AN ALGORITHM TO COMPUTE THE SET OF MANY-TO-MANY STABLE MATCHINGS*

by

R. Martínez[†], J. Massó[‡], A. Neme[†], and J. Oviedo[†]

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Abstract: The paper proposes an algorithm to compute the set of many-to-many stable matchings when agents have substitutable preferences. The algorithm starts by calculating the two optimal-stable matchings using the deferred-acceptance algorithm. Then, it computes each remaining stable matching as the firm-optimal stable matching corresponding to a new preference profile which is obtained after modifying the preferences of a previously identified sequence of firms.

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[†] Instituto de Matemática Aplicada. Universidad Nacional de San Luis and CONICET. Ejército de los Andes 950. 5700, San Luis, Argentina. E-mails: martinez@unsl.edu.ar, aneme@unsl.edu.ar, and joviedo@unsl.edu.ar

[‡] Departament d'Economia i d'Història Econòmica and CODE. Universitat Autònoma de Barcelona. 08193, Bellaterra (Barcelona), Spain. E-mail: jordi.masso@uab.es

1 Introduction

The paper proposes an algorithm to compute the set of many-to-many stable matchings when agents have substitutable preferences.

Many-to-many matching models have been useful for studying assignment problems with the distinctive feature that agents can be divided from the very beginning into two disjoint subsets: the set of firms and the set of workers.[1] The nature of the assignment problem consists of matching each agent (firms and workers) with a subset of agents from the other side of the market. Thus, each firm will hire a subset of workers while each worker may work for a number of different firms.

The problem becomes interesting because agents have preferences on the subsets of potential partners. Stability has been considered the main property to be satisfied by any sensible matching. A matching is called stable if all agents have an acceptable subset of partners and there is no unmatched worker-firm pair who both would prefer to add the other to their current subset of partners. To give all blocking power to only individual agents and worker-firm pairs seems a very weak requirement in terms of the durability of the matching.

Unfortunately the set of stable matchings may be empty. Substitutability is the weakest condition imposed on agents preferences under which the existence of stable matchings is guaranteed. An agent has substitutable preferences if he continues to want to partner an agent of the other side of the market even if other agents become unavailable.[2]

Surprisingly, the set of stable matchings under substitutable preferences is very-well structured. It contains two distinctive matchings: the firm-optimal stable matching (denoted by μ_F) and the worker-optimal stable matching (denoted by μ_W). The matching μ_F is unanimously considered by all firms to be the best among all stable matchings and by all workers to be the worst among all stable matchings. Symmetrically, the matching μ_W is unanimously considered by all workers to be the best among all stable matchings and by all firms to be the worst among all stable matchings. They can be obtained by the so-called deferred-acceptance algorithm (originally defined by Gale and Shapley (1962) for the one-to-one case and later adapted by Roth (1984) to the many-to-many case). Additionally, Blair (1988) shows that the set of

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[1] We will be using as a reference (and as a source of terminology) labor markets with part-time jobs and we will generically refer to these two sets as the two sides of the market.

[2] See Definition 3 for a formal statement of this property. Kelso and Crawford (1982) were the first to use it to show the existence of stable matchings in a many-to-one model with money. Roth (1984) shows that if all agents have substitutable preferences the set of many-to-many stable matchings is non-empty.

stable matchings has a lattice structure.[3] In particular, Roth (1984) and Blair (1988) show that this unanimity and opposition of interests of the two sides of the market is even stronger in the sense that all firms, if they had to choose the best subset from the set of workers made up of the union of the firm-optimal stable matching and any other stable matching, would choose the firm-optimal stable matching. Also, all firms, if they had to choose the best subset from the set of workers made up of the union of the worker-optimal stable matching and any other stable matching, would choose the stable matching. And symmetrically, the two properties also hold interchanging the roles of firms and workers.[4]

While there are many algorithms designed to compute the full set of one-to-one stable matchings as well as the two-optimal stable matchings (for the many-to-many model) we are not aware of any algorithm which can compute the *full* set of matchings for this more general many-to-many case.[5] This paper provides such an algorithm.

Roughly, our algorithm works by applying successively the following procedure. First, and given as input an original profile of substitutable preferences, it computes by the deferred-acceptance algorithm the two optimal stable matchings μ_F and μ_W. Second, it identifies all firm-worker pairs (f, w) where firm f hires the worker w in μ_F but *not* in μ_W. Successively, for each of these pairs, it modifies the preference of firm f by declaring all subsets of workers containing worker w unacceptable but leaving the orderings among all subsets not containing w unchanged. This is called an (f, w)-truncation of the original preference. By the deferred-acceptance algorithm it computes (for each pair) the firm-optimal stable matching corresponding to the preference profile where all agents have the original preferences except that firm f has the (f, w)-truncated preference. Third, although this new firm-optimal stable matching may not be stable relative to the original preference profile it is stable provided that all workers, if they had to choose the best subset from the set of firms made up of the union of the two firm-optimal stable matchings (the original and the new one) they would choose the new one. If it passes this test (and hence, if it is stable relative to the original profile of preferences) we keep it and proceed again from the very beginning using this modified profile as an input.[6] The algorithm stops when there is no

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[3] Roth (1985), Sotomayor (1999), Alkan (1999), and Martínez, Massó, Neme, and Oviedo (1999) also study the lattice structure of the set of stable matchings in different models.

[4] See Remark 1 in Section 2 for a formal statement of these four properties.

[5] See Gusfield and Irving (1989) for an algorithmic approach to the one-to-one and roommate models.

[6] In the formal definition of the algorithm the reader will find an additional (but dis-

firm-worker pair with the above property.

The paper is organized as follows. In Section 2 we present the preliminary notation, definitions, and results. Section 3 contains the definition of the algorithm, the Theorem stating that the outcome of the algorithm is equal to the set of stable matchings, and an example illustrating how the algorithm works. In Section 4 we prove the Theorem. Finally, an Appendix at the end of the paper illustrates by means of an example the deferred-acceptance algorithm of Gale and Shapley adapted to the many-to-many case.

2 Preliminaries

There are two disjoint sets of *agents*, the set of n *firms* $F = \{f_1, ..., f_n\}$ and the set of m *workers* $W = \{w_1, ..., w_m\}$. Generic elements of both sets will be denoted, respectively, by f, f_i, f_{i_k}, $\bar{f}$, and $\tilde{f}$, and by w, w_j, w_{j_k}, $\bar{w}$, and $\tilde{w}$. A generic agent will be denoted by a and we will refer to a *set of partners of* a as a subset of agents of the set not containing a. Each agent $a \in F \cup W$ has a strict, transitive, and complete preference relation $P(a)$ over the set of all subsets of partners (over 2^F if a is a worker and over 2^W if a is a firm). *Preference profiles* are $(n+m)$-tuples of preference relations and they are represented by $P = (P(f_1), ..., P(f_n); P(w_1), ..., P(w_m))$. Given a preference relation of an agent $P(a)$ the subsets of partners preferred to the empty set by a are called *acceptable*; therefore, we are allowing for the possibility that firm f may prefer not hiring any worker rather than hiring unacceptable subsets of workers and that worker w may prefer to remain unemployed rather than working for an unacceptable subset of firms.

To express preference relations in a concise manner, and since only acceptable partners will matter, we will represent preference relations as lists of acceptable partners. For instance,

$$\begin{aligned} P(f_i) &= w_1 w_3, w_2, w_1, w_3 \\ P(w_j) &= f_1 f_3, f_1, f_3 \end{aligned}$$

indicate that $\{w_1, w_3\} P(f_i) \{w_2\} P(f_i) \{w_1\} P(f_i) \{w_3\} P(f_i) \emptyset$ and $\{f_1, f_3\} P(w_j) \{f_1\} P(w_j) \{f_3\} P(w_j) \emptyset$.

The assignment problem consists of matching workers with firms keeping the bilateral nature of their relationship and allowing for the possibility that both, firms and workers, may remain unmatched. Formally,

Definition 1. *A* ***matching*** μ *is a mapping from the set* $F \cup W$ *into the set of all subsets of* $F \cup W$ *such that for all* $w \in W$ *and* $f \in F$*:*

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pensable) step only used to speed up the algorithm.

1. $\mu(w) \subseteq F$ *or else* $\mu(w) = \emptyset$.
2. $\mu(f) \subseteq W$ *or else* $\mu(f) = \emptyset$.
3. $f \in \mu(w)$ *if and only if* $w \in \mu(f)$.

Condition *1* says that a worker can be either matched to a subset of firms or remain unmatched. Condition *2* says that a firm can either hire a subset of workers or be unmatched. Finally, condition *3* states the bilateral nature of a matching in the sense that firm f hires worker w if and only if worker w works form firm f. We say that w and f are *single* in a matching μ if $\mu(w) = \emptyset$ and $\mu(f) = \emptyset$. Otherwise, they are matched. A matching μ is said to be *one-to-one* if firms can hire at most one worker and workers can work for at most one firm. The model in which all matchings are one-to-one is also known in the literature as the *marriage model.* A matching μ is said to be *many-to-one* if workers can work for at most one firm but firms may hire many workers. The model in which all matchings are many-to-one, and firms have responsive preferences,[7] is also known in the literature as the *college admissions model.* To represent matchings concisely we will follow the widespread notation where, for instance, given $F = \{f_1, f_2, f_3\}$ and $W = \{w_1, w_2, w_3, w_4\}$

$$\begin{array}{ccccc} & f_1 & f_2 & f_3 & \emptyset \\ \mu_1 & w_3w_4 & w_1 & w_1w_3w_4 & w_2 \\ \mu_2 & \emptyset & w_1w_2 & w_3 & w_4 \end{array}$$

represents two matchings where in matching μ_1 firm f_1 is matched to workers w_3 and w_4, firm f_2 is matched to worker w_1, firm f_3 is matched to workers w_1, w_3, and w_4, and worker w_2 is single and in matching μ_2 firm f_1 and worker w_4 are single, firm f_2 is matched to workers w_1 and w_2, and firm f_3 is matched to worker w_3. Notice that we could equivalently represent the two matchings as

$$\begin{array}{cccccc} & w_1 & w_2 & w_3 & w_4 & \emptyset \\ \mu_1 & f_2f_3 & \emptyset & f_1f_3 & f_1f_3 & \emptyset \\ \mu_2 & f_2 & f_2 & f_3 & \emptyset & f_1. \end{array}$$

Let P be a preference profile. Given a set of partners S, let $Ch(S, P(a))$ denote agent a's *most-preferred* subset of S according to a's preference ordering $P(a)$. A matching μ is *blocked by agent a* if $\mu(a) \neq Ch(\mu(a), P(a))$.

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[7] Namely, for any two subsets of workers that differ in only one worker a firm prefers the subset containing the most-preferred worker. See Roth and Sotomayor (1990) for a precise and formal definition of responsive preferences as well as for a masterful and illuminating analysis of these models and an exhaustive bibliography.

We say that a matching is *individually rational* if it is not blocked by any agent. We will denote by $IR(P)$ the set of individually rational matchings. A matching μ is *blocked by a worker-firm pair* (w, f) if $w \notin \mu(f)$, $w \in Ch(\mu(f) \cup \{w\}, P(f))$, and $f \in Ch(\mu(w) \cup \{f\}, P(w))$.

Definition 2. *A matching μ is* ***stable*** *if it is not blocked by any individual agent or any worker-firm pair.*

Given a preference profile P, denote the set of stable matchings by $S(P)$. It is easy to construct examples of preference profiles with the property that the set of stable matchings is empty. Those examples share the feature that at least one agent regards a subset of partners as being complements. This is the reason why the literature has focused on the restriction where partners are regarded as substitutes.

Definition 3. *An agent a's preference ordering $P(a)$ satisfies* ***substitutability*** *if for any set S of partners containing agents b and c $(b \neq c)$, if $b \in Ch(S, P(a))$ then $b \in Ch(S \setminus \{c\}, P(a))$.*

A preference profile P is *substitutable* if for each agent a, the preference ordering $P(a)$ satisfies substitutability.

Roth (1984) shows that if all agents have substitutable preferences then: (1) the set of stable matchings is non-empty, (2) firms (workers) unanimously agree that a stable matching μ_F (μ_W) is the best stable matching,[8] and (3) the optimal stable matching for one side is the worst stable matching for the other side (that is, for all $\mu \in S(P)$ we have that $\mu R(f) \mu_W$ for all $f \in F$ and $\mu R(w) \mu_F$ for all $w \in W$).

The deferred-acceptance algorithm, originally defined by Gale and Shapley (1962) for the one-to-one case, produces either μ_F or μ_W depending on who makes the offers. At any step of the algorithm in which firms make offers, a firm proposes itself to the choice set of the set of workers that have not already rejected it during the previous steps, while a worker accepts the choice set of the set of current offers plus those of the firms provisionally matched in the previous step (if any). The algorithm stops at the step at which all offers are accepted; the (provisional) matching becomes then definite and it

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[8] The matchings μ_F and μ_W are called, respectively, the firms-optimal stable matching and the workers-optimal stable matching. We are following the convention of extending preferences from the original sets (2^W and 2^F) to the set of matchings. However, we now have to consider weak orderings since the matchings μ and μ' may associate the same set of partners to an agent. These orderings will be denoted by $R(f)$ and $R(w)$. For instance, to say that all firms prefer μ_F to any stable μ means that for every $f \in F$ we have that $\mu_F R(f) \mu$ for all stable μ (that is, either $\mu_F(f) = \mu(f)$ or else $\mu_F(f) P(f) \mu(f)$).

is the stable matching μ_F. Symmetrically, if workers make offers, the outcome of the algorithm is the stable matching μ_W. The Appendix at the end of the paper illustrates by means of an example how the deferred-acceptance algorithm works for the many-to-many case.

Our algorithm will consist of applying the deferred-acceptance algorithm where firms make offers to preferences profiles that are obtained after modifying the preference of a firm by making all subsets containing a particular worker unacceptable.[9] Formally,

Definition 4. *We say that a preference $P^{(f,w)}(f)$ is an $(f,w)-$**truncation of** $P(f)$ if:*

1. *There exists an acceptable subset of workers according to $P(f)$ containing worker w; that is, $\exists S \in 2^W$ such that $w \in S$ and $SP(f)\emptyset$.*
2. *The preferences $P(f)$ and $P^{(f,w)}(f)$ coincide on all subsets that not contain w; that is, if $w \notin S_1 \cup S_2$ then $S_1 P(f) S_2$ if and only if $S_1 P^{(f,w)}(f) S_2$.*
3. *All subsets containing w are unacceptable according to $P^{(f,w)}(f)$; that is, if $w \in S$ then $\emptyset P^{(f,w)}(f) S$.*

Given a preference profile P and an $(f,w)-$truncation of $P(f)$ we denote by $P^{(f,w)}$ the preference profile obtained by replacing $P(f)$ in P by $P^{(f,w)}(f)$. We denote by $\mu_F^{(f,w)}$ and $\mu_W^{(f,w)}$ the firm and worker-optimal stable matchings corresponding to the preference profile $P^{(f,w)}$. Moreover, given a preference profile P and a sequence of pairs $(f_{i_1}, w_{j_1}) \dots (f_{i_k}, w_{j_k})$ we will represent by $P^{(f_{i_1}, w_{j_1}) \dots (f_{i_k}, w_{j_k})}$ the preference profile obtained from P after successively truncating the corresponding preference; we will also denote by $\mu_F^{(f_{i_1}, w_{j_1}) \dots (f_{i_k}, w_{j_k})}$ and $\mu_W^{(f_{i_1}, w_{j_1}) \dots (f_{i_k}, w_{j_k})}$ its corresponding optimal-stable matchings. The following lemma states that the property of substitutability is preserved by truncations and therefore $\mu_F^{(f,w)}$ and $\mu_W^{(f,w)}$ exist provided that P is substitutable.

Lemma 1. *If $P(f)$ is substitutable then $P^{(f,w)}(f)$ is substitutable.*

Proof. Let $\bar{w}, w' \in S$ be arbitrary and assume that $\bar{w} \in Ch(S, P^{(f,w)}(f))$. If $w \notin S$, then $\bar{w} \in Ch(S \backslash \{w'\}, P^{(f,w)}(f))$ because $Ch(S, P^{(f,w)}(f)) = Ch(S, P(f))$,

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[9] Given the symmetric role of firms and workers it will become clear that the construction that follows could be equivalently done by interchanging the roles of workers and firms.

$Ch(S\backslash\{w'\}, P^{(f,w)}(f)) = Ch(S\backslash\{w'\}, P(f))$, and because of the substitutability of $P(f)$. If $w \in S$, then we have that $Ch(S, P^{(f,w)}(f)) = Ch(S\backslash\{w\}, P(f))$; therefore, by assumption $\bar{w} \in Ch(S\backslash\{w\}, P(f))$. By the substitutability of $P(f)$ we have that $\bar{w} \in Ch([S\backslash\{w\}]\backslash\{w'\}, P(f))$ but the two equalities $Ch([S\backslash\{w\}]\backslash\{w'\}, P(f)) = Ch([S\backslash\{w\}]\backslash\{w'\}, P^{(f,w)}(f)) = Ch(S\backslash\{w'\}, P^{(f,w)}(f))$ imply that worker $\bar{w} \in Ch(S\backslash\{w'\}, P^{(f,w)}(f))$. $\square$

Before finishing this section we present, as a Remark below, four properties of stable matchings.

Remark 1 Assume P is substitutable and let $\mu \in S(P)$. Then, for all f and w:

1. $Ch(\mu_F(f) \cup \mu(f), P(f)) = \mu_F(f)$.
2. $Ch(\mu_W(w) \cup \mu(w), P(w)) = \mu_W(w)$.
3. $Ch(\mu_W(f) \cup \mu(f), P(f)) = \mu(f)$.
4. $Ch(\mu_F(w) \cup \mu(w), P(w)) = \mu(w)$.

Properties 1 and 2 are due to Roth (1984) while properties 3 and 4 follow from 1, 2, and Theorem 4.5 in Blair (1988). They can be interpreted as an strengthening of the optimality of μ_F and μ_W. Since Property 4 will play a crucial role in the construction of our algorithm we will some times refer to it as the *Choice Property*. Example 1 below shows that, although necessary, they are far from being a characterization of stable matchings.

Example 1 Let $F = \{f_1, f_2, f_3, f_4\}$ and $W = \{w_1, w_2, w_3, w_4\}$ be the two sets of agents with the preference profile P, where

$$\begin{aligned}
P(f_1) &= w_1, w_2, w_3, w_4\\
P(f_2) &= w_2, w_4, w_1\\
P(f_3) &= w_3, w_1, w_2\\
P(f_4) &= w_4, w_2, w_3\\
P(w_1) &= f_2, f_3, f_1\\
P(w_2) &= f_3, f_1, f_4, f_2\\
P(w_3) &= f_4, f_1, f_3\\
P(w_4) &= f_1, f_2, f_4.
\end{aligned}$$

The two optimal-stable matchings are

$$\begin{array}{ccccc}
 & f_1 & f_2 & f_3 & f_4\\
\mu_F & w_1 & w_2 & w_3 & w_4\\
\mu_W & w_4 & w_1 & w_2 & w_3.
\end{array}$$

The matching

$$\begin{array}{ccccc} & f_1 & f_2 & f_3 & f_4 \\ \mu & w_3 & w_4 & w_1 & w_2 \end{array}$$

is not stable since (w_2, f_1) blocks it. However, it can be verified that μ satisfies the four properties of Remark 1.

3 An Algorithm to compute the set of stable matchings

3.1 The Algorithm and the Theorem

Given a preference profile P, we define an algorithm to compute the set of stable matchings $S(P)$.

Stage 1: Input P. By the deferred-acceptance algorithm obtain μ_F and μ_W. Set $T^0(P) = P$ and $S^0(P) = \{\mu_F\}$.
Step 1: Define $T(T^0(P)) = \left\{P^{(f,w)} \mid w \in \mu_F(f) \setminus \mu_W(f)\right\}$.
Step 2: (a) If $T(T^0(P)) = \emptyset$ set $T^1(P) = \emptyset$ and $S^1(P) = S^0(P)$.
(b) If not, for each truncation $P^{(f,w)} \in T(T^0(P))$ obtain $\mu_F^{(f,w)}$.
Step 3: Define

$$T^*\left(T^0(P)\right) = \left\{ \begin{array}{l} P^{(f,w)} \in T(T^0(P)) \mid \forall w' \in W, \\ Ch\left(\mu_F^{(f,w)}(w') \cup \mu_F(w'), P(w')\right) = \mu_F^{(f,w)}(w') \end{array} \right\}.$$

Order the set $T^*(T^0(P))$ in an arbitrary way and let $\prec^1$ denote this ordering.
Step 4: Define

$$\widehat{T}\left(T^0(P)\right) = \left\{ \begin{array}{l} P^{(f,w)} \in T^*(T^0(P)) \mid \forall P^{(f',w')} \in T^*(T^0(P)) \\ \text{such that } P^{(f,w)} \prec^1 P^{(f',w')}, w' \in \mu_F^{(f,w)}(f') \end{array} \right\}.$$

Set

$$T^1(P) = \widehat{T}\left(T^0(P)\right)$$

and

$$S^1(P) = S^0(P) \cup \left\{\mu_F^{(f,w)} \mid P^{(f,w)} \in T^1(P)\right\}.$$

End of Stage 1.

Stage $k+1$: Input $T^k(P)$ and $S^k(P)$. By the deferred-acceptance algorithm where firms make offers we have obtained, for each $P^{\left(f_{i_1},w_{j_1}\right)\cdots\left(f_{i_k},w_{j_k}\right)} \in T^k(P)$, its corresponding $\mu_F^{\left(f_{i_1},w_{j_1}\right)\cdots\left(f_{i_k},w_{j_k}\right)}$.

Step 1: Define

$$T\left(T^k\left(P\right)\right)=\left\{P^{\left(f_{i_1},w_{j_1}\right)\cdots\left(f_{i_k},w_{j_k}\right)(f,w)}\mid w\in\mu_F^{\left(f_{i_1},w_{j_1}\right)\cdots\left(f_{i_k},w_{j_k}\right)}\left(f\right)\backslash\mu_W\left(f\right)\right\}.$$

Step 2: (a) If $T\left(T^k\left(P\right)\right)=\emptyset$ set $T^{k+1}\left(P\right)=\emptyset$ and $S^{k+1}\left(P\right)=S^k\left(P\right)$.

(b) If not, for each truncation $P^{\left(f_{i_1},w_{j_1}\right)\cdots\left(f_{i_k},w_{j_k}\right)(f,w)}\in T\left(T^k\left(P\right)\right)$ obtain $\mu_F^{\left(f_{i_1},w_{j_1}\right)\cdots\left(f_{i_k},w_{j_k}\right)(f,w)}$.

Step 3: Define

$$T^*\left(T^k\left(P\right)\right)=\left\{\begin{array}{l}P^{\left(f_{i_1},w_{j_1}\right)\cdots\left(f_{i_k},w_{j_k}\right)(f,w)}\in T\left(T^k\left(P\right)\right)\mid\forall w'\in W,\\ Ch\left(\mu_F^{\left(f_{i_1},w_{j_1}\right)\cdots\left(f_{i_k},w_{j_k}\right)(f,w)}\left(w'\right)\cup\mu_F^{\left(f_{i_1},w_{j_1}\right)\cdots\left(f_{i_k},w_{j_k}\right)}\left(w'\right),P\left(w'\right)\right)=\\ =\mu_F^{\left(f_{i_1},w_{j_1}\right)\cdots\left(f_{i_k},w_{j_k}\right)(f,w)}\left(w'\right)\end{array}\right\}.$$

Order the set $T^*\left(T^k\left(P\right)\right)$ in an arbitrary way and let $\prec^{k+1}$ denote this ordering.

Step 4: Define

$$\widehat{T}\left(T^k\left(P\right)\right)=\left\{\begin{array}{l}P^{\left(f_{i_1},w_{j_1}\right)\cdots\left(f_{i_k},w_{j_k}\right)(f,w)}\in T^*\left(T^k\left(P\right)\right)\mid\\ \forall P^{\left(f_{i_1},w_{j_1}\right)\cdots\left(f_{i_k},w_{j_k}\right)(f',w')}\in T^*\left(T^k\left(P\right)\right)\text{ such that}\\ P^{\left(f_{i_1},w_{j_1}\right)\cdots\left(f_{i_k},w_{j_k}\right)(f,w)}\prec^{k+1}P^{\left(f_{i_1},w_{j_1}\right)\cdots\left(f_{i_k},w_{j_k}\right)(f',w')},\\ w'\in\mu_F^{\left(f_{i_1},w_{j_1}\right)\cdots\left(f_{i_k},w_{j_k}\right)(f,w)}\left(f'\right)\end{array}\right\}.$$

Set

$$T^{k+1}\left(P\right)=\widehat{T}\left(T^k\left(P\right)\right)$$

and

$$S^{k+1}\left(P\right)=S^k\left(P\right)\cup\left\{\mu_F^{\left(f_{i_1},w_{j_1}\right)\cdots\left(f_{i_k},w_{j_k}\right)(f,w)}\mid P^{\left(f_{i_1},w_{j_1}\right)\cdots\left(f_{i_k},w_{j_k}\right)(f,w)}\in T^{k+1}\left(P\right)\right\}.$$

End of Stage $k+1$.

The algorithm stops at the stage K where $T^K\left(P\right)$ is empty.

Theorem 1. *Assume P is substitutable and let K be the stage where the algorithm stops. Then $S^K\left(P\right)=S\left(P\right)$.*

3.2 An Example

We illustrate how the algorithm works with the following example.

Example 2 Let $F = \{f_1, f_2, f_3, f_4\}$ and $W = \{w_1, w_2, w_3, w_4\}$ be the two sets of agents with the substitutable profile of preferences P, where

$$\begin{array}{rl}
P(f_1) = & w_1w_2, w_1w_3, w_2w_4, w_3w_4, w_1w_4, w_2w_3, w_1, w_2, w_3, w_4 \\
P(f_2) = & w_1w_2, w_2w_3, w_1w_4, w_3w_4, w_1w_3, w_2w_4, w_1, w_2, w_3, w_4 \\
P(f_3) = & w_3w_4, w_2w_3, w_1w_4, w_1w_2, w_2w_4, w_1w_3, w_1, w_2, w_3, w_4 \\
P(f_4) = & w_3w_4, w_2w_4, w_1w_3, w_1w_2, w_2w_3, w_1w_4, w_1, w_2, w_3, w_4 \\
P(w_1) = & f_3f_4, f_2f_3, f_2f_4, f_1f_4, f_1f_3, f_1f_2, f_1, f_2, f_3, f_4 \\
P(w_2) = & f_3f_4, f_2f_3, f_1f_4, f_2f_4, f_1f_3, f_1f_2, f_1, f_2, f_3, f_4 \\
P(w_3) = & f_1f_2, f_2f_3, f_1f_3, f_2f_4, f_1f_4, f_3f_4, f_1, f_2, f_3, f_4 \\
P(w_4) = & f_1f_2, f_1f_3, f_1f_4, f_2f_3, f_2f_4, f_3f_4, f_1, f_2, f_3, f_4.
\end{array}$$

Stage 1: By the deferred-acceptance algorithm we obtain the two optimal-stable matchings

$$\begin{array}{ccccc}
 & f_1 & f_2 & f_3 & f_4 \\
\mu_F & w_1w_2 & w_1w_2 & w_3w_4 & w_3w_4 \\
\mu_W & w_3w_4 & w_3w_4 & w_1w_2 & w_1w_2.
\end{array}$$

Set $T^0(P) = P$ and $S^0(P) = \{\mu_F\}$. The set $T(T^0(P))$ of Step 1 consists of the following truncations of P:

$$T\left(T^0(P)\right) = \left\{P^{(f_1,w_1)}, P^{(f_1,w_2)}, P^{(f_2,w_1)}, P^{(f_2,w_2)}, P^{(f_3,w_3)}, P^{(f_3,w_4)}, P^{(f_4,w_3)}, P^{(f_4,w_4)}\right\}$$

where in all profiles firms and workers have the same preference as in P, except

$$\begin{array}{rcl}
P^{(f_1,w_1)}(f_1) & = & w_2w_4, w_3w_4, w_2w_3, w_2, w_3, w_4 \\
P^{(f_1,w_2)}(f_1) & = & w_1w_3, w_3w_4, w_1w_4, w_1, w_3, w_4 \\
P^{(f_2,w_1)}(f_2) & = & w_2w_3, w_3w_4, w_2w_4, w_2, w_3, w_4 \\
P^{(f_2,w_2)}(f_2) & = & w_1w_4, w_3w_4, w_1w_3, w_1, w_3, w_4 \\
P^{(f_3,w_3)}(f_3) & = & w_1w_4, w_1w_2, w_2w_4, w_1, w_2, w_4 \\
P^{(f_3,w_4)}(f_3) & = & w_2w_3, w_1w_2, w_1w_3, w_1, w_2, w_3 \\
P^{(f_4,w_3)}(f_4) & = & w_2w_4, w_1w_2, w_1w_4, w_1, w_2, w_4 \\
P^{(f_4,w_4)}(f_4) & = & w_1w_3, w_1w_2, w_2w_3, w_1, w_2, w_3.
\end{array}$$

In Step 2, and since the set $T\left(T^0\left(P\right)\right)$ is non-empty, we obtain for each of its truncations the corresponding firm-optimal stable matching

$$\begin{array}{lcccc}
 & f_1 & f_2 & f_3 & f_4 \\
\mu_F^{(f_1,w_1)} & w_2w_4 & w_1w_2 & w_3w_4 & w_1w_3 \\
\mu_F^{(f_1,w_2)} & w_1w_3 & w_1w_2 & w_3w_4 & w_2w_4 \\
\mu_F^{(f_2,w_1)} & w_1w_2 & w_3w_4 & w_3w_4 & w_1w_2 \\
\mu_F^{(f_2,w_2)} & w_2w_4 & w_1w_4 & w_2w_3 & w_1w_3 \\
\mu_F^{(f_3,w_3)} & w_2w_4 & w_2w_3 & w_1w_4 & w_1w_3 \\
\mu_F^{(f_3,w_4)} & w_1w_3 & w_1w_4 & w_2w_3 & w_2w_4 \\
\mu_F^{(f_4,w_3)} & w_3w_4 & w_1w_4 & w_2w_3 & w_1w_2 \\
\mu_F^{(f_4,w_4)} & w_2w_4 & w_1w_2 & w_3w_4 & w_1w_3.
\end{array}$$

Notice that $\mu_F^{(f_1,w_1)} = \mu_F^{(f_4,w_4)}$. In Step 3 we obtain the set $T^*\left(T^0\left(P\right)\right) = \left\{P^{(f_1,w_1)}, P^{(f_4,w_3)}, P^{(f_4,w_4)}\right\}$. For instance, the truncation $P^{(f_1,w_2)}$ does not belong to this set because

$$\begin{aligned}
Ch\left(\mu_F\left(w_2\right) \cup \mu_F^{(f_1,w_2)}\left(w_2\right), P\left(w_2\right)\right) &= Ch\left(\left\{f_1, f_2\right\} \cup \left\{f_2, f_4\right\}, P\left(w_2\right)\right) \\
&= Ch\left(\left\{f_1, f_2, f_4\right\}, P\left(w_2\right)\right) \\
&= \left\{f_1, f_4\right\} \\
&\neq \left\{f_2, f_4\right\} \\
&= \mu_F^{(f_1,w_2)}\left(w_2\right),
\end{aligned}$$

but this is not a problem since $\mu_F^{(f_1,w_2)}$ is not stable because the pair (w_2, f_1) blocks it. Considering the ordering $P^{(f_1,w_1)} \prec^1 P^{(f_4,w_3)} \prec^1 P^{(f_4,w_4)}$ we have that $\widehat{T}\left(T^0\left(P\right)\right) = \left\{P^{(f_4,w_4)}\right\}$ since $P^{(f_1,w_1)}$ does not belong to it because $w_4 \notin \mu_F^{(f_1,w_1)}\left(f_4\right)$ and $P^{(f_1,w_1)} \prec^1 P^{(f_4,w_4)}$ and $P^{(f_4,w_3)}$ does not belong to it either because $w_4 \notin \mu_F^{(f_4,w_3)}\left(f_4\right)$ and $P^{(f_4,w_3)} \prec^1 P^{(f_4,w_4)}$. Set $T^1\left(P\right) = \left\{P^{(f_4,w_4)}\right\}$ and $S^1(P) = \{\mu_F, \mu_1\}$ where $\mu_1 = \mu_F^{(f_1,w_1)} = \mu_F^{(f_4,w_4)}$. This finishes Stage 1.

STAGE 2: In Step 1, we obtain for the truncation $P^{(f_4,w_4)}$ (the unique one belonging to the set $T^1\left(P\right)$) the corresponding set of truncations using $\mu_F^{(f_4,w_4)}$ and μ_W:

$$T\left(T^1\left(P\right)\right) = \left\{\begin{array}{c} P^{(f_4,w_4)(f_1,w_2)}, P^{(f_4,w_4)(f_2,w_1)}, P^{(f_4,w_4)(f_2,w_2)}, \\ P^{(f_4,w_4)(f_3,w_3)}, P^{(f_4,w_4)(f_3,w_4)}, P^{(f_4,w_4)(f_4,w_3)} \end{array}\right\}.$$

Now, in Step 2 and since $T\left(T^1\left(P\right)\right) \neq \emptyset$, for each truncation in $T\left(T^1\left(P\right)\right)$

we compute its corresponding firms–optimal stable matching

$$\begin{array}{lcccc} & f_1 & f_2 & f_3 & f_4 \\ \mu_F^{(f_4,w_4)(f_1,w_2)} & w_3w_4 & w_1w_2 & w_3w_4 & w_1w_2 \\ \mu_F^{(f_4,w_4)(f_2,w_1)} & w_1w_2 & w_3w_4 & w_3w_4 & w_1w_2 \\ \mu_F^{(f_4,w_4)(f_2,w_2)} & w_2w_4 & w_1w_4 & w_2w_3 & w_1w_3 \\ \mu_F^{(f_4,w_4)(f_3,w_3)} & w_2w_4 & w_2w_3 & w_1w_4 & w_1w_3 \\ \mu_F^{(f_4,w_4)(f_3,w_4)} & w_3w_4 & w_1w_4 & w_2w_3 & w_1w_2 \\ \mu_F^{(f_4,w_4)(f_4,w_3)} & w_3w_4 & w_1w_4 & w_2w_3 & w_1w_2. \end{array}$$

In Step 3 we obtain the set

$$T^*\left(T^1\left(P\right)\right) = \left\{P^{(f_4,w_4)(f_3,w_4)}, P^{(f_4,w_4)(f_4,w_3)}\right\}$$

and consider the ordering $P^{(f_4,w_4)(f_3,w_4)} \prec^2 P^{(f_4,w_4)(f_4,w_3)}$. In Step 4 the set $\widehat{T}\left(T^1\left(P\right)\right)$ is the singleton $\left\{P^{(f_4,w_4)(f_4,w_3)}\right\}$ since $w_3 \notin \mu_F^{(f_4,w_4)(f_3,w_4)}\left(f_4\right)$. Set $T^2(P) = \left\{P^{(f_4,w_4)(f_4,w_3)}\right\}$ and $S^2(P) = \{\mu_F, \mu_1, \mu_2\}$ where $\mu_2 = \mu_F^{(f_4,w_4)(f_4,w_3)}$.

STAGE 3: In Step 1, we obtain for the truncation $P^{(f_4,w_4)(f_4,w_3)}$ its corresponding truncations using $\mu_F^{(f_4,w_4)(f_4,w_3)}$ and μ_W:

$$T\left(T^2\left(P\right)\right) = \left\{P^{(f_4,w_4)(f_4,w_3)(f_2,w_1)}, P^{(f_4,w_4)(f_4,w_3)(f_3,w_3)}\right\}.$$

Since it is non-empty we compute, in Step 2, the corresponding firm-optimal stable matchings

$$\begin{array}{lcccc} & f_1 & f_2 & f_3 & f_4 \\ \mu_F^{(f_4,w_4)(f_4,w_3)(f_2,w_1)} & w_1w_2 & w_3w_4 & w_3w_4 & w_1w_2 \\ \mu_F^{(f_4,w_4)(f_4,w_3)(f_3,w_3)} & w_3w_4 & w_3w_4 & w_1w_2 & w_1w_2. \end{array}$$

In Step 3 we obtain the set

$$T^*\left(T^2\left(P\right)\right) = \left\{P^{(f_4,w_4)(f_4,w_3)(f_3,w_3)}\right\}.$$

Notice that $P^{(f_4,w_4)(f_4,w_3)(f_2,w_1)}$ does not belong to it because

$$\begin{aligned} Ch\left(\mu_F^{(f_4,w_4)(f_4,w_3)(f_2,w_1)}\left(w_3\right) \cup \mu_F^{(f_4,w_4)(f_4,w_3)}\left(w_3\right), P\left(w_3\right)\right) &= \{f_1, f_2\} \\ &\neq \{f_2, f_3\} \\ &= \mu_F^{(f_4,w_4)(f_4,w_3)(f_2,w_1)}\left(w_3\right). \end{aligned}$$

Since $T^*\left(T^2\left(P\right)\right)$ is a singleton we set $T^3\left(P\right) = \widehat{T}\left(T^2\left(P\right)\right) = \left\{P^{(f_4,w_4)(f_4,w_3)(f_3,w_3)}\right\}$ and $S^3\left(P\right) = \{\mu_F, \mu_1, \mu_2, \mu_W\}$ because $\mu_F^{(f_4,w_4)(f_4,w_3)(f_3,w_3)} = \mu_W$.

Stage 4: Finally, the algorithm stops (that is, $K = 4$) because $T\left(T^3(P)\right) = \emptyset$. Therefore $S(P) = \{\mu_F, \mu_1, \mu_2, \mu_W\}$, where

$$\begin{array}{ccccc} & f_1 & f_2 & f_3 & f_4 \\ \mu_F & w_1 w_2 & w_1 w_2 & w_3 w_4 & w_3 w_4 \\ \mu_1 & w_2 w_4 & w_1 w_2 & w_3 w_4 & w_1 w_3 \\ \mu_2 & w_3 w_4 & w_1 w_4 & w_2 w_3 & w_1 w_2 \\ \mu_W & w_3 w_4 & w_3 w_4 & w_1 w_2 & w_1 w_2 . \end{array}$$

3.3 Comments

Before moving to the next section to prove the Theorem few comments about the algorithm are in order.

First, for all truncations the worker-optimal stable matching coincides with the worker-optimal stable matching of the original preference profile P; that is, $\mu_W = \mu_W^{\left(f_{i_1}, w_{j_1}\right) \ldots \left(f_{i_k}, w_{j_k}\right)}$ for all $P^{\left(f_{i_1}, w_{j_1}\right) \ldots \left(f_{i_k}, w_{j_k}\right)}$.

Second, to make sure that the firm-optimal stable matching corresponding to a truncation is indeed stable it is sufficient to check *only* that Property 4 of Remark 1 holds; that is, all workers would choose it if confronted with the union of itself and the firm-optimal stable matching of the original profile. This is what Step 3 does in each stage. At the light of Example 1 this is surprising, although Lemma 2 in Section 4 states that this is the case. However, the fact that a truncation only changes one firm's preference guarantees that the other three properties also hold.

Third, the algorithm would also work without Step 4. However, it helps very much to speed up the algorithm (see Corollary 1 in Section 4) because, by adding it, we avoid carrying to subsequent stages all truncations (and all others obtained from them) whose corresponding firm-optimal stable matching will be identified later on.

Fourth, the particular ordering on the set $T^*\left(T^k\left(P\right)\right)$ is irrelevant but necessary. Namely, it is necessary because we can not ask for individual rationality of each truncation against all other truncations. To see this consider in Stage 1 of Example 2, the set $T^*\left(T^0\left(P\right)\right) = \left\{P^{(f_1, w_1)}, P^{(f_4, w_3)}, P^{(f_4, w_4)}\right\}$. If we had defined it without the restriction of the ordering, i.e.

$$\widehat{T}_{\not<}\left(T^0\left(P\right)\right) = \left\{P^{(f,w)} \in T^*\left(T^0\left(P\right)\right) \mid \forall P^{(f',w')} \in T^*\left(T^0\left(P\right)\right), w' \in \mu_F^{(f,w)}\left(f'\right)\right\}$$

this set would have been empty since $P^{(f_1, w_1)} \notin \widehat{T}_{\not<}\left(T^0\left(P\right)\right)$ because $w_4 \notin \mu_F^{(f_1, w_1)}\left(f_4\right)$, $P^{(f_4, w_3)} \notin \widehat{T}_{\not<}\left(T^0\left(P\right)\right)$ because $w_4 \notin \mu_F^{(f_4, w_3)}\left(f_4\right)$, and (in contrast with the correct definition of $\widehat{T}\left(T^0\left(P\right)\right)$) $P^{(f_4, w_4)} \notin \widehat{T}_{\not<}\left(T^0\left(P\right)\right)$ because $w_1 \notin \mu_F^{(f_4, w_4)}\left(f_1\right)$. Moreover, it is irrelevant because the outcome of the algorithm does not depend on the specific ordering on the set $T^*(T^k(P))$. For instance, in Stage 1 of Example 2 we could have used (instead of $\prec^1$) the ordering $P^{(f_4,w_4)} \prec^{1'} P^{(f_4,w_3)} \prec^{1'} P^{(f_1,w_1)}$ without altering the final outcome of the algorithm.

4 The Proof of the Theorem

Let P be a substitutable preference profile and let μ_F and μ_W be its corresponding optimal-stable matchings. Given an $(f,w)-$truncation of P where $w \in \mu_F(f) \backslash \mu_W(f)$, denote by $S^{(f,w)}(P)$ the set of stable matchings relative to the truncated profile $P^{(f,w)}$ that satisfy the Choice Property; namely,

$$S^{(f,w)}(P) = \left\{\mu \in S\left(P^{(f,w)}\right) \mid \forall w', Ch\left(\mu_F(w') \cup \mu(w'), P\left(w'\right)\right) = \mu(w')\right\}. \tag{1}$$

Lemma 2 below says that $S^{(f,w)}(P)$ is a subset of $S(P)$. Hence, the Choice Property is sufficient to guarantee stability of a matching which is stable relative to a truncation.

Lemma 2. *Let μ be a matching such that $\mu \in S^{(f,w)}(P)$. Then $\mu \in S(P)$.*

Proof. Assume that $\mu \notin S(P)$. If μ is not individually rational for preference profile P then it is not individually rational for preference profile $P^{(f,w)}$ and hence $\mu \notin S^{f,w}(P)$. Therefore, let $(\widetilde{w}, \widetilde{f})$ be a blocking pair of μ; namely,

$$\widetilde{f} \notin \mu(\widetilde{w}), \tag{2}$$

$$\widetilde{w} \in Ch(\mu(\widetilde{f}) \cup \{\widetilde{w}\}, P(\widetilde{f})), \text{ and} \tag{3}$$

$$\widetilde{f} \in Ch(\mu(\widetilde{w}) \cup \{\widetilde{f}\}, P(\widetilde{w})). \tag{4}$$

Consider the following two cases:

1. $\widetilde{f} \neq f$. In this case $P^{(f,w)}\left(\widetilde{f}\right) = P\left(\widetilde{f}\right)$ and $P^{(f,w)}(\widetilde{w}) = P(\widetilde{w})$ implying that the pair $(\widetilde{w}, \widetilde{f})$ also blocks the matching μ in the preference profile $P^{(f,w)}$. Therefore $\mu \notin S\left(P^{(f,w)}\right)$. Hence $\mu \notin S^{(f,w)}(P)$.

2. $\widetilde{f} = f$. Then by conditions (3) and (4)

$$\widetilde{w} \in Ch(\mu(f) \cup \{\widetilde{w}\}, P(f)) \text{ and} \tag{5}$$

$$f \in Ch(\mu(\widetilde{w}) \cup \{f\}, P(\widetilde{w})). \tag{6}$$

Assume that $\mu \in S\left(P^{(f,w)}\right)$, otherwise $\mu \notin S^{f,w}(P)$ and the Lemma is proved. Therefore,

$$\widetilde{w} \notin Ch(\mu(f) \cup \{\widetilde{w}\}, P^{(f,w)}(f)). \tag{7}$$

The definition of $P^{(f,w)}(f)$ and conditions (5) and (7) imply

$$w \in \mu(f) \cup \{\widetilde{w}\}.$$

But, by the definition of $P^{(f,w)}(f)$ again, $w \notin \mu(f)$. Then $\widetilde{w} = w$. Now, we can rewrite conditions (3) and (4) as

$$w \in Ch(\mu(f) \cup \{w\}, P(f)) \text{ and}$$

$$f \in Ch(\mu(w) \cup \{f\}, P(w)). \tag{8}$$

Notice that by hypothesis $f \in \mu_F(w)$ and by condition (2) $f \notin \mu(w)$. Also, by hypothesis, $\mu \in S^{(f,w)}(P)$ which means that, in particular,

$$Ch\left(\mu_F(w) \cup \mu(w), P(w)\right) = \mu(w)$$

holds. But this contradicts (8) because

$$Ch\left(\mu(w) \cup \{f\}, P(w)\right) P(w)\, \mu(w) = Ch\left(\mu_F(w) \cup \mu(w), P(w)\right).$$

□

Next Lemma establishes two useful properties of the choice set.

Lemma 3. *For all subsets of partners A, B, and C of agent $a \in F \cup W$:*
(a) $Ch(A \cup B, P(a)) = Ch(Ch(A) \cup B, P(a))$.
(b) $Ch(A \cup B, P(a)) = A$ and $Ch(B \cup C, P(a)) = B$ imply $Ch(A \cup C, P(a)) = A$.

Proof. Property (a) follows from Proposition 2.3 in Blair (1988). To prove (b), consider the following equalities:

$$\begin{aligned}
Ch(A \cup C, P(a)) &= Ch(Ch(A \cup B, P(a)) \cup C, P(a)) && \text{by hypothesis}\\
&= Ch(A \cup B \cup C, P(a)) && \text{by (a)}\\
&= Ch(A \cup Ch(B \cup C, P(a)), P(a)) && \text{by (a)}\\
&= Ch(A \cup B, P(a)) && \text{by hypothesis}\\
&= A && \text{by hypothesis.}
\end{aligned}$$

□

Lemma 4 below can be understood as an strengthening of Lemma 2. It says that to check the Choice Property *only* for the firm-optimal stable matching it is sufficient to guarantee that *all* stable matchings relative to the truncated profile are indeed stable for the original profile.

Lemma 4. *Let* $P^{(f,w)}$ *be a truncation such that*

$$Ch\left(\mu_F(w') \cup \mu_F^{(f,w)}(w'), P\left(w'\right)\right) = \mu_F^{(f,w)}(w')$$

holds for all w'. *Then,* $\mu \in S(P^{(f,w)})$ *implies* $\mu \in S(P)$.

Proof. Let μ be a matching such that $\mu \in S(P^{(f,w)})$. Then by Property 4 in Remark 1, for all w',

$$Ch\left(\mu(w') \cup \mu_F^{(f,w)}(w'), P^{(f,w)}\left(w'\right)\right) = \mu(w').$$

However, for all w', preferences $P^{(f,w)}\left(w'\right)$ and $P\left(w'\right)$ coincide. Therefore,

$$Ch\left(\mu(w') \cup \mu_F^{(f,w)}(w'), P\left(w'\right)\right) = \mu(w') \tag{9}$$

also holds. By hypothesis, for all w'

$$Ch\left(\mu_F(w') \cup \mu_F^{(f,w)}(w'), P\left(w'\right)\right) = \mu_F^{(f,w)}(w'). \tag{10}$$

By Lemma 3 we have that conditions (9) and (10) imply that for all w'

$$Ch\left(\mu_F(w') \cup \mu(w'), P\left(w'\right)\right) = \mu(w').$$

Then, by Lemma 2, $\mu \in S^{(f,w)}\left(P\right)$. Hence, $\mu \in S(P)$. □

Lemma 5 says that only adding the individual rationality condition of a stable matching relative to a truncation ensures that the matching is stable relative to the truncated profile. This will immediately imply Corollary 1 which will be crucial to the justification of Step 4 in the algorithm.

Lemma 5. *Let* μ *be a matching such that* $\mu \in S(P) \cap IR\left(P^{(f,w)}\right)$. *Then* $\mu \in S(P^{(f,w)})$.

Proof. Assume that $\mu \notin S(P^{(f,w)})$ and $\mu \in IR\left(P^{(f,w)}\right)$. Then, there exists a blocking pair $(\widetilde{w}, \widetilde{f})$ of μ; namely,

$$\widetilde{w} \in Ch(\mu(\widetilde{f}) \cup \{\widetilde{w}\}, P^{(f,w)}(\widetilde{f})) \text{ and} \tag{11}$$

$$\widetilde{f} \in Ch(\mu(\widetilde{w}) \cup \{\widetilde{f}\}, P^{(f,w)}(\widetilde{w})). \tag{12}$$

Consider the following two cases:

1. $\widetilde{f} \neq f$. Because $P^{(f,w)}(\widetilde{w}) = P(\widetilde{w})$ and $P^{(f,w)}\left(\widetilde{f}\right) = P\left(\widetilde{f}\right)$ the pair $(\widetilde{w}, \widetilde{f})$ also blocks the matching μ in the preference profile P. Hence, $\mu \notin S(P)$.

2. $\widetilde{f} = f$. Then by conditions (11) and (12)

$$\widetilde{w} \in Ch(\mu(f) \cup \{\widetilde{w}\}, P^{(f,w)}(f)) \text{ and } \tag{13}$$

$$f \in Ch(\mu(\widetilde{w}) \cup \{f\}, P(\widetilde{w})).$$

   The hypothesis that $\mu \in IR\left(P^{(f,w)}\right)$ implies that $w \notin \mu(f)$. Therefore, condition (13) can be rewritten as $\widetilde{w} \in Ch(\mu(f) \cup \{\widetilde{w}\}, P(f))$, implying that the pair $(\widetilde{w}, f)$ blocks μ in the preference P. Hence $\mu \notin S(P)$. □

As we have just said, Corollary 1 below justifies the insertion of Step 4 at each stage of the algorithm. If we have two truncations $P^{(f,w)}$ and $P^{(f',w')}$ with the properties that (1) their corresponding firm-optimal stable matchings $\mu_F^{(f,w)}$ and $\mu_F^{(f',w')}$ satisfy the Choice Property (that is, they are stable relative to the original profile) and (2) the matching $\mu_F^{(f',w')}$ is individually rational relative to $P^{(f,w)}$ (that is, $w \notin \mu_F^{(f',w')}(f)$) then we may not add at this stage $\mu_F^{(f',w')}$ (with the subsequent computational savings) because we will find it later on (and add it to the provisional set of stable matchings) as a firm-optimal stable matching of a subsequent truncation of $P^{(f,w)}$.

Corollary 1. *Let $P^{(f,w)}, P^{(f',w')}$ be two truncations such that $\mu_F^{(f',w')} \in S(P)$. If $w \notin \mu_F^{(f',w')}(f)$ then $\mu_F^{(f',w')} \in S\left(P^{(f,w)}\right)$.*

Proof. Notice that $w \notin \mu_F^{(f',w')}(f)$ implies that $\mu_F^{(f',w')} \in IR\left(P^{(f,w)}\right)$. Hence, by Lemma 5, $\mu_F^{(f',w')} \in S(P^{(f,w)})$. □

Next lemma establishes a useful fact about the set of stable matchings: a worker who is matched to the same firm in the two optimal-stable matchings has also to be matched to the same firm in all stable matchings.

Lemma 6. *Assume $w \in \mu_F(f) \cap \mu_W(f)$. Then, $w \in \mu(f)$ for all $\mu \in S(P)$.*

Proof. Assume otherwise; that is, we can find w, f, and $\mu \in S(P)$ such that $w \in \mu_F(f) \cap \mu_W(f)$ and $w \notin \mu(f)$. By Remark 1,

$$Ch(\mu_F(f) \cup \mu(f), P(f)) = \mu_F(f) \tag{14}$$

and

$$Ch\left(\mu_W(f) \cup \mu(f), P(f)\right) = \mu(f). \tag{15}$$

Since $w \in \mu_F(f)$ condition (14) implies that $w \in Ch\left(\mu_F(f) \cap \mu(f), P(f)\right)$.
□

Lemma 7 and its Corollary 2 guarantee that any non-optimal stable matching μ will eventually be identified and selected as the firm-optimal stable matching corresponding to a preference profile which will be obtained after truncating the preferences of a sequence of firms.

Lemma 7. *Let $\mu \in S(P)$ be such that $\mu_F \neq \mu \neq \mu_W$. Then there exists $P^{(f,w)}$ such that $\mu \in S(P^{(f,w)})$.*

Proof. Since $\mu_F \neq \mu$ then there exists w and f such that $w \in \mu_F(f) \backslash \mu(f)$. Therefore, by Lemma 6, $w \notin \mu_W(f)$. Consider the preference profile $P^{(f,w)}$ and notice that $\mu_W^{(f,w)} = \mu_W$. Because $w \notin \mu(f)$ we have that $\mu \in IR\left(P^{(f,w)}\right)$. Hence, Lemma 5 implies that $\mu \in S\left(P^{(f,w)}\right)$. □

Remark 2 As a consequence of Lemma 7 and the fact that $\mu_F \notin S(P^{(f,w)})$ we have that $\#S(P) > \#S(P^{(f,w)})$ whenever $w \in \mu_F(f) \backslash \mu_W(f)$.

Corollary 2. *Let $\mu \in S(P)$ be such that $\mu_F \neq \mu \neq \mu_W$. Then there exists a sequence of pairs $(f_{i_1}, w_{j_1}) \ldots (f_{i_k}, w_{j_k})$ such that $\mu = \mu_F^{(f_{i_1},w_{j_1})\ldots(f_{i_k},w_{j_k})} \in S(P^{(f_{i_1},w_{j_1})\ldots(f_{i_k},w_{j_k})})$.*

Proof. Let $\mu \in S(P)$ be such that $\mu_F \neq \mu \neq \mu_W$. By Lemma 5 there exists $P^{(f,w)}$ such that $\mu \in S(P^{(f,w)})$. If $\mu = \mu_F^{(f,w)}$ the statement follows. Otherwise, since $\mu_W = \mu_W^{(f,w)}$, we apply again Lemma 5 replacing the roles of P and μ_F by $P^{(f,w)}$and $\mu_F^{(f,w)}$, respectively. □

Now, we are ready to show that the outcome of the algorithm is the set of stable matchings.

Proof of the Theorem. First, from Lemma 4, we have $S^1(P) \subseteq S(P)$. Applying iteratively Lemma 4 to successive stages we obtain

$$S^K(P) \subseteq S(P).$$

Second, assume that $\mu \in S(P)$. By Corollary 2, there exists $k \leq K$ such that $\mu \in S^k(P)$. Therefore,

$$S(P) \subseteq S^K(P). \quad \square$$

5 Appendix

To illustrate the deferred-acceptance algorithm in which firms make offers we use the preference profile $P^{(f_4,w_4)(f_4,w_3)(f_3,w_3)}$ of Example 2 to compute $\mu_F^{(f_4,w_4)(f_4,w_3)(f_3,w_3)}$; that is, $F = \{f_1, f_2, f_3, f_4\}$ and $W = \{w_1, w_2, w_3, w_4\}$ are the two sets of agents with the following substitutable profile of preferences

$$\begin{aligned}
P(f_1) &= \; w_1w_2, w_1w_3, w_2w_4, w_3w_4, w_1w_4, w_2w_3, w_1, w_2, w_3, w_4\\
P(f_2) &= \; w_1w_2, w_2w_3, w_1w_4, w_3w_4, w_1w_3, w_2w_4, w_1, w_2, w_3, w_4\\
P(f_3) &= \; w_1w_4, w_1w_2, w_2w_4, w_1, w_2, w_4\\
P(f_4) &= \; w_1w_2, w_1w_4, w_1, w_2\\
P(w_1) &= \; f_3f_4, f_2f_3, f_2f_4, f_1f_4, f_1f_3, f_1f_2, f_1, f_2, f_3, f_4\\
P(w_2) &= \; f_3f_4, f_2f_3, f_1f_4, f_2f_4, f_1f_3, f_1f_2, f_1, f_2, f_3, f_4\\
P(w_3) &= \; f_1f_2, f_2f_3, f_1f_3, f_2f_4, f_1f_4, f_3f_4, f_1, f_2, f_3, f_4\\
P(w_4) &= \; f_1f_2, f_1f_3, f_1f_4, f_2f_3, f_2f_4, f_3f_4, f_1, f_2, f_3, f_4.
\end{aligned}$$

The offers made by firms, and received and accepted by workers, in Step 1 are:

f_1	f_2	f_3	f_4	w_1	w_2	w_3	w_4
w_1w_2	w_1w_2	w_1w_4	w_1w_2	$f_1f_2f_3f_4$	$f_1f_2f_4$	$\emptyset$	f_3
				f_3f_4	f_1f_4	$\emptyset$	f_3.

The provisional matching μ^1 after Step 1 is:

	f_1	f_2	f_3	f_4
μ^1	w_2	$\emptyset$	w_1w_4	w_1w_2.

The offers made by firms, and received and accepted by workers, in Step 2 are:

f_1	f_2	f_3	f_4	w_1	w_2	w_3	w_4
w_2w_4	w_3w_4	w_1w_4	w_1w_2	f_3f_4	f_1f_4	f_2	$f_1f_2f_3$
				f_3f_4	f_1f_4	f_2	f_1f_2.

The provisional matching μ^2 after Step 2 is:

	f_1	f_2	f_3	f_4
μ^2	w_2w_4	w_3w_4	w_1	w_1w_2.

The offers made by firms, and received and accepted by workers, in Step 3 are:

f_1	f_2	f_3	f_4	w_1	w_2	w_3	w_4
w_2w_4	w_3w_4	w_1w_2	w_1w_2	f_3f_4	$f_1f_3f_4$	f_2	f_1f_2
				f_3f_4	f_3f_4	f_2	f_1f_2.

The provisional matching μ^3 after Step 3 is:

$$\begin{array}{ccccc} & f_1 & f_2 & f_3 & f_4 \\ \mu^3 & w_4 & w_3w_4 & w_1w_2 & w_1w_2. \end{array}$$

The offers made by firms, and received and accepted by workers, in Step 4 are:

$$\begin{array}{cccccccc} f_1 & f_2 & f_3 & f_4 & w_1 & w_2 & w_3 & w_4 \\ w_3w_4 & w_3w_4 & w_1w_2 & w_1w_2 & f_3f_4 & f_3f_4 & f_1f_2 & f_1f_2 \\ & & & & f_3f_4 & f_3f_4 & f_1f_2 & f_1f_2. \end{array}$$

the provisional matching μ^4 after Step 4 is:

$$\begin{array}{ccccc} & f_1 & f_2 & f_3 & f_4 \\ \mu^4 & w_3w_4 & w_3w_4 & w_1w_2 & w_1w_2. \end{array}$$

The algorithm stops after Step 4 because all offers have been accepted. The provisional matching μ^4, becomes definite, and it is the firm-optimal stable matching.

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