_1), L_2^{\varepsilon})$ and $\mathcal{P}((x_2, b), L_1) \subseteq \mathcal{P}((x_2, b - \varepsilon), L_1^{\varepsilon})$ (see Figure 12). Thus any median line intersecting both $\mathcal{P}((a, y_1), L_2)$ and $\mathcal{P}((x_2, b), L_1)$ intersects both $\mathcal{P}((a - \varepsilon, y_1)), L_2^{\varepsilon})$ and $\mathcal{P}((x_2, b - \varepsilon), L_1^{\varepsilon})$ as well.

Hence, by Lemma 1, (y_1, x_2) is a Nash equilibrium of game $\Gamma^{\text{i,j}}_{a-\varepsilon,b-\varepsilon}$. Moreover, there exists a minimal distance $D^l_f \geq 0$ between $\bar{p}_0(m^+) \downarrow \mathrm{i}$ and a such that $\mathcal{P}((a, y_1), L_2) \cup \mathcal{P}((x_2, b), L_1)$ intersects all the median lines. Value of D^l_f depends on the distribution f only as $\bar{p}_0(m^+)$ depends on the distribution f.

If $a > \bar{p}_0(m^+) \downarrow \mathrm{i}$, then we can show analogical result for game $\Gamma^{\mathrm{i,j}}_{a+\varepsilon,b+\varepsilon}$, using similar arguments. Hence there exists a minimal distance $D^r_f \geq 0$ between a and $\bar{p}_0(m^+) \downarrow \mathrm{i}$ such that $\mathcal{P}((a, y_1), L_2) \cup \mathcal{P}((x_2, b), L_1)$ intersects all the median lines. Again, value of D^r_f depends on the distribution f only. $\square$

As mentioned before, an analogical theorem holds for the symmetric case where $(a, b) \in m^-$.

Equilibria with a non-empty core. If the distribution function f has non-empty core, then we are able to extend Theorem 5 to provide the necessary and sufficient conditions for existence of divergent full participation Nash equilibrium and its full characterisation. The main feature of the following theorem is that *such a DFPE can be truly divergent, that is each candidate can propose a unique policy in each issue.* In particular, the chosen value of the free issue by one candidate is the projection of the core on the opponent's committed issue.

Theorem 6. *Consider a game $\Gamma^{\mathrm{i,j}}_{a,b}$ with $\mathrm{i} \neq \mathrm{j}$. If the distribution function f has a non-empty core $\mathcal{C}_f = \{\bar{c}_f\}$ then a DFPE exists if and only if either $(a, b) \in m^+$ or $(a, b) \in m^-$. Moreover, (y_1, x_2) is a DFPE if and only if $y_1 = \bar{c}_f \downarrow \mathrm{j}$ and $x_2 = \bar{c}_f \downarrow \mathrm{i}$.*

Proof. It follows from Lemma 2 that if a DFPE of game $\Gamma^{\mathrm{i,j}}_{a,b}$ exists, then either $(a, b) \in m^+$ or $(a, b) \in m^-$.

Suppose that $(a,b) \in m^+$. If (y_1, x_2) is a DFPE then $y_1 = \bar{c}_f \downarrow \mathrm{j}$ and $x_2 = \bar{c}_f \downarrow \mathrm{i}$ follows immediately from Theorem 5.

On the other hand we will show that a strategy profile (y_1, x_2) such that $y_1 = \bar{c}_f \downarrow \mathrm{j}$ and $x_2 = \bar{c}_f \downarrow \mathrm{i}$ is a Nash equilibrium of game $\Gamma^{\mathrm{i,j}}_{a,b}$. Let $\bar{x}_1$ and $\bar{x}_2$ denote the policies proposed by the candidates 1 and 2, respectively, under the strategy profile (y_1, x_2). Notice that $\mathcal{P}(\bar{x}_1, L_2)$ and $\mathcal{P}(\bar{x}_2, L_1)$ intersect all the median lines. Hence, by Lemma 1, no profitable deviation from (y_2, x_1) is possible by any candidate and so (y_1, x_2) is a Nash equilibrium. This shows also that if $(a,b) \in m^+$, then a DFPE of $\Gamma^{\mathrm{i,j}}_{a,b}$ exists.

If $(a,b) \in m^-$ then the theorem can be proved by similar arguments, using the analogue of Theorem 5 for $(a,b) \in m^-$. □

Equilibrium with an empty core. As in the case of homogeneous commitment, we are unable to give full conditions for existence of a DFPE and complete characterisation of Nash equilibria if the core of f is empty. However we provide the following example with an empty core distribution where a DFPE exists.

Example 3 (Existence of equilibrium)**.** *Consider the distribution function f presented in Figure 13, which support is a (closed) square with side of length* 2*. Moreover the square is divided into* 8 *parts of equal area. The density function is constant on each of these areas (including the clockwise border, and excluding the centre) and at the centre its value is the same as in areas labelled with B. The mass of each area labelled with A is μ_A and the mass of each area labelled with B is μ_B. Moreover $\mu_B > \mu_A > 0$ and $\mu_A + \mu_B = 1/4$. As before, we will also refer to a difference $\theta = \mu_B - \mu_A$ between the two*

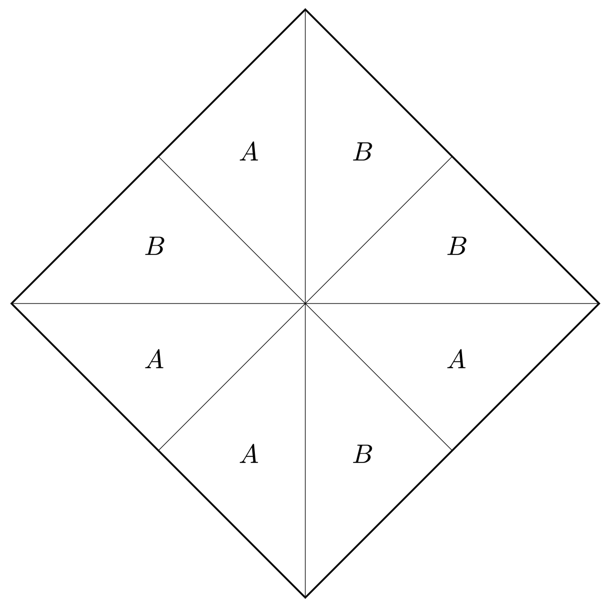

Figure 13: Distribution function for Example 3.

masses (notice that $0 < \theta < 1/4$*). Notice that the distribution is a rotated distribution from Example 1 and so again the core is empty.*

Let the coordinate system be re-orientated so that the horizontal (x*) axis is the horizontal axis of symmetry of the square and the vertical (*y*) axis is the vertical axis of symmetry of the square. Notice that the lines* $y = x$ *and* $y = -x$ *are median lines* m^+ *and* m^-*, respectively.*

Since the distribution f *is a rotated distribution from Example 1 so the distance from the origin to the intersection point of* m^+ *and a median line* $[m^+]_r$ *is given by the function* $x(r)$*, as computed in Example 1. Hence* $x(r) \in [0, 2\theta)$*,* $x(r) = x(-r)$ *and for* $0 < r \leq 1/2$*,* $x(r)$ *is given by equation (2). Moreover it holds that* $\bar{p}_0^-(m^+) = \bar{p}_0^+(m^+) = \bar{p}_0(m^+) = (\sqrt{2}\theta, \sqrt{2}\theta)$*.*

Consider a game $\Gamma^{\mathrm{i,j}}_{-c,-c}$ *with the committed issue* i *of candidate* 1 *being vertical dimension, the committed issue* j *of candidate* 2 *being horizontal di-*

mension and the values to which candidate 1 *and* 2 *are committed being* c *and* $-c$*, respectively. We prove the following:*

Claim 3. *If* $c \geq 2\sqrt{2}\theta$*, then* $(\sqrt{2}\theta, \sqrt{2}\theta)$ *is a Nash equilibrium of game* $\Gamma^{\mathrm{i,j}}_{-c,-c}$.

Proof. Notice that the region

$$\mathcal{P}((-c, \sqrt{2}\theta), L_2) = \left\{ (x, y) \in \mathbb{R}^2 : y \geq \frac{(x+c)^2}{2(c+\sqrt{2}\theta)} + \frac{\sqrt{2}\theta - c}{2} \right\}$$

and the region

$$\mathcal{P}((\sqrt{2}\theta, -c), L_1) = \left\{ (x, y) \in \mathbb{R}^2 : x \geq \frac{(y+c)^2}{2(c+\sqrt{2}\theta)} + \frac{\sqrt{2}\theta - c}{2} \right\}.$$

Both the regions intersect the median line m^- and are tangent to each other and to median line m^+ at point $(\sqrt{2}\theta, \sqrt{2}\theta)$ (see Figure 14).

We will show that for all $r \in \mathbb{R} \setminus \{0\}$ both these regions intersect the median line $[m^+]_r$. Consider region $\mathcal{P}((\sqrt{2}\theta, -c), L_1)$. Since all median lines intersect the median line m^+ between points $(0, 0)$ and $(\sqrt{2}\theta, \sqrt{2}\theta)$, so for all $r < 0$ the median line $[m^+]_r$ intersects the region. For $r > 0$ consider the line t_r parallel to $[m^+]_r$ and tangent to the region $\mathcal{P}((\sqrt{2}\theta, -c), L_1)$ and let $x_p(r)$ be the signed[4] distance between the origin and the intersection point of t_r and m^+. Then $x_p(r) = (2\theta - \sqrt{2}rc)/(r+1)$ and if we show that $x(r) \geq x_p(r)$, for $r > 0$, then it will imply that associated median lines intersect the region. As we observed above, it must be that $x(r) \geq 0$, for $r > 0$. Thus if $x_p(r) \leq 0$,

[4]By the signed distance of a point from the origin we mean the distance between the point and origin, if horizontal coordinates of the point are positive and minus the distance between the point and origin, otherwise.

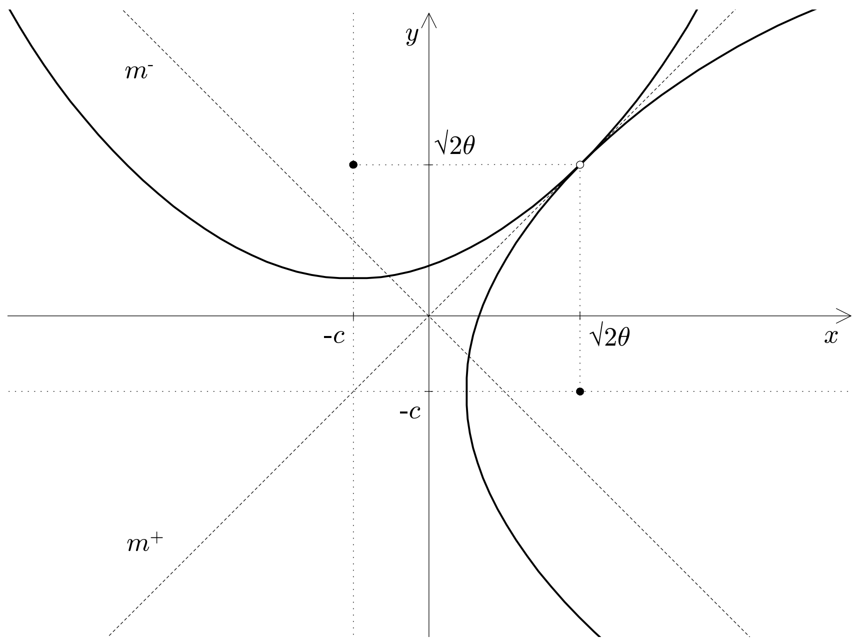

Figure 14: Regions $\mathcal{P}((-c,\sqrt{2}\theta),L_2)$ and $\mathcal{P}((\sqrt{2}\theta,-c),L_1)$.

that is $r \geq \sqrt{2}\theta/c$, then the inequality is satisfied. Suppose that $r < \sqrt{2}\theta/c$. Then $r \leq 1/2$, as $c \geq 2\sqrt{2}\theta$, and we have to show that

$$x_p(r) \leq \frac{\sqrt{(1-r^2)(32r^2\theta^2+1-r^2)}-(1-r^2)}{8r^2\theta}$$

as formula (2) for $x(r)$ applies. Notice that

$$x_p(r) \leq \frac{2\theta-4r\theta}{r+1} \leq 2\theta-4r\theta$$

for $0<r\leq 1/2$ and $0<\theta\leq 1/4$ and, as we have shown in Example 1

$$2\theta-4r\theta \leq \frac{\sqrt{(1-r^2)(32r^2\theta^2+1-r^2)}-(1-r^2)}{8r^2\theta}$$

for $0<r\leq 1/2$ and $0<\theta\leq 1/4$.

Thus we have shown that all median lines intersect the region $\mathcal{P}((\sqrt{2}\theta,-c),L_1)$. Showing that all median lines intersect the region $\mathcal{P}((-c,\sqrt{2}\theta),L_2)$ can be done by symmetrical arguments, due to the fact that f is symmetrical about the median line m^+. Hence if $c \geq 2\sqrt{2}\theta$, then $(\sqrt{2}\theta, \sqrt{2}\theta)$ is a Nash equilibrium of $\Gamma^{\mathrm{i,j}}_{-c,-c}$. $\square$

4.3. One-party equilibria

Although the focus of this paper is on full participation Nash equilibria, or in other words on two party equilibria, we would like to make some comments on Nash equilibria where one of the players stays out of the competition. Like in the case of full participation Nash equilibrium, the key to analysis of this case is Lemma 1. Clearly if the game is such that none of the bisectors of feasible sets of policies of the players is a median line, then one of the players cannot have an unbeatable policy to propose. For a one party Nash equilibrium to exist it is necessary that another player can propose a policy $\bar{x}$ such that $\mathcal{P}(\bar{x}, L)$ intersects all the median lines (where L denotes the feasible sets of policies of the opponent). Notice that if this condition is satisfied, then it is possible that it is satisfied for other policies in the close neighbourhood of $\bar{x}$ and so there can be more than one one-party equilibrium for such game. As a passing remark, we would like to mention that in our model the only class of equilibria which are robust to small perturbations to the distribution of voters preferences is where a single party stands uncontested. This is because full participation equilibria imply that the two parties are tied in their vote support. However, this critique holds for all models with sincere voting.

5. Discussion

Existence of full participation equilibrium depends on the existence of some special point in the ideology space, identified in our analysis by p_0. To understand the role of the voters with ideal policy as p_0, we first recall the definition of the constrained core from Beeler Asay (2008): the *constrained core* of candidate i is simply the set of policies available to candidate i which remain unbeatable by any policy available to candidate j in our restricted voting scenario.

We have shown that when the set of voters is a continuum, the constrained cores of each candidate, if non-empty, contain unique policy points – *constrained core points*. In our game, any full participation Nash equilibrium must involve each candidate proposing unbeatable strategies. Hence such an equilibrium either does not exist (when the constrained core of some candidate is empty) or policies proposed by the players in equilibrium are their respective constrained core points. Point p_0 is then the projection of these constrained core points of each candidate. A voter at p_0 is indifferent between the constrained core points of each candidate while he strictly prefers a constrained core point of each player to any other policy of that player. Hence, any voter at p_0 enjoys the following decisive power: if he weakly prefers the policy proposed by candidate 1 to any policy that is feasible to candidate 2, then 1 cannot be strictly beaten by 2, and the same holds for candidate 2. Hence each candidate wants to secure the support of these voters. In a one-dimensional competition with two competing candidates, the median voter plays exactly this role. However, we would like to point out that voters on the bisecting median line in our model are not in this sense median voters. To see this, pick an arbitrary median line m and two arbitrary policy choices of the candidates. Suppose some voter i with ideal policy on m strictly prefers the policy proposed by candidate 1 to any other policy proposed by candidate 2. If voter i is not located at p_0, then the policy of candidate 1 cannot be 1's constrained core point and hence candidate 2 can defeat this policy.

Given this discussion we would like to identify voters with ideal policy p_0 as *strict median voters.* Of course there may be distributions where there is no strict median voter and in every such situation a divergent full-participation equilibrium will not exist. In case of homogeneous commitment, the strict median, if it exists is always unique. This is true for the heterogeneous case as well, barring a very special case: one where L_1, L_2, m^+ and m^- have a common intersection and we have a pair of strict medians (notice that in this case the projections of strict medians on L_i will coincide for one of the candidates i).

Although we know that even if core of the distribution is non-empty, then for any median line m point $p_0(m)$ may still exist, we are unable to obtain any properties of the distribution that would guarantee its existence in general. First of all it is unclear what properties of the distribution guarantee that points $p_0^+(m)$ and $p_0^-(m)$ coincide. We conjecture that a sufficient condition for this to happen is continuity of the density function at almost every point of m. However we are unable to prove this result. Second, and a more primitive problem is existence of these limit points, that is what properties of the distribution guarantee that $p_0^+(m) \in \mathbb{R}^2$ and $p_0^-(m) \in \mathbb{R}^2$.

We would like to note that existence of point $p_0(m)$, where m is a median line which is a bisector of feasible sets of policies of the candidates is still not sufficient for guaranteeing existence of a DFPE. Another problem is the behaviour of median lines $[m]_r$ in the neighbourhood of $r = 0$. The question is whether it is always possible to have parabolic regions of ideal positions of loyal voters of the candidates large enough to "capture" all the median lines.

Despite the problems mentioned above it is still possible to have distributions with empty core for which a DFPE exist, even in the heterogeneous settings, as presented in Examples 1 and 3.

Our results show that in case of Downsian competition with unidimensional commitment some sort of convergence is still in place. In case of a CFPE, this leads either to the median voter theorem and convergence to the median point (in homogeneous setting) of the free issue or convergence to the policy in the union of the candidates' feasible sets of policies which is weakly majority preferred to all other policies in the union of these sets. In case of a DFPE, the policies proposed by the candidates in equilibrium converge to the projection of the point $p_0(m)$ on their respective feasible sets of policies. Point $p_0(m)$ belongs to the set of points equidistant from the feasible sets of policies of both candidates (a bisector of these sets) m. Moreover m must be a median line. Despite this convergence the policies proposed in equilibrium by both candidates are indeed different, even in the heterogeneous settings, where feasible sets of policies of both candidates have non-empty intersection.

6. Conclusions

In this paper we have studied Downsian competition in a two dimensional ideology space between two candidates whose sets of feasible policies are restricted by unidimensional commitment. We have shown that this kind of restriction allows for existence of Nash equilibria under larger class of distributions than in the case of unrestricted competition. Moreover we provided necessary conditions for existence of Nash equilibria where both candidates enter the competition. We gave necessary and sufficient conditions together with full characterisation in the case of convergent Nash equilibria (where both candidates propose the same policy in equilibrium) as well as in the case of divergent Nash equilibria under distributions with non-empty core. For distributions with empty core, we have given necessary conditions for the existence of such equilibria, linking their existence with the existence of the strict median that depends only on the distribution. Moreover, we provided examples showing situations where divergent Nash equilibria exist even though the distribution has an empty core. Interestingly, it is possible to have such equilibria even in the case where feasible sets of policies of both candidates have non-empty intersection. Two interesting features of divergent equilibria with two parties are: (a) when the two share the same committed issue, they tend to have extreme opposite stands on that issue while agreeing completely on the free issue while (b) when the committed issues are different for different parties, their final positions in the two-dimensional space tend to be significantly far apart from each other and truly divergent in every issue.

We are so far unable to provide properties of the distribution function that would guarantee existence of the strict median that is closely related to the existence of divergent Nash equilibria and the interesting research question is whether the strict median exists for large enough class of distributions.

Another interesting direction of research is to check whether the results obtained here would extend to more than two dimensions. More precisely, what would the effect of unidimensional commitment be when the ideology space has three or more dimensions. It seems that the regions containing ideal points of loyal voters of the candidates would be multidimensional parabolic quadratic curves and that existence of divergent Nash equilibria would depend on the existence of the strict median in $d-1$-dimensional hyperspace being a bisector of hyperspaces of feasible polices of the candidates. It seems also that existence of divergent Nash equilibria in this setting will remain closely related to existence of the strict median in the bisector of candidates' feasible sets of policies, like in two dimensional case.

We did not address the issue of more than two candidates competing in the elections or a setting with endogenous entry. These extensions are yet another interesting direction to explore.

Appendix

Proof of Fact 1. Take any $\vec{v} \in \mathbb{R}^2$ and let $l = span\{\vec{v}\}$ be the line spanned on this vector. Since f is non-atomic, so the distribution f projected on l, f_l is non-atomic as well and has unique median. Let $\bar{x}$ be the median of f_l. Then the line $l^{\perp}(\bar{x})$ is a median line of f and is perpendicular to $\vec{v}$, that is $l^{\perp}(\bar{x}) = m^f_{\vec{v}}$. □

Proof of Fact 2. Notice that since f is non-atomic, then, by Fact 1, for any vector $\vec{v} \in \mathbb{R}^2$ there exists a unique median line $m^f_{\vec{v}}$. Hence there exists a unique intersection point of all median lines from $\mathcal{M}_f$.

For the left to right implication suppose that $\mathcal{C}_f \neq \varnothing$ and let $\bar{x} \in \mathcal{C}_f$. We will show that for all $m \in \mathcal{M}_f$ it must hold that $\bar{x} \in m$. For assume the opposite and let $m \in \mathcal{M}_f$ be such that $\bar{x} \notin m$. Let $\bar{z}$ be the projection of $\bar{x}$ on m. Then $\bar{z}M\bar{x}$, which contradict the assumption that $\bar{x} \in \mathcal{C}_f$. Thus it must be that for any $\bar{x} \in \mathcal{C}_f$ and for all $m \in \mathcal{M}_f$ it holds that $\bar{x} \in m$. Hence $\mathcal{C}_f = \{\bar{c}_f\}$, where $\bar{c}_f$ is the intersection point of all median lines.

For the right to left implication we will show that if $\bar{c}_f$ is the intersection point of all median lines, then for all $\bar{x} \in \mathbb{R}^2$, $\bar{c}_f \bar{M} \bar{x}$, that is $\bar{c}_f \in \mathcal{C}_f$. Assume the opposite and let $\bar{x} \in \mathbb{R}^2$ be such that $\bar{x}M\bar{c}_f$. But this is impossible as $m \in D(\bar{c}_f, \bar{x})$, where m is the median line parallel to the bisector $B(\bar{c}_f, \bar{x})$ of $\bar{c}_f$ and $\bar{x}$, and so $\mu(D(\bar{c}_f, \bar{x})) > 1/2 > \mu(D(\bar{x}, \bar{c}_f))$. Hence it must be that $\bar{c}_f \bar{M} \bar{x}$, for all $\bar{x} \in \mathbb{R}^2$, and consequently $\bar{c}_f \in \mathcal{C}_f$. $\square$

Proof of Fact 3. Take any $m \in \mathcal{M}_f$ and any $r \in \mathbb{R} \setminus \{0\}$. Let $\bar{x} = \gamma^m(r)$ and let $l_s(\bar{x})$ be the line intersecting m at $\bar{x}$ and such that the gradient of it with respect to m is s. Let $\beta^m_{\bar{x}}(s)$ be the difference between the mass of the right half plane and the mass of the left half plane defined by $l_s(\bar{x})$. Then $\beta^m_{\bar{x}}$ is a continuous function of s.

Let $\alpha^m_{\bar{x},s}(d)$ be the mass of the region between the line $l_s(\bar{x})$ and the line $l_s(\bar{z})$, where $\bar{z} \in m$ is a point to the right of $\bar{x}$ and such that $\delta(\bar{x}, \bar{z}) = d$, if $d \geq 0$, or a point to the left of $\bar{x}$ and such that $\delta(\bar{x}, \bar{z}) = -d$, if $d < 0$. Notice that $\alpha^m_{\bar{x},s}$ is continuous and, since f is non-atomic, it is strictly increasing with d increasing, for any s. Thus its reverse function $(\alpha^m_{\bar{x},s})^{-1}$ is well defined and continuous for any s.

Observe that $\delta(\bar{x}, \gamma^m(s)) = |(\alpha^m_{\bar{x},s})^{-1}(\beta^m_{\bar{x}(s)})|$. Notice that since $r \neq 0$, so

$\lim_{s\to r}(\alpha^m_{\bar{x},s})^{-1}(t) = (\alpha^m_{\bar{x},r})^{-1}(t)$, for all t. It also holds that $\lim_{s\to r}\beta^m_{\bar{x}}(s) = 0$. Hence $\lim_{s\to r}|(\alpha^m_{\bar{x},s})^{-1}(\beta^m_{\bar{x}}(s))| = 0$ and so $\lim_{s\to r}\gamma^m(s) = \gamma^m(r)$. Hence γ^m is continuous on $\mathbb{R}\setminus\{0\}$. $\square$

References

Aurenhammer, F., Klein, R., 2000. Voronoi diagrams. In: Handbook of Computational Geometry. Elsevier Science Publishers B.V. North-Holland, pp. 201–290.

Beeler Asay, G. R., 2008. How does ideology matter in the spatial model of voting? Public Choice 135, 109–123.

Downs, A., 1957. An Economic Theory Of Democracy. New York: Harper and Row.

Duggan, J., 2005. A survey of equilibrium analysis in spatial models of elections.

Duggan, J., Jackson, M., 2005. Mixed strategy equilibrium and deep covering in multidimensional electoral competition, mimeo.

Ferejohn, J. A., McKelvey, R. D., Packel, E., 1984. Limiting distributions for continuous state markov models. Social Choice and Welfare 1 (1), 45–67.

Hinich, M. J., Munger, M. C., 2008. The dynamics of issue introduction: A model based on the politics of ideology. Mathematical and Computer Modelling 48, 1510–1518.

Hotelling, A., 1929. Stability in competition. Economic Journal 39 (1), 41–57.

Owen, G., Shapley, L. S., 1989. Optimal location of candidates in ideological space. International Journal of Game Theory 18, 339–356.

Plott, C. R., 1967. A notion of equilibrium and its possibility under majority rule. American Economic Review 57, 787–806.

Wuffle, A., Feld, S. L., Owen, G., Grofman, B., 1989. Finagle's law and the finagle point, a new solution concept for two-candidate competition in spatial voting games without a core. American Journal of Political Science 33 (2).