Individual versus group strategy-proofness: when do they coincide?*

Salvador Barberà† Dolors Berga‡ Bernardo Moreno§

February 4, 2009

Abstract: A social choice function is group strategy-proof on a domain if no group of agents can manipulate its final outcome to their own benefit by declaring false preferences on that domain. Group strategy-proofness is a very attractive requirement of incentive compatibility. But in many cases it is hard or impossible to find nontrivial social choice functions satisfying even the weakest condition of individual strategy-proofness. However, there are a number of economically significant domains where interesting rules satisfying individual strategy-proofness can be defined, and for some of them, all these rules turn out to also satisfy the stronger requirement of group strategy-proofness. This is the case, for example, when preferences are single-peaked or single-dipped. In other cases, this equivalence does not hold. We provide sufficient conditions defining domains of preferences guaranteeing that individual and group strategy-proofness are equivalent for all rules defined on these domains. Our results extend to intermediate versions of strategy-proofness, defined to exclude manipulations by small group of agents. They also provide guidelines on how to restrict the ranges of functions defined on domains that only satisfy our condition partially. Finally, we provide a partial answer regarding the necessity of our conditions.

JEL Classification Number: D71.
Keywords: Strategy-proofness, Group strategy-proofness, k-size strategy-proofness, Sequential inclusion, Single-peaked preferences, Single-dipped preferences, Separable preferences.

*Salvador Barberà acknowledges the support of the Barcelona Graduate School of Economics where he is an affiliated professor, through its Research Recognition Programme. His work is also supported through grants SEJ2006-2789-E and ECO2008-04756 (Grupo Consolidado-C) from the Spanish Ministry of Science and Innovation, and by the Distinció per la Promoció de la Recerca Universitària from the Generalitat de Catalunya. Dolors Berga acknowledges the support of the Spanish Ministry of Science and Innovation through grant SEJ2007-60671 and of Generalitat de Catalunya through grant 2005SGR-213. She also acknowledges the Research Recognition Programme of the Barcelona GSE. Bernardo Moreno acknowledges financial support from Junta de Andalucía through Proyecto de Excelencia 522 and the ECO2008-03674 from the Spanish Ministry of Science and Innovation.

†*Corresponding author*: Departament d'Economia i d'Història Econòmica, Universitat Autònoma de Barcelona, Edifici B, 08193 Bellaterra, Spain. E-mail: salvador.barbera@uab.cat

‡Departament d'Economia, Campus de Montilivi, Universitat de Girona, 17071 Girona, Spain. E-mail: dolors.berga@udg.edu.

§Departamento de Teoría e Historia Económica, Facultad de Ciencias Económicas y Empresariales, Campus de El Ejido, 29071 Málaga, Spain. E-mail: bernardo@uma.es

1 Introduction

Strategy-proofness is a demanding condition that most mechanisms will fail to satisfy, unless they are defined on properly restricted environments. Group strategy-proofness is a more stringent requirement, but also a much more attractive one. Indeed, what use would it be to guarantee that no single individual could cheat, if a handful of them could jointly manipulate the system? Yet, the literature on mechanism design has concentrated mostly in analyzing the weakest of these two conditions: after all, it is hard enough to meet, and often impossible except for trivial procedures.

We are now well beyond the disquieting negative message of the Gibbard-Satterthwaite theorem (Gibbard, 1973 and Satterthwaite, 1975). After years of research, we know of a considerable number of instances where non-trivial mechanisms can be found to be strategy-proof when defined on some domains of interest. They include the case of voting when preferences are single-peaked or separable, or when the outcomes are lotteries; they also include families of cost sharing, matching and allocation procedures, again for the case where the preferences of agents are conveniently restricted. When looking at the rich literature on strategy-proof rules over restricted domains, one discovers, somewhat surprisingly, that some of the non-trivial strategy-proof mechanisms that arise in these environments are, indeed, also group strategy-proof! It is as if, after a hard search for a solution to the challenge of strategy-proofness, the additional blessing of group strategy-proofness would arise automatically, as an extra gift. Of course, this does not happen in all cases, but it occurs often enough as to warrant the following question. When is it that, once a strategy-proof social choice function is defined, it is necessarily also group strategy-proof? More specifically, what are the domains of preferences under which these two conditions collapse into one and the same requirement?

We provide a rather complete solution to that question in voting models, by providing conditions on families of preference profiles such that any strategy-proof social choice function whose domain is one of these families will also be necessarily group strategy-proof. Of course, there could exist rules defined on domains that do not meet our conditions, that are still strategy-proof but not group strategy-proof.

How restrictive are our conditions, and what ground do we cover? In some special situations group strategy-proofness follows from individual strategy-proofness without any further restrictions. These include the cases where there are only two or three social alternatives and the trivial domains where only one agent has more than one possible preferences.

For general cases, we provide essentially two conditions for domains to imply the equivalence between our two incentive compatibility requirements. The first one, called sequential inclusion, is satisfied by very classical domains, including the set of all preferences that are single-peaked (or single-dipped), for a given order of the alternatives. It is important to remark that sequential inclusion is a condition that applies to each preference profile individually, rather than a requirement on the family of profiles that are included in a domain. Because of that, subsets of profiles satisfying sequential inclusion do also meet the condition. This allows our results to hold not only for sufficiently large sets of profiles, but also for all of their subsets. Notice that, as we shrink a given domain to a smaller one, containing only part of the original profiles, new rules may start satisfying the requirement of strategy-proofness. Our condition guarantees that these emerging rules will also be group strategy-proof.

Our second condition holds for larger domains where not all individual profiles satisfy sequential inclusion, but where an indirect version of this condition holds. One of these is the universal domain, but we have other examples. Again this condition is proven to be sufficient for the equivalence between group and individual strategy-proofness, but it is no longer the case that the result applies to all subdomains.

Filling the gap between individual and group strategy-proofness is important, because it would be hard to avoid manipulations if groups of agents could distort the rules to their advantage, even if no single individual was able to. For this same reason, it is also interesting to examine those cases where groups could only manipulate if they were large enough. In these second best worlds, one could still hope that coordination costs and other restrictions might avoid actual manipulation. In Section 5 we discuss an adaptation of our conditions guaranteeing that, when strategy-proof rules exist on given domains, they are also immune to manipulations by groups smaller than a given size.

Similarly, we know that on any given domain of preferences, individual and group strategy-proofness can be obtained at the cost of reducing the range of social choices to only include two alternatives. And we also prove in this paper that both conditions become equivalent if we restrict the range of any function to only have three elements. In Section 6 we discuss how our conditions may guide the choice of larger but still restricted ranges for social choice functions, in order to maintain this equivalence.

The importance of considering deviations by coalitions, and not only by individuals was already remarked in the early literature on strategy-proofness and implementation. See, for example, the early works of Pattanaik (1978), the seminal contributions of Green and Laffont (1979) and of Dasgupta, Hammond, and Maskin (1979, especially section 4.5) or the short note by Barberà (1979). All of these papers refer to the characteristics of functions which may be immune to manipulation by coalitions. They either explore conditions for this property to hold in some cases, or they exhibit by means of examples the difficulties involved in their design.

The question we address is related, but different. We investigate conditions on domains of preferences under which individual and group strategy-proofness become equivalent. This new approach is a natural consequence of the many remarks by different authors on the coincidence between the two conditions in different settings (Barberà and Jackson, 1995, Moulin, 1999 and Pápai, 2000; see also Peleg, 1998 and Peleg and Sudholter, 1999 for the connections with the related concept of coalition-proofness). A systematic analysis of this question has only one antecedent we know of, a recent paper by Le Breton and Zaporozhets (2008) where they prove the equivalence in the case of rich domains. Our results cover several domains that are rich but also many others that do not satisfy this requirement.[1] This will become apparent in the text. In addition, we also study the connection between strategy-proofness and vulnerability to manipulation by small groups of individuals, a phenomenon whose importance has been remarked by several authors (Pattanaik, 1978, Barberà, 1979,

[1] Richness is a property that appears under different versions in many papers since Dasgupta, Hammond, and Maskin used it extensively in their 1979 paper. Its spirit is to guarantee that domains contain "enough" preferences, connected in appropriate ways, as to be able to establish links between several profiles in the domain. It is an "interprofile" condition. Our main conditions (sequential inclusion and k-size sequential inclusion) can be defined in reference to only one profile at a time: they are "intraprofile" conditions.

Barberà, Sonnenschein, and Zhou, 1991, and Serizawa, 2006). To our knowledge, these connections had never been explored systematically in the terms that we use here.

The paper is organized as follows. Section 2 contains the model, the definition of our basic condition of sequential inclusion and the proof that, in domains where this condition is satisfied, individual and group strategy-proofness become equivalent. Section 3 provides examples of domains that satisfy sequential inclusion. Section 4 introduces the weaker condition of indirect sequential inclusion, and proves it to still be sufficient to guarantee the equivalence of our two incentive compatibility requirements. Examples of domains where this new condition holds are also provided. In Section 5 we analyze a natural weakening of group strategy-proofness, one that precludes manipulation by "small enough" groups of agents. We then prove that the newly defined notion of k-group strategy-proofness is equivalent to individual strategy-proofness under a convenient weakening of the original notion of sequential inclusion. Section 6 considers the possibility of attaining equivalence by restricting the range of our social choice functions. Section 7 provides a partial result on the necessity of sequential inclusion to achieve equivalence. Specifically, we can prove that, if the condition is violated by some domain of preferences, there are subdomains of the initial one where some strategy-proof rules are no longer group strategy-proof. Section 8 concludes with some directions for further work. Finally, an Appendix collects some proofs.

2 The model and the basic result

Let A be the set of *alternatives* and $N = \{1, 2, ..., n\}$ be the set of *agents* (with $n \geqslant 2$). Let capital letters $S, T \subset N$ denote subsets of agents while lower case letters s, t denote their cardinality.

Let $\mathcal{R}$ be the set of complete, reflexive, and transitive orderings on A and $\mathcal{R}_i \subset \mathcal{R}$ be *the set of admissible preferences for agent* $i \in N$. A *preference profile,* denoted by $R_N = (R_1, .., R_n)$, is an element of $\times_{i\in N}\mathcal{R}_i$. Preferences will be denoted in two possible ways. When they are part of a given preference profile, we'll use the notation $R_i \in \mathcal{R}_i$ to denote that these are the preferences of agent i in the profile. In other cases, we want to have a name for a given preference, and our notation must be independent of who holds the preference in question at what profile. Then we will use superscripts, and let R^l stand for a specific relation. We may later on attribute that preference to some agents, and then we will write that $R^l = R_i$ or equivalently, R^l_i.[2] As usual, we denote by P_i and I_i the strict and the indifference part of R_i, respectively. Let $C, S \subset N$ be two coalitions such that $C \subset S$. We will write the subprofile $R_S = (R_C, R_{S\backslash C}) \in \times_{i\in S}\mathcal{R}_i$ when we want to stress the role of coalition C in S. Then the subprofiles $R_C \in \times_{i\in C}\mathcal{R}_i$ and $R_{S\backslash C} \in \times_{i\in S\backslash C}\mathcal{R}_i$ denote the preferences of agents in C and in $S\backslash C$, respectively. In the case, where we denote full preference profile (that is, when $S = N$), we simplify notation by using $(R_C, R_{N\backslash C})$ as (R_C, R_{-C}).

The following concept is crucial in all our analysis. For any $x \in A$ and $R_i \in \mathcal{R}_i$, define *the lower contour set of* R_i *at* x as $L(R_i, x) = \{y \in A : xR_iy\}$. Similarly, the strict lower contour set at x is $\overline{L}(R_i, x) = \{y \in A : xP_iy\}$.

[2]The latter notation will be used extensively in the Appendix, proof of Theorem 4.

A *social choice function* (or a rule) is a function $f : \times_{i \in N} \mathcal{R}_i \to A$. Let A_f denote the range of the social choice function f.

We will focus on rules that are nonmanipulable, either by a single agent or by a coalition of agents. We first define what we mean by a manipulation and then we introduce the well known concepts of *strategy-proofness* and *group strategy-proofness.*

Definition 1 *A social choice function f is group manipulable on $\times_{i \in N} \mathcal{R}_i$ at $R_N \in \times_{i \in N} \mathcal{R}_i$ if there exists a coalition $C \subset N$ and $R'_C \in \times_{i \in C} \mathcal{R}_i$ ($R'_i \neq R_i$ for any $i \in C$) such that $f(R'_C, R_{-C}) P_i f(R_N)$ for all $i \in C$. We say that f is individually manipulable if there exists a possible manipulation where coalition C is a singleton.*

Definition 2 *A social choice function f is group strategy-proof on $\times_{i \in N} \mathcal{R}_i$ if f is not group manipulable for any $R_N \in \times_{i \in N} \mathcal{R}_i$. Similarly, f is strategy-proof if it is not individually manipulable.*

The reader will notice that our definition requires that all agents in a coalition that manipulates should obtain a strictly positive benefit from doing so. We consider this requirement compelling, since it leaves no doubt regarding the incentives for each member of the coalition to participate in a collective deviation from truthful revelation.

Notice that the domains of our social choice functions will always have the form of a cartesian product. This is necessary to give meaning to our definitions of individual and group strategy-proofness. Also notice that, although the notion of a domain is attached to that of a given function, we shall also refer to any cartesian family of preference profiles as a domain, and to its cartesian subsets as its subdomains. This is consistent with tradition, although it would be more precise to call them potential domains, as we shall in fact consider sets of functions that could be defined on them.

We shall not dwell on the importance of achieving either of these two incentive compatibility conditions. Let us just insist, as we already did in the introduction, that even the weakest one of individually strategy-proofness is hard to meet, but that both have been shown to be achievable under a variety of domain restrictions. Our focus will be on these specific cases where it is possible to at least define satisfactory strategy-proof rules. What is needed then to hope for the stronger and much more reassuring property of group strategy-proofness to also hold? This is the main question we address.

Specifically, in this section we define our first condition on preference profiles, called *sequential inclusion*, and we establish the equivalence between individual and group strategy-proofness for social choice functions defined on domains satisfying that condition. Before that, let us introduce some notation that will be important and will often appear along the paper.

Let $R_N \in \times_{i \in N} \mathcal{R}_i$, and y, z be a pair of alternatives. Denote by $S(R_N; y, z) \equiv \{i \in N : yP_i z\}$, that is, the set of agents who strictly prefer y to z according to their individual preferences in R_N.

Definition 3 *Given a preference profile* $R_N \in \times_{i\in N}\mathcal{R}_i$ *and a pair of alternatives* $y, z \in A$, *we define a binary relation* $\succsim (R_N; y, z)$ *on* $S(R_N; y, z)$ *as follows:*[3]

$$i \succsim (R_N; y, z) j \text{ if } L(R_i, z) \subset \overline{L}(R_j, y).$$

Note that the binary relation $\succsim$ must be reflexive but not necessarily complete. As usual, we can define the strict and the indifference binary relations associated to $\succsim$. Formally, $i \sim j$ if $L(R_i, z) \subset \overline{L}(R_j, y)$ and $L(R_j, z) \subset \overline{L}(R_i, y)$. We say that $i \succ j$ if $L(R_i, z) \subset \overline{L}(R_j, y)$ and $\neg[L(R_j, z) \subset \overline{L}(R_i, y)]$.

We now suggest an interpretation of the relation in terms that will be useful for the proof of our main result. As we shall see, our reasoning will involve orderings of individuals, and the binary relation we just defined will be closely connected to the possibility or the impossibility of constructing such orderings. Here is how we should interpret it. If $i \succsim j$, we'll say that i may precede j. If $i \succ j$, then we'll say that i must precede j.

We can now define our main condition.

Definition 4 *A preference profile* $R_N \in \times_{i\in N}\mathcal{R}_i$ *satisfies sequential inclusion for* $y, z \in A$ *if the binary relation* $\succsim (R_N; y, z)$ *on* $S(R_N; y, z)$ *is complete and acyclic.*

Definition 5 *A preference profile* $R_N \in \times_{i\in N}\mathcal{R}_i$ *satisfies sequential inclusion if for any pair* $y, z \in A$ *the binary relation* $\succsim (R_N; y, z)$ *on* $S(R_N; y, z)$ *is complete and acyclic. A domain* $\times_{i\in N}\mathcal{R}_i$ *satisfies sequential inclusion if any preference profile in this domain satisfies it.*

Since sequential inclusion is a property on preference profiles, it follows that if a domain satisfies sequential inclusion each subdomain inherits the same property. This is interesting to note: our condition does not require domains to be large in the size contrary with other conditions, like that of "richness" (see Dasgupta, Hammond, and Maskin, 1979, Le Breton and Zaporozhets, 2008).

To illustrate sequential inclusion consider the following examples.

Example 1 *Let* $N = \{1, 2\}$, $A = \{x, y, z, w\}$, *and* $\mathcal{R}_1, \mathcal{R}_2 \supset \{R, R', \widetilde{R}\}$, *where* $xPyPzPw$, $xP'yP'wP'z$, *and* $w\widetilde{P}y\widetilde{P}z\widetilde{P}x$ *as represented in the following table*[4]*:*

R	R'	$\widetilde{R}$
x	x	w
$\boldsymbol{y}$	$\boldsymbol{y}$	$\boldsymbol{y}$
$\boldsymbol{z}$	w	$\boldsymbol{z}$
w	$\boldsymbol{z}$	x

Let $R_N = (R, R')$ *and take* $y, z \in A$. *By definition,* $S(R_N; y, z) = \{1, 2\}$. *Observe also that* $L(R, z) \subset \overline{L}(R', y)$ *and* $L(R', z) \subset \overline{L}(R, y)$, *that is,* $1 \sim (R_N; y, z) 2$. *Thus,* $\succsim (R_N; y, z)$ *is complete (trivially acyclic since there are only two agents). The same argument works for*

[3] In what follows, and when this does not induce to error, we may omit the arguments R_N, y and z and just write $\succsim$.

[4] From now on, we will use in some examples the same table representation for individual preferences.

any pair of alternatives and thus we can conclude that R_N satisfies sequential inclusion.
Let $R_N = (R, \widetilde{R})$ and take $y, z \in A$. By definition, $S(R_N; y, z) = \{1, 2\}$. Observe that neither $L(R, z) \subset \overline{L}(\widetilde{R}, y)$ nor $L(\widetilde{R}, z) \subset \overline{L}(R, y)$ holds. Then $\succsim (R_N; y, z)$ is not complete which means that R_N violates sequential inclusion.

Example 2 *Let $N = \{1, 2, 3\}$, $A = \{x, y, z, w, t\}$, and $\mathcal{R}_1, \mathcal{R}_2 \supset \{R, R', \widehat{R}\}$ such that the binary relations stated in the following table hold:*

R	R'	$\widehat{R}$
x	w	t
$\boldsymbol{y}$	$\boldsymbol{y}$	$\boldsymbol{y}$
w	t	x
$\boldsymbol{z}$	$\boldsymbol{z}$	$\boldsymbol{z}$
t	x	w

.

Let $R_N = (R, R', \widehat{R})$, y, z: $S(R_N; y, z) = \{1, 2, 3\}$. Observe that $L(R, z) \subset \overline{L}(R', y)$ but $\neg \left[L(R', z) \subset \overline{L}(R, y) \right]$, then $1 \succ 2$. Similarly, $2 \succ 3$ and $3 \succ 1$. Thus, $\succsim (R_N; y, z)$ is complete but it has a cycle: R_N violates sequential inclusion.

Remark 1 *Notice that there are two distinct ways to violate sequential inclusion: by lack of completeness and because of cycles. Both aspects of the definition are essential in what follows, but could be factored out for other purposes, as their implications are different.*[5]

We can now present our main result.

Theorem 1 *Let $\times_{i \in N} \mathcal{R}_i$ be a domain satisfying sequential inclusion. Then, any strategy-proof social choice function on $\times_{i \in N} \mathcal{R}_i$ is group strategy-proof.*

Proof. We will use the following fact several times: for any reflexive, complete and acyclic binary relation T on a finite set B, the set $C(B, T) = \{x \in B : xTy \text{ for any } y \in B\}$ of best elements of T on B is non-empty.
Let f be a strategy-proof social choice function on $\times_{i \in N} \mathcal{R}_i$ satisfying sequential inclusion. By contradiction, suppose that f is manipulable by some coalition $C \subset N$. That is, there exists a coalition C, $R_N \in \times_{i \in N} \mathcal{R}_i$, and $\widetilde{R}_C \in \times_{i \in C} \mathcal{R}_i$, such that for any agent $i \in C$, $f(\widetilde{R}_C, R_{-C}) P_i f(R_N)$. Let $y = f(\widetilde{R}_C, R_{-C})$ and $z = f(R_N)$. Note that $C \subseteq S(R_N; y, z)$. Now we will go from R_N to $(\widetilde{R}_C, R_{-C})$ by steps, replacing each time the preference of one agent in C. The order in C will be determined by sequential inclusion.
By sequential inclusion, $\succsim (R_N; y, z)$ is complete and acyclic on $S(R_N; y, z)$, thus, complete and acyclic on C. Choose an agent in $C(C; \succsim (R_N; y, z))$, without loss of generality, say it is agent 1. Thus, $L(R_1, z) \subset \overline{L}(R_h, y)$ for $h \in C \backslash \{1\}$. Change preferences of agent 1 and let $z_1 = f(\widetilde{R}_1, R_{-\{1\}})$. By strategy-proofness, $z_1 \in L(R_1, z)$ and thus $z_1 \in \overline{L}(R_h, y)$ for any $h \in C \backslash \{1\}$. Observe that $S((\widetilde{R}_1, R_{-\{1\}}); y, z_1) \supset C \backslash \{1\}$.

[5] We use this factorization in the proof of Theorem 4 sketched in Section 7.

By sequential inclusion, $\succsim(R_N; y, z_1)$ is complete and acyclic on $S(R_N; y, z_1)$, thus, complete and acyclic on $C\backslash\{1\}$. Choose an agent in $C(C\backslash\{1\}; \succsim(R_N; y, z_1))$, without loss of generality, say it is agent 2. Thus, $L(R_2, z_1) \subset \overline{L}(R_h, y)$ for $h \in C\backslash\{1,2\}$. Change preferences of agent 2 and let $z_2 = f(\widetilde{R}_{\{1,2\}}, R_{-\{1,2\}})$. By strategy-proofness, $z_2 \in L(R_2, z_1)$ and thus $z_2 \in \overline{L}(R_h, y)$ for any $h \in C\backslash\{1,2\}$. Observe that $S((\widetilde{R}_{\{1,2\}}, R_{-\{1,2\}}); y, z_2) \supset C\backslash\{1,2\}$.
Repeat the same argument changing from R_i to $\widetilde{R}_i$ the preferences of all agents in C. Suppose that we have repeated this process $c-2$ times, that is, without loss of generality, we have changed preferences of agent $c-2$.
By sequential inclusion, $\succsim(R_N; y, z_{c-2})$ is complete and acyclic on $S(R_N; y, z_{c-2})$, thus, complete and acyclic on $C\backslash\{1, ..., c-2\}$. Choose an agent in $C(C\backslash\{1, ..., c-2\}; \succsim(R_N; y, z_{c-2}))$, without loss of generality, say it is agent $c-1$. Thus, $L(R_{c-1}, z_{c-2}) \subset \overline{L}(R_h, y)$ for $h \in C\backslash\{1, ..., c-1\} \equiv c$. Change preferences of agent $c-1$ and let $z_{c-1} = f(\widetilde{R}_{\{1,...,c-1\}}, R_{-\{1,...,c-1\}})$. By strategy-proofness, $z_{c-1} \in L(R_{c-1}, z_{c-2})$ and thus $z_{c-1} \in \overline{L}(R_c, y)$.
Finally, change preferences of agent c and let $z_c = f(\widetilde{R}_C, R_{-C})$. Note that $z_c = y$. By strategy-proofness, $z_c = y \in L(R_c, z_{c-1})$ which is the desired contradiction. ■

3 Examples and special cases

We provide here some examples of domains in which sequential inclusion holds and of others where it does not.

3.1 Examples

Single-peaked preferences

Single-peakedness arises as a natural restriction on the preferences of agents facing many relevant problems: determining the level of a pure public good without transfers, locating a facility on a line, deciding on a tax level, choosing among candidates, among others.

Definition 6 *A preference profile R_N is single-peaked iff there exists a linear order $>$ of the set of alternatives such that*
(1) Each of the voters' preferences has a unique maximal element $p_i(A)$, called the peak of i, and
(2) For all $i \in N$, for all $p_i(A)$, and for all $y, z \in A$

$$[z < y \leqslant p_i(A) \text{ or } z > y \geqslant p_i(A)] \rightarrow \quad yP_iz.$$

Proposition 1 *Any single-peaked profile R_N satisfies sequential inclusion.*

Proof. Take any single-peaked profile R_N and any pair of alternatives $y, z \in A$ such that, without loss of generality, $y < z$. Let us show that $\succsim(R_N; y, z)$ on $S(R_N; y, z)$ is complete and acyclic. Remark that for any pair of agents i, j such that yP_iz and yP_jz, then, either $L(R_i, z) \subset L(R_j, z)$ or $L(R_j, z) \subset L(R_i, z)$. To see this, notice that both lower contour sets are unions of two sets (the set containing alternatives to the right side of z and the set of

alternatives to the left side of y). One of the two sets (the one to the right side of z) is common to all individual preferences. The other sets are specific to each z and they are necessarily contained in each other.
By this remark, agents in $S(R_N; y, z)$ can always be ordered according to the increasing order of lower contour sets at z. Without loss of generality, say $L(R_1, z) \subset ... \subset L(R_s, z)$. Since yP_iz for any $i \in S(R_N; y, z)$, $1 \succsim 2, 3, ..., s$; $2 \succsim 3, ..., s$; ...; and $s - 1 \succsim s$. This shows that $\succsim (R_N; y, z)$ is complete and acyclic (there is no cycle since any agent may precede any other following him given the order stated by the increasing order of lower contour sets at z). ■

In Figure 1 below, we depict three preferences of a single-peaked profile corresponding to agents who prefer y to z and where A is a closed interval in the real line. The picture may help with the general argument in the proof.

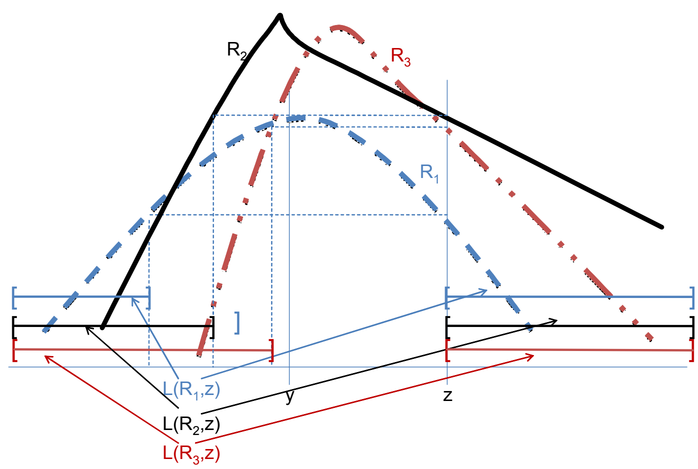

Figure 1. Illustration that any single-peaked preference profile satisfies sequential inclusion.

Single-dipped preferences

We consider now the domain of single dipped preferences. They allow us to analyze cases where distance to a reference "worse" point is preferred, as it is the case when one must allocate a public bad. This is in contrast with the opposite motivation of single peaked preferences, where being closer to the reference "best" point is preferred.

Definition 7 *A preference profile R_N is single-dipped iff there exists a linear order $>$ of the set of alternatives such that*
(1) Each of the voters' preferences has a unique minimal element $d_i(A)$, called the dip of i, and
(2) For all $i \in N$, for all $d_i(A)$, and for all $y, z \in A$

$$[z < y \leqslant d_i(A) \text{ or } z > y \geqslant d_i(A)] \rightarrow \quad zP_iy.$$

Proposition 2 *Any single-dipped profile R_N satisfies sequential inclusion.*

Proof. Take any single-dipped profile R_N and any pair of alternatives $y, z \in A$ such that, without loss of generality, $y < z$. Let us show that $\succsim (R_N; y, z)$ on $S(R_N; y, z)$ is complete and acyclic. We can always consider a partition of $S(R_N; y, z) = S_1(R_N; y, z) \cup S_2(R_N; y, z)$ such that $S_1(R_N; y, z) = \{i \in S(R_N; y, z)$ such that $d_i(A) \leq z\}$ and $S_2((R_N; y, z)) = \{i \in S(R_N; y, z)$ such that $d_i(A) > z\}$. Note that agents in $S_1(R_N; y, z)$ are such that $L(R_i, z) \subset \overline{L}(R_l, y)$ for any $i, l \in S_1(R_N; y, z)$. Thus, for any $i, l \in S_1(R_N; y, z)$, $i \sim (R_N; y, z) l$. Observe also that for any pair of agents $i, j \in S_2(R_N; y, z)$, then either $L(R_i, z) \subset L(R_j, z)$ or $L(R_j, z) \subset L(R_i, z)$. To see this, notice that the lower contour sets of agents with dip above z are specific to each z and they are necessarily contained in each other. Thus, we can always order agents in $S_2(R_N; y, z)$ according to the increasing order of lower contour sets at z. Without loss of generality, say $L(R_{s_1+1}, z) \subset ... \subset L(R_{s_1+s_2}, z)$. Thus, $s_1 + 1 \succsim s_1 + 2, ..., s_1 + s_2$; $s_1 + 2 \succsim s_1 + 3, ..., s_1 + s_2$; ...; and $s_2 - 1 \succsim s_2$. Finally, observe that $L(R_i, z) \subset \overline{L}(R_j, y)$ for any $i \in S_1(R_N; y, z)$ and $j \in S_2(R_N; y, z)$ (since $y < z$ and $d_j(A) > z$ for all $j \in S_2(R_N; y, z)$). Thus, $i \succsim (R_N; y, z) j$ for any $i \in S_1(R_N; y, z)$ and $j \in S_2(R_N; y, z)$. This shows that $\succsim (R_N; y, z)$ is complete and acyclic (first note that any possible cycle should contain agents in S_2 since agents in S_1 are all indifferent. However, among agents in S_2 there is no cycle given the order stated by the increasing order of lower contour sets at z). ■

In Figure 2 below, we depict four preferences of a single-dipped profile corresponding to agents who prefer y to z and where A is a closed interval in the real line. The example may help with the general argument in the proof.

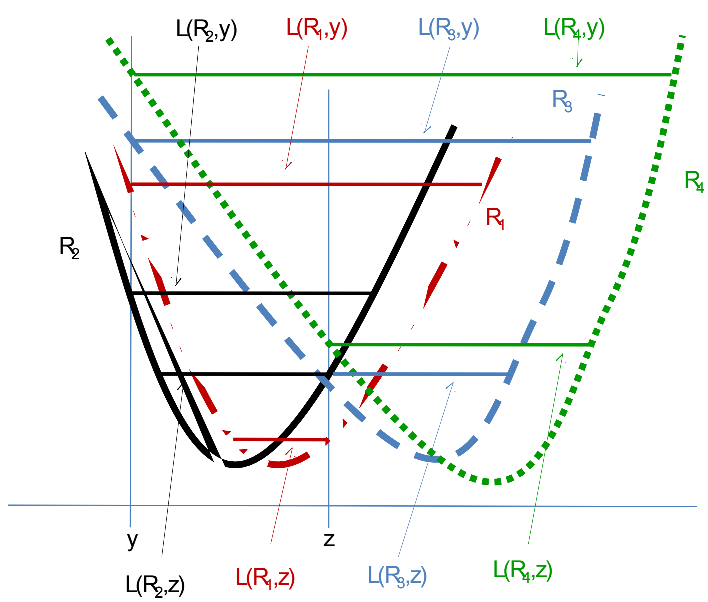

Figure 2. Illustration that any single-dipped preference profile satisfies sequential inclusion.

Separable preferences and its subdomains

The domain of separable preferences, which we describe below, provides an example of a domain admitting strategy-proof rules that are not group strategy-proof. Yet, we will also define a subdomain of these preferences where the equivalence still holds.

Alternatives are vectors of length k, with a zero or a one in each component. They can be interpreted in a variety of manners. They can stand for a characteristic function describing subsets of a basic set of k alternatives. They can also be understood as representing collections of decisions on whether or not to accept each of k independent projects. They can also be interpreted as descriptions of alternatives that may or may not have some of k characteristics, each one associated with one of the dimensions.

Separability is a restriction on the class of linear orders on the family of vectors in $\{0,1\}^k$. Let v denote a generic vector in $\{0,1\}^k$, and v_l be its $l-$component for any $l = 1, ..., k$. Given any linear order P on that set, let $B(P)$ denote the set of unit vectors (vectors with only a 1) that are worse than the zero vector, according to P, and let $G(P)$ denote the set of unit vectors that are better than the zero vector. Denote by 1_h the unit vector with a one in dimension h, and by V_h^0 (respectively, V_h^1) the set of vectors that have a zero (respectively, a 1) in component h. Then, P is separable if, for all h, and any $v \in V_h^0$,

$$\begin{array}{ll} (v + 1_h)\, Pv & \text{if } 1_h \in G(P) \text{, and} \\ vP\,(v + 1_h) & \text{if } 1_h \in B(P). \end{array}$$

Barberà, Sonnenschein, and Zhou (1991) have characterized the family of social choice functions on the domain $\mathcal{S}_k$ of separable preferences on $\{0,1\}^k$ that are strategy-proof.

An example of these rules, which are in addition neutral and anonymous, is given by quota rules. When society consists of n agents, a quota is a number between 0 an n. Then, given the preferences of the agents, the rule with quota q chooses a vector that has a 1 in position h if 1_h belongs to at least q of the sets $G(P_i)$ of agents $i \in N$, and has a 0 in position h, otherwise.

Consider the case $k = 2$, $n = 2$, $q = 1$. The following preferences are separable.

P_1	P_2
$(1,0)$	$(0,1)$
$(0,0)$	$(0,0)$
$(1,1)$	$(1,1)$
$(0,1)$	$(1,0)$

Under the quota 1 rule, the outcome is $(1,1)$, and no individual can manipulate. Yet, both agents would prefer $(0,0)$, and they can obtain this result if they both declare $(0,0)$ to be their best alternative. Observe that this preference profile does not satisfy sequential inclusion.

This example extends to larger $k's$, $n's$ and different $q's$. It not only proves that one can have strategy-proof rules that are not group strategy-proof on that domain, but it also suggests that the rules may be extremely fragile, as they can be manipulated by groups composed of two agents alone. We return to that subject in Section 5.

We now provide a new example of a domain where all strategy-proof rules are group strategy-proof: the class of separable preferences on $\{0,1\}^k$ with a common lexicographic order $\sigma(k)$ on the dimensions. This domain, that we denote by $\mathcal{L}_k^{\sigma(k)}$, is defined as follows. Preferences are still separable. In addition, there is an ordering $\sigma(k)$ of the dimensions (without loss of generality, we take it to be the natural order $1,...,n$) such that preferences satisfy the following condition:

$$vPv' \text{ if } \exists h \in N \text{ such that } v_l = v'_l \text{ for } l < h, \text{ and}$$
$$v \in V_h^0,\ v' \in V_h^1, \text{ and } 1_h \in B(P), \text{ or else } v \in V_h^1,\ v' \in V_h^0, \text{ and } 1_h \in G(P).$$

We provide an argument to show that for the case $k = 2$, any profile of preferences that are separable and lexicographic relative to a common order of the dimensions satisfies sequential inclusion. Our argument runs as follows. Fix the order 1<2 on both dimensions. Note that the domain of all individual preferences $\mathcal{L}_2^{1<2}$ is the following:

P	P'	P''	P'''
$(0,0)$	$(1,0)$	$(0,1)$	$(1,1)$
$(0,1)$	$(1,1)$	$(0,0)$	$(1,0)$
$(1,0)$	$(0,0)$	$(1,1)$	$(0,1)$
$(1,1)$	$(0,1)$	$(1,0)$	$(0,0)$

Observe that for any pair of alternatives y, $z \in A$ there only exist two preferences in $\mathcal{L}_2^{1<2}$ such that yPz. Thus, for any preference profile R_N, $S(R_N; y, z)$ will always consist of, at most, two agents. Consider the following example where $y = (0,0)$, $z = (0,1)$, and take R_N such that it contains agents of the two different types, say 1 and 2, with $R_1 = P'$ and $R_2 = P$. Without loss of generality, there is one agent of each type, that is, $S(R_N; y, z) = \{1,2\}$. Remark that $L(R_1, z) = \{(0,1)\} \subseteq L(R_2, z) = \{(0,1),(1,0),(1,1)\}$. We will choose the order of agents $1 < 2$ suggested by this inclusion (again, this remark about inclusion and its construction generalizes to all such preference profiles). Observe that given such order of agents, $L(R_1, z) \subset \overline{L}(R_h, y)$ for any h, $h > 1$, and therefore sequential inclusion holds.

Notice that the profile we have used in our example showing that a separable domain may admit strategy-proof rules that are not group strategy-proof does not satisfy our new requirement. For instance, given that (1,0) is best for agent 1, and that (0,1) is best for agent 2, (1,1) should be better than (0,0) for both.

3.2 Some special cases

As a corollary of Theorem 1, we obtain some implications of our equivalence result related with the set of alternatives and with the set of admissible preferences, respectively.

The number of alternatives

We first show that when there are at most three alternatives at stake, any preference profile $R_N \in \times_{i \in N} \mathcal{R}_i$ satisfies sequential inclusion and moreover, individual and group strategy-proofness are equivalent.

Proposition 3 *Let $\#A \leq 3$. Then, any profile of preferences $R_N \in \times_{i\in N}\mathcal{R}_i$ satisfies sequential inclusion and any strategy-proof social choice function on $\times_{i\in N}\mathcal{R}_i$ is also group strategy-proof.*[6]

Proof. If $\#A = 2$ or $\#A = 1$ sequential inclusion trivially holds. Therefore, we concentrate on the case where A consists of three distinct alternatives x, y, and z and $R_N \in \times_{i\in N}\mathcal{R}_i$. Without loss of generality, choose two of these three alternatives, say y and z. Define the following partition: $S(R_N; y, z) = S_z(R_N; y, z) \cup S_{\{x,z\}}(R_N; y, z)$ where $S_z(R_N; y, z) = \{j \in S(R_N; y, z)$ such that $L(R_j, z) = \{z\}\}$ and $S_{\{x,z\}}(R_N; y, z) = \{k \in S(R_N; y, z)$ such that $L(R_k, z) = \{x, z\}\}$, respectively. Since their lower contour set at z coincide, for any $j, l \in S_z(R_N; y, z)$, $j \sim (R_N; y, z)l$. Similarly, for any $k, h \in S_{\{x,z\}}(R_N; y, z)$, $k \sim (R_N; y, z)h$. Moreover, for any $j \in S_z(R_N; y, z)$, $\{z\} = L(R_j, z) \subset \overline{L}(R_k, y)$ for any $k \in S_{\{x,z\}}(R_N; y, z)$, thus $j \succsim (R_N; y, z)k$. Thus, $\succsim (R_N; y, z)$ is complete and acyclic showing that sequential inclusion holds.
Finally, by Theorem 1, any strategy-proof social choice function on a domain satisfying sequential inclusion is also group strategy-proof. ■

The size of admissible sets of preferences

Concerning the set of admissible preferences, observe that if there only exists one agent whose domain of preferences has cardinality greater than one then strategy-proofness and group strategy-proofness are obviously the same. Notice that as a consequence of this observation and the statement in Proposition 3, in order to have non equivalence between individual and group strategy-proofness, there must be at least four alternatives to choose from and two agents with $\#\mathcal{R}_i \geq 2$.

In Example 3 we illustrate the simplest case where there exist social choice functions that are individual strategy-proof but not group strategy-proof.

Example 3 *Let $N = \{1, 2\}$, $A = \{x, y, z, w\}$, and $\mathcal{R}_i = \{R_i, R'_i\}$ for any $i \in N$ such that $xP_iyP_izP_iw$ and $wP'_iyP'_izP'_ix$.*

R_i	R'_i
x	w
y	y
z	z
w	x

Let f and f_M be the social choice function described by the following tables:

f	R_2	R'_2
R_1	x	z
R'_1	y	w

,

f_M	R_2	R'_2
R_1	x	x
R'_1	x	w

where f_M is the majority rule between x and w choosing x when any tie-break is required.

[6] The same proof works to show that any strategy-proof social choice function with $\#A_f \leq 3$ is group strategy-proof.

Note that f is strategy-proof but it is not group strategy-proof (in particular f is not efficient). However, observe that there also exist non-constant rules (f_M) that are both strategy-proof and group strategy-proof.

4 Indirect sequential inclusion

4.1 The condition and its implications

In this section we provide a new condition, weaker than sequential inclusion, that still guarantees the equivalence between individual and group strategy-proofness. It is no longer a condition on individual profiles. Rather, it requires that, given a profile within the domain, some other profile, conveniently related to the first one (see below), does indeed satisfy our previous requirement. That is why we say that profiles that meet our new condition satisfy indirect sequential inclusion. The new definition allows us to incorporate new and interesting domains into our list of those guaranteeing equivalence.

One of them is the universal domain. Since we know from the Gibbard-Satterthwaite theorem that in that case the only strategy-proof rules are the dictatorial ones, which are also obviously group strategy-proof, it is nice to have this clear case of equivalence incorporated into our framework. Moreover, as we shall see, there are also other relevant domains where sequential inclusion holds indirectly, while not in its original form. Why don't we emphasize the new condition, which is weaker and satisfies the same purpose, rather than its direct version? Essentially, we want to keep both on the same foot, because each one has its own attractive features. Indirect sequential inclusion is indeed weaker but it is a condition on domains taken as a whole. Therefore, it may fail if some preferences in a domain are removed from it. Sequential inclusion is also a domain restriction, but it relies on the properties of each profile. Therefore, it has the very nice feature of holding for all non-trivial subdomains of domains satisfying it. Indirect sequential inclusion does not meet this requirement.

Definition 8 *For preferences R_i,$R_i' \in \mathcal{R}_i$ and alternative $z \in A$, R_i' is a strict monotonic transformation of R_i at z if R_i' is such that for all $x \in A\backslash\{z\}$ such that zR_ix, $zP_i'x$.*

Definition 9 *Let $R_N', R_N \in \times_{i\in N}\mathcal{R}_i$ be two preference profiles and let $z \in A$. We say that R_N' is a strict monotonic transformation of R_N at alternative z if for any $i \in N$, either $R_i' = R_i$ or else R_i' is a strict monotonic transformation of R_i at z.*

Definition 10 *A domain $\times_{i\in N}\mathcal{R}_i$ satisfies indirect sequential inclusion if, for all profiles $R_N \in \times_{i\in N}\mathcal{R}_i$, either (a) the profile R_N satisfies sequential inclusion; or else (b) for each pair $y, z \in A$ there exists $R_N' \in \times_{i\in N}\mathcal{R}_i$ where $R'_{N\backslash S(R_N;y,z)} = R_{N\backslash S(R_N;y,z)}$, such that*
(1) R_N' is a strict monotonic transformation of R_N at z,
(2) for any $i \in S(R_N; y, z)$, $yP_i'z$ and
(3) $\succsim (R_N'; y, z)$ is complete and acyclic.

Theorem 2 *Let $\times_{i\in N}\mathcal{R}_i$ be a domain satisfying indirect sequential inclusion. Then, any strategy-proof social choice function on $\times_{i\in N}\mathcal{R}_i$ is group strategy-proof.*

See the proof of Theorem 2 in the Appendix.

4.2 Examples: three domains satisfying indirect but not direct sequential inclusion

The Universal domain

Consider the universal domain: for any agent $i \in N$, $\mathcal{R}_i = \mathcal{R}$.

First, we show that *the universal domain violates sequential inclusion.* To do that, we will show the existence of a preference profile R_N and a pair of alternatives y, z, for which sequential inclusion does not hold. Let $R, R' \in \mathcal{R}$ such that $wPyPzPx$ and $xP'yP'zP'w$, respectively.[7] Let R_N be such that for any $i \neq n$, $R_i = R$ and $R_n = R'$. Then, $S(R_N; y, z) = N$. Observe that $\neg \left[L(R_1, z) \subset \overline{L}(R_n, y)\right]$ and $\neg \left[L(R_n, z) \subset \overline{L}(R_1, y)\right]$ hold, which means that the binary relation $\succsim (R_N; y, z)$ is not complete. Thus, R_N does not satisfy sequential inclusion for y, z.

Now, we show that *the universal domain satisfies indirect sequential inclusion.* Let R_N be a preference profile and let $y, z \in A$. If $\succsim (R_N; y, z)$ is complete and acyclic, let $R'_N = R_N$. Otherwise, define R'_N as follows: for any $j \in N \backslash S(R_N; y, z)$, $R'_j = R_j$. And for any $i \in S(R_N; y, z)$, R'_i and R_i coincide in the ranking of most pairs of alternatives, with two exceptions. One is that yP'_it for any $t \in A \backslash \{y\}$ (that is, y is pushed up to be the unique best alternative for R'_i). The other change from R_i to R'_i is that for any $t \in A$ such that tI_iz, then zP'_it (that is, alternatives indifferent to z in R_i are strictly worse than z under R'_i). Observe that R'_N satisfies the following: for any $i \in S(R_N; y, z)$, R'_N is a strict monotonic transformation of R_N at z. Also, for any agent $i \in S(R_N; y, z)$ the ranking between y and z has not changed going from R_i to R'_i. And finally, $\succsim (R'_N; y, z)$ is complete and acyclic: Observe that $L(R'_l, z) \subset \overline{L}(R'_i, y) = A \backslash \{y\}$ for any $i, l \in S(R_N; y, z)$. Thus, $i \sim (R_N; y, z) l$ for any $i, l \in S(R_N; y, z)$. This shows that the universal domain satisfies indirect sequential inclusion.

Mixed single-peaked / single-dipped domains

There are frameworks where the combination of single-peaked and single-dipped preferences, relative to the same linear order of alternatives $>$, makes sense. In particular, suppose that there is a non-empty set of agents in N, say $N_{\mathcal{S}}$, with all single-peaked preferences relative to $>$ as admissible set. And the complementary set, say $N_{\mathcal{D}} = N \backslash N_{\mathcal{S}}$, the set of agents with all single-dipped preferences relative to $>$ as admissible set.[8] It is worth mentioning that for such domain of preferences, there exist non-dictatorial strategy-proof rules. Furthermore, this domain violates *sequential inclusion*.[9]

We can construct R'_N similarly to the case of the universal domain, to show that *the domain of mixed single-peaked / single-dipped preferences satisfies indirect sequential inclusion.* Let R_N be a preference profile and let $y, z \in A$. If $\succsim (R_N; y, z)$ is complete and

[7] Note that by Theorem 1 and Proposition 3, the interesting cases are those where there exist at least four alternatives.

[8] This is a domain that has been recently considered by Thomson (2008). Note that any subdomain where agents in $N_{\mathcal{D}}$ have as set of admissible preferences a proper subset of single-dipped ones, would also satisfy indirect sequential inclusion.

[9] Consider Example 3 and take the preference profile $R_N = (R_1, R'_2)$. Note that R_1 is single-peaked and R'_2 is single-dipped relative to the order $y < x < z < w$. R_N violates sequential inclusion for y, z. This argument can be easily generalized for any A.

acyclic, let $R'_N = R_N$ (note that if either $S(R_N; y, z) \subset N_{\mathcal{S}}$ or $S(R_N; y, z) \subset N_{\mathcal{D}}$ this will hold). Otherwise, define R'_N as follows: for any $i \in N \backslash S(R_N; y, z)$, $R'_i = R_i$, for any $i \in S(R_N; y, z) \cap N_{\mathcal{D}}$, $R'_i = R_i$, and for any $i \in S(R_N; y, z) \cap N_{\mathcal{S}}$, R'_i is a single-peaked preference which is a monotonic transformation of R_i at z with $p_i(A) = y$. Note that R'_i exists since agents have all single-peaked preferences available. Observe that R'_N is a strict monotonic transformation of R_N at z and for any agent $i \in S(R_N; y, z)$ the ranking between y and z has not changed going from R_i to R'_i. Moreover, $\succsim (R'_N; y, z)$ is complete and acyclic, where $S(R'_N; y, z) = S(R_N; y, z)$ by construction of R'_N. To see that R'_N satisfies sequential inclusion for y, z, observe first that $L(R'_i, z) \subset \overline{L}(R'_j, y) = A \backslash \{y\}$ for any $i \in S(R'_N; y, z)$ and $j \in S(R'_N; y, z) \cap N_{\mathcal{S}}$. Thus, $i \succsim (R'_N; y, z) j$ for any $i \in S(R'_N; y, z) \cap N_{\mathcal{D}}$ and $j \in S(R'_N; y, z) \cap N_{\mathcal{S}}$ and $l \sim (R'_N; y, z) j$ for any $l, j \in S(R'_N; y, z) \cap N_{\mathcal{S}}$. Using the same argument as in the proof of Proposition 2 we can also show that $\succsim (R'_N; y, z)$ is complete and acyclic over any pair of agents in $S(R'_N; y, z) \cap N_{\mathcal{D}}$. Thus, the domain of mixed single-peaked / single-dipped preferences satisfies indirect sequential inclusion.

Separable lexicographic preferences with a common ordering of dimensions

The domain where all agents have separable and lexicographic preferences relative to a common order of dimensions satisfies indirect sequential inclusion, though not sequential inclusion when $k \geq 3$.[10]

We provide an argument for the case $k = 3$ (this can be generalized for any $k > 3$) showing that *preference profiles on this domain do satisfy indirect sequential inclusion*. In Subsection 3.1 we have already shown that for $k = 2$ such preference profiles satisfy sequential inclusion.

Let the number of components be $k = 3$, and fix the order 1<2<3 on all dimensions. Note that the domain of all individual preferences $\mathcal{L}_3^{1<2<3}$ is the following:

R^1	R^2	R^3	R^4	R^5	R^6	R^7	R^8
$(0,0,0)$	$(1,0,0)$	$(\mathbf{0},\mathbf{1},\mathbf{0})$	$(1,1,0)$	$(0,0,1)$	$(1,0,1)$	$(0,1,1)$	$(1,1,1)$
$(0,0,1)$	$(1,0,1)$	$(\mathbf{0},\mathbf{1},\mathbf{1})$	$(1,1,1)$	$(0,0,0)$	$(1,0,0)$	$(0,1,0)$	$(1,1,0)$
$(\mathbf{0},\mathbf{1},\mathbf{0})$	$(1,1,0)$	$(0,0,0)$	$(1,0,0)$	$(0,1,1)$	$(1,1,1)$	$(0,0,1)$	$(1,0,1)$
$(\mathbf{0},\mathbf{1},\mathbf{1})$	$(1,1,1)$	$(0,0,1)$	$(1,0,1)$	$(0,1,0)$	$(1,1,0)$	$(0,0,0)$	$(1,0,0)$
$(1,0,0)$	$(0,0,0)$	$(1,1,0)$	$(\mathbf{0},\mathbf{1},\mathbf{0})$	$(1,0,1)$	$(0,0,1)$	$(1,1,1)$	$(0,1,1)$
$(1,0,1)$	$(0,0,1)$	$(1,1,1)$	$(\mathbf{0},\mathbf{1},\mathbf{1})$	$(1,0,0)$	$(0,0,0)$	$(1,1,0)$	$(0,1,0)$
$(1,1,0)$	$(\mathbf{0},\mathbf{1},\mathbf{0})$	$(1,0,0)$	$(0,0,0)$	$(1,1,1)$	$(0,1,1)$	$(1,0,1)$	$(0,0,1)$
$(1,1,1)$	$(\mathbf{0},\mathbf{1},\mathbf{1})$	$(1,0,1)$	$(0,0,1)$	$(1,1,0)$	$(0,1,0)$	$(1,0,0)$	$(0,0,0)$

Observe that for any pair of alternatives y, z there only exist four preferences in $\mathcal{L}_3^{1<2<3}$ such that yPz. Thus, for any preference profile R_N, $S(R_N; y, z)$ will always consist of, at most, four types of agents. Consider the following example where $y = (0, 1, 0)$, $z = (0, 1, 1)$, and suppose that R_N is such that agents' preferences are of four different types: say agents 1, 2, 3, and 4, with $R_1 = R^1$, $R_2 = R^2$, $R_3 = R^3$, and $R_4 = R^4$.[11] Without loss of generality,

[10] Note that the set of all separable preferences does not satisfy indirect sequential inclusion.

[11] A similar argument would work for cases where not all types of agents coexist.

suppose that there is one agent of each type, that is, $S(R_N; y, z) = \{1, 2, 3, 4\}$. Remark that this profile does not satisfy sequential inclusion, $\succsim (R_N; y, z)$ is not complete since neither $L(R_1, z) \subset \overline{L}(R_4, y)$ nor $L(R_4, z) \subset \overline{L}(R_1, y)$ hold. However, define $R'_1 = R_3$ and $R'_j = R_j$ for any $j \neq 1$. Clearly, R'_N is a strict monotonic transformation of R_N at z and for any $i \in S(R_N; y, z)$, $yP'_i z$ (in fact, $S(R'_N; y, z) = S(R_N; y, z)$). Moreover, R'_N satisfies sequential inclusion for y, z. To see that, observe that $L(R'_2, z) \subset L(R'_4, z) \subset L(R'_3, z) = L(R'_1, z)$. Thus, according to the binary relation $\succsim (R'_N; y, z)$ on $S(R'_N; y, z)$ we obtain that $2 \succsim 3, 4, 1$; $4 \succsim 3, 1$; and $3 \sim 1$. Thus, $\succsim (R'_N; y, z)$ is complete and acyclic. Thus, the domain where all agents have separable and lexicographic preferences relative to a common order of dimensions satisfies indirect sequential inclusion.[12]

5 Immunity to manipulation by groups that are not "too large"

By requiring group strategy-proofness we avoid manipulations by means of coalitions of any size. However, we could be interested in a strategic concept avoiding manipulations by coalitions of particular sizes. Specifically, we could weaken the requirement of group strategy-proofness by just imposing that we want to avoid manipulations of coalitions of size less or equal than k, for a fixed $k < n$, that is, imposing "k-group strategy-proofness".

In this section we define a property that is weaker than sequential inclusion, and we show that it is a sufficient condition on domains of preferences that guarantee the equivalence between strategy-proofness and "k-group strategy-proofness".

First, we formally define k-group strategy-proofness.[13] From now on, let $k \in \mathbb{Z}$ such that $k < n$.

Definition 11 *A social choice function f is k-group strategy-proof on $\times_{i \in N} \mathcal{R}_i$ if for any $R_N \in \times_{i \in N} \mathcal{R}_i$, there is no coalition $C \subseteq N$ with $\#C \leq k$ that manipulates f on $\times_{i \in N} \mathcal{R}_i$ at R_N.*

Note that if f is k-group strategy-proof then f is also l-group strategy-proof for $l < k$. The converse is not true (see Example 4 below). When $k = 2$, we use the terms pairwise strategy-proofness and 2-size strategy-proofness indistinctly.

Now, we define k-size sequential inclusion, a weaker version of sequential inclusion.

Definition 12 *A preference profile $R_N \in \times_{i \in N} \mathcal{R}_i$ satisfies the k-size sequential inclusion condition if for any pair $y, z \in A$, $\succsim (R_N; y, z)$ on $S(R_N; y, z)$ is complete and there is no cycle of l agents, for any $l \leq k$. A domain $\times_{i \in N} \mathcal{R}_i$ satisfies k-size sequential inclusion if any preference profile $R_N \in \times_{i \in N} \mathcal{R}_i$ satisfies k-size sequential inclusion.*

[12] Another example: Consider a framework where each agent has separable and lexicographic preferences relative to one of two fixed orders of dimensions. Each one of such preference profiles satisfies the indirect sequential inclusion (we leave the proof to the reader). Thus, all strategy-proof rules in this new framework are also group strategy-proof.

[13] This generalizes Serizawa's (2006) concept of "effectively pairwise strategy-proofness". His concept requires that not only agents, but also pairs of agents should not be able to manipulate. In our case, we require that no group of size less than k can do it.

Note that if a preference profile R_N satisfies k-size sequential inclusion it also satisfies the l-size version, where $l \leq k$. Notice that if a preference profile satisfies 2-size sequential inclusion, this means that for any pair $y, z \in A$, $\succsim (R_N; y, z)$ on $S(R_N; y, z)$ is complete. If a preference profile satisfies k-size sequential inclusion, for $k > 2$, this means that for any pair $y, z \in A$, $\succsim (R_N; y, z)$ on $S(R_N; y, z)$ is not only complete but also that there are no cycles of any number of agents lower or equal than k.

In the following result we show that if the k version of sequential inclusion holds then strategy-proofness implies k-group strategy-proofness.

Theorem 3 *Let $\times_{i \in N} \mathcal{R}_i$ be a domain satisfying the k-size sequential inclusion condition. Then, any strategy-proof social choice function on $\times_{i \in N} \mathcal{R}_i$ is k-group strategy-proof.*

The proof of this fact is parallel to that of Theorem 1, and is left to the reader.

The following example illustrates the nature of the result in Theorem 3. We exhibit a domain where pairwise sequential inclusion holds, but 3-size sequential inclusion fails. We then exhibit a strategy proof social choice function that is also pairwise strategy-proof on that domain, but fails to satisfy full group strategy-proofness.

Example 4 *Let $N = \{1, 2, 3\}$ and $A = \{y, z, a, b, c\}$. Suppose that $\mathcal{R}_i = \{R_i^1, R_i^2, R_i^3\}$ for each agent $i \in N$ and is given by:*

R_i^1	R_i^2	R_i^3
a_1	a_2	a_3
y	y	y
a_2	a_3	a_1
z	z	z
a_3	a_1	a_2

It is easy to see that $\times_{i \in N} \mathcal{R}_i$ satisfies pairwise sequential inclusion. To see a violation of 3-size sequential inclusion, consider the following profile $R_N = (R_1^1, R_2^2, R_3^3)$ and the pair of alternatives y, z. Note that $S(R_N; y, z) = \{1, 2, 3\}$. Since $L(R_1^1, z) \subset \overline{L}(R_2^2, y)$ but $\neg[L(R_2^2, z) \subset \overline{L}(R_1^1, y)]$ then $1 \succ 2$. Since $L(R_2^2, z) \subset \overline{L}(R_3^3, y)$ but $\neg[L(R_3^3, z) \subset \overline{L}(R_2^2, y)]$ then $2 \succ 3$. Since $L(R_3^3, z) \subset \overline{L}(R_1^1, y)$ but $\neg[L(R_1^1, z) \subset \overline{L}(R_3^3, y)]$ then $3 \succ 1$. Therefore, $1 \succ 2 \succ 3$ and $3 \succ 1$: there is a cycle involving three agents and thus R_N violates 3-size sequential inclusion implying that $\times_{i \in N} \mathcal{R}_i$ violates(3-size) sequential inclusion.[14]
Define a social choice function f as follows:

R_3^1				R_3^2				R_3^3			
	R_2^1	R_2^2	R_2^3		R_2^1	R_2^2	R_2^3		R_2^1	R_2^2	R_2^3
R_1^1	a_1	a_1	a_1	R_1^1	a_1	a_2	a_3	R_1^1	a_1	y	a_3
R_1^2	a_1	a_2	z	R_1^2	a_2	a_2	a_3	R_1^2	y	a_2	a_3
R_1^3	a_1	a_1	a_1	R_1^3	y	a_2	a_3	R_1^3	a_3	a_3	a_3

[14]The way to construct this example is the following: there is a pair of alternatives y, z that are the second best and second worst alternatives, respectively in all individual preferences. Moreover, we need additional k alternatives, each one being the best for some individual preference and the worst for some other. The other $k - 2$ alternatives are worse than y and better than z.

We can check that f is pairwise strategy-proof (thus, strategy-proof) but not group strategy-proof. To see the latter, consider the profile $R_N = (R_1^2, R_2^3, R_3^1)$ and $R'_N = (R_1^3, R_2^1, R_3^2)$. Observe that $f(R_N) = z$ while $f(R'_N) = y$. Since y is strictly preferred to z by all agents, the whole coalition manipulates f at R_N via R'_N.

6 Controlling for the alternatives in the range

In this section we provide a result that could be used by a designer to eventually decide whether or not to limit the range of a social choice function, as a method to enhance some of the good properties of a mechanism. In our case, limiting the range could help avoiding manipulation by groups, but the type of analysis we suggest here might be applicable for other purposes as well.

When there are only two social alternatives to choose from, it is possible to design non-dictatorial and (group) strategy-proof social choice functions on the universal domain of all possible preference profiles. The same is also true if society faces more alternatives, but the range of the social choice function is restricted to only contain two of them. Of course, artificially restricting the range, so that it does not contain all conceivable alternatives, may have some negative consequences, especially in terms of efficiency. But it may also have the advantage of limiting the strategic behavior of agents. Therefore, it is useful, in this and other contexts, to study the trade-offs that a designer may face when deciding whether or not to allow all alternatives to be in the range of a social choice function.

The literature has seldom mentioned the possibility of imposing limits to the range as part of a deliberate action of design (except, of course, that it has spent much effort to study the case where there are only two possible social choices). Our specific results suggest that we may gain by paying attention to such possibility.

Our next result suggests possible limitations of the range as a tool to strengthen the resilience of functions to be manipulated by large groups. It extends the remark we already made in Section 3, that with three alternatives all strategy-proof rules are also group strategy-proof. This time what will matter is no longer the number of alternatives that may be chosen, but their specific names. The idea is that, if our condition of sequential inclusion holds for a given subset of alternatives, then the range independent functions defined on this subset will satisfy our equivalence result.

Definition 13 *Given a preference profile $R_N \in \times_{i \in N} \mathcal{R}_i$ and a pair of alternatives $y, z \in B$, we define a binary relation $\succsim (R_N; y, z)$ on $S(R_N; y, z)$ as follows:*

$$i \succsim (R_N; y, z) j \text{ if } L(R_i, z) \cap B \subset \overline{L}(R_j, y) \cap B.$$

Definition 14 *A profile of preferences $R_N \in \times_{i \in N} \mathcal{R}_i$ satisfies sequential inclusion on $B \subset A$ if for any $y, z \in B$ the binary relation $\succsim (R_N; y, z)$ on $S(R_N; y, z)$ is complete and acyclic. A domain $\times_{i \in N} \mathcal{R}_i$ satisfies sequential inclusion on B if the condition holds for all profiles in it.*

Proposition 4 *Let $\times_{i \in N} \mathcal{R}_i$ be a domain of preferences satisfying sequential inclusion on $B \subset A$. Then, any strategy-proof social choice function with range B is also group strategy-proof.*

The proof of Proposition 4 follows the same lines of the proof of Theorem 1, where we need to replace any expression of lower contour sets $L(\cdot,\cdot)$ by $L(\cdot,\cdot) \cap A_f$.

To conclude this section, notice that we could have complicated the equivalence result in Proposition 4 by combining the requirement that the condition holds on a subset of alternatives and for a subgroup of agents (as in Section 5), and obtaining a result on $k-$group strategy-proofness for restricted ranges.

7 A partial result on necessity

In this section we provide a result that establishes the partial necessity of sequential inclusion to guarantee that individual and group strategy-proofness become equivalent. We have already remarked that, if sequential inclusion holds in some domain, it also holds in all subdomains, and so does equivalence. What we can prove is that if equivalence holds for a domain and all of its subdomains, then sequential inclusion must be satisfied by all of them. Or, equivalently, that if sequential inclusion fails for some domain, then it is possible to find a subdomain where to define a rule that is individually but not group strategy proof. Since this rule will be defined on a subdomain, rather than on the full domain we start with, we do not have a full proof of necessity. But we are close, and our decision to stop here is mostly technical. Since our proof is constructive, it becomes rather involved even if we choose to work on a very small subdomain where the equivalence fails. In particular, we must choose the subdomain in question carefully enough to prove that individually strategy-proof rules are well-defined on it.[15] In this section, we concentrate on *domains that allow for opposite preferences*. That is, on domains $\times_{i\in N}\mathcal{R}_i$ such that for any i, if $R_i \in \mathcal{R}_i$, its opposite $\overline{R}_i$ also belongs to $\mathcal{R}_i$ (R_i and $\overline{R}_i$ are *opposite* if for any $x, y \in A$, $xR_iy \Leftrightarrow y\overline{R}_ix$).[16]

We first state the basic theorem.

Theorem 4 *Let $\times_{i\in N}\mathcal{R}_i$ be a domain that allows for opposite preferences and such that any strategy-proof social choice function on $\times_{i\in N}\mathcal{D}_i \subset \times_{i\in N}\mathcal{R}_i$ is also group strategy-proof. Then, $\times_{i\in N}\mathcal{R}_i$ satisfies sequential inclusion.*

In fact, we can state the result in more generality, in the spirit of Section 5.

Theorem 5 *Let $2 \leq k \leq n$. Let $\times_{i\in N}\mathcal{R}_i$ be a domain that allows for opposite preferences and such that any strategy-proof social choice function on $\times_{i\in N}\mathcal{D}_i \subset \times_{i\in N}\mathcal{R}_i$ is also k-group strategy-proof. Then, $\times_{i\in N}\mathcal{R}_i$ satisfies k-size sequential inclusion.*

The proof of Theorem 4 is long and quite involved. We sketch it here and we provide all the remaining details in the Appendix. The proof of Theorem 5 is left to the reader as it follows exactly the same lines.[17]

---

[15] Existence may be an issue for small domains, where even the dictatorial rule may not be well-defined.

[16] Notice that all the domains we have discussed in this paper (single-peaked, single-dipped, separable, etc) admit opposite preferences.

[17] In fact, we do prove the $k = 2$ version of Theorem 5 as part of the proof of Theorem 4 (see Appendix, Lemma 3).

Sketch of the proof of Theorem 4.

Our starting point is given by a domain of preferences containing some profile where sequential inclusion is violated. We first use this profile in order to identify a specific set of alternatives and a specific set of preferences on these alternatives that belong to the domain and satisfy a number of properties (Lemmas 1 and 2). We use these preferences to create a subdomain of the original one, and then exploit their characteristics in order to construct a specific rule that is strategy-proof but not group strategy-proof, within the subdomain. In the case where sequential inclusion is violated by lack of completeness Lemma 3 provides a simple construction. When the violation is due to a cycle our choice of domain is more involved and the rule is more complex. It is essentially a modification of the Borda rule with two extra features: ties are always avoided and the rule does not follow Borda in two crucial profiles. This rule turns out to be individually but not group strategy-proof on our domain and this completes the proof.

8 Concluding Remarks

In this paper we have identified conditions on domains where *all* rules satisfying strategy-proofness must also be immune to manipulations by groups. We have not imposed any further conditions on the nature and properties of such rules. One could think of further refinements of our results, where the equivalence between individual and group strategy-proofness might hold (in given domains) between classes of rules which satisfy some additional condition. As an example, consider social choice functions defined on classical exchange economies, with the natural domain restrictions implied by the fact that agents' preferences are selfish, strictly convex and continuous. In this scenario, Barberà and Jackson (1995) establishes the equivalence between individual and group strategy-proofness for those rules satisfying the added condition of non-bossyness.

This example also allows us to comment on the type of domain restrictions we may be interested in. Those that we have used as examples arise very naturally in voting contexts. But there are also other domains that allow for non-trivial strategy-proof rules and that appear in connection with economic problems like those of matching, cost allocation, or the assignment of indivisibles. We conjecture that one would also be able to prove interesting equivalence results within these domains, especially if concentrating in subclasses of functions satisfying appropriate additional conditions.

Finally, we also notice that our sequential inclusion condition can be factored out into two pieces. One requires that the binary relation in our condition is complete; the other demands its acyclicity. Because of that, in the presence of domain restrictions arising from other considerations, we may not need to impose the full form of sequential inclusion in order to get equivalence. For example, we know that in cartesian domains where preferences satisfy single-crossing, the first half of our condition will be sufficient.

In summary, our work has studied the pure case of equivalence without further restrictions. What we suggest is that imposing added structure on domains and/or the functions defined upon them is a natural next step.

9 Appendix

Proof of Theorem 2. We begin the proof by stating and proving an auxiliary Lemma and a Remark.

Lemma Let f be a strategy-proof social choice function. For any $R_N \in \times_{i\in N}\mathcal{R}_i$, any $i \in N$, if $R'_i \in \mathcal{R}_i$ is a strict monotonic transformation of R_i at $f(R_N)$ we have that $f(R_N) = f(R'_i, R_{-i})$.

Proof of the Lemma Let f be any strategy-proof social choice function. Let $R_N \in \times_{i\in N}\mathcal{R}_i$ and let $R'_i \in \mathcal{R}_i$ be a strict monotonic transformation of R_i at $f(R_N)$. Suppose that $f(R_N) \neq f(R'_i, R_{-i})$. By strategy-proofness, $f(R_N)R_i f(R'_i, R_{-i})$ and $f(R'_i, R_{-i})R'_i f(R_N)$ for all $R_{-i} \in \mathcal{R}_{-i}$. Since R'_i is a strict monotonic transformation of R_i at $f(R_N)$, we have that $f(R_N)P'_i f(R'_i, R_{-i})$, a contradiction. This ends the proof of the Lemma.

Remark Notice that condition (b) in Definition 10 can be restated as follows: (b') for each pair $y, z \in A$ and for each subset T in $S(R_N; y, z)$, there exists $R'_N \in \times_{i\in N}\mathcal{R}_i$ where $R'_{N\backslash T} = R_{N\backslash T}$, such that
(1') R'_N is a strict monotonic transformation of R_N at z,
(2') for any $i \in T$, $yP'_i z$ and
(3') $\succsim (R'_N; y, z)$ is complete and acyclic on T.

Proof of the Remark Clearly condition (b') implies condition (b). Now, consider $R_N \in \times_{i\in N}\mathcal{R}_i$ and let y, z, and R'_N such that (b) holds. Let $T \subset S(R_N; y, z)$ and define $\overline{R}'_N$ such that $\overline{R}'_{N\backslash T} = R_{N\backslash T}$ and $\overline{R}'_T = R'_T$. Note that (1'), (2'), and (3') hold for $\overline{R}'_N$ on T since (1), (2), and (3) hold for R'_N on the larger set $S(R_N; y, z)$. This ends the proof of the Remark.

We now complete the proof of Theorem 2.

Let f be a strategy-proof social choice function on $\times_{i\in N}\mathcal{R}_i$ and by contradiction suppose that f is manipulable by some coalition $C \subset N$, that is, there exists $\widetilde{R}_C \in \times_{i\in C}\mathcal{R}_i$, such that for any agent $i \in C$, $f(\widetilde{R}_C, R_{-C})P_i f(R_N)$. Let us denote $y = f(\widetilde{R}_C, R_{-C})$ and $z = f(R_N)$. Note that $C \subset S(R_N; y, z)$. The proof consists of c steps. Mainly, in each one, we change the preference of a single agent $i \in C$ to be $\widetilde{R}_i$.

Step 1:

<u>*Case 1.1*</u>. If R_N satisfies sequential inclusion. We get a contradiction using the same argument as in the proof of Theorem 1.

<u>*Case 1.2*</u>. If R_N does not satisfy sequential inclusion, since the domain satisfies indirect sequential inclusion, by the above Remark applied to y, z and $T = C$, there exists R'_N such that $R'_{N\backslash C} = R_{N\backslash C}$ and conditions (1'), (2'), and (3') in the above Remark hold. By the above Lemma and (1'), we obtain that $f(R'_C, R_{-C}) = z$. By condition (2'), $y = f(\widetilde{R}_C, R_{-C})P'_i f(R'_C, R_{-C}) = z$. Thus, C manipulates f at (R'_C, R_{-C}) via $\widetilde{R}_C$. By (3'), $\succsim (R'_N; y, z)$ is complete and acyclic on C. Note that $S((R'_C, R_{-C}); y, z) = S(R_N; y, z)$. Choose an agent in $C(C, \succsim (R'_N; y, z))$, without loss of generality, say agent 1. Then, $L(R'_1, z) \subset \overline{L}(R'_h, y)$ for any $h \in C\backslash\{1\}$. Change preferences of agent 1 and let $z_2 = f(\widetilde{R}_1, R'_{C\backslash\{1\}}, R_{-C})$. By strategy-proofness, $z_2 \in L(R'_1, z)$ and thus $z_2 \in \overline{L}(R'_h, y)$ for any

$h \in C\backslash\{1\}$. Moreover, observe that $S((\widetilde{R}_1, R'_{C\backslash\{1\}}, R_{-C}); y, z_2) \supset C\backslash\{1\}$. Go to Step 2.

Step 2: Redefine $R_N \equiv (\widetilde{R}_1, R'_{C\backslash\{1\}}, R_{-C})$.
Case 2.1. If R_N satisfies sequential inclusion we get a contradiction using the same argument as in the proof of Theorem 1 changing preferences of all agents except 1 going from $R_N \equiv (\widetilde{R}_1, R'_{C\backslash\{1\}}, R_{-C})$ to $(\widetilde{R}_C, R_{-C})$ and obtaining a contradiction that C manipulates f at (R'_C, R_{-C}) via $\widetilde{R}_C$.
Case 2.2. If R_N does not satisfy sequential inclusion, since the domain satisfies indirect sequential inclusion, by the above Remark applied to y, z_2 and $T = C\backslash\{1\} \subset S(R_N; y, z_2) = S((\widetilde{R}_1, R'_{C\backslash\{1\}}, R_{-C}); y, z_2)$, there exists R'_N such that $R'_{N\backslash[C\backslash\{1\}]} = R_{N\backslash[C\backslash\{1\}]}$ (that is, $R'_{\{1\}\cup N\backslash C} = (\widetilde{R}_1, R_{N\backslash C})$) such that conditions (1'), (2'), and (3') in the above Remark hold. By the above Lemma and (1'), we obtain that $f(R'_{C\backslash\{1\}}, R_{\{1\}\cup N\backslash C}) = z_2$. By condition (2'), $y = f(\widetilde{R}_C, R_{-C}) P'_i f(R'_{C\backslash\{1\}}, R_{\{1\}\cup N\backslash C}) = z_2$. Thus, $C\backslash\{1\}$ manipulates f at $(R'_{C\backslash\{1\}}, R_{\{1\}\cup N\backslash C}) = (\widetilde{R}_1, R'_{C\backslash\{1\}}, R_{-C})$ via $\widetilde{R}_{C\backslash\{1\}}$. By (3'), $\succsim (R'_N; y, z_2)$ is complete and acyclic on $C\backslash\{1\}$. Choose an agent in $C(C\backslash\{1\}, \succsim (R'_N; y, z_2))$, without loss of generality, say agent 2. Then, $L(R'_2, z_2) \subset \overline{L}(R'_h, y)$ for any $h \in C\backslash\{1, 2\}$. Change preferences of agent 2 and let $z_3 = f(\widetilde{R}_{\{1,2\}}, R'_{C\backslash\{1,2\}}, R_{-C})$. By strategy-proofness, $z_3 \in L(R'_2, z_2)$. Thus $z_3 \in \overline{L}(R'_h, y)$ for any $h \in C\backslash\{1, 2\}$. Also note that $S((\widetilde{R}_{\{1,2\}}, R'_{C\backslash\{1,2\}}, R_{-C}); y, z_3) \supset C\backslash\{1, 2\}$. Go to Step 3.

Step 3: Redefine $R_N \equiv (\widetilde{R}_{\{1,2\}}, R'_{C\backslash\{1,2\}}, R_{-C})$.
Case 3.1. If R_N satisfies sequential inclusion we get a contradiction using the same argument as in the proof of Theorem 1 applied to $C\backslash\{1\}$ changing preferences of all agents except 2 going from $R_N \equiv (\widetilde{R}_{\{1,2\}}, R'_{C\backslash\{1,2\}}, R_{-C})$ to $(\widetilde{R}_C, R_{-C})$ and obtaining a contradiction that $C\backslash\{1\}$ manipulates f at $(\widetilde{R}_1, R'_{C\backslash\{1\}}, R_{-C})$ via $\widetilde{R}_{C\backslash\{1\}}$.
Case 3.2. If R_N does not satisfy sequential inclusion, since the domain satisfies indirect sequential inclusion, by the above Remark applied to y, z_3 and $T = C\backslash\{1, 2\} \subset S(R_N; y, z_3) = S((\widetilde{R}_{\{1,2\}}, R'_{C\backslash\{1,2\}}, R_{-C}); y, z_3)$, there exists R'_N such that $R'_{N\backslash[C\backslash\{1,2\}]} = R_{N\backslash[C\backslash\{1,2\}]}$ (that is, $R'_{\{1,2\}\cup N\backslash C} = (\widetilde{R}_{\{1,2\}}, R_{N\backslash C})$) such that conditions (1'), (2'), and (3') in the above Remark hold. By the above Lemma and (1'), we obtain that $f(R'_{C\backslash\{1,2\}}, R_{\{1,2\}\cup N\backslash C}) = z_3$. By condition (2'), for $i \in C\backslash\{1, 2\}$, $y = f(\widetilde{R}_C, R_{-C}) P'_i f(R'_{C\backslash\{1,2\}}, R_{\{1,2\}\cup N\backslash C}) = z_3$. Thus, $C\backslash\{1, 2\}$ manipulates f at $(R'_{C\backslash\{1,2\}}, R_{\{1,2\}\cup N\backslash C}) = (\widetilde{R}_{\{1,2\}}, R'_{C\backslash\{1,2\}}, R_{-C})$ via $\widetilde{R}_{C\backslash\{1,2\}}$. By (3'), $\succsim (R'_N; y, z_3)$ is complete and acyclic on $C\backslash\{1, 2\}$. Choose an agent in $C(C\backslash\{1, 2\}, \succsim (R'_N; y, z_3))$, without loss of generality, say agent 3. Then, $L(R'_3, z_3) \subset \overline{L}(R'_h, y)$ for any $h \in C\backslash\{1, 2, 3\}$. Change preferences of agent 3 and let $z_4 = f(\widetilde{R}_{\{1,2,3\}}, R'_{C\backslash\{1,2,3\}}, R_{-C})$. By strategy-proofness, $z_4 \in L(R'_3, z_3)$. Thus $z_4 \in \overline{L}(R'_h, y)$ for any $h \in C\backslash\{1, 2, 3\}$. Also note that $S((\widetilde{R}_{\{1,2,3\}}, R'_{C\backslash\{1,2,3\}}, R_{-C}); y, z_4) \supset C\backslash\{1, 2, 3\}$. Go to Step 4.

We repeat a similar argument till Step $(c-1)$. Note that by the recursive argument when analyzing Step $(c-1)$, from Step $(c-2)$ we know that: $C\backslash\{1, ..., c-3\}$ manipulate f at $(\widetilde{R}_{\{1,...,c-3\}}, R'_{C\backslash\{1,...,c-3\}}, R_{-C})$ via $\widetilde{R}_{C\backslash\{1,...,c-3\}}$. Let $z_{c-1} = f(\widetilde{R}_{\{1,...,c-2\}}, R'_{C\backslash\{1,...,c-2\}}, R_{-C})$.By strategy-proofness, $z_{c-1} \in L(R'_{c-2}, z_{c-2})$. Thus $z_{c-1} \in \overline{L}(R'_h, y)$ for any agent in $\{c-1, c\}$.

Moreover, $S((\widetilde{R}_{\{1,...,c-2\}}, R'_{C\backslash\{1,...,c-2\}}, R_{-C}); y, z_{c-1}) \supset \{c-1, -c\}$.

Step (c-1): Redefine $R_N \equiv (\widetilde{R}_{\{1,...,c-2\}}, R'_{C\backslash\{1,...,c-2\}}, R_{-C})$.

Case (c-1).1. If R_N satisfies sequential inclusion we get a contradiction using the same argument as in the proof of Theorem 1 applied to $C\backslash\{1,...,c-3\}$ changing only preferences of agents $c-1$ and c going from $R_N \equiv (\widetilde{R}_{\{1,...,c-2\}}, R'_{\{c-1,c\}}, R_{-C})$ to $(\widetilde{R}_C, R_{-C})$ and obtaining a contradiction that $C\backslash\{1,...,c-3\}$ manipulate f at $(\widetilde{R}_{\{1,...,c-3\}}, R'_{C\backslash\{1,...,c-3\}}, R_{-C})$ via $\widetilde{R}_{C\backslash\{1,...,c-3\}}$.

Case (c-1).2. If R_N does not satisfy sequential inclusion, since the domain satisfies indirect sequential inclusion, by the above Remark applied to y, z_{c-1} and $T = \{c-1, c\} \subset S(R_N; y, z_{c-1}) = S((\widetilde{R}_{\{1,...,c-2\}}, R'_{C\backslash\{1,...,c-2\}}, R_{-C}); y, z_{c-1})$, there exists R'_N such that $R'_{N\backslash\{c-1,c\}} = R_{N\backslash\{c-1,c\}}$ such that conditions (1'), (2'), and (3') in the above Remark hold. By the above Lemma and (1'), we obtain that $f(R'_{\{c-1,c\}}, R_{N\backslash\{c-1,c\}}) = z_{c-1}$. By condition (2'), for $c-1$ and c, $y = f(\widetilde{R}_C, R_{-C}) P'_i f(R'_{\{c-1,c\}}, R_{N\backslash\{c-1,c\}}) = z_{c-1}$. Thus, $\{c-1, c\}$ manipulates f at $(R'_{\{c-1,c\}}, R_{N\backslash\{c-1,c\}}) = (\widetilde{R}_{\{1,...,c-2\}}, R'_{\{c-1,c\}}, R_{-C})$ via $\widetilde{R}_{\{c-1,c\}}$. By (3'), $\succsim (R'_N; y, z_{c-1})$ is complete and acyclic on $\{c-1, c\}$. Choose an agent in $C(\{c-1, c\}, \succsim (R'_N; y, z_{c-1}))$, without loss of generality, say agent $c-1$. Then, $L(R'_{c-1}, z_{c-1}) \subset \overline{L}(R'_h, y)$ for $h = c$. Change preferences of agent $c-1$ and let $z_c = f(\widetilde{R}_{\{1,...,c-1\}}, R'_c, R_{-C})$. By strategy-proofness, $z_c \in L(R'_{c-1}, z_{c-1})$. Thus $z_c \in \overline{L}(R'_c, y)$, which implies that $L(R'_c, z_c) \subset \overline{L}(R'_c, y)$. Go to Step c.

Step c: Let $z_{c+1} = f(\widetilde{R}_C, R_{-C})$ (that is, $z_{c+1} = y$). By strategy-proofness and definition of z_c, $z_{c+1} \in L(R'_c, z_c)$. Thus $z_{c+1} = y \in \overline{L}(R'_c, y)$ which is the desired contradiction. This ends the proof of Theorem 2. ■

Proof of Theorem 4. Our strategy of proof is as follows. Let $\times_{i\in N}\mathcal{R}_i$ be a domain where sequential inclusion is violated. We shall construct a subdomain and exhibit a rule on it which is strategy-proof but not group strategy-proof.

We will structure our argument as follows. We first present three lemmas, and then provide our construction as a final step.

We begin by Lemma 1. Remember that sequential inclusion requires our binary relation $\succsim$ to be complete and acyclic for all pairs of alternatives y, z. Therefore, its violation can come from two sources. If $\succsim$ is incomplete on a pair y, z, then it is easy to show that there are four distinct alternatives and two preference relations in the domain satisfying enough properties to allow for our construction. The starting point for this reasoning is the following Lemma.

Lemma 1 Let $R_N \in \times_{i\in N}\mathcal{R}_i$ and $y, z \in A$. If $\succsim (R_N; y, z)$ on $S(R_N; y, z)$ is not complete then there exist at least four alternatives in A, say y, z, a_1, and a_2. Moreover, there exist 2 individual preference relations in R_N, say R^1, R^2 such that $a_1 R^1 y$, $y P^1 z$, $z R^1 a_2$, and $a_2 R^2 y$, $y P^2 z$, $z R^2 a_1$.

Proof of Lemma 1 For completeness to fail, there must exist two alternatives y and z and two individuals i and j such that $\neg[L(R_i, z) \subset \overline{L}(R_j, y)]$ and $\neg[L(R_j, z) \subset \overline{L}(R_i, y)]$ hold. There will also be $a_1, a_2 \in A\backslash\{y, z\}$ such that $a_2 \in L(R_i, z)$ but $a_2 \notin \overline{L}(R_j, y)$ and $a_1 \in L(R_j, z)$ but $a_1 \notin \overline{L}(R_i, y)$ (note that $a_1 \neq z$ since $z \in \overline{L}(R_i, y)$ but $a_1 \notin \overline{L}(R_i, y)$.

Similarly, $a_1 \neq y$ since $a_1 \in L(R_j, z)$ but $y \notin L(R_j, z)$). Finally, observe that $a_1 \neq a_2$: since $a_1 \in L(R_j, z)$ and yP_jz then $a_1 \in \overline{L}(R_j, y)$. But, $a_2 \notin \overline{L}(R_j, y)$. We have then shown that the violation of completeness requires at least four distinct alternatives. Furthermore, observe that the preferences of the two agents i and j satisfy the conditions predicated by the lemma. Then, define $R^1 = R_i$ and $R^2 = R_j$. This ends the proof of Lemma 1.

Lemma 1 identifies the alternatives and the preferences on them that will help us determine a subdomain where equivalence will not hold, given that $\succsim$ is not acyclic. This construction of such a subdomain and of a function that is strategy-proof on it, but group manipulable (in fact, by only two-agent coalitions) is the essence of the proof of Lemma 3.

We now turn to the case where $\succsim$ is complete for all pairs and all profiles, and the violation of sequential inclusion arises because it is cyclical for some profile and some pair. We concentrate on one cycle of minimal length k, and prove that its existence requires that a number of distinct preferences on at least $k+2$ distinct alternatives exist in the initial domain. We shall use these preferences to construct a subdomain and to define a social choice function on it that is strategy-proof but not group strategy-proof. The starting point for the construction is provided by Lemma 2.

Lemma 2 *Let $R_N \in \times_{i \in N} \mathcal{R}_i$ and $y, z \in A$ such that $\succsim (R_N; y, z)$ on $S(R_N; y, z)$ has a cycle of size k, $3 \leq k \leq s$ but $\succsim (R_N; y, z)$ does not have any cycle of lower size. Then, there exist at least $k+2$ alternatives in A, say y, z, a_1, a_2,...,$a_k \in A$. There also exist k individual preference relations in R_N, say R^1, R^2, ..., R^k, such that for any $l = 1, ..., k \in N$, the following holds: $[(1)\ a_l R^l y,\ (2)\ z R^l a_{l-1}$, for any $j = 1, ..., k-2$, $(3)\ y P^l a_{l+j}$, and $(4)\ a_{l+j} P^l z]$. Moreover, if for any alternative $w \in A$, the binary relation $\succsim (R_N; y, w)$ on $S(R_N; y, w)$ is complete, then (5) for any $l = 1, ..., k$, $[a_{l+j} P^l a_{l+1+j}$, for $j = 1, ...k-3]$.*[18]

Again, Lemma 2 allows us to identify alternatives and specific preferences that will be used to construct a subdomain where to define a function that proves the lack of equivalence.

Before turning to the proof we introduce a piece of notation and we present an example of the type of reasoning that we shall be using.
Concerning notation, we define the following equivalence relation on the set of integers. Let $k \in \mathbb{Z}$ ($\mathbb{Z}$ is the set of integers), $k \leq n$, and $i, j \in \mathbb{Z}$ we say that j and i are equivalent modulo k, say $j \equiv i$ (mod k), if either $j = i$ or else $j = i + vk$, for some $v \in \mathbb{Z}$.
In what follows, whenever we have $j \equiv i$ (mod k), then we will write $R_j = R_i$. We will do the same and $R^j = R^i$, when we refer to a list of specific preference relations labelled by their superindex, or $a_j = a_i$, where these are the subindices in a list of alternatives.
Now, Example 5 shows how, given a profile that generates a cycle in our binary relation $\succsim$, this allows us to single out some set of preferences that will later be used to construct the domain we are looking for. The argument in the proof of Lemma 2 extends the intuition in this example.

Example 5 Take $N = \{1, 2, 3, 4, 5\}$ and $A = \{a_1, a_2, a_3, a_4, a_5, y, z\}$. Let $R_N \in \times_{i \in N} \mathcal{R}_i$ such that $\succsim (R_N; y, z)$ on $S(R_N; y, z)$ has a cycle of size four, but none of lower size. Note that this means that $\#S(R_N; y, z) \geq 4$. We single out four preference relations in R_N and six

[18] The latter condition has some bite only for $k \geq 4$.

alternatives in A in such a way that those preferences, when restricted to the six alternatives, are as follows:

R^1	R^2	R^3	R^4
a_1	a_2	a_3	a_4
y	y	y	y
a_2	a_3	a_4	a_1
a_3	a_4	a_1	a_2
z	z	z	z
a_4	a_1	a_2	a_3

.

Note that we do not say anything about the relative position of a_5 with respect to any other alternative: it could be any. Observe that the relation $\succsim (R_N; y, z)$ over agents having the preferences in the table has a cycle of size four, is complete and has no cycle of size 3. In particular, observe that the following binary relations hold: $1 \succ 2$, $2 \succ 3$, $3 \succ 4$ and $4 \succ 1$. Moreover, $1 \sim 3$, $2 \sim 4$.

Proof of Lemma 2 We first prove that there must exist alternatives and individuals whose preferences satisfy (1) and (2). Without loss of generality, let the cycle of k agents be $1 \succ 2 \succ 3 \succ ... \succ k$ and $k \succ 1$. Thus, their individual preferences in R_N, say R_1, R_2, ..., R_k, are such that $L(R_l, z) \subset \overline{L}(R_{l+1}, y)$ and $\neg \left[L(R_{l+1}, z) \subset \overline{L}(R_l, y)\right]$ for $l = 1, ..., k$. From the latter expression, for any $l = 1, ..., k$, there exists $a_l \in A \backslash \{z, y\}$ such that $zR_{l+1}a_l$ and $a_l R_l y$. Note that for any $l = 1, ..., k$, $a_l \neq z$ and $a_l \neq y$ since $yP_l z$ and $yP_{l+1}z$, respectively. Thus, there exist k individual preference relations in R_N, in particular: R_1, R_2, ..., R_k, such that expressions (1) and (2) in the lemma hold. We now show that the alternatives a_l so defined are all distinct.

For any i, $1 \leq i \leq k-1$, $[a_i \neq a_j$, for any j, $k \geq j > i]$.
Let i, j such that $j - i = 1$. Given the above cycle of k agents, $j \succ j+1$, thus $a_{j-1} \in L(R_j, z) \subset \overline{L}(R_j y)$. But we also proved that $a_j \notin \overline{L}(R_j, y)$. Thus, $a_j \neq a_{j-1}$.
Let i, j such that $j - i = 2$. Given the above cycle of k agents, $j - 1 \succ j$, thus $a_{j-2} \in L(R_{j-1}, z) \subset \overline{L}(R_j, y)$. But since $j \succ j+1$, by $a_j \notin \overline{L}(R_j, y)$. Thus, $a_j \neq a_{j-1}$.
Let i, j such that $j - i = t$, for $3 \leq t \leq k-1$. By assumption, there is no cycle of agents in $S(R_N; y, z)$ of size t. Thus, given the above cycle of k agents, $i+1 \succ i+2 \succ ... \succ i+(t-1) \succ j$ and $i+1 \succsim j$. Thus, $a_i \in L(R_{i+1}, z) \subset \overline{L}(R_j, y)$. But we also proved that $a_j \notin \overline{L}(R_j, y)$. Thus, $a_i \neq a_j$.
We now prove that the preferences in question satisfy expression (3).
For any $l = 1, ..., k$, $[a_{l+j} P_l z$ for any $j = 1, ..., k-2]$.
Take any j. By assumption, there is no cycle of agents in $S(R_N; y, z)$ of size $j+1$ (since $2 \leq j+1 \leq k-1$). Thus, given the above defined cycle of k agents, $l \succ l+1 \succ ... \succ l+j \succ j$ and $l \succsim l+j$. Thus, $L(R_l, z) \subset \overline{L}(R_{l+j}, y)$. If $zR_l a_{l+j}$ then $a_{l+j} \in \overline{L}(R_{l+j}, y)$ which contradicts expression (1) above proved.

We now show that the preferences also satisfy expression (4).
For any $l = 1, ..., k$, $[yP_l a_{l+j}$ for any $j = 1, ..., k-2]$.
Take any j. By assumption, there is no cycle of agents in $S(R_N; y, z)$ of size $k - j$ (since $2 \leq k - j \leq k - 1$). Thus, given the above defined cycle of k agents $l - (k-j-1) \succ l - (k-j) \succ l - (k-j+1) \succ ... \succ l-3 \succ l-2 \succ l-1 \succ l$ and $l - (k-j-1) \succsim l$.

Using our equivalence, $l+j+1 \equiv l+j+1-k$ (mod k) and thus $R_{l+j+1} = R_{l+j+1-k}$. Thus, $L(R_{l+j+1}, z) \subset \overline{L}(R_l, y)$. By the previous lemma, we know that $a_{l+j} \in L(R_{l+j+1}, z)$, and thus $a_{l+j} \in \overline{L}(R_l, y)$.

For $k = 3$ the Lemma is proved.

For $k \geq 4$, we finally prove that the preferences also satisfy expression (5).
For any $l = 1, ..., k$, $[a_{l+j} P_l a_{l+1+j}$, for $j = 1, ..., k-3]$.

We first show that for any $l = 1, ..., k$, $a_{l+1} P_l a_{l+1+j}$ for $j = 1, ..., k-3$. (*)
Fix l and j. By contradiction, suppose that $a_{l+1+j} R_l a_{l+1}$. Consider the pair of alternatives (y, a_{l+1+j}). By hypothesis, the binary relation $\succsim (R_N; y, a_{l+1+j})$ on $S(R_N; y, a_{l+1+j})$ is complete. Since $l, l+1 \in S(R_N; y, a_{l+1+j})$, either $l \succsim l+1$ or $l+1 \succsim l$ should hold. By hypothesis, $a_{l+1} \in L(R_l, a_{l+1+j})$ and, by conditions (1) to (4) already proved, $a_{l+1} \notin \overline{L}(R_{l+1}, y)$. Then, $\neg \left[L(R_l, a_{l+1+j}) \subset \overline{L}(R_{l+1}, y)\right]$, or equivalently, $\neg [l \succsim l+1]$. On the other hand, by conditions (1) to (4), $a_l \notin \overline{L}(R_l, y)$, but also $a_l \in L(R_{l+1}, z)$ and $a_{l+1+j} P_{l+1} z$ holds. The latter two expressions imply that $a_l \in L(R_{l+1}, a_{l+1+j})$. Then, $\neg \left[L(R_{l+1}, a_{l+1+j}) \subset \overline{L}(R_l, y)\right]$, or equivalently, $\neg [l+1 \succsim l]$ which is the desired contradiction.

For $k = 4$ the Lemma is proved.

Now, for $k > 4$, we show that for any $l = 1, ..., k$, $a_{l+t} P_l a_{l+t+1}$ for any t, $t = 2, ..., k-3$.
Fix l and t. By contradiction, suppose that $a_{l+t+1} R_l a_{l+t}$. Consider the pair of alternatives (y, a_{l+t+1}). By hypothesis, the binary relation $\succsim (R_N; y, a_{l+t+1})$ on $S(R_N; y, a_{l+t+1})$ is complete. Since $l, l+t \in S(R_N; y, a_{l+t+1})$, either $l \succsim l+t$ or $l+t \succsim l$ should hold. By hypothesis, $a_{l+t} \in L(R_l, a_{l+t+1})$ and, by conditions (1) to (4), $a_{l+t} \notin \overline{L}(R_{l+t}, y)$. Then, $\neg \left[L(R_l, a_{l+t+1}) \subset \overline{L}(R_{l+t}, y)\right]$, or equivalently, $\neg [l \succsim l+t]$. On the other hand, by conditions (1) to (4), $a_l \notin \overline{L}(R_l, y)$. Using the modulo k equivalence, and by the previous statement (*) applied for $j = k-1-t$ and for $l \equiv l+t$, $a_{l+t+1} P_{l+t} a_l$ (that is, $a_l \in L(R_{l+t}, a_{l+t+1})$). Then, $\neg \left[L(R_{l+t}, a_{l+t+1}) \subset \overline{L}(R_l, y)\right]$, or equivalently, $\neg [l+t \succsim l]$ which is the desired contradiction.

Observe that the preferences of the k agents, $1, ..., k$, satisfy the conditions predicated by the lemma. Again define $R^i = R_i$ for all $i = 1, ..., k$. This ends the proof of Lemma 2.

Next we prove a Lemma that is, in fact, the $k = 2$ version of our Theorem 5. This Lemma already establishes our result for the case where sequential inclusion is violated by the lack of completeness of our binary relation.[19]

Lemma 3 *Let $\times_{i \in N} R_i$ be a domain on which any strategy-proof social choice function on $\times_{i \in N} D_i \subset \times_{i \in N} R_i$ is also pairwise strategy-proof on $\times_{i \in N} D_i$. Then, $\times_{i \in N} R_i$ satisfies 2-size sequential inclusion.*

Proof of Lemma 3 Suppose that $\times_{i \in N} \mathcal{R}_i$ violates 2-size sequential inclusion. Then, there exists $R_N \in \times_{i \in N} \mathcal{R}_i$, a pair of alternatives $y, z \in A$, such that the binary relation $\succsim (R_N; y, z)$ defined on $S(R_N; y, z)$ is not complete.
Therefore there exists a pair of agents $i, j \in S(R_N; y, z)$ such that neither $i \succsim j$ nor $j \succsim i$.

[19] This is the first instance where we use the assumption that the domain admits opposite preferences. Lemmas 1 and 2 do not require it and could therefore be used to explore similar partial necessity results on domains satisfying alternative requirements.

By Lemma 1 we know that there exist $a_1, a_2 \in A\backslash\{y,z\}$ and $a_1 \neq a_2$, and there exists two individual preference relations in R_N, say R^1, R^2 such that $a_1 R^1 y$, $yP^1 z$, $zR^1 a_2$, and $a_2 R^2 y$, $yP^2 z$, $zR^2 a_1$. Define $\mathcal{D}_i = \{R^1, \overline{R}^1\}$, $\mathcal{D}_j = \{R^2, \overline{R}^2\}$ for i, j, $\mathcal{D}_l = \mathcal{R}_l$ for any $l \in N\backslash\{i,j\}$. Note that $\overline{R}^1$ and $\overline{R}^2$ are the opposite preferences of R^1 and R^2, respectively. Now construct f as follows: for any $\widehat{R}_{-\{i,j\}} \times_{l\in N\backslash\{i,j\}} \mathcal{R}_l$, $f(R^1_i, \overline{R}^2_j; \widehat{R}_{-\{i,j\}}) = a_1$, $f(\overline{R}^1_i, R^2_j; \widehat{R}_{-\{i,j\}}) = a_2$, $f(R^1_i, R^2_j; \widehat{R}_{-\{i,j\}}) = z$, $f(\overline{R}^1_i, \overline{R}^2_j; \widehat{R}_{-\{i,j\}}) = y$. Note that f is strategy-proof but not pairwise strategy-proof. This ends the proof of Lemma 3.

A final step: identifying a domain and defining a rule

Suppose, without loss of generality, that $\times_{i\in N}\mathcal{R}_i$ is a domain such that for any $R_N \in \times_{i\in N}\mathcal{R}_i$ and any pair of alternatives $y, z \in A$, $\succsim (R_N; y, z)$ on $S(R_N; y, z)$ is complete but sequential inclusion does not hold. Then, there exists $\widetilde{R}_N \in \times_{i\in N}\mathcal{R}_i$, a pair of alternatives $y, z \in A$, such that $\succsim (\widetilde{R}_N; y, z)$ defined on $S(\widetilde{R}_N; y, z) = \{i \in N : y\widetilde{P}_i z\}$ has cycles. Let $k \geq 3$ be the minimal size of these cycles. Therefore there exists a set of k agents, say $S_k = \{1, 2, ..., k\}$ included in $S(\widetilde{R}_N; y, z)$ such that $1 \succ 2 \succ ... \succ k$ and $k \succ 1$. Thus, there exist at least $k+2$ alternatives, say y, z, a_1, a_2,...,$a_k \in A$, and k individual preference relations in $\widetilde{R}_N$, say R^1,..., R^k, satisfying conditions (1) to (5) in Lemma 2.

We now identify a subdomain and a rule on it that is strategy-proof but not group strategy-proof.[20]
Each individual will have two possible preferences. Then, out of all possible profiles we will single out two of them: the one where each agent i has preferences R^i, and the one where each agent i has opposite preferences $\overline{R}^i$. All other profiles will be classified in a way that depends on the sets of agents that have preferences of one type and of the other. Formally, consider the following subdomain of $\times_{i\in N}\mathcal{R}_i$. For any $l \in N\backslash\{1, ..., k\}$, define $\mathcal{D}_l = \mathcal{R}_l$. For any $i = 1, ..., k$, define $\mathcal{D}_i = \{R^i, \overline{R}^i\}$ where $\overline{R}^i$ is the opposite preference of R^i. Clearly, $\times_{i\in N}\mathcal{D}_i \subset \times_{i\in N}\mathcal{R}_i$. Let $\widetilde{R}_{S_k} = (R^1,..., R^k)$ and call $\overline{R}_{S_k} = (\overline{R}^1,..., \overline{R}^k)$.
Let $R_{S_k} \in \times_{i\in S_k}\mathcal{D}_i$. We define a partition of S_k that depends on R_{S_k}: Let $S_{k,z}(R_{S_k}) = \{i \in S_k : R_i \neq R^i\}$ and $S_{k,y}(R_{S_k}) = \{j \in S_k : R_j = R^j\}$.
Now construct f as follows: for any $\widehat{R}_{-S_k} \in \times_{l\in N\backslash S_k}\mathcal{D}_l$,

$$f(R^1_1, ..., R^k_k, \widehat{R}_{-S_k}) = f(\widetilde{R}_{S_k}, \widehat{R}_{-S_k}) = z,$$
$$f(R^{1+t}_1, ..., R^{k+t}_k, \widehat{R}_{-S_k}) = f(\overline{R}_{S_k}, \widehat{R}_{-S_k}) = y, \text{ and}$$
$$f(R_{S_k}, \widehat{R}_{-S_k}) = \arg\max_{a_l\in\{a_1,...,a_k\}} Borda^{\varepsilon}((R_{S_k}; \widehat{R}_{-S_k}); a_l), \text{ for any other } R_{S_k} \in \times_{i\in S_k}\mathcal{D}_i,$$

where $Borda^{\varepsilon}((R_{S_k}; \widehat{R}_{-S_k}); a_l) \equiv \sum_{j\in S_{k,y}(R_{S_k})} Borda^{\varepsilon}(R_j; a_l)$, with $Borda^{\varepsilon}(R_j; a_l)$ the score that alternative a_l receives under individual preference R_j for $j \in S_{k,y}(R_{S_k})$ as given in the

[20] In particular this rule happens to be not only strategy-proof but also (k-1)-group strategy-proof. This follows from the fact that our subdomain satisfies (k-1)-size sequential inclusion.

following table

$a_i \backslash R^j$	R^1	R^2	R^3	...	R^{k-2}	R^{k-1}	R^k
$Borda^\varepsilon(R^j; a_1)$	$k-1+\varepsilon$	0	1		$k-4$	$k-3$	$k-2+\varepsilon$
$Borda^\varepsilon(R^j; a_2)$	$k-2+\varepsilon$	$k-1+\varepsilon$	0		$k-5$	$k-4$	$k-3+\varepsilon$
$Borda^\varepsilon(R^j; a_3)$	$k-3+\varepsilon$	$k-2+\varepsilon$	$k-1+\varepsilon$		$k-6$	$k-5$	$k-4+\varepsilon$
.							
$Borda^\varepsilon(R^j; a_{k-2})$	$2+\varepsilon$	$3+\varepsilon$	$4+\varepsilon$		$k-1+\varepsilon$	0	$1+\varepsilon$
$Borda^\varepsilon(R^j; a_{k-1})$	$1+\varepsilon$	$2+\varepsilon$	$3+\varepsilon$		$k-2+\varepsilon$	$k-1+\varepsilon$	0
$Borda^\varepsilon(R^j; a_k)$	0	1	2		$k-3$	$k-2$	$k-1+\varepsilon$

and with $\varepsilon > 0$ and $\varepsilon < \frac{1}{k-2}$.
Notice first that only agents in $S_{k,y}$ play an effective role in the social choice function. The rest are dummies. Next, we should explain what we mean by the expression above: $Borda^\varepsilon(\cdot)$. What we do is to compute a variant of the Borda count for all those profiles that are different from the two we singled out to produce z and y, respectively. In this version of the Borda count, each alternative receives the points that correspond to it because of its position, plus eventually some value ε. The alternatives and positions that get this extra epsilon are chosen in such a way that no ties arise among winners within the domain. Therefore, we are defining a function, not a correspondence. The specific values that apply under this rule for our profiles are given in the above table.
Note that for any $\widehat{R}_{-S_k} \in \times_{l\in N\backslash S_k}\mathcal{R}_l$ and any $i \in \{1,...,k\}$, $f(\widehat{R}_{S_k}, \widehat{R}_{-S_k}) = a_i$, where $\widehat{R}_{S_k} \in \times_{l\in S_k}\mathcal{D}_l\backslash\{\widetilde{R}_{S_k}, \overline{R}_{S_k}\}$ is such that $S_{k,y}(\widehat{R}_{S_k}) = \{i\}$. Thus, the range of the social choice function is $A_f = \{a_1, ..., a_k, y, z\}$.

We now show the following two claims for any $R_{S_k} \in \times_{l\in S_k}\mathcal{D}_l\backslash\{\widetilde{R}_{S_k}, \overline{R}_{S_k}\}$, and $\widehat{R}_{-S_k} \in \times_{l\in N\backslash S_k}\mathcal{R}_l$. Note that $S_{k,z}(R_{S_k})$ and $S_{k,y}(R_{S_k})$ are non-empty sets. In words, Claim 1 requires that at least one agent belonging to the set $S_{k,y}$, of agents i with preference R^i, has no incentive to deviate to alternative y. Claim 2 requires that at least one agent belonging to the set $S_{k,z}$, of agents i with preference $R^{i\prime}$, has no incentive to deviate from alternative z.

Claim 1 Let $R_{S_k} \in \times_{l\in S_k}\mathcal{D}_l\backslash\{\widetilde{R}_{S_k}, \overline{R}_{S_k}\}$ and $\widehat{R}_{-S_k} \in \times_{l\in N\backslash S_k}\mathcal{R}_l$. Then,

$$f(R_{S_k}; \widehat{R}_{-S_k}) \in \bigcup_{j\in S_{k,y}(R_{S_k})} U(R_j, y),$$

where $U(R_j, y) = \{x \in A : xR_jy\}$.

Proof of Claim 1 By contradiction, suppose that, without loss of generality, $f(R_{S_k}; \widehat{R}_{-S_k}) = a_i$ such that yP_ja_i for all $j \in S_{k,y}(R_{S_k})$. Thus, $i \notin S_{k,y}(R_{S_k})$. Therefore,

$$\sum_{j\in S_{k,y}(R_{S_k})} Borda^\varepsilon\left(R_j; a_{(i-1)\mathrm{mod}k}\right) > \sum_{j\in S_{k,y}(R_{S_k})} Borda^\varepsilon(R_j; a_i),$$

and thus $f(R_{S_k}; \widehat{R}_{-S_k}) \neq a_i$ which is the desired contradiction. This ends the proof of Claim 1.

Claim 2 Let $R_{S_k} \in \times_{l \in S_k} \mathcal{D}_l \backslash \{\widetilde{R}_{S_k}, \overline{R}_{S_k}\}$ and $\widehat{R}_{-S_k} \in \times_{l \in N \backslash S_k} \mathcal{R}_l$. Then,

$$f(R_{S_k}; \widehat{R}_{-S_k}) \in \bigcup_{i \in S_{k,z}(R_{S_k})} L(R_i, z).$$

Proof of Claim 2 By contradiction, suppose that, without loss of generality, $f(R_{S_k}; \widehat{R}_{-S_k}) = a_l$ such that $a_l \notin \bigcup_{i \in S_{k,z}(R_{S_k})} L(R_i, z)$. Therefore, $(l+1) \bmod k \in S_{k,y}(R_{S_k})$. For notational simplicity, we will write $(l+1)$ instead. By Claim 1, $l \in S_{k,y}(R_{S_k})$. Note that for any $l = 1, ..., k$,

$$Borda^{\varepsilon}(R_{l+1}; a_{l+1}) - Borda^{\varepsilon}(R_{l+1}; a_l) = k - 1 + \varepsilon.$$

Since $S_{k,z}(R_{S_k})$ is not empty, if $l \neq k-1$ we have

$$\sum_{j \in S_{k,y}(R_{S_k}) \backslash \{l+1\}} Borda^{\varepsilon}(R_j; a_l) - \sum_{j \in S_{k,y}(R_{S_k}) \backslash \{l+1\}} Borda^{\varepsilon}(R_j; a_{l+1}) < k-1.$$

Then,

$$Borda^{\varepsilon}\left((R_{S_k}; \widehat{R}_{-S_k}); a_{l+1}\right) - Borda^{\varepsilon}\left((R_{S_k}; \widehat{R}_{-S_k}); a_l\right) > k - 1 + \varepsilon + 1 - k = \varepsilon > 0,$$

and thus $f(R_{S_k}; \widehat{R}_{-S_k}) \neq a_l$ which is the desired contradiction.
If $l = k-1$ and since $S_{k,z}(R_{S_k})$ is not empty,

$$\sum_{j \in S_{k,y}(R_{S_k}) \backslash \{k\}} Borda^{\varepsilon}(R_j; a_{k-1}) - \sum_{j \in S_{k,y}(R_{S_k}) \backslash \{k\}} Borda^{\varepsilon}(R_j; a_k) \leq (k-2)(1+\varepsilon).$$

Therefore,

$$Borda^{\varepsilon}\left((R_{S_k}; \widehat{R}_{-S_k}); a_k\right) - Borda^{\varepsilon}\left((R_{S_k}; \widehat{R}_{-S_k}); a_{k-1}\right) \geq 1 - (k-3)\varepsilon$$

where $1 - (k-3)\varepsilon > 0$ since $\varepsilon < \frac{1}{k-2}$. Thus $f(R_{S_k}; \widehat{R}_{-S_k}) \neq a_l = a_{k-1}$ which is the desired contradiction. This ends the proof of Claim 2.

We now show that f is strategy-proof but it is not k-group strategy-proof. Observe that by construction of f, the only relevant deviating coalitions are those $C \subset S_k$. Since each agent in coalition S_k strictly gains by deviating from $f(R_1^1, R_2^2, ..., R_k^k, ; \widehat{R}_{-S_k}) = z$ to $f(\overline{R}_1^1, \overline{R}_2^2, ..., \overline{R}_k^k; \widehat{R}_{-S_k}) = y$, then f is not k-group strategy-proof.

We now check that no single individual $l \in S_k$ can manipulate f at any profile. Let $(R_{S_k}; \widehat{R}_{-S_k}) \in \times_{i \in N} \mathcal{D}_i$ be any profile. Remember that for any $l \in S_k$, $\mathcal{D}_l = \{R^l, \overline{R}^l\}$. We distinguish two cases:

Case 1 If $l \in S_{k,y}(R_{S_k})$. By definition of f, $f(R_{S_k}, \widehat{R}_{-S_k}) \neq y$. Suppose that $f(R_{S_k}; \widehat{R}_{-S_k}) = a_i$ for some $i \in S_k$ and thus $S_{k,y}(R_{S_k}) \subsetneq S_k \backslash \{l\}$. If agent l announces $\overline{R}_l^l$ instead of R_l^l, in the definition of f agent l deducts more score to those alternatives he likes most. Therefore either the initial alternative a_i or else a less preferred alternative for agent l is chosen and agent l does not manipulate f at $(R_{S_k}, \widehat{R}_{-S_k})$ via $\overline{R}^l$. An identical argument would apply if $f(R_{S_k}; \widehat{R}_{-S_k}) = z$ instead of a_i. This shows that no agent $l \in S_{k,y}(R_{S_k})$ can manipulate f at any profile $(R_{S_k}; \widehat{R}_{-S_k})$ via $\overline{R}_l^l$.

Case 2 If $l \in S_{k,z}(R_{S_k})$. A similar argument to Case 1 works: By definition of f, $f(R_{S_k}, \widehat{R}_{-S_k}) \neq z$. Suppose that $f(R_{S_k}; \widehat{R}_{-S_k}) = a_i$ for some $i \in S_k$ and thus $S_{k,y}(R_{S_k}) \subset S_k \backslash \{l\}$. If agent l announces R_l^l instead of $\overline{R}_l^l$, in the definition of f agent l adds more score to those alternatives he likes less (when announcing R_l^l agent $l \in S_{k,y}((R_l^l, R_{S_k \backslash \{l\}}; \widehat{R}_{-S_k})$. Therefore either the initial alternative a_i or else a less preferred alternative for agent l is chosen and agent l does not manipulate f at $(R_{S_k}, \widehat{R}_{-S_k})$ via R^l. An identical argument would apply if $f(R_{S_k}; \widehat{R}_{-S_k}) = y$ instead of a_i. This shows that no agent $l \in S_{k,y}(R_{S_k})$ can manipulate f at any profile $(R_{S_k}; \widehat{R}_{-S_k})$ via R_l^l. ■

References

Barberà, S. (1979). "A Note on Group Strategy-Proof Decision Schemes". *Econometrica,* 47(3), 637-640.

Barberà, S., H. Sonnenschein and L. Zhou (1991). "Voting by committees". *Econometrica,* 59, 595-609.

Barberà, S. and M. Jackson (1995). "Strategy-proof Exchange". *Econometrica,* 63 (1): 51-87.

Dasgupta, P., P. Hammond, and E. Maskin (1979). "The Implementation of Social Choice Rules: Some General Results on Incentive Compatibility". *Review of Economic Studies,* 46 (2): 185-216.

Gibbard, A. (1973). "Manipulation of Voting Schemes: A General Result". *Econometrica,* 41 (4): 587-601.

Green, J. and J.-J. Laffont (1979). "On Coalition Incentive Compatibility". *Review of Economic Studies*, 46 (2): 243-254.

Le Breton, M. and V. Zaporozhets (2008). "On the Equivalence of Coalitional and Individual Strategy-Proofness Properties". Forthcoming in *Social Choice and Welfare.*

Moulin, H. (1999). "Incremental cost-sharing: Characterization by coalition strategy-proofness". *Social Choice and Welfare*, 16 (2): 279-320.

Pápai, S. (2000). "Strategyproof Assignment by Hierarchical Exchange". *Econometrica,* 68: 1403-1433.

Pattanaik, P. K. (1978). *Strategy and Group Choice.* North-Holland Publishing Company.

Peleg, B. (1998). "Almost all equilibria in dominant strategies are coalition-proof". *Economics letters* 60: 157-162.

Peleg, B. and P. Sudholter (1999). "Single-peakedness and coalition-proofness". *Review of Economic Design* 4: 381-387.

Satterthwaite, M. (1975). "Strategy-Proofness and Arrow's Conditions: Existence and Correspondence Theorems for Voting Procedures and Social Welfare Functions*". Journal of Economic Theory* 10, 187-217.

Serizawa, S. (2006). "Pairwise strategy-proofness and self-enforcing manipulation". *Social Choice and Welfare*, 26: 305-331.

Thomson, W. (2008). "Where to send your daughter to college?". Mimeo University of Rochester.