

The optimal behaviour of firms facing stochastic costs *

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Abstract

This paper aims at assessing the optimal behavior of a firm facing stochastic costs of production. In an imperfectly competitive setting, we evaluate to what extent a firm may decide to locate part of its production in other markets different from which it is actually settled. This decision is taken in a stochastic environment. Portfolio theory is used to derive the optimal solution for the intertemporal profit maximization problem. In such a framework, splitting production between different locations may be optimal when a firm is able to charge different prices in the different local markets.

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Key words: Firm behaviour, Portfolio theory, Risk aversion, Uncertainty.

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1 Introduction

Profit maximization in a fully deterministic setting is the standard framework that epitomizes the rational behavior of a firm. This setting underpins the hypothesis of the risk neutrality of the firm and most of cases it is not at odds with the fact. Nevertheless, when evaluating, for instance, the location choice in markets different from the domestic one, a fully deterministic setting does not seem to be suitable. The lack of full information about the destination market can strongly affect the structure of costs of the firm once it decides to delocate. This is a source of uncertainty and this fact makes it difficult to fully consider such a problem in a simple deterministic setting. When taking its investment decisions, a firm has to face uncertainty and, thus, a stochastic setting would better describe this framework. In this sense, some empirical evidence supports this statement. A few empirical studies about retail location decisions point out that firms always need to face uncertainty in demand or market conditions once they decide to enter new markets (see for instance, Hernandez and Biasiotto, 2001). In an applied study concerning the determinants of location by French firms in Eastern and Western countries, Disdier and Mayer (2004) note that such firms apply different strategies depending on the destination market. The uncertainty about the possible issues in political, economic and legal matters affects the investment decisions for Eastern countries. Furthermore, the authors find that French firms display important risk-averse behaviour. They succeed in proving that the uncertainty related to the investment return (embodied in the exchange rate volatility) inhibits the investment flows towards that destination.

In spite of such evidence, the economic literature usually deals with the location problem from a strictly deterministic point of view and location strategies are always taken for granted. In economic geography (which recently proposed a new approach to the location theory), location choices are driven by centripetal and centrifugal forces and most of the attention is addressed to the fragmentation problem (Fujita and Thisse, 2003). Another branch of the economic literature analyses the investment decision from the perspective of foreign direct investments for which firm decisions are mainly taken with the purpose of exploiting fiscal advantages as well as specific local resources (Devereux and Griffith, 1998).

Accordingly, the study of firm decision making in a stochastic setting seems to have been partially glossed over from a very strictly microeconomic viewpoint. An exception can be found in the seminal papers by Sandmo (1971) and Leland (1972) where a theory of firm decision making under uncertainty is proposed.

Sandmo (1971) focuses on a typical competitive setting where a firm performs as a price taker in a stochastic framework. The firm maximizes the expected utility of its profits. The firm's attitude towards risk is captured by a standard Von Newman-Morgenstern utility function. At the equilibrium under uncertainty, Sandmo proves that the level of the output is affected even by an infinitesimal increase in fixed costs. Furthermore, such a level is lower than in the deterministic case. In order to increase the level of the output, Sandmo argues

that a social planner should provide the firm with a per-unit subsidy whose optimal level heavily depends on the value of its (constant absolute) risk aversion index. In the same manner, he proves that an increase in the imposition (tax) rate does affect the level of firm's output which is proportional to its relative risk aversion.

Leland (1972) expands part of the analysis presented in Sandmo by formalizing the assumption of demand uncertainty. His results are in the spirit of Sandmo's and extend his conclusions. He confirms that: (i) fixed costs affect the optimal value of the control variable (prices and/or output), (ii) the firm is not neutral between quantity-setting and price-setting behaviour, (iii) the optimal output is smaller when uncertainty increases, and (iv) risk averse firms set lower quantity and price when risks decrease and marginal costs are non decreasing.

Besides, one novelty of the approach by Leland (1972) is to stress a particular link between his setting and the optimal portfolio literature. He asserts that in the theory of the firm, changes in fixed costs may have the same effects as do changes in the initial wealth invested in the risky assets in portfolio theory.¹ This effect depends not only on the absolute level of risk aversion, but also on the change in risk aversion due to profit or wealth increases.²

The present study recovers part of the aim of Sandmo and Leland's analyses. We consider a setting where firms behave in a monopolistic competition way. Our purpose is to describe the optimal behaviour of a firm that decides to partially delocate its production, focusing on the way it chooses its location strategies. This is typically the situation of the firm that switches from an export to a direct investment mode in order to penetrate into new markets.³ Our framework is built around a two-stage approach since a firm is supposed to decide the optimal mark-up that must be charged on its products, and, then, the quantity that must be produced in each local market. The usual technique of the backward induction leads us to solve the problem by deciding: firstly the optimal quantity that must be sold in a deterministic framework of imperfect competition, and secondly the prices that maximize the expected value of the firm's intertemporal profits in a stochastic environment. In this part, we assume, as it is usual in the literature about stochastic optimization,⁴ that the firm is risk averse, since firms bear risk in destination markets.

We use the mathematical techniques applied in finance to solve a microeconomic problem (namely, the delocation choice of a firm) in a dynamic and stochastic framework. That is,

¹A very general presentation of continuous-time finance and portfolio theory can be found in Merton (1990).

²Recently, other theoretical developments (see for instance, Lögfren et al., 1987, or Head et al., 2002) have emphasized the role played by uncertainty in the behavioural models of firms in duopolistic setting by concentrating on the interactive dimension.

³This is a very common strategy of firms (in industrialized countries), when they decide, for instance, to locate part of their production in ex-socialist or less developed countries.

⁴Usually, the literature concerning bank regulation assumes risk averse behaviour for describing banks (see, for instance, Koehn and Santomero, 1980; Kim and Santomero, 1988; and Keeley, 1990).

in accord with Leland's idea we use the stochastic dynamic optimization technique (largely used in finance) to analyse the investment decision of a (risk averse) firm.

In our setting, uncertainty is embedded in production costs (a firm faces in foreign markets) that are modelled as time varying stochastic processes. This means that we implicitly assume that uncertainty concerns the technology adopted by the firm in the destination markets. Furthermore, as time passes, firms may acquire experiences in dealing with such random environments, and thus reduce the risk they face.

A dynamic optimization framework supposes that firms maximize the expected intertemporal utility of their profits when incurring stochastic production costs in the foreign markets (while they face deterministic costs in the home market). Following Leland's (1972) intuition, we model the production costs through a Wiener process (as stock prices in portfolio theory).

In order to take into account the hypothesis of risk aversion, another distinguishing feature of our setting is to model a *utility* function for the firm as a HARA (i.e. Hyperbolic Absolute Risk Aversion) function. Thus, our model encompasses all the most widely used utility functions. In such a way we are able to replicate the result of Sandmo (1971), but in a more general setting. Indeed, Sandmo used CARA preferences which are a special case of the HARAs.⁵

The relevance of our contribution is twofold: on one hand we introduce uncertainty in the cost function firms face involving a time dimension. Furthermore, we allow firms experience a learning process by letting the stochastic part of costs reduce while time is running. On the other hand we are able to identify the key elements that make the location strategy consistent and efficient: discrimination and mark-ups. The economic intuition is straightforward: firms manage the risk by charging higher prices in less risky markets. The importance of being able to discriminate among markets follows. This is the core issue of our contribution and makes our result closer to the existing empirical evidence cited above. Indeed, in a monopolistic competitive (and deterministic) setting the mark-up level is basically constant. Here, firms elaborate their strategies in a different manner by applying different mark-ups and leaving room for dumping strategies.

Moreover, by applying HARA preferences we obtain an optimal mark-up as an affine transformation of profits, while with CARA preferences we could have got an optimal mark-up which is independent of the profit level, and with CRRA preferences an optimal mark-up which is linear in the level of profits.

This paper is organized as follows. In Section 2 we introduce the main features of the theoretical setting by presenting the behaviour of stochastic marginal costs and firm's preferences. In Section 3 we solve the optimization problem and we select the conditions under which the mark-up discriminating strategy can arise. Finally, we compute the behaviour of

⁵Recall that Constant Relative Risk Aversion (CRRA), Constant Absolute Risk Aversion (CARA), quadratic, and log-utility functions are all particular cases of the HARA.

both the optimal mark up and profit. Section 4 concludes.

2 The model

In order to assess to what extent uncertainty affects the delocation choice at firm level, we develop a two-stage model. We evaluate the delocation choice of a group of firms when they decide to enter n *foreign* symmetric markets rather than staying in the local (*home*) market. Uncertainty is related to the choice of the investment and firms take their decision on the basis of their utility function which takes account of uncertainty because of the lack of knowledge about the social, economic, and political features of the destination markets. Once firms have decided to locate in one or more of these markets, they start competing as in a deterministic setting, namely they all maximize their profits by fixing their prices as a mark-up on the costs and supplying a fixed quantity of goods (as in Dixit and Stiglitz, 1977). In the first step of our two-stage problem, a delocating firm chooses the mark-ups on costs and then decides the quantities of goods that must be produced in each sub-market. In order to solve this problem we apply a backward induction procedure. The first part of the problem is solved in a fully deterministic environment, while the stochastic optimization techniques will be applied in order to solve the second step of the problem (i.e. the profit optimization with respect to prices).

We concentrate on a monopolistically competitive framework. At each moment in time, a firm may simultaneously serve $n + 1$ markets: the local (*home*) one and the n foreign ones. We allow firms to serve local markets by setting up their plants there. Hence, once a firm enters one of the n symmetric foreign markets, it starts competing with the other $N - 2$ foreign firms and with the local one.⁶ We peg our analysis on a monopolistic competition framework by embracing a Dixit-Stiglitz's view. Thus, the results obtained for one firm remain valid for all of them.

Our setting lies on two main assumptions: (i) there exists an exogenously given number of symmetric firms (N), and (ii) each of them gets can get losses or profits in each moment in time and profits are fully shared among shareholders. In our economy, in each local market, there are H shareholders that are also workers and consumers. In each local market, the utility function (D) of a consumer j , D_j ($j = 1...H$), is a CES function:

$$D_j = \left[\sum_{i=1}^N d_i^{\frac{\sigma-1}{\sigma}} \right]^{\frac{\sigma}{\sigma-1}}, \quad \sigma > 1. \quad (1)$$

This function consists of a composite index which is generated by the combination of the consumption of the N different varieties of goods. Each consumer has a preference for

⁶In this setting we are assuming that all firms are symmetric. It means that if one firm decides to locate in a foreign market as an issue of a maximization process all the other $N-1$ *firms* will do the same.

variety and consumes d_j units of each variety according to the elasticity of substitution equal to σ . This elasticity is the same for all consumers in a same market, but it can be different across the various local markets. As mentioned above, each consumer is also simultaneously worker and shareholder. The whole income of each consumer/workers (Y_j) is given by:

$$Y_j = w + N \frac{\pi_i}{H}. \quad (2)$$

where π_i is the profit function of a firm.

Each consumer devotes all his income to consumption, and by maximizing (1) under revenue constraint (2), for any pair of (l, k) of varieties we obtain :

$$d_k = \left[\frac{p_l}{p_k} \right]^\sigma d_l. \quad (3)$$

The first order condition of this maximization problem give equality of the marginal rate of substitution to the price ratio. We define the general price index of a market P as :

$$P = \left[\sum_{i=1}^N p_i^{1-\sigma} \right]^{\frac{1}{1-\sigma}}.$$

The price index P corresponds to the minimum cost of purchasing a unite of the composite good D_j . Hence, knowing that consumers devote all revenue to consumption we can rewrite (2) as:

$$Y_j = w + N \frac{\pi_i}{H} = P D_j, \quad (4)$$

and by replacing (3) and (4) into (1), the demand function for the k th variety of goods becomes:

$$d_k = \frac{p_k^{-\sigma}}{P^{1-\sigma}} Y_j. \quad (5)$$

Let us consider a particular firm producing one variety. Each firm produces one variety of good by hiring workers. All firms are identical and each of them is expected to maximize the following profit function facing a given wage. In each market :

$$\pi_i = p_i q_i - g_i, \quad (6)$$

and we know that the cost function g_i is related to the technology applied by the firm and it is equal to

$$g_i = w L_i, \quad (7)$$

where L_i is the number of worker hired by the firm.⁷ The technology of the firm is given by

$$L_i = f_i + c_i q_i, \quad (8)$$

where the workers L_i produce q_i units of good with a marginal cost c_i and fixed costs f_i . Plugging (7) and (8) into (6), we maximize the resulting profits following Dixit-Stiglitz (1977) and considering nominal wages as numéraire (that is $w = 1$). Each firm fixes its selling prices as mark up over marginal costs:

$$p_i = m_i c_i, \quad \text{with} \quad m_i = \frac{\sigma}{\sigma - 1} . \quad (9)$$

Knowing the demand function (5) and applying (4), the market clearing condition for one variety (and by symmetry for all the others) may be written as:

$$q_i = H d_i = H \left[\frac{p_i^{-\sigma}}{P^{1-\sigma}} Y_j \right] .$$

From the general index of prices and equation (9), we deduce that, at the equilibrium, the price index coincide with the price fixed by a single firm. By this assumption, we are able to rewrite the previous equation as:

$$q_i = \frac{1}{p_i} [H + p_i q_i - c_i q_i - f_i]$$

and, then, reduce it to:

$$q_i = \frac{H - f_i}{c_i}, \quad (10)$$

which is a valid solution for $H > f_i$.

Once the demand faced by each firm is known, one can move to the first stage of the problem: the location choice. Firms know that, once in the foreign local market, each of them should provide a quantity q_i of final good. However, when into one foreign market, a firm must pay fixed costs (f_i) in order to set up plants there and furthermore, there is uncertainty related to the production conditions.⁸

We need to model the uncertainty linked to the foreign production costs. For each firm, we assume that the local marginal cost is a deterministic value (c_H), while the foreign marginal cost (c) is described by a stochastic differential equation whose form is discussed in detail in the next subsection. The rationale of this assumption is the following: the lack of knowledge about the economic and political foreign environment makes a firm's choices

⁷We assume that the labor force is homogeneous and there is not unemployment in this market.

⁸We mean that a firm usually has to face bureaucratic, institutional, and financial problems in the foreign environment that make the profitability of her investment more uncertain.

more complicated and only firms founded there have a deep knowledge of the environment. This status of uncertainty affects the investment returns.

An alternative way to model the uncertainty is to assume that both domestic and foreign marginal costs are constant if they are expressed in their own currencies. So, the only risk would be embedded in the exchange rate. In this way the foreign costs become stochastic once they have been expressed in domestic currency.

In economic terms, most of the uncertainty related to foreign investments relies on the problem repatriating part of the benefits earned in foreign markets. In particular, if the exchange rate E follows a stochastic equation, while the foreign marginal cost (c_E) is constant, then the foreign marginal cost in domestic currency ($c = c_E E$) behaves stochastically. In this work we prefer to concentrate on modeling the foreign marginal costs instead of the exchange rate in order to avoid any macroeconomic problem about the equilibrium value of the exchange rate.

Before treating the structure of marginal costs, for sake of simplicity, we introduce some notational simplifications. We call $p \in \mathbb{R}^n$ the vector of prices for the N local markets (countries or regions) while p_H is the price for the domestic market.

According to the results we previously cited, prices are defined as

$$p = Mc, \tag{11}$$

where $M \in \mathbb{R}^{n \times n}$ is the diagonal matrix containing all the mark-up parameters m_i , $\forall i \in \{1, \dots, n\}$. The corresponding mark-up equation for the domestic production implies

$$p_H = m_H c_H. \tag{12}$$

The firms are assumed to maximize their objective functions (i.e. intertemporal utility of profits) with respect to both foreign and domestic mark-ups. The objective function accounting for firm risk aversion will be widely described in Subsection 2.2.

2.1 The marginal costs

In our framework, the only source of risk for a firm is the marginal cost in the foreign markets (henceforth foreign marginal costs) whereas the domestic marginal cost is constant. When a firm locates part of its production abroad, the corresponding marginal cost follows a stochastic differential equation with a direct dependence on time and space. Quite reasonably, we are assuming that such costs are not constant across time, since a firm may develop learning procedures as well as accumulating experiences in dealing with production in an international setting. In such a way, it may adapt better to the local environment, and can manage (or control) the state of the risk more easily. In the same manner, the

various local markets may display different levels of risk involving different factors that can affect the firm's choices. We model the dynamics of marginal costs as⁹

$$\underset{n \times 1}{dc} = \underset{n \times n}{\Omega(c, t)} \underset{n \times 1}{\mu(t)} dt + \underset{n \times n}{\Omega(c, t)} \underset{n \times k}{\Sigma(t)'} \underset{k \times 1}{dW}, \quad c(t_0) = c_0, \quad (13)$$

where W is k -dimensional Wiener process.¹⁰ Hereafter, the prime denotes transposition. Equation (13) implies that the marginal costs follow a stochastic process made up of two components that vary across time and space and may depend on all the foreign marginal costs. This general setting allows us to account for the case where the marginal cost in market i is affected by the costs of producing in all the other $N - 1$ foreign markets. The values of all the costs are known in t_0 (namely the initial period) and so c_0 is a deterministic vector of real (positive) variables.

The product $\Omega\mu$ is called "drift" and represents the expected instantaneous variation in costs through time. Instead, $\Omega\Sigma'$ is called "diffusion" and measures the instantaneous standard deviation of costs. Actually:¹¹

$$\begin{aligned} \mathbb{E}_t [dc] &= \Omega\mu dt, \\ \text{Var}_t [dc] &= \Omega\Sigma'\Sigma\Omega dt. \end{aligned}$$

The vector c contains all the costs c_i ($\forall i \in \{1, \dots, n\}$) of producing in each foreign sub-market and each marginal cost c_i is affected by a set of k risk sources (country risk, sector risk, etc..) embedded in W .

The particular form we assumed for marginal costs in (13) implies that the drift and diffusion terms must have a symmetric behavior with respect to c . Indeed, the same matrix $\Omega(c, t)$ appears in both the drift and the diffusion component. This functional form simplifies the mathematical computations; however, further extensions of our model will focus on more general forms for the drift and the diffusion coefficients in (13).

Remark 1 *The model presented in Equation (13) allows us to take into account a learning process. As soon as firm's knowledge of the economic conditions in foreign markets starts improving, the volatility component decreases (i.e. the derivative of $\Sigma(t)$ with respect to time should be negative).*

⁹The drift and diffusion terms $\Omega(c, t)\mu(t)$ and $\Omega(c, t)\Sigma(t)'$ are assumed to satisfy all the usual conditions necessary for having a unique *strong* solution of this differential equation (see Karatzas and Shreve, 1991).

¹⁰As usual, all the stochastic processes belonging to W are assumed to be independent. This hypothesis is made without loss of generality because the independent case can be easily traced back to the dependent case by means of the Cholesky matrix.

¹¹Here, \mathbb{E}_t is the expected value operator with respect to the information in t . The notation $\mathbb{E}_t[\bullet]$ is sometimes replaced by the notation $\mathbb{E}[\bullet | \mathcal{F}_t]$ where the σ -algebra \mathcal{F}_t embeds all the relevant information. Furthermore, Var_t is the variance with respect to the information in t .

The matrix $\Omega(c, t)$ is a diagonal matrix¹² containing n functions, one for each foreign market. Thus, Equation (13) can be written in a less compact form as follows:

$$d \begin{bmatrix} c_1 \\ c_2 \\ \dots \\ c_n \end{bmatrix} = \begin{bmatrix} \omega_1 \mu_1 \\ \omega_2 \mu_2 \\ \dots \\ \omega_n \mu_n \end{bmatrix} dt + \begin{bmatrix} \omega_1 \sigma_{11} & \omega_1 \sigma_{12} & \dots & \omega_1 \sigma_{1k} \\ \omega_2 \sigma_{21} & \omega_2 \sigma_{22} & \dots & \omega_2 \sigma_{2k} \\ \dots & \dots & \dots & \dots \\ \omega_n \sigma_{n1} & \omega_n \sigma_{n2} & \dots & \omega_n \sigma_{nk} \end{bmatrix} d \begin{bmatrix} W_1 \\ W_2 \\ \dots \\ W_k \end{bmatrix},$$

where $\sigma_{ij}(t)$, $i, j \in \{1, \dots, n\}$ are elements of matrix Σ and the functional dependence on costs and time have been omitted for the sake of simplicity. This extended form shows that our model is sufficiently general to allow the embedding also of the case of correlated costs of production among the different foreign markets. This correlation appears via the matrix Σ . When Σ is a diagonal matrix (i.e. $\sigma_{ij}(t) = 0, \forall i \neq j$) then the marginal costs of production in different markets are independent.

Particular examples may be generated by choosing particular forms for the matrix $\Omega(c, t)$.

Case 1 (The geometric Brownian costs). When the matrix $\Omega(c, t)$ appears as

$$\Omega = \begin{bmatrix} c_1 & 0 & \dots & 0 \\ 0 & c_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & c_n \end{bmatrix},$$

then each marginal cost c_i in (13) follows a geometric Brownian motion, i.e.

$$dc_i = c_i \mu_i(t) dt + c_i \Sigma_i(t)' dW,$$

where Σ_i is the i^{th} column of matrix Σ . In this case, the production costs in a sub-market are not affected by the production costs in all the other sub-markets.

Case 2 (The mean reverting costs). When the matrix $\Omega(c, t)$ appears as

$$\Omega = \begin{bmatrix} \xi_1 - c_1 & 0 & \dots & 0 \\ 0 & \xi_2 - c_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \xi_n - c_n \end{bmatrix},$$

then Equation (13) for the element c_i can be written as

$$dc_i = \mu_i(t) (\xi_i - c_i) dt + (\xi_i - c_i) \Sigma(t)'_i dW,$$

¹²Its elements $\omega_{ij}(c, t)$, $i, j \in \{1, \dots, n\}$ are such that $\omega_{ij}(c, t) = 0, \forall i \neq j$.

where, as in the previous case, Σ_i is the i^{th} column of matrix Σ . In this model the parameters ξ_i play the role of the “equilibrium values” for the marginal costs. When c_i equals ξ_i , the marginal cost becomes deterministic and constant. By contrast, when c_i is higher (lower) than its equilibrium value, the market forces (whose strength is measured by μ_i) bring c_i back towards the value ξ_i . In both the cases, the farther c_i from its equilibrium value, the higher its volatility (which is proportional to $(\xi_i - c_i)^2$).

As we have already stated, the domestic marginal cost (c_H) is constant, therefore, by equation (10), the quantity of good a firm can sell in the domestic market (q_H) is a constant as well. The analogy with the asset allocation problem suggests that the domestic marginal cost plays the same role as the riskless asset. This is a particular result stemming from the adoption of risk aversion hypothesis: any risk averse investor always includes in her portfolio a quota of the riskless asset. In this setting, any entrepreneur chooses to set up part of the production in the home market. In case we are referring to foreign markets, we assume that the firm knows the value of marginal costs at time it enters in the market ($t = 0$), that is c_0 (according to equation (13)). Hence, at that moment, each firm can fix the quantity it can produce and sell in each foreign markets as a constant as well. In our framework, we make the assumption that when a firm starts producing in a market, it builds a new plant that is able to produce the quantity q_i (stated at $t = 0$) for ever. In case of demand fluctuations, the exceeding quantity can be stocked with no additional costs.¹³

Let us define the vectors of prices and quantities (produced and sold in the foreign markets) as follows:

$$p_{n \times 1} \equiv \begin{bmatrix} p_1 \\ p_2 \\ \dots \\ p_n \end{bmatrix}, \quad q_{n \times 1} \equiv \begin{bmatrix} q_1 \\ q_2 \\ \dots \\ q_n \end{bmatrix}.$$

The corresponding domestic variables are p_H and q_H .

By our previous assumption and by the outcomes of the second stage, we know that both the final level of demand q_i ($\forall i \in \{1, \dots, n\}$) and the domestic demand q_H are constant.

When a firm locates part of its production in foreign local markets, we assume that its total profits (earned in local and foreign markets) turn out to be

$$\pi = q_H (p_H - c_H) + q'(p - c) - F, \quad (14)$$

where F is the the amount of fixed costs faced by the firm for building the new plants.¹⁴

¹³We are implicitly considering the situation of not fashionable or not perishable goods.

¹⁴For the sake of simplicity, we apply a linear transformation of the technology previously described in order to get an expression displaying the marginal cost of output. In that sense, F stands for the whole sum of fixed costs a firm incurs in each destination.

By plugging p_H and p from (11) and (12) into (14), we get

$$\pi = q_H (m_H - 1) c_H + q' (M - I) c - F,$$

where $I \in \mathbb{R}^{n \times n}$ is an identity matrix. The profit is maximized with respect to the mark-ups. Nevertheless, since the quantities q and q_H are constant, then for the sake of simplicity we define the new following control variables

$$\begin{aligned} x &\equiv (M - I) q, \\ x_H &\equiv (m_H - 1) q_H. \end{aligned} \tag{15}$$

It will be straightforward to compute the optimal values of the mark-ups after solving the optimal control problem for x and x_H . As a consequence of this change of variables, the profit may be rewritten as

$$\pi = x_H c_H + x' c - F, \tag{16}$$

whose Itô's differential is given by ¹⁵

$$d\pi = x' dc + c_H dx_H + (dx)' (c + dc).$$

The usual self-financing condition implies

$$c_H dx_H + (dx)' (c + dc) = 0,$$

which means that there are neither external contributions nor external withdrawals from firm's profit. Finally, after expressing costs in terms of profits, the profit dynamic behavior is given by the stochastic differential equation

$$d\pi = x' \Omega \mu dt + x' \Omega \Sigma' dW, \tag{17}$$

where the functional dependences have been omitted for the sake of simplicity. The only control variable is now x while x_H will be easily recovered from Equation (16).

2.2 The firm risk aversion

In this subsection we model the risk aversion of the firm. Thus, we need a formal device to describe how firms cope with the risk. According to Asplund (2002) there are some reasons which cause firms to act in a risk-averse way, for instance non-diversified owners, liquidity constraints or non-linear tax systems. Even the delegation of control decisions to a risk-averse manager who is paid according to firm's performance may produce risk averse

¹⁵We recall that c and x are stochastic variables. Accordingly, the product of their differentials $(dx)' (dc)$ cannot be neglected as in the usual chain rule for the differentiation.

behaviour of the firm. Banal and Ottaviani (2004) add also that risk aversion makes firms more concerned about low-profits states due to low level of demand or high costs. They try to have good performances in these hard times at the cost of reducing profits in good times.

In a dynamic setting, a risk neutral firm would simply maximize its intertemporal expected profit. Therefore, under the risk neutral assumption, each firm would impose an infinite mark-up on the foreign market with the lowest cost (as we describe below). However, when thinking of the foreign investment option, under the strong uncertainty assumption, such behaviour is not rational. The best way to take into account this new dimension is by modelling the firm attitude towards risk by an increasing and strictly concave transformation of its profits. This kind of transformation is the standard von Neumann- Morgenstern *utility function* applied to firm profits.

In this analysis, we select a general form of utility function: the so-called HARA (Hyperbolic Absolute Risk Aversion) family. In algebraic terms the HARA utility function can be written as function of firm profits (π)

$$U(\pi) = \delta (\gamma\pi - \alpha)^{1-\frac{\beta}{\gamma}}.$$

In order to have a well defined maximization problem, we need the utility function to be increasing and concave in its argument (here, profits). These conditions lead the following restrictions on the parameters:

$$\begin{aligned} \frac{\partial U}{\partial \pi} &> 0 \implies \delta(\gamma - \beta) > 0, \\ \frac{\partial^2 U}{\partial \pi^2} &< 0 \implies -\beta\delta(\gamma - \beta) < 0, \end{aligned}$$

and we obtain

$$\beta > 0, \quad \delta(\gamma - \beta) > 0. \quad (18)$$

The Arrow-Pratt absolute risk aversion index \mathcal{R} computed for U is a hyperbolic function of π :

$$\mathcal{R} \equiv -\frac{\partial^2 U}{\partial \pi^2} \left(\frac{\partial U}{\partial \pi} \right)^{-1} = \frac{\beta}{\gamma\pi - \alpha}. \quad (19)$$

Notice that the index \mathcal{R} is inversely proportional to the firm profit. It implies that a higher level of profits yields a lower level of \mathcal{R} . This means that firms with high profits are less risk averse.

If we want to avoid negative profits at any point in time, then parameter α plays a crucial role. In fact, if $\alpha \geq 0$, then it turns out to be a measure of the minimum acceptable profit level (as set out in Karatzas and Shreve, 1998, for the subsistence consumption level in the case of consumption and investment optimization). This result can be easily derived by computing the marginal utility of the profit:

$$\frac{\partial U}{\partial \pi} = \delta(\gamma - \beta)(\gamma\pi - \alpha)^{-\frac{\beta}{\gamma}}.$$

When $\beta/\gamma > 0$, there exists a level of profit (equal to α/γ) leading to an infinite marginal utility. As a consequence of that, if the profit reaches the value α/γ , the firm would get an infinite increase in its utility by augmenting its profits (even by a very small amount).¹⁶ This means that the optimal profit will never fall to the value α/γ . Thus, in this analysis we exclude the non-negativity constraint on profits by assuming that $\alpha/\gamma \geq 0$.

One can think about other ways to model the risk aversion of firms, but the HARA utility function is the most general one since it takes into account a quite large range of preferences, and most of the other functions applied in economic theory can be derived directly from it. For instance:

1. By imposing $\alpha = 0$ and $\gamma = 1$, the HARA function becomes a CRRA (Constant Relative Risk Aversion) utility function in the form $U(R) = \delta c^{1-\beta}$; with the minimal profit level equal to zero;
2. by assuming $\alpha = -1$ and with γ approaching zero, the HARA function converts into the CARA (Constant Absolute Risk Aversion) utility function in the form $U(R) = \delta e^{-\beta c}$.¹⁷ In such a case, there does not exist any finite and non-negative level of profits giving an infinite marginal utility; thus, for CARA preferences the non-negativity constraint on profit should be explicitly imposed;
3. in case $\alpha = 0$, $\gamma = 1$, $\delta = (1 - \beta)^{-1}$ and β approaches to one, the HARA function provides the same results as the log utility function;
4. when $1 - \frac{\beta}{\gamma} = 2$, the HARA reduces to the quadratic utility function.

3 The profit optimization

Each firm intertemporally maximizes the expected utility of its profits. This is a dynamic problem where the intertemporal utility function is maximized with respect to the mark-up given the dynamic behaviours of both costs and profits (represented by two vector differential equations). Keeping in mind the contents of the previous section, the firm's programme can be written as¹⁸

$$\begin{cases} \max_x \mathbb{E}_{t_0} \left[\int_{t_0}^{\infty} e^{-\rho(t-t_0)} (\gamma\pi(t) - \alpha)^{1-\frac{\beta}{\gamma}} dt \right] \\ dc(t) = \Omega(c, t) \mu(t) dt + \Omega(c, t) \Sigma(t)' dW, \\ d\pi(t) = x' \Omega(c, t) \mu dt + x' \Omega(c, t) \Sigma(t)' dW, \end{cases} \quad (20)$$

¹⁶This would be possible by investing only in the domestic production, for instance.

¹⁷In this case, according to Conditions (18), the parameter δ must be negative.

¹⁸Recall that the variable x is a linear transformation of the mark-ups, as in (15).

where ρ is the constant subjective discount rate. The optimal mark-ups (m) solving Problem (20) are shown in Proposition 1.

Proposition 1 *The optimal foreign mark-ups solving Problem (20) in presence of constant domestic marginal costs, are*

$$m^* = \mathbf{1} + \frac{\gamma\pi - \alpha}{\beta} I_q^{-1} \Omega^{-1} (\Sigma' \Sigma)^{-1} \mu, \quad (21)$$

where $\mathbf{1} \in \mathbb{R}^n$ is a vector of 1s. The optimal domestic mark-up is

$$m_H^* = 1 + \frac{1}{c_H q_H} \left(\pi + F - \frac{\gamma\pi - \alpha}{\beta} c' \Omega^{-1} (\Sigma' \Sigma)^{-1} \mu \right). \quad (22)$$

Proof. See Appendix A.1. ■

Remark 2 *The result of the optimization problem does not hold when $\Sigma(t)$ approaches zero (see Equations (21) and (22)). Thus, even taking into account the case with a decreasing volatility of foreign marginal costs (i.e. of a learning process), the result stated in Proposition 1 cannot be extended to the case where some costs become fully deterministic. In this last situation the optimal strategy would consist of choosing the lowest costs among the deterministic ones and investing in the remaining countries according to Proposition 1.*

The mark-ups in (21) and (22) rely on the profit level. This is an implication of choosing a HARA utility function. It embeds the assumption that firms display different risk aversion according to their profit levels. A firm earning high profits has a tendency to apply higher mark-ups than firm with low profits. In addition, the endogeneity that relates mark-ups and profits is just apparent because the optimal profit is a function containing only parameters (as we are going to show in the next section).

Also note that:

1. The mark-up is increasing in the profit. This means that more profitable firms impose higher mark-ups. This outcome is more precise than that proposed by Sandmo (1971) when using a constant absolute risk aversion utility function (for firms). It is possible to recover his result by fixing $\alpha = -1$ and letting γ tend to zero. One can immediately check from Equations (21) and (22) that when γ tends to zero, the optimal mark-ups do not depend on the level of profits. Nevertheless, our model also includes the case of a firm exhibiting a decreasing risk aversion index. When profits increase, the firm becomes less risk averse and so it can charge a higher level of mark-up for goods produced and sold in the foreign markets.

2. The only mark-up depending on fixed costs is the domestic one. Because of its risk aversion, the firm prefers to charge these costs just on the domestic mark-up since the domestic market has no risk.

3.1 The mark-up discrimination

As shown in Proposition 1 there exists an optimal level of mark-up for each foreign market. Thus, the usual result of a constant mark-up across markets does not hold. In this subsection, we consider to what extent the discrimination between the domestic and foreign mark ups can actually be a profitable strategy for firms. In particular, we concentrate on the situation where the firm charges a higher mark up in the foreign markets than in the local one. This situation happens if and only if the inequality

$$m^* > m_H^* \mathbf{1}, \quad (23)$$

holds, where $\mathbf{1}$ is a vector of 1s. As shown in Appendix A.2 inequality (23) holds when

$$m^* - \mathbf{1} > \frac{\pi + F}{c_H q_H + c' q} \mathbf{1}. \quad (24)$$

Since $\pi + F$ is usually known as *contribution margin*, we have the following.

Proposition 2 *The firm solving the maximization problem (20) successfully applies a strategy imposing foreign mark-ups higher than domestic ones, if and only if the net foreign mark-ups (i.e. $m^* - \mathbf{1}$) are greater than the ratio between the contribution margin (namely the sum of profits and fixed costs) and total variable costs.*

Proof. See Appendix A.2. ■

Proposition 2 implies that, under likely conditions, dumping¹⁹ is an optimal strategy for firms discriminating mark-ups. Now, we are interested in finding an easy way for checking whether firms profitably apply a dumping strategy. In particular, inequality (24) may be rewritten as

$$m^* > \frac{\pi + c_H q_H + c' q + F}{c_H q_H + c' q} \mathbf{1}, \quad (25)$$

so that a corollary follows easily.

¹⁹Remember that dumping is defined as charging higher prices in the local market than in the foreign ones.

Corollary 3 *The firm solving Problem (20) may select a discriminating strategy when the foreign mark-ups (i.e. m^*) are greater than the ratio of total revenues to total variable costs.*

From a policy viewpoint, the results stated in Proposition 2 and Corollary 3 can be useful for implementing a regulation strategy. A discriminating mark-up strategy yields price discrimination which may take the appearance of dumping. In order to improve the contents and effectiveness of an anti-dumping regulation, the regulator should just gather information about the following data: (i) the firm contribution margin (or total revenues), (ii) the firm total variable costs, and (iii) the firm mark-ups on foreign markets. If the reverse of condition (24) holds, then there is room for a dumping strategy. Hence, the need of a regulation policy appears.

Remark 4 *When firms solve Problem (20), a regulator could prevent firms from applying a dumping policy by stating the following rule: the mark-up on foreign markets must not be lower than the ratio between total returns and total variable costs.*

3.2 The behaviour of optimal mark-ups

Even if the total amount of profits is always positive (as we will show in the next section), we are interested in checking whether it is optimal for firms to allow losses in some markets that are compensated by higher profits in other markets. This is equivalent to look for mark-ups higher (lower) than 1 generating a profit (loss). Hence, we investigate the behaviour of the optimal mark-up and we check in Equation (21) if the elements of the vector

$$\frac{\gamma\pi - \alpha}{\beta} I_q^{-1} \Omega^{-1} (\Sigma' \Sigma)^{-1} \mu,$$

are positive or negative. The form of the utility function guarantees that the optimal value of profits never falls below α/γ (i.e. $\gamma\pi(t) - \alpha > 0, \forall t \geq t_0$). Moreover, the quadratic form $\Sigma' \Sigma$ is always positive semi-definite and μ can be assumed to be a vector of positive functions with respect to time without loss of generality. Finally, q is a vector of positive constants. This means that the elements of the vector m^* are higher (lower) than 1 depending on the sign of the elements in the matrix Ω .

Let us go back to the previous cases in order to check whether, in those cases, the mark ups are higher or lower than 1.

Case 1 *(The geometric Brownian costs). A variable following a geometric Brownian motion has always positive values, so that Ω (containing the elements of c) is always positive definite and the optimal mark-ups given in (21) are always greater than one.*

Accordingly, we can conclude.

Proposition 3 *When the foreign marginal costs follow geometric Brownian motions (i.e. $\omega_{i,j} = 0, \forall i \neq j$ and $\omega_{i,i} = c_i$) and the domestic marginal cost is constant, then the optimal foreign mark-ups are all greater than 1.*

We now turn to the case of mean reverting costs.

Case 2 *(The mean reverting cost). When it is possible to invest only in one foreign country (i.e. $n = 1$), the optimal mark-up (21) is*

$$m^* = 1 + \frac{\gamma\pi - \alpha}{\beta} \frac{\mu}{(\xi - c)q\sigma^2}.$$

Thus, when c is under its equilibrium level (i.e. $c < \xi$) the optimal mark-up is greater than one and vice versa, when c is over its equilibrium value (i.e. $c > \xi$) the optimal mark-up is lower than one. We stress that once the costs reach their equilibrium values (i.e. $c = \xi$), the optimal mark-up tends towards infinity because, under this assumption, there is no more uncertainty. This issue is strongly related to the assumption of fixed quantities. Actually, if one allowed for variable quantities, this result would be different.

As will be argued in detail in the following subsection, for any mark-up, the optimal profit always remains positive. Thus, we will not be concerned with the case where the mark-ups are less than one, since, even in this case, the optimal strategy will always lead to the sum of profits over all locations being positive. When the foreign mark-ups are lower than one ($m^* < \mathbf{1}$),²⁰ it is easy to check from Proposition 1 that the value of the domestic mark-up is higher than the value it takes when $m^* > \mathbf{1}$. Hence, the loss incurred in the foreign market is compensated for by the additional earnings in the domestic one.

3.3 The optimal profit

Once we have found the optimal mark-up strategy we are able to compute the optimal level of profits a firm gets under the delocation assumption. The behaviour of the optimal profit can be easily obtained by plugging the value of x^* from (31) into the differential equation (17) for π . We obtain

$$d\pi = \frac{\gamma\pi - \alpha}{\beta} \mu' (\Sigma' \Sigma)^{-1} \mu dt + \frac{\gamma\pi - \alpha}{\beta} \mu' (\Sigma' \Sigma)^{-1} \Sigma' dW,$$

²⁰In this case the inequality between vectors means that each element of m^* must be less than 1.

which is a linear stochastic differential equation in π whose solution exists in a closed form according to the following proposition.

Proposition 4 *The optimal profit is given by*

$$\begin{aligned} \pi^*(t) &= \Phi(t_0, t) \pi(t_0) + \frac{\alpha}{\beta} \left(\frac{\gamma}{\beta} - 1 \right) \int_{t_0}^t \lambda(s)' \lambda(s) \Phi(s, t) ds \\ &\quad - \frac{\alpha}{\beta} \int_{t_0}^t \lambda(s)' \Phi(s, t) dW(s), \end{aligned} \quad (26)$$

where

$$\begin{aligned} \Phi(t_0, t) &= \exp \left\{ \frac{\gamma}{\beta} \left(1 - \frac{\gamma}{2\beta} \right) \int_{t_0}^t \lambda(\tau)' \lambda(\tau) d\tau + \frac{\gamma}{\beta} \int_{t_0}^t \lambda(\tau)' dW(\tau) \right\}, \\ \lambda(t) &\equiv \Sigma(t) (\Sigma(t)' \Sigma(t))^{-1} \mu(t). \end{aligned}$$

Proof. See Kloeden and Platen (1992, Paragraph 4.2). ■

As one can check in Equation (26), a relevant measure of the optimal location choice (that replicates the portfolio choice) is given by λ . In order to give it an economic interpretation, we rewrite its value in scalar terms:

$$\lambda = \frac{\mu}{\sigma}.$$

As μ is the expected value (mean value) of c and σ is its standard deviation, λ is the inverse of the variation coefficient of the marginal cost. Therefore, λ measures the expected cost in terms of risk and it represents an easy way of taking into account, simultaneously, the first and second moment of the stochastic variable c .

The optimal profit in (26) contains one component proportional to the initial profit $\pi(t_0)$, and two components proportional to α which is a measure of the minimum level of profit (α/γ). It is easy to show that the expected optimal profit never falls below the value α/γ . After stressing that the expected value of the particular function $\Phi(t, t_0)$ is given by²¹

$$\mathbb{E}_{t_0} [\Phi(t, t_0)] = \exp \left\{ \frac{\gamma}{\beta} \int_{t_0}^t \lambda(\tau)' \lambda(\tau) d\tau \right\},$$

the expected value of the optimal profit is

$$\begin{aligned} \mathbb{E}_{t_0} [\pi^*(t)] &= \pi(t_0) \mathbb{E}_{t_0} [\Phi(t_0, t)] \\ &\quad + \frac{\alpha}{\beta} \left(\frac{\gamma}{\beta} - 1 \right) \int_{t_0}^t \lambda(s)' \lambda(s) \mathbb{E}_{t_0} \left[\frac{\Phi(t_0, t)}{\Phi(t_0, s)} \right] ds. \end{aligned} \quad (27)$$

²¹Recall that Φ is log-normally distributed.

Remark 5 *In order to have a strictly concave utility function, we need to assume $\alpha \geq 0$, and $\beta > 0$. Now, let us add $\gamma/\beta > 1$ (which is compatible with Conditions (18) when $\delta > 0$). Hence, since $\Phi(t_0, t) > 0, \forall t \geq t_0$, we can conclude that the expected optimal profit never falls below the initial value $\pi(t_0)$. Nevertheless, the initial value of profit must be greater than α/γ for the objective function to be well behaved. This means that $\mathbb{E}_{t_0}[\pi^*(t)]$ never falls below α/γ , or put differently, when delocating, firms always experience a minimum positive level of profits as a whole.*

When the minimum profit stands at zero (i.e. $\alpha = 0$ and firm's preferences belong to the CRRA family) the optimal profit function reduces to

$$\pi^*(t) = \Phi(t_0, t) \pi(t_0),$$

and the optimal profit turns out to be just a geometric Brownian motion. Under this hypothesis, $\Phi(t_0, t)$ plays the role of a stochastic capitalization factor.

By contrast, when firm's preferences are of the CARA type (i.e. $\alpha = -1$ and $\gamma \rightarrow 0$) we get

$$\pi^*(t) = \pi(t_0) + \frac{1}{\beta} \int_{t_0}^t \lambda(s)' \lambda(s) ds + \frac{1}{\beta} \int_{t_0}^t \lambda(s)' dW(s), \quad (28)$$

and its expected value is

$$\mathbb{E}_{t_0}[\pi^*(t)] = \pi(t_0) + \frac{1}{\beta} \int_{t_0}^t \lambda(s)' \lambda(s) ds.$$

Nevertheless, even if in the case of CARA functions the expected optimal profit never falls below its initial value, the same property does not hold for the optimal profit π^* that could take even negative values, as one can easily see from Equation (28). As previously mentioned, when using CARA preferences a non negative constraint on π should be explicitly imposed. This is the reason for CRRA preferences to prevail in the literature.

4 Conclusions

In this study we focused on the intertemporal optimal behavior of a firm that decides to locate part of its production in different (foreign) markets rather than concentrate all the production in its headquarters. We developed a model where a delocating firm meets stochastic foreign production costs since it faces uncertainty in the new destination markets. The domestic marginal costs are assumed to be constant. Firms are assumed to be risk averse and operate in a monopolistic competition setting. By exploiting portfolio choice techniques, the optimal behaviour of the firm (facing such a stochastic environment) is

to charge different mark-ups for different destinations. In the general case of hyperbolic absolute risk aversion preferences, the mark-up is increasing in the level of profits.

Because of the constant domestic marginal cost hypothesis, all the fixed costs for relocating the firm's plants are charged to the domestic mark-up. The firm's risk aversion makes the firm itself charge higher mark-ups in the less uncertain markets (and the domestic market is fully deterministic).

We also show that under a suitable condition on the firm's contribution margin (or total revenues) and total variable costs, a mark-up discrimination policy turns out to be an optimal strategy. Moreover, and in contrast with the deterministic results, in the case of mean reverting costs, a firm can get optimal positive profits even when charging a mark-up less than one (consequently the domestic mark-up will be higher).

This analysis can be extended in a number of directions. On one hand, it could be worth generalizing the setting to other forms of competition among firms as well as introducing a more general demand function. On the other hand, still under the assumption of monopolistic competition, it could be interesting to compare the location option (in the sense of foreign investment option) with the export one, but accounting for transport costs (and so, including a spatial dimension).

A Appendix

A.1 Proof of Proposition 1

The Hamiltonian of Problem (20) is²²

$$\begin{aligned} \mathcal{H} &= e^{-\rho(t-t_0)} (\gamma\pi - \alpha)^{1-\frac{\beta}{\gamma}} + J'_c \Omega \mu + J_\pi x' \Omega \mu \\ &\quad + \frac{1}{2} \text{tr} (\Omega \Sigma' \Sigma \Omega J_{cc}) + \frac{1}{2} J_{\pi\pi} x' \Omega \Sigma' \Sigma \Omega x + x' \Omega \Sigma' \Sigma \Omega J_{c\pi}, \end{aligned}$$

where $J(c, \pi, t)$ is the value function solving Problem (20). The subscripts on J mean partial derivatives.

The first order condition for x on \mathcal{H} gives the system

$$\begin{aligned} \frac{\partial \mathcal{H}}{\partial x} &= J_\pi \Omega \mu + J_{\pi\pi} \Omega \Sigma' \Sigma \Omega x + \Omega \Sigma' \Sigma \Omega J_{c\pi} = 0 \\ \Rightarrow x^* &= -\frac{J_\pi}{J_{\pi\pi}} \Omega^{-1} (\Sigma' \Sigma)^{-1} \mu - \frac{1}{J_{\pi\pi}} J_{c\pi}, \end{aligned} \tag{29}$$

and the second order condition holds if the matrix

$$\frac{\partial^2 \mathcal{H}}{\partial c' \partial c} = J_{\pi\pi} \Omega \Sigma' \Sigma \Omega,$$

²²Recall that Ω is symmetric (i.e. $\Omega = \Omega'$).

is negative definite. Since $\Omega\Sigma'\Sigma\Omega$ is a quadratic form and always positive semidefinite, then the second order condition holds if the value function J is concave in π . As the objective function is strictly concave in π , J will be too. This property will be evident in what follows.²³

After substituting the optimal value of x into the Hamiltonian, we obtain the following partial differential equation called Hamilton-Jacobi-Bellman equation (hereafter HJB):²⁴

$$\begin{aligned} 0 = & J_t + e^{-\rho(t-t_0)} (\gamma\pi - \alpha)^{1-\frac{\beta}{\gamma}} + J'_c \Omega \mu - \frac{J_\pi}{J_{\pi\pi}} \Omega \mu' J_{c\pi} \\ & - \frac{1}{2} \frac{J_\pi^2}{J_{\pi\pi}} \mu' (\Sigma'\Sigma)^{-1} \mu + \frac{1}{2} \text{tr} (\Omega \Sigma' \Sigma \Omega J_{cc}) - \frac{1}{2} \frac{1}{J_{\pi\pi}} J'_{c\pi} \Omega \Sigma' \Sigma \Omega J_{c\pi}, \end{aligned}$$

whose solution is the value function. Furthermore, the boundary (transversality) condition

$$\lim_{t \rightarrow \infty} J(t, \pi) = 0,$$

must hold.

The value function often inherits its functional form from the utility function. Thus, we now try the functional form

$$J(\pi, c, t) = e^{-\rho(t-t_0)} V(c, t) (\gamma\pi - \alpha)^{1-\frac{\beta}{\gamma}}, \quad (30)$$

where $V(c, t)$ is a function whose form must be determined. After replacing this expression into the HJB equation and carrying out a few algebraic simplifications, we obtain

$$\begin{aligned} 0 = & V_t + \frac{\gamma}{\beta} \mu' \Omega V_c + \frac{1}{2} \text{tr} (\Omega \Sigma' \Sigma \Omega V_{cc}) \\ & + \left(\frac{1}{2} \left(\frac{\gamma}{\beta} - 1 \right) \mu' (\Sigma'\Sigma)^{-1} \mu - \rho \right) V + 1 \\ & + \frac{1}{2} \left(\frac{\gamma}{\beta} - 1 \right) \frac{1}{V} V'_c \Omega \Sigma' \Sigma \Omega V_c. \end{aligned}$$

Recall that the scalar $\mu' (\Sigma'\Sigma)^{-1} \mu$ does not depend on c . Hence, the suitable function $V(c, t)$ solving this partial differential equation must be independent of c as well. In this case the matrices V_c and V_{cc} contain only zeros and $V(t)$ solves

$$0 = V_t + \left(\frac{1}{2} \left(\frac{\gamma}{\beta} - 1 \right) \mu' (\Sigma'\Sigma)^{-1} \mu - \rho \right) V + 1,$$

²³The reader is referred to Fleming and Soner (1993) for the assumptions that must hold on the function $U(\pi)$ for having a strictly concave value function.

²⁴For a complete exposition of the derivation of the Hamilton-Jacobi-Bellman equation the reader is referred to Duffie (1996), Björk (1998), and Øksendal (2000).

which must satisfy the boundary condition

$$\lim_{t \rightarrow \infty} V(t) = 0.$$

This ordinary differential equation has one and only one solution. Nevertheless, the form of this solution is not relevant for our purpose. Indeed, when the value function (30) is plugged into (29) we get ²⁵

$$x^* = \frac{\gamma\pi - \alpha}{\beta} \Omega^{-1} (\Sigma' \Sigma)^{-1} \mu, \quad (31)$$

where the function V does not play any role.

Remember that the form of x defined in (15) to obtain the optimal value of the mark-up (M^*) is given by

$$x^* = M^* q - q.$$

The matrix product $M^* q$ can be written as $m^* I_q$ where $m^* \in \mathbb{R}^n$ is a vector containing the optimal values of the mark-ups while $I_q \in \mathbb{R}^{n \times n}$ is a diagonal matrix containing the elements of vector q . Finally, the optimal value for the domestic mark-up can be obtained from (16).

A.2 Proof of Proposition 2

Inequality (23) implies

$$\mathbf{1} + \frac{\gamma\pi - \alpha}{\beta} I_q^{-1} \Omega^{-1} (\Sigma' \Sigma)^{-1} \mu > \mathbf{1} + \frac{1}{c_{HQH}} \left(\pi + F - \frac{\gamma\pi - \alpha}{\beta} c' \Omega^{-1} (\Sigma' \Sigma)^{-1} \mu \right) \mathbf{1},$$

which can be written as

$$\frac{\gamma\pi - \alpha}{\beta} \left(I_q^{-1} \Omega^{-1} (\Sigma' \Sigma)^{-1} \mu + \frac{1}{c_{HQH}} c' \Omega^{-1} (\Sigma' \Sigma)^{-1} \mu \mathbf{1} \right) > \frac{\pi + F}{c_{HQH}} \mathbf{1}.$$

Now we use the properties of the Kronecker product (\otimes)²⁶ in order to write

$$\frac{\gamma\pi - \alpha}{\beta} \left(I_q^{-1} + \frac{1}{c_{HQH}} (c' \otimes \mathbf{1}) \right) \Omega^{-1} (\Sigma' \Sigma)^{-1} \mu > \frac{\pi + F}{c_{HQH}} \mathbf{1},$$

²⁵ Recall that $\frac{\partial V}{\partial c} = \mathbf{0}$ and thus $\frac{\partial J}{\partial c} = \mathbf{0}$.

²⁶ Kronecker product (\otimes) has the following properties

$$\begin{aligned} A \otimes B &= C \equiv \begin{bmatrix} a_{11}B & a_{12}B & \dots & a_{1m}B \\ a_{21}B & a_{22}B & \dots & a_{2m}B \\ \dots & \dots & \dots & \dots \\ a_{n1}B & a_{n2}B & \dots & a_{nm}B \end{bmatrix}, \\ (A \otimes B)^{-1} &= A^{-1} \otimes B^{-1}, \quad (A \otimes B)' = A' \otimes B', \\ (A \otimes B)(C \otimes D) &= AC \otimes BD. \end{aligned}$$

and, finally

$$\frac{\gamma\pi - \alpha}{\beta} I_q^{-1} \Omega^{-1} (\Sigma' \Sigma)^{-1} \mu > (\pi + F) (c_{HQH} I + (c' \otimes \mathbf{1}) I_q)^{-1} \mathbf{1}.$$

It is easy to prove that the following equality holds

$$(c_{HQH} I + (c' \otimes \mathbf{1}) I_q)^{-1} \mathbf{1} = (c_{HQH} + c' q)^{-1} \mathbf{1}.$$

Now, since $c' \Omega^{-1} (\Sigma' \Sigma)^{-1} \mu$ is a scalar, we can write

$$c' \Omega^{-1} (\Sigma' \Sigma)^{-1} \mu \otimes \mathbf{1},$$

and

$$c' \Omega^{-1} (\Sigma' \Sigma)^{-1} \mu \otimes \mathbf{1} = (c' \otimes \mathbf{1}) \Omega^{-1} (\Sigma' \Sigma)^{-1} \mu.$$

References

- [1] Asplund M., 2002, Risk-averse firms in oligopoly, *International Journal of Industrial Organization*, vol.20, pp. 995-1012,
- [2] Banal-Estañol A., Ottaviani M., 2004, *Mergers with Product Market Risk*, mimeo,
- [3] Björk T.,1998, *Arbitrage Theory in Continuous Time*, Oxford University Press, New York,
- [4] Devereux M.P., Griffith R., 1998, Taxes and the location of production: evidence from a panel of US multinationals, *Journal of Public Economics*, vol. 68, pp. 335-367,
- [5] Disdier A.C., Mayer Th., 2004, How different is Eastern Europe ? Structure and determinants of location choices by French firms in Eastern and Western Europe, *Journal of Comparative Economics*, vol. 32, pp. 280-296,
- [6] Dixit A.K., Stiglitz J.E.,1977, Monopolistic Competition and Optimum Product Diversity, *American Economic Review*, 67(3), pp. 297-308,
- [7] Duffie D.,1996, *Dynamic Asset Pricing Theory* (Second edition), Princeton University Press,
- [8] Fujita M., Thisse J.F., 2003, *Globalization and the Evolution of the Supply Chain: who gains and who loses ?*, CORE DP 2003-74,
- [9] Fleming W. H., Soner H. M., 1993, *Controlled Markov Processes and Viscosity Solutions*. New York: Springer-Verlag,
- [10] Head K., Mayer Th., Ries J., 2002, Revisiting Oligopolistic Reaction: Are Decisions on Foreign Direct Investment Strategic Complements ?, *Journal of Economics & Management Strategy*, 11(3), pp. 453-472,
- [11] Hernandez T., Biasiotto M., 2001, Retail Location Decision-Making and Store Portfolio Management, *Canadian Journal of Regional Science/Revue canadienne des sciences régionales*, vol.XXIV,3, pp.399-418,
- [12] Karatzas I., Shreve E. S., 1991, *Brownian Motion and Stochastic Calculus*, (Second edition), Springer, New York,
- [13] Karatzas I., Shreve E. S., 1998, *Methods of Mathematical Finance*, Springer-Verlag, New York,
- [14] Keeley M. C., 1990, Deposit Insurance, Risk, and Market Power in Banking, *American Economic Review*, 80(5), pp. 1183-1200,

- [15] Kim D., Santomero, A. M., 1988, Risk in Banking and Capital Regulation, *Journal of Finance*, 43(5), pp. 1219-33,
- [16] Kloeden P. E., Platen E., 1992, *Numerical Solution of Stochastic Differential Equations*, Springer, Berlin,
- [17] Koehn M., Santomero A. M., 1980, Regulation of Bank Capital and Portfolio Risk, *Journal of Finance*, 35(5), pp. 1235-44,
- [18] Leland H.E., 1972, Theory of the Firm Facing Uncertain Demand, *American Economic Review*, 62(3), pp. 278-291,
- [19] Löfgren K.-G., Ranneby, B., 1987, Behavioural Modes for a Firm Facing an Uncertain Supply and Demand Curve, *Scandinavian Journal of Economics*, 89(1), pp. 39-54,
- [20] Merton R. C., 1990, *Continuous-time finance*. Blackwell, Cambridge, MA.
- [21] Øksendal B., 2000, *Stochastic Differential Equations - An Introduction with Applications*, Fifth edition, Springer, Berlin,
- [22] Sandmo A., 1971, On the Theory of the Competitive Firms Under Price Uncertainty, *American Economic Review*, 61(1), pp. 65-73.