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Abstract

Two players are endowed with resources for setting up N locations on K identical circles, with $N > K \geq 1$. The players alternately choose these locations (possibly in batches of more than one in each round) in order to secure the area closer to their locations than that of their rival's. They face a resource mobility constraint such that not all N locations can be placed in the first round. The player with the highest secured area wins the game and otherwise the game ends in a tie. Earlier research has shown that for $K = 1$, the second mover always has a winning strategy in this game. In this paper we show that with $K > 1$, the second mover advantage disappears as in this case both players have a tying strategy. We also study a natural variant of this game where the resource mobility constraint is more stringent so that in each round each player chooses a single location where we show that the second mover advantage re-appears. We suggest some Nash equilibrium configurations of locations in both versions of the game.

Keywords: Competitive locations, Disjoint spaces, Winning/Tying strategies, Equilibrium configurations.

JEL Classification: C72, D21, D72.

1 Introduction

The possibility and practice of choosing locations competitively in order to maximize influence over sources that generate payoffs is widespread in economics and politics. For example, retail firms compete over geographic location of chain stores in order to capture a larger share of the market. Political parties may set up party-offices or affiliated bodies in order to spread political influence over the electorate. In such situations, the ability of players to maximize influence by choosing locations may depend upon the order in which such locations are chosen, thereby bringing up issues concerning the first and second mover advantages.

Games involving choice of locations has long been an important area of study in economics. The corresponding literature centers around the seminal work by Hotelling [1929] which considers a profit maximizing firm's decision about optimal location when the consumers are located uniformly on a line segment. Subsequently, this was extended to the celebrated circular city model in Chamberlin [1953] and later by Salop [1979]. While in Hotelling [1929], Chamberlin [1953] and Salop [1979] simultaneous-move games are considered, Prescott and Visscher [1977] and Economides [1986] study the problem when firms are allowed to enter sequentially on a line segment and circular city respectively to show that the outcomes of a sequential location game can differ significantly from those obtained in a simultaneous-move scenario.

In some environments involving location games, players may have the sole objective of being the one with the highest influence, as for example, in a competition to win the race for establishing its product as the standard product in the market, a firm may set shops to acquire patronage from a majority of customers (like popularizing a software) which may then have long term benefits for the firm. In politics, having the highest ideological

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influence is a natural objective under plurality rules and political parties locate their representatives in order to spread this influence. It is of particular interest in situations where voters have but a small cost of voting in which case it is hard to justify voter participation and several papers try to explain large voter turnouts by assuming that parties are able to influence individual voters to join ideological groups. As put in Martinelli and Herrera [2006], voters are to belong to groups and groups are formed by leading party activists (see also for example Shachar and Nalebuff [1999] and Coate and Conlin [2004]). While Martinelli and Herrera [2006] extend the existing literature on how parties influence voters by forming groups through group leaders to the case where these leader arrive endogenously from the population, the game we study can be applied to situations where two existing parties locate respective party leaders across the electorate to do the same. With such objectives, it is important to find specifically a winning or tying strategy for a player.

A recent work in this respect is a game of influence studied by Ahn et al. [2004], where there are two players (firms or political parties) who are each endowed with the same number of facilities (resources to set up a number of shops or finance a number of party leaders) to locate (possibly in batches of more than one facilities) on a circle in a sequential manner. In order to win the game, a player must try to secure as much area as possible that is closer to its locations than those of its competitor. Each player faces a resource mobility constraint such that not all facilities can be located in the first round. They show that in such a game (to be described precisely in section 2) where play must involve at least two rounds, the second mover always has a winning strategy and the game would always result in a tie if players were forced to end the game in a single round. In Cheong et al. [2002] the existence of a winning strategy for the second mover is shown,

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even for a single round location game played on a two dimensional closed plane. In Chawla et al. [2003] an upper bound for the size of the first mover disadvantage is provided in a game where firms compete to maximize market shares and consumers are distributed over a d-dimensional Euclidean space.

A variant of the the above mentioned games of influence is where players compete over a collection of disjoint areas in which locations can be placed. To the best of our knowledge, this variant has not been studied so far and there are many real life situations that suggest its importance. For example, retail chains set up stores in different cities or countries. In politics, these disjoint areas can represent different sections of the citizens with distinct group-identities (like workers, students, or simply electorally disconnected geographic neighbourhoods like districts and states) and to set up locations in a given region can be viewed as an attempt by the political parties to open political units (like politically motivated trade unions, district party offices or students unions in academic institutions wth designated leaders) to spread influence among target groups and increase favorable voter participation. This paper adresses such location games on disjoint areas by extending Ahn et al. [2004] to a family of disjoint circles. In what follows we shall abstract away from parties and firms and simply refer to them as players. We are not interested in studying any particular model in politics or industry, but rather analyze the issue of stratgic influence in abstract. It is also important to mention that all our results can be easily extended to any closed curves rather than just circles.

We show that the second mover advantage as in Ahn et al. [2004] disappears and the first mover always has a tying strategy. We also show that in any Nash equilibrim of the game, there must be a tie. We then extend this game by making the resource mobility constraint more stringent so that in each round, each player places exactly one location. In this extended game

we show that the second mover advantage as in Ahn et al. [2004] reappears. We also provide some characterizations of final equilibrium configurations. The rest of the paper is structured as follows. In section 2 we define the game. Section 3 states and proves our results. Examples of final equilibrium configurations are depicted in section 4 and the paper concludes in section 5.

2 The Multiple Circle game

The circle game studied in Ahn et al. [2004] has two players, called[1] Red (R) and Green (G) each having N points to place (or locations to choose) alternately on a circle with R making the first move. Moreover, (i) each player must place at least one point in each round, (ii) in the first round when play begins, R cannot place all N points (perhaps because not all resources are available at the beginning of the game), [2] (iii) the game ends only after all players have placed all $2N$ points, (iv) at any round, the total points placed so far by G cannot exceed that of R[3], and (v) a location on the circle cannot serve more than one points. This results in a sequential game where roles (that is first and second mover identities) cannot be reversed and the number of rounds is endogenous and can be controlled by R subject to the restriction that there must be at least 2 rounds. The objective of each player, as in Voronoi games (a term coined by Ahn et al. [2004]), is to maximize the total length of the curve that is closer to that player than to

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[1]originally called White and Black in Ahn et al. [2004]

[2]requirements (i) and (ii) imply that $N \geq 2$.

[3]This is basically a condition required to preserve the first and second mover identities over any play. These identites could be preserved even with the assumption that players place equal number of points in each period. In this sense, the condition given in Ahn et al. [2004] and used here is general and hence weaker. In Subsection 3.1 we shall study a natural variant of this game where each player must place exactly one point in each round.

the other so that a player wins if and only if the area it secures is strictly the largest one. Otherwise there is a tie. It is shown in Ahn et al. [2004] that in this game G always has a winning strategy, though R can bring its length of influence as close as that of G's. Our objective is to check if such a second mover advantage prevails when there are more that one disjoint identical circles. We now present these ideas and the finding in Ahn et al. [2004] formally.

Let $\{R, G\}$ be a set of players, where R stands for Red and G stands for Green. *The game on the family of disjoint circles* is defined by a pair $\langle N, \{C_j\}_{j=1}^K \rangle$, such that $N > K \geq 1$ and $\{C_j\}_{j=1}^K$ is a family of K disjoint circles. Notice that the game studied in Ahn et al. [2004] is the special case where $K = 1$. Throughout the game each player $p \in \{R, G\}$ will select a *total* of N points on K circles. The set of points selected by R is $\Gamma \subseteq \bigcup_{j=1}^K C_j$ and the set of points selected by G is $\Omega \subseteq \bigcup_{j=1}^K C_j$. Players re-arrive in alternating sequence with R moving first, and are in principle allowed to place points in batches. Let Γ_r be the set of points that R places in round $r \geq 1$ while Ω_r be the same for G. The game ends when all $2N$ points are placed on the circles.[4] We will use $w \in \Gamma$ ($b \in \Omega$) to denote a point placed by R (G) during the game. We will call points placed by the player R *red points* and those placed by the player G *green points*.

As discussed above, the game has the following conditions:

1. $|\Gamma_r|, |\Omega_r| \geq 1$ for every $r \geq 1$.

2. $|\Gamma_1| < N$.

3. $\sum_{i=1}^r |\Gamma_i| \geq \sum_{i=1}^r |\Omega_i|$ for every $r \geq 1$.

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[4]Please note that we put no restriction on how players distribute these points across the circles (some circles are allowed to remain empty in which case it is ignored while computing payoffs).

4. $\sum_{i \geq 1} |\Gamma_i| = \sum_{i \geq 1} |\Omega_i| = N$.

The endogenously determined number of rounds in a given play of the game will be denoted by Z. Obviously $\Gamma = \bigcup_{r=1}^{Z} \Gamma_r$ and $\Omega = \bigcup_{r=1}^{Z} \Omega_r$. Notice that the restrictions of the game imply that $Z \geq 2$.

Let C be any circle and let (x, y) be an ordered pair of elements of C. We will use (x, y) to denote the arc of the circle between x and y in clockwise direction. Let $a(x, y) \in [0, 1]$ denote an angle in clockwise direction between halflines starting from the center of the circle and going through x and y (we normalize an angle, so that a full circle has angle equal to 1). Then $d(x, y) = \min\{a(x, y), a(y, x)\}$ is the angular distance between x and y. Notice that $d(x, y) = d(y, x) \in [0, 1/2]$. Given an arc (x, y), the length (or a volume) of (x, y) is $a(x, y)$.

Given a circle C_k, let

$$A_R(C_k) = \left\{ x \in C_k : \min_{w \in C_k \cap \Gamma} d(x, w) < \min_{b \in C_k \cap \Omega} d(x, b) \right\}$$

be a set of points of C_k that are closer to points placed by R on C_k than to points placed there by G. Let $A_G(C_k)$ be the analogical set defined for G. Notice that each of these sets is a finite set of arcs of a circle C_k. Let A be a finite set of arcs and let $V(A)$ denote the volume (sum of lengths, in angular terms) of arcs in A. When the game ends, each player p receives a score S_p equal to the volume of the set of arcs constituting the set of points closest to a position chosen by that player over all circles, that is

$$S_p = \sum_{k=1}^{K} V(A_p(C_k))$$

for $p \in \{R, G\}$. Given these scores, the payoff of the players is $u_p(S_p, S_q) = S_p - S_q$, where $\{p, q\} = \{R, G\}$. We say that the game is *a tie* if $S_p = S_q$, while player p *wins* if $S_p > S_q$. A strategy in general will be a contingent plan for every possible history of the game. We do not need to define this

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general notion formally although we lay out complete specifications of the strategies we report. We will use uppercase letters S, T, X, Y to denote pure strategies. Strategy X is called *a winning strategy (a tying strategy)* for player p if no matter what player q does, by using X player p guarantees that $S_p > S_q$ ($S_p \geq S_q$). Throughout the paper we will use the standard notation $m \mid n$, to denote the fact that m divides n and $m \nmid n$, to denote its negation.

2.1   Some definitions and existing results

We first develop some concepts and notations. Let C be a circle and let $P \subseteq C$ be a finite set of points on the circle. Then an arc $(x, y) \subseteq C$ such that $\{x, y\} \subseteq P$ and $(x, y) \cap P = \varnothing$ is called an *interval*. Now let P_R and P_G such that $P_R \cup P_G = P$ be sets of red and green points of P, respectively. Then an arc $(x, y) \subseteq C$ such that $\{x, y\} \subseteq P_R$ ($\{x, y\} \subseteq P_G$) and and $(x, y) \cap P_R = \varnothing$ ($(x, y) \cap P_G = \varnothing$) is called a *red (green) interval.* An interval that is neither red nor green is called a *bichromatic interval* and an interval which is not bichromatic shall be at times referred to as a *monochromatic interval* in general. We will use r^C (g^C) to denote the number of red (green) points placed on the circle C. We will also use $I^R(C)$ ($I^G(C)$) to denote the number of red (green) intervals on the circle C.

Given a circle C and a point $x \in C$, an *antipode* of x is the point $y \in C$ such that $d(x, y) = 1/2$. The pair of points $\{x, y\}$ is called a *pair of antipodes* of C. Let m be a positive natural number. Then the set of *key positions*[5] on C determined by point x and m is the set

$$\kappa(C, x, m) = \{p \in C : a(p, x) = l/m, \text{ where } l \in \{0, \dots, m - 1\}\}.$$

By the set of key positions determined by m we mean a set of key positions

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[5]We use a term key position here for what was called a key point in the paper Ahn et al. [2004]. We found the name key position somewhat more apropriate in our context.

determined by m and some point in C. A point placed in a key position will be called a *key point* and an interval formed by two key points will be called a *key interval*.

Before presenting our results in the next section, we report the main result from Ahn et al. [2004] for the case $K = 1$. Consider the following strategy, S^* used by player G (where key positions are simply N equidistant points on the circle:

Strategy S^*
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    if *there is an empty key position left* **then**

(a) | place a point on s key position

    else if *if $r < Z$* **then**

(b) | place a point in the middle of a maximal interval of the opponent

    else

(c)   **if** *there is more than one interval of the opponent* **then**
            place a point in the middle of a maximal interval of the
            opponent
        else if *there is exactly one interval of the opponent and its length*
        is l **then**
            place a point in a bichromatic key interval at distance less
            than $1/N - l$ from endpoint of the opponent

Theorem 1 (Ahn et al. [2004]) *Let $\langle N, \{C_j\}_{j=1}^{K}\rangle$ define a game on a single circle such that $K = 1$. Then S^* is a winning strategy for G although R can always bring the difference $S_G - S_R$ as close as possible to zero.*

3   Results

We first show that for any game $\langle N, \{C_j\}_{j=1}^K \rangle$ with $K \geq 2$ and $N \geq K$, R, i.e. the first mover, has a tying strategy. We will consider two cases separately: $K \nmid N$ and $K \mid N$. We start by demonstrating a tying strategy for R for the first case. The general idea of this strategy is for R to capture key positions on the circles. Key positions on each circle will be determined by the first point placed on the circle and either $\lceil N/K \rceil$ or $\lfloor N/K \rfloor$, depending on the situation (and G's play, in particular). Let r be a round and let $L(r)$ be the number of circles occupied after R places the first of the points he is to place in round r. Key positions on the occupied circles are determined with respect to $\lceil N/K \rceil$ and the first point placed on the circle. The number of total key positions on these circles is $L(r)\lceil N/K \rceil$ and the number of vacant key positions is

$$V(r) = L(r)\lceil N/K \rceil - Y(r),$$

where $Y(r)$ is the number of key position already occupied after R places the first point in round r. Let $\varphi(r)$ stand for the number of points R is left with if after placing his first point in round r he would have covered all vacant key positions in the occupied circles, that is

$$\varphi(r) = N - r - V(r).$$

We first prove the following lemmas and a corollary, which are generalizations of the lemmas presented in Ahn et al. [2004] for more than one circle.

Lemma 1 *Let $\{C_k\}_{k=1}^K$ be a family of circles. Then*

$$\sum_{k=1}^K I^R(C_k) - \sum_{k=1}^K I^G(C_k)) = \sum_{k=1}^K r^{C_k} - \sum_{k=1}^K g^{C_k},$$

Strategy T^*

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if $r = 1$ **then**
 place one point in some circle

else if $r \geq 2$ **then**

(a)   **if** *at $r - 1$ G played in a free circle* **then**
 place one point in that circle taking a key position (defined by

 the position of the green point and $\lceil N/K \rceil$

 if $(K - L(r))\lceil N/K \rceil = \varphi(r)$ **then**
 take all key positions in the occupied circles and then

 divide the free circles equally by taking key positions on

 them with all remaining points

(b)   **else if** *at $r - 1$ G played in the circle with at least one red point*

 and vacant key position, not taking a key position **then**
 place one point in that circle taking a key position (defined by

 the position of the red point and $\lceil N/K \rceil$

(c)   **else if** *there is a circle with red points only and vacant key*

 positions **then**
 place one point in that circle taking a key position (defined by

 the position of the red points and $\lceil N/K \rceil$

(d)   **else**
 place one point in a free circle

 if $(K - L(r))\lceil N/K \rceil = N - r$ *or* $(K - L(r))\lfloor N/K \rfloor = N - r$

 then
 divide the free circles equally by taking key positions on

 them with all remaining points

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i.e. the difference between the number if red points and green points placed on the family of circles is equal to the difference between the number of red and green intervals on that family of circles.

Proof. In Ahn et al. [2004] it is shown that for any circle C it holds that $I^R(C) - I^G(C) = r^C - g^C$. Then $\sum_{k=1}^{K}(I^R(C_k) - I^G(C_k)) = \sum_{k=1}^{K}(r^{C_k} - g^{C_k})$ and, so $\sum_{k=1}^{K} I^R(C_k) - \sum_{k=1}^{K} I^G(C_k)) = \sum_{k=1}^{K} r^{C_k} - \sum_{k=1}^{K} g^{C_k}$. ∎

The following corollary is immediate from the above lemma.

Corollary 1 *Let $\{C_j\}_{j=1}^{K}$ be a family of circles where each of the players R and G placed the same number of points. Then the number of red and green intervals is the same.*

Lemma 2 *Let $\{C_k\}_{k=1}^{K}$ be a family of circles with key positions with respect to some M for each circle. Assume that (i) there are $r = \sum_{k=1}^{K} r^{C_k} \leq KM$ red and $g = \sum_{k=1}^{K} g^{C_k} < r$ green points on the family of circles covering all KM key positions and (ii) there is only one red interval which is not a key interval. Then there exists a bichromatic key interval.*

Proof. The argument used in Ahn et al. [2004] for an analogical lemma for one circle works for this lemma as well. We have $r + g \leq 2KM - 1$ points on the circles and KM of them are key points. Thus there are at most $KM - 1$ points lying within some key intervals. This leaves one key interval without a point. This key interval cannot be red, as there is only one red interval which is not a key interval. The key interval cannot be green, as by Lemma 1, this would mean that there are more then one red intervals (notice that $r > g$). Thus the key interval must be bichromatic. ∎

We now prove our first main result by identifying a tying strategy called T^* for R, the first mover.

Theorem 2 *Let $\langle N, \{C_j\}_{j=1}^K \rangle$ define a game on the family of disjoint circles with $K \geq 2$. If $K \nmid N$, then T^* is a tying strategy for R.*

Proof. We start by showing that strategy T^* is implementable. The only situation, where the strategy may not be applied is the one where at some round r, R faces the situation where there is no key position left (and he is having $e > 0$ points left). This means that in each of the K circles, $\lceil N/K \rceil$ key positions are taken (notice that if G does not take any key position up to the round r, then this situation cannot appear). Let $r' \leq r$ be the last round such that at the round $r' - 1$, G took a key position. Observe that it must be that $r' = r$, as R is able to cover all remaining key positions at r. Consider R's move at the round $r - 1$. Since R has enough points to cover all remaining key positions at this stage, he would do that and this contradicts the assumption that G places a point in a key position at this round. Thus the strategy is implementable.

Now we will show that using this strategy either R wins or ties. Firstly, observe that after R's move in the last round all key positions must be covered. For assume throughout the game G did not take any key position (this means in particular that G has never placed a point in an empty circle, as such point is always a key position). Then at round $(N \bmod K)\lceil N/K \rceil + 1^6$ R plays in an empty circle and he can cover $\lfloor N/K \rfloor$ key positions in each of the remaining empty circles, so the game finishes and all key positions are covered. Now assume that at some round r, G places a point in a key position. Then R is still capable of covering all remaining key positions towards the end of the game. Moreover after G's move, the number of circles where R would have to take $\lfloor N/K \rfloor$ key positions decreases by 1. After G takes $K - (N \bmod K)$ key positions, R is capable of covering all remaining key positions and at the and of the game $\lceil N/K \rceil$ key positions will be taken

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[6]Notice that $N = (N \bmod K)\lceil N/K \rceil + (K - N \bmod K)\lfloor N/K \rfloor$.

in each of K circles.

Secondly, observe that if G and R covered all key positions in L circles (k key positions on each of the circles) using the same number of points, and so that R's points are placed on key positions only then G cannot be winning on these circles. This is because if G is to be winning then there must be some green intervals on the circles (for if there are not then there is a tie, as each bichromatic interval is divided equally between G and R). According to the fact 1, there must be the same amount of red intervals. Since each red interval is of size $1/k$ and each green interval is of size $\leq 1/k$, so G cannot be winning.

Assume that in the last round R played according to (d). Then before R's move some $1 \leq L < K$ circles were covered by G and R so that the same amount of red and green points where placed there, all key positions are taken and all red points are placed on key positions (and there are $\lceil N/K \rceil$ key positions on each circle). Thus there is a tie on these circles. There are two possible numbers of key positions on each of the $K - L$ circles: (i) $\lceil N/K \rceil$ and (ii) $\lfloor N/K \rfloor$. In case (i) the situation analogous to the one on L circles will be created, resulting in a tie. For case (ii) observe that answering R's move G has to place $(K - L)\lfloor N/K \rfloor$ points. Observe also that after R's move $(K - L)\lfloor N/K \rfloor$ red intervals where created, each of the size $1/\lfloor N/K \rfloor$. Since R cannot gain[7] by playing in any of the L circles that where occupied before R's move (as he can gain $< 1/\lceil N/K \rceil$, and by playing within newly created red interval he can gain $1/\lceil N/K \rceil$), he has to place his points within newly created red intervals. Moreover by placing more than one point within such interval he can gain $< 1/\lfloor N/K \rfloor$ while placing one red point in each of the newly created red intervals he gains $1/\lfloor N/K \rfloor$ and in this case there is a tie.

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[7]By "gain" we mean the area acquired by the player plus the area lost by the opponent.

Now assume that in the last round R did not play according to (d). Then at the end of the game we will have the situation where all circles are covered with the same amount of green and red points, all key positions are taken ($\lceil N/K \rceil$ key positions on each circle) and R placed his points on key positions only. Then there is a tie. ∎

Our next result deals with the case where $K \mid N$ where we show that R has a tying strategy in this case as well. The strategy is simple and is therefore defined directly in the proof of the theorem. We shall refer to it as T^* as well since its identity will be clear from whether $K \mid N$ or not.

Theorem 3 *Let $\langle N, \{C_j\}_{j=1}^{K} \rangle$ define a game on the family of disjoint circles with $K \geq 2$. If $K \mid N$, then R has a tying strategy (which will be also called T^*).*

Proof. Observe that R can implement K/N red arcs of equal size on each of the K circles by placing N points. To see this, consider the following strategy of R (called T^*): place exactly N/K equidistant points in round r on circle C_r. Obviously since C_r is continuous, G can always do that.

Thus at the end of the game both R and G placed the same amount N of points in K circles. Moreover R points take N/K key positions determined by a red point and N/K in each of these circles. If there is no green interval on the circles then there is a tie. Observe that if there is a green interval, then its size is always $\leq K/N$ while each red interval has size K/N. Thus if there are any green intervals, then by the fact 1 it cannot be that G won. ∎

By Theroem 2 and Theroem 3 we have shown that if $K \geq 2$, then R, the first mover, has a tying strategy in the game and hence the second mover advantage present under $K = 1$ disappears. The question that arises now is: can the first mover do better? The answer is no as we show the existence

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of a tying strategy for the second mover as well, as stated in the theorem below. This strategy, which we call T', is simple and is as follows:

Strategy T'

place exactly one point in each red interval created by R in any given round

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Theorem 4 *Let $\langle N, \{C_j\}_{j=1}^{K} \rangle$ define a game on a family of K disjoint circles. Then T' is a tying strategy for G.*

Proof. Strategy T' requires G to place exactly one point in each red interval created in a given round. It is easy to see that this will ensure that at the end of the game there is no monochromatic intervals, which is then a tie. So what requires to be proved is that this strategy is implementable which we do by induction on the number of the current round. We will show two things: after R plays in round r, G can place exactly one point in a red interval and there is no monochromatic interval after G's move. Consider the first round. There are no intervals before players move. Assume that R placed m points. Then, by Lemma 1 there are m red intervals created and G can place m points, one within each interval. Thus there is no monochromatic interval after the first round. Now consider a round $r > 1$. By induction, there is no monochromatic interval before R's move and, by similar argument as in the case of the first round, there is exactly the same number of newly created red intervals as the number of red points placed. Thus G can place exactly one point in each of the newly created red intervals and there are no monochromatic intervals after the round r. This shows that G's tying strategy is implementable. ∎

The following theorem is an immediate consequence of the results we have proved so far.

Theorem 5 *Let $\langle N, \{C_j\}_{j=1}^{K} \rangle$ define a game on the family of K disjoint circles. Then strategy profile (T^*, T'), where player R plays T^* while player G plays T', is a pure strategy Nash equilibrium. Moreover, in every final configuration resulting from this Nash equilibrium profile of strategies, (i) there is no monochromatic interval on any circle and (ii) all red points lie on key positions.*

There may be other equilibrium points in this game. We shall use the above equilibrium in the examples we set in section 4 to study equilibrium configurations. We now deal with a variant of this game where the resource mobility constraint becomes most binding.

3.1 "One-by-one" variant of the game

A natural variant of the game studied above is the one where players face very strict resource mobilization constraints so that each places a single point in each round. Recall that in the tying strategy T^* used by R, it was crucial for R to place more than one point at some rounds. It turns out that if players face such a strict resource mobility constraint as the one we are dealing with now, then G, the second mover has a winning strategy.

The strategy, which we shall call Y^*, is a generalization of the winning strategy S^* of the second mover in the one circle case presented in Theorem 1. Player G first tries to take key positions with respect to $\lceil N/K \rceil$ or $\lceil N/K \rceil$ (depending on the situation described precisely below) and the first point placed on that circle. Then he breaks biggest red intervals, by placing a point inside them. In his last move he either breaks the biggest red interval or plays in a bichromatic interval that is bigger than the biggest red interval.

Strategy Y^*

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if *there is an empty key position left* **then**

(a)   **if** *there is an empty key position on the circle where the opponent took a key position in his last move* **then**

    if *there is only one point in that circle* **then**
        assign the number of key positions for this circle to $\lceil N/K \rceil$

    place a point on a key position in that circle next to the point placed by the opponent on the clockwise side of that point (if possible), otherwise on the anti-clockwise side (if possible)

    otherwise anywhere else

else if *there is an empty key position in a non empty circle* **then**
     place a point on a key position in that circle

else if *number of empty circles* $L \le K - N$ mod K **then**
    place a point on an empty circle assigning the number of key positions for this circle to $\lfloor N/K \rfloor$

else
    place a point on an empty circle assigning the number of key positions for this circle to $\lceil N/K \rceil$

else if *if* $r < Z$ **then**

(b)   place a point in the middle of a maximal interval of the opponent

else

(c)   **if** *there is more than one interval of the opponent* **then**
    place a point in the middle of a maximal interval of the opponent

else if *there is exactly one interval of the opponent and its length is* l **then**
    place a point in a bichromatic key interval at distance less than $1/\lceil N/K \rceil - l$ from endpoint of the opponent

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Theorem 6 *Let $\langle N, \{C_j\}_{j=1}^K \rangle$ define a game on the family of disjoint circles with $N > K \geq 2$ and assume that players face a very strict resource mobility constraint so that they are allowed to place exactly one point at a time. Then Y^* is a winning strategy for G.*

Proof. Notice that, just as in the case of S^*, the use of strategy Y^* leads to the following three stages of the game for player G. First the option (a) is excercised, until all key positions are covered (some of them with respect to $\lceil N/K \rceil$ and others with respect to $\lfloor N/K \rfloor$). Then the option (b) is excercised, until G reaches a round where he has only one point left (since the game is restricted, so that both players play exactly one point at each round this will be at the round $r = Z$). Finally the last stage is reached, where G plays according to (c). We start with two claims.

Claim 1 *After the end of round (a), G has at least one point left.*

Proof. Observe first that key positions are taken either with respect to $\lfloor N/K \rfloor$ or $\lceil N/K \rceil$ and $N = (N \bmod K)\lceil N/K \rceil + (K - N \bmod K)\lfloor N/K \rfloor$, so a player is capable of capturing all key positions on all circles, taking key positions with respect to $\lceil N/K \rceil$ on $(N \bmod K)$ circles, the remaining key positions with respect to $\lfloor N/K \rfloor$ on the remaining $K - N \bmod K$ circles. Since R places at least one point in a key position (which is the first point placed by him) and throughout the game G never assigns $\lceil N/K \rceil$ as the number of key positions to more than $N \bmod K$ circles that do not contain a red key point (by checking each time when the green point is placed on free circles as to whether the number of free circles $L \leq K - N \bmod K$), so G will have at least one point left after all key positions are taken. $\blacksquare$

Claim 2 *The number of red key intervals of size $1/\lfloor N/K \rfloor$ is never greater than the number of green key intervals of that size.*

Proof. This is because whenever R places a point in an empty circle, G places a point there in the same round, assigning the number of key positions for that circle to $\lceil N/K \rceil$. Thus red key intervals of the size $1/\lfloor N/K \rfloor$ can be created only on the circles where G placed the first point, and so after all key points in each such circle are taken, G will take no less key positions there than R. Moreover strategy Y^* ensures that each red key point has at most one neighbouring red key point (because G places a point next to a newly placed red point, if possible, and starts by placing it on the same side (clockwise in the case of strategy Y^*)).[8] Thus the number of green key intervals in such circles cannot be smaller than the number of red key intervals there. ■

Now assume that all key positions are taken and R places a point. Then there is one more red points than green points on the circles, and by Lemma 1, there is at least one red interval. Thus the next move of G is implementable and the game is either in stage (b) or (c).

Observe that in stage (b) G will place a point in all red key intervals of size $1/\lfloor N/K \rfloor$ (as he has at least twice the number of such intervals of points left, and these intervals are being broken first). Moreover G will place a point in all red key intervals of size $1/\lceil N/K \rceil$ in the stage (b) (as he saves at least one point each time such interval is created by R). Notice also that whenever R creates a red interval of the size $\geq 1/\lceil N/K \rceil$, this interval is created by placing two red points within a green key interval of the size $1/\lfloor N/K \rfloor$ and at most one red interval of this size can be created in such green key interval. Since the stage (b) ends when both players have only one point left, so all such intervals created in the stage (b) will have been broken by G by the end of that stage. Thus after the stage (b) there is no

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[8]The restriction on the game, so that players move one-by-one is crucial for this property. Notice that this issue arises only when $\lfloor N/K \rfloor \neq \lceil N/K \rceil$, i.e. $K \nmid N$.

red interval of the size $\geq 1/\lceil N/K \rceil$.

Consider now stage (c), where both players place their last points. The following situations are possible after R places his point in that round: (i) there are two or more red intervals, or (ii) there is one red interval. Assume that case (i) holds. Then G places a point in the largest red interval. If the size of the newly created red interval was $\geq \lceil N/K \rceil$, then the green point was placed within this interval, as after the stage (b) there is no red interval of the size $\geq 1/\lceil N/K \rceil$. Thus after the last round there is no red interval of the size $\geq 1/\lceil N/K \rceil$. Since, also, by Lemma 1, there is the same amount of green and red intervals and, moreover, each of green intervals is a key interval (as by strategy Y^*, G never created a green interval apart from the first stage), so G must be winning (recall that the size of a key interval is $\geq 1/\lceil N/K \rceil$).[9]

Now assume that case (ii) holds. Consider the situation before R's move. It must be that there is no red interval. Consider the group of circles for which $\lceil N/K \rceil$ key positions are assigned. It cannot be that R has more points than G on these circles (as otherwise, by Lemma 1 there would be a red interval there). Similarly R cannot have more points on the group of the circles with $\lfloor N/K \rfloor$ key positions assigned. Thus on each such group of circles, classified by the number of key positions, G and R have the same number of points. Suppose that after R's move, a red interval is created in the group of circles for which $\lfloor N/K \rfloor$ key positions are assigned (notice that such red interval may have the size $\geq 1/\lceil N/K \rceil$). Then, by Lemma 2, there must be a bichromatic key interval in that group of circles (i.e. a bichromatic key interval of the size $1/\lfloor N/K \rfloor$). Thus G can win by placing a point in that key interval and creating a green interval of the size bigger than the

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[9]Notice that G's advantage may be arbitrarily small, as non-key intervals created by R may be arbitrarily close in size to that of key intervals.

size of the newly created red interval (which has the size $< 1/\lfloor N/K \rfloor$).[10] Analogically it can be shown that G wins when R creates a red interval on the group of circles for which $\lceil N/K \rceil$ key positions are assigned.[11] This completes the proof. ∎

Some footnotes used in the proof of the above theorem suggest that although G, the second mover can win the game, R may be able to make the difference between their scores arbitrarily small. In the following theorem we show that the winning strategy Y^* for the second mover, with a slight modification and called Y', can be used by the first mover R to achieve this independent of the strategy used by G.

A strategy X is *a virtually tying* strategy for player p if X is not a winning strategy and for any $\varepsilon > 0$, if player p uses X, then no matter what player q does, player p can guarantee that $S_q - S_p < \varepsilon$.

Theorem 7 *Let $\langle N, \{C_j\}_{j=1}^K \rangle$ define a game on the family of disjoint circles with $N > K \geq 2$ and assume that players face a very strict resource mobility constraint so that they are allowed to place exactly one point at a time. Then there is a virtually tying strategy for G.*

Proof. Consider strategy Y^* as defined before with option (c) replaced by option (c') (this modified strategy will be called Y').

Y': Modification of Y^* for player R
(c')　place a point in a maximal bichromatic interval at distance ε from its green endpoint

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[10]Notice that the advantage of G may be arbitrarily small and depends on how big the red interval is.

[11]Analogical remark on the G's advantage applies here.

We will show that if R plays according to Y', he achieves the required outcome. Similarly to the case where Y^* is used by player G, the use of strategy Y' leads to three stages of the game for player R, though the stages are slightly different. At first, option (a) is exercised. After all key positions become occupied, option (b) is used as long as there is a green interval. This is the second stage. When there is no green interval, options (c') and (b) are selected depending on what player G does. If in his move G breaks the red interval created by the use of option (c') by R, in the next round R applies option (c') again and creates another red interval. Otherwise (which means that G created a green interval) R applies option (b) and breaks the newly created green interval.

After the first stage, where option (a) is exercised, R is not loosing. This is because all his points lie in key positions. Moreover the number of red key intervals of the size $1/\lfloor N/K \rfloor$ cannot be greater than the number of key intervals of this size (as red key intervals of the size $1/\lfloor N/K \rfloor$ can be created only on the circles where R placed the first point, cf proof of Claim 2). Hence if there are monochromatic interval after the first stage, then all red intervals are at least as big as the existing green intervals. Observe that since $N > K$, so option (a) will be applied at least once and there will be at least one bichromatic interval after the first stage.

In the second stage, where option (b) is exercised, R breaks maximal green intervals. Observe that throughout this stage after each move of player R he has an advantage of size of a key interval (either of the size of $1/\lceil N/K \rceil$ or of the size $1/\lfloor N/K \rfloor$). Moreover G cannot create green intervals of size $\geq 1/\lceil N/K \rceil$, as all key positions are occupied after the firts stage. Thus after each round of the second stage player R cannot be loosing. This means in particular that if G is a payoff maximizer, the game will always enter the third stage.

In the third stage, whenever player R creates a red interval he is gaining an advantage of the size of this interval. If G breaks the interval, the game is in a tie again. Otherwise in the next round (if there is a next round) R breaks the green interval, regaining his advantage. Now assume the game is in its last round. Assume the length of the bichromatic interval within which R created his last red interval is l. If G is to win, he must create a green interval within a bichromatic interval of the size $> l - \varepsilon$ and the created interval cannot be bigger than the red one by more than a margin $< \varepsilon$ (as a maximal remaining bichromatic interval has size $\leq l$). $\blacksquare$

3.1.1   Nash equilibrium of the ε-adjusted game

Since G wins for sure in the one-by-one variant of the game, that points are be placed on continuous curves and no single point on a circle can be served by more than one loctions, R does not really have an optimal strategy. Hence, a Nash equilibrium in this version of the game does not exist. However, strategy Y' becomes a dominant strategy for player R if we restrict attention further to one-by-one games where R is not allowed to place his points within a distance smaller than $1/\lceil N/K \rceil \gg \varepsilon > 0$ to a green point. This is because, as follows from proof of Theroem 7, just before the last round R is not loosing and in the last round G can create an interval of the size greater than the interval created by R, by at most ε. With this observation, the following theorem is immediate.

Theorem 8 *Let $\langle N, \{C_j\}_{j=1}^K \rangle$ define a game on the family of disjoint circles with $N > K \geq 2$ and assume that players face a very strict resource mobility constraint so that they are allowed to place exactly one point at a time. Suppose also that R, the first mover, is not allowed to place his points within a distance smaller than $1/\lceil N/K \rceil \gg \varepsilon > 0$ to a green point. Then strat-*

egy profile (Y', Y^*), *where the first mover uses strategy* Y' *while the second player uses strategy* Y^* *is a Nash equilibrium. Moreover, in every final configuration resulting from this Nash equilibrium there exists a monochromatic interval.*

We are obviously interested in the case where ε is arbitrarily close to zero. This theorem will be used to produce equilibrium final configurations in the one-by-one variant.

4   Examples

In this section we present examples illustrating the game when the players use the strategies presented above. We start with general game where R plays according to T^* while G plays according to T'. There are two cases here: $K \mid N$ and $K \nmid N$. We illustrate only the second case, which is more involved. Final configurations in the first case are similar to those of the second case if both players play these tying strategies.

Let $N = 11$ and $K = 3$. Player R starts by placing a point in an empty circle (which defines key positions for this circle with respect to the red point and $\lceil 11/3 \rceil = 4$) and G answers by placing a point in an empty circle (taking a key position and defining remaining key positions for this circle with respect to 4). The configurations created during the game are presented in Fig. 1. We use empty discs to depict red points and filled discs to depict green points. Key positions are depicted with short dashes intersecting the circles. In the rounds 2–4 player R plays according to option (b) taking free key positions in the first circle while G places his points within red intervals. In the round 5 player R plays according to option (d) taking a key position in a new circle. This determines key positions in the new circle which are taken with respect to the red point and the number 4 obtained as above. Player G

responds by playing within the newly created red interval, R takes another key position in the circle and G responds in the same manner. Since in round 6 player G placed his point in a key position, so player R plays according to option (a). After he places his point, the number of occupied key positions at round 7 is $Y(7) = 8$, the number of vacant key positions on non empty circles $V(7) = 0$ and the number of points R would have left if he had covered all vacant key positions in the occupied circles is $\varphi(7) = 11 - 7 - 0 = 4$. Since this is equal to $(11 - L(7))\lceil 11/3 \rceil = 4$, player R can cover key positions with respect to 4 in the remaining one circle, which he does. Player G answers placing exactly one point within each newly created red interval. The game is hence tied.

For the ε-restricted one-by-one version of the game, we take the same parameters and present an example where players apply their respective Nah equilibrium strategies Y' (used by the first mover Red) and Y^* (used by the second mover Green) (see Fig. 2). Player R starts by placing a point in an empty circle (which defines key positions for this circle with respect to the red point and $\lceil 11/3 \rceil = 4$) and G answers by placing a point in a clockwise neighbouring key position (also assigning key positions for this circle with respect to the position of the red point and 4). Then both players continue with taking key positions. When key positions are taken (4 of them on each circle) player R applies (c') of his strategy Y' and G responds applying option (b) of his strategy Y^* by breaking the red interval created by R. The game goes on in this manner until the last round is reached. In the last round player R applies option (c') of his strategy Y' again and player G responds by applying option (c) of his strategy Y^* and creating a green interval slightly bigger than the one created by R in this turn. This ends the game and G wins by the margin ε, the difference between intervals created in the last round.

5   Concluding remarks

We have studied an extension of the two-player Voronoi game of Ahn et al. [2004] to a playing arena involving multiple disjoint closed curves. Such games can be used to model important real life situations as highlighted in the introduction. We have shown that the second mover advantage, albeit arbitrarily small as shown in Ahn et al. [2004], disappears as we find tying strategies for both the first and the second mover, thereby enabling us to demonstrate Nash equilibrium configurations of locations. A general property of all such equilibrium configurations is that locations on each circle alternate in colour. We then study a natural variant of this game where players face very strict resource mobility constraints to show that the second mover advantage, again though arbitrarily small, re-appears. In the resulting equilibrium configurations of this version of the game, we show that there exists monochromatic intervals, an interesting difference vis-a-vis equilibrium configurations in the original game. One may think of the rules of the game as a mechanism by which distributions of influence between the two acting players can be affected and in that sense we have shown that a "literally fair" division is always Nash implementable. Ahn et al. [2004] has also studied such location games on line segments and it would be interesting to study our game on a family of disjoint line segments. Also, it would be important to generalize our games to those involving more than two players. Note also that the tying strategy of the first mover that we demonstrate depends crucially on the fact the the total number of points N is known. It would be interesting to extend these environments to incomplete information.

Acknowledgments

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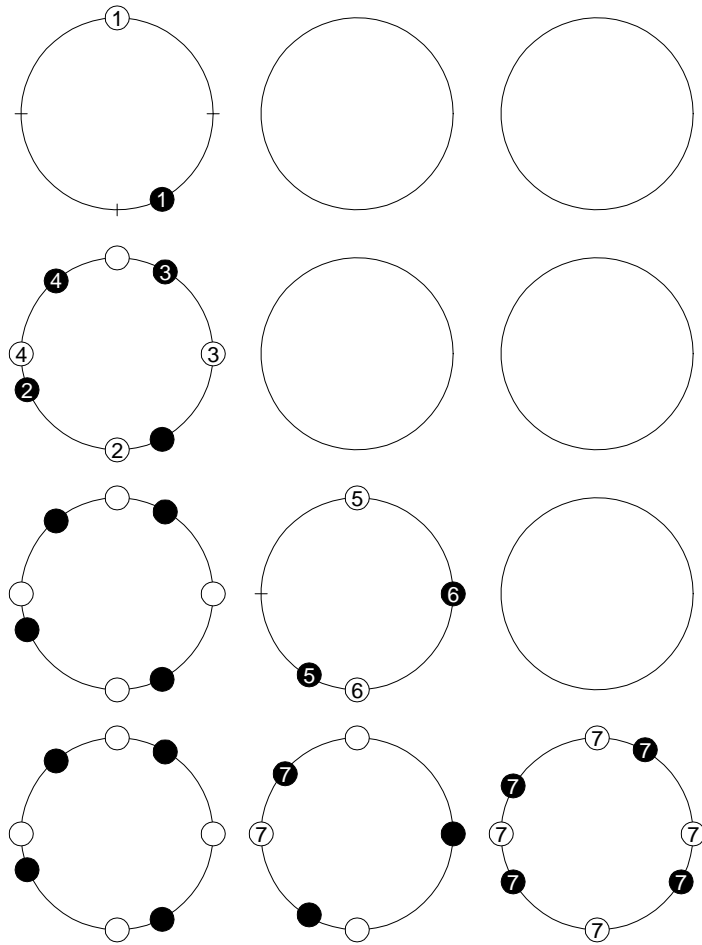
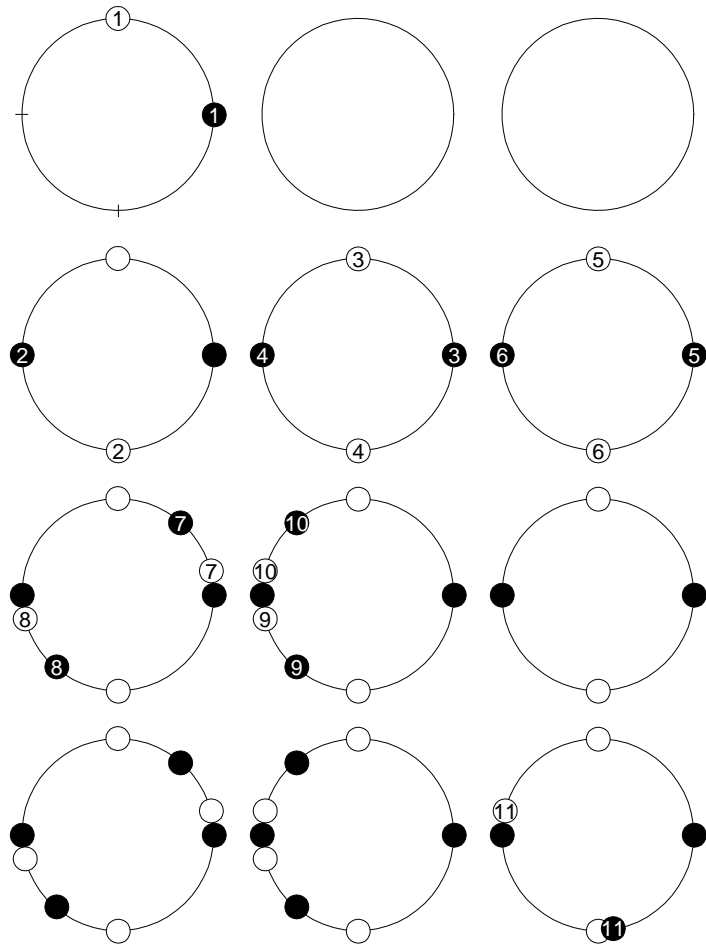Figure 1: Unrestricted game, $N = 11$, $K = 3$

Figure 2: "One-by-one" game, $N = 11$, $K = 3$