An Ordinal Shapley Value for Economic Environments (Revised Version)[1]

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Abstract

We propose a new solution concept to address the problem of sharing a surplus among the agents generating it. The problem is formulated in the preferences-endowments space. The solution is defined recursively, incorporating notions of consistency and fairness and relying on properties satisfied by the Shapley value for Transferable Utility (TU) games. We show a solution exists, and call it the Ordinal Shapley value (OSV). We characterize the OSV using the notion of coalitional dividends, and furthermore show it is monotone and anonymous. Finally, similarly to the weighted Shapely value for TU games, we construct a weighted OSV as well.

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1 Introduction

A feature common to most economic environments is that the interaction among agents, be it through exchange, production or both, generates benefits shared among the participating individuals. The question of what would be the resulting distribution of gains has been central to economic theory. In this paper, we propose and analyze a new solution concept (sharing method) that satisfies appealing properties in economic environments.

In economic environments characterized by transferable utility (TU), where there exists a "numeraire" commodity that all agents value the same in terms of utility, there are several popular notions of the distribution of gains, the most well-known of which are the Core and the Shapley value. These satisfy several desirable properties such as efficiency and group stability in the case of the core, and efficiency, fairness and consistency for the Shapley value.

Extending the notion of the Core to more general environments with non-transferable utility (NTU) is straightforward. However, the extension of the central concept of the Shapley value turns out to be a much more demanding task. The three known extensions describe the environment in the utility space, i.e., specifying feasible utility tuples, abstracting from the physical environment generating the tuples. They associate with each environment one or more TU games, and use their Shapley value to generate a surplus sharing method. To define such a method, Shapley (1969) associates with each environment a TU game, by means of a weights vector, giving the "worth" of each utility tuple. This TU game has a well-defined Shapley value. If this value is feasible for the original game, it is a utility profile associated with this environment. Harsanyi (1959) suggests a different extension, by stressing the idea of equity. His solution contains the notion of coalitional "dividends" and each agent must end up with a payoff corresponding to the sum of his dividends. Finally, Maschler and Owen (1989) and (1992), using a TU game associated with the grand coalition, provide an extension preserving the consistency properties of the Shapley value.

A major shortcoming of the extensions of the Shapley value is that the solutions are not invariant to order-preserving transformations of the agents' utilities. The notion of invariance has been addressed in the literature in two different ways. One approach considers *bargaining problems*, where the environment is given by the utility possibilities frontier for the whole set of agents and the disagreement point. A solution is then said to be ordinal, if it is invariant with respect to strictly increasing monotonic transformations of these entities. Shapley (1969) shows that there does not exist an ordinal, efficient and anonymous solution for the case of two agents, and constructs one for the three-agent case. Safra and Samet (2004) provide a family of ordinal, efficient and anonymous solutions for bargaining problems with any number of agents greater than two.

The second approach towards the ordinality issue considers the underlying physical environment generating the utility possibilities frontier. This approach better captures the basic structure of the environment since identical economic environments may lead to drastically different bargaining problems, by appropriate choices of utility functions that represent the same preferences. In this approach the solution is defined in terms of the physical environment, i.e., in terms of allocations of commodity bundles.

To clarify the difference between the two approaches, take the example of a two-agent exchange economy. Consider the representation of this economy as an NTU game. Following Shapley (1969) there is no ordinal, efficient and anonymous solution concept for this game. However, there are several ordinal, efficient and anonymous solution concepts for the exchange economy such as the competitive equilibrium, the core and others. Therefore, an ordinal solution for the economic environment need not be an ordinal solution for the NTU game. Similarly, an ordinal NTU solution need not be ordinal if analyzed as a solution for the economic environment.

Pazner and Schmeidler (1978) provide a family of ordinal solutions given by Pareto-Efficient Egalitarian-Equivalent ($PEEE$) allocations for exchange economies. They consider the problem of allocating a bundle of goods among a set of agents. In their environment, each of the agents has the same *a priori* rights. An allocation is $PEEE$ if it is Pareto efficient and fair, in the sense that there exists a fixed commodity bundle (the same for each agent) such that each agent is indifferent between this bundle and what he gets in the allocation. McLean and Postlewaite (1989) consider pure exchange economies as well, and define an ordinal solution given by nucleolus allocations, extending the notion of the nucleolus defined for TU games in Schmeidler (1969). Nicolò and Perea (2004) also start from the physical environment, and provide ordinal solutions for the case of two agents that, under some conditions, also extend to environments with any number of agents.

Our work continues this line of research by proposing an *ordinal solution* based on the physical environment. This new solution incorporates several of the principles underlying the Shapley value in TU environments, and will be referred to as the *Ordinal Shapley Value* (OSV). It generalizes the fairness notion (of $PEEE$) by considering possibly different *a priori* rights (i.e., different initial endowments), and also the options agents have in any possible subgroup, and not just their own initial endowments. It is consistent in the sense that agents' payoffs are based on what they would get according to this rule when applied to sub-environments. In addition to these properties of equity and consistency, the solution is efficient, monotonic, anonymous, and satisfies individual rationality. Also, the OSV is characterized through the use of "coalitional dividends" similar to the characterization of the Shapley value by the use of Harsanyi dividends (Harsanyi, 1959).

The OSV exists whenever preferences are continuous and monotonic. No convexity restrictions common in the specification of NTU games are necessary. It provides a reasonable outcome for a large class of environments even where competitive equilibria or core allocations may fail to exist.

In the next Section we start by reviewing the Shapley value in TU environments. In Section 3 we describe the pure exchange economy underlying the NTU environment and introduce the OSV, building on the characterization of the Shapley value for TU environments provided in the previous section. In Section 4 we analyze the OSV for two-agent economies and compare it to exiting constructions. In Section 5, we prove that the OSV exists and furthermore it is individually rational. In Section 6, we start by proving the construction of the OSV satisfies a symmetry property. We then proceed to characterize the OSV via coalitional dividends, and provide further properties of the solution. In Section 7, we show how to generate a family of weighted OSVs, providing an ordinal analogue to the weighted Shapley values for TU environments. In Section 8, we conclude and discuss further directions of research.

2 The Shapley Value in *TU* environments: A New Characterization

Consider a *Transferable Utility* (*TU*) game (N, v), where $N = \{1, ..., n\}$ is the set of players, and $v : 2^N \to R$ is a characteristic function satisfying $v(\varnothing) = 0$, where $\varnothing$ is the empty set. For a coalition $S \subseteq N$,[1] $v(S)$ represents the total payoff that the partners in S can jointly obtain if this coalition is formed. We define a *value* as a mapping ξ which associates with every game (N, v) a vector in R^n that satisfies $\sum_{i \in N} \xi_i(N, v) = v(N)$.[2]

The *Shapley value* (Shapley, 1953a) of every agent $i \in N$ in the *TU* game (N, v) is (denoting $|S|$ the cardinality of the subset S):

$$\phi_i(N, v) = \sum_{S \subseteq N \backslash i} \frac{|S|!(n - |S| - 1)!}{n!} [v(S \cup \{i\}) - v(S)].$$

The next theorem provides a new characterization of the Shapley value.[3]

Theorem 1 *A value ξ is the Shapley value if and only if it satisfies:*

$$\sum_{i \in N \backslash j} (\xi_i(N, v) - \xi_i(N \backslash j, v)) = \sum_{i \in N \backslash j} (\xi_j(N, v) - \xi_j(N \backslash i, v)) \tag{1}$$

for all (N, v) with $|N| \geq 2$ and for all $j \in N$.

Proof. To prove that the Shapley value satisfies the equality note that (1) is equivalent (rearranging terms and using $\sum_{i \in N} \xi_i(N, v) = v(N)$) to:

$$\xi_j(N, v) = \frac{1}{n} [v(N) - v(N \backslash j)] + \frac{1}{n} \sum_{i \in N \backslash j} \xi_j(N \backslash i, v). \tag{2}$$

It is easy to check that the Shapley value satisfies (2). (This equality has been previously used by Maschler and Owen (1989) and Hart and Mas-Colell (1989).)

Furthermore suppose that equality (1), equivalently (2), is satisfied by the value ξ, for all $j \in N$ and for all (N, v). Since (2) provides a unique recursive way of calculating ξ

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[1] Throughout the paper, we use $\subseteq$ to denote the weak inclusion and $\subset$ to denote the strict inclusion.

[2] Thus we require efficiciency as part of the definition of a value.

[3] When using the symbol (M, v) where v is *a priori* defined on $N \supseteq M$, v is taken to be the restriction of the original v to 2^M.

starting with $\xi_i(\{i\},v) = v(\{i\})$, it characterizes the Shapley value, which completes the proof. ■

The expression $\phi_i(N,v) - \phi_i(N\backslash j,v)$ is usually referred to as the contribution of player j to the Shapley value of player i. It corresponds to the amount that makes player i indifferent between receiving the value suggested to him in the game (N,v), or receiving this payment and reapplying the value concept to the game without player j. Theorem 1 states that a value is the Shapley value if and only if, for any player j, the sum of the contributions of player j to the other players is equal to the sum of the contributions of the other players to player j.

We refer to the difference $\phi_i(N,v) - \phi_i(N\backslash j,v)$ as a *concession*, what player j concedes to player i, and denote it by c_i^j.[4]

Corollary 1 *A value ξ is the Shapley value if and only if for each game (N,v) with $|N| \geq 2$ there exists a matrix of concessions $c(N,v) \equiv (c_j^i(N,v))_{i,j\in N, i\neq j}$, with $c_j^i(N,v)$ in R for all $i,j \in N, i \neq j$, such that:*

(1) $\xi_i(N,v) = \xi_i(N\backslash j,v) + c_i^j(N,v)$ for all $i,j \in N, i \neq j$, and

(2) $\sum\limits_{i\in N\backslash j} c_i^j(N,v) = \sum\limits_{i\in N\backslash j} c_j^i(N,v)$ for all $j \in N$.

We can view part (1) in Corollary 1 as a *consistency* property of the Shapley value. When the $n-1$ players other than j consider the value offered to them by the solution concept, they contemplate what might happen if they decide to go on their own. However, the resources at their disposal should incorporate rents they could conceivably achieve by cooperating with j. We call these rents the concessions of j to the other players. Part (2) can be interpreted as a *fairness* requirement: the concessions balance out, the sum of concessions one player makes to the others equals the sum of concessions the others make to him.

We now briefly describe some characteristics of the concessions.

For a *TU* game (N,v), for any coalition $S \subseteq N$, let the game w_S be the unanimity game (i.e., $w_S(T) = 1$ if $T \supseteq S, w_S(T) = 0$ otherwise). It is well known that the characteristic function v can be written as linear combination of unanimity games: $v = \sum_{S\subseteq N} \alpha_S w_S$.

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[4] See also Pérez-Castrillo and Wettstein (2001), where concessions are interpreted as bids.

Denoting $\lambda_S = \frac{\alpha_S}{|S|}$ for all $S \subseteq N$, the Shapley value can be written (see Harsanyi, 1959) as:

$$\phi_i(N, v) = \sum_{\substack{S \ni i \\ S \subseteq N}} \lambda_S \text{ for all } i \in N. \tag{3}$$

It follows that:

$$c_i^j(N, v) = \sum_{\substack{S \ni i,j \\ S \subseteq N}} \lambda_S \text{ for all } i, j \in N, i \neq j.$$

An immediate implication of the previous equality is that, in TU games, the concessions are symmetric in the sense that what player j concedes to i is the same as what player i concedes to j. The symmetry of the concessions corresponds to the balanced contributions property (see Myerson (1980)).

Another interesting property of the concessions is that, although they can in general be positive of negative, they are always non-negative if the game is convex. The game (N, v) is convex if, for all $S, T \subseteq N$ with $S \subset T$ and $i \notin T$ we have:

$$v(S \cup \{i\}) - v(S) \leq v(T \cup \{i\}) - v(T).$$

Next proposition states the result.

Proposition 1 *If the TU game (N, v) is convex, all the concessions $c_i^j(N, v)$ are non-negative.*

Proof. The concession $c_i^j(N, v) = \phi_i(N, v) - \phi_i(N \backslash j, v)$ is the difference between the Shapley value of agent i in the game with all the agents and agent i's Shapley value in the game without agent j. Sprumont (1990) showed that for convex games the Shapley value is a population monotonic allocation scheme. Each agent's Shapley value increases as the coalition to which he belongs expands. Thus, $\phi_i(N, v) - \phi_i(N \backslash j, v) \geq 0$ and hence the concessions are non-negative. ■

To complete the section, we point out that a value can be expressed in terms of the "Harsanyi dividends" (they are also called coalitional dividends), given in equation (3) if and only if it is the Shapley value. We return to this characterization when analyzing the properties of our proposal.

Proposition 2 *A value ξ is the Shapley value if and only if, for any game (N, v) there exists $\lambda_S \in R$ for all $S \subseteq N$ such that,*

$$\xi_i(T, v) = \sum_{\substack{S \ni i \\ S \subseteq T}} \lambda_S \text{ for all } i \in T, \text{ for all } T \subseteq N. \tag{4}$$

Proof. The fact that the Shapley value satisfies this property was shown by Harsanyi (1959) and it is stated in (3). To show the sufficiency we note that (4) implies that ξ is an egalitarian solution and hence must be the Shapley value (see Mas-Colell, Whinston and Green (1995, pp. 680-681) for the definition of an egalitarian solution and the fact it coincides with the Shapley value). ■

3 The Environment and the Solution

We consider a pure exchange economy with a set $N = \{1, 2, ..., n\}$ of agents and $k \geq 2$ commodities. Agent $i \in N$ is described by $\{\succeq^i, w^i\}$, where $w^i \in R^k$ is the vector of initial endowments and $\succeq^i$ is the preference relation defined over R^k. An economy (usually denoted by E) is thus given by $E = \{\succeq^i, w^i\}_{i=1}^n$. We denote by $\succ^i$ and $\sim^i$ the strict preference and indifference relationships associated with $\succeq^i$. For each $i \in N$, the preference relation $\succeq^i$ is assumed to be continuous and monotonic on R^k (i.e., if $y_l > x_l$ for all $l = 1, ..., k$, then $y \succ^i x$). To simplify the notation in several definitions and proofs it would be convenient to refer to a utility function representing the preferences of agent i, denoted by u^i. For concreteness we map each commodity bundle x to the (unique) number $u^i(x)$ that satisfies $x \sim^i u^i(x) \cdot e$, where $e \equiv (1, ..., 1) \in R^k$. Such a number exists since preferences are monotonic and continuous. As we define an ordinal solution concept, the solution itself will, of course, not depend upon this particular choice of a utility function.

We let $w \equiv \sum_{i \in N} w^i$. The set of feasible utility profiles in R^n for an economy E is denoted by $A(E)$ and defined by:

$$A(E) = \left\{ u \in R^n | \exists \left(x^i\right)_{i=1,...,n} \in R^{kn}, \text{ such that } u^i(x^i) = u^i,\ i = 1, ..., n \text{ and } \sum_{i \in N} x^i \leq w \right\}.$$

Agents can conceivably be better off by reallocating their initial endowments. However, it should not be possible for the utility of one agent to grow arbitrarily large if the utilities of the other agents are bounded from below. To capture this idea, we assume that, for any $u \in A(E)$ and $i \in N$, the set $A_i(u) \equiv \{\overline{u} \in A | \overline{u}_{-i} = u_{-i}\}$ is bounded from above.[5] We note that this property is ordinal, if it is satisfied for the u^i we have constructed it also holds for any strictly monotone transformation of it. In this paper, any pure exchange economy that satisfies the previous requirements is referred to as an *economic environment.*

We propose a solution concept, called the *Ordinal Shapley Value* (OSV), for pure exchange economies, the construction of which relies on the notion of concessions. However, since these economies constitute NTU environments, which are described in terms of the underlying physical structure, concessions cannot be in the form of utility transfers. Concessions are expressed in terms of commodities. We measure them in terms of a "reference bundle" which we take to be e. The main characteristic of the concept proposed is that it is *ordinal.* That is, the solution associates with each economy a set of allocations that does not depend on the numerical representation of the underlying preferences of the agents. Moreover, the solution proposed is *efficient* and satisfies *consistency* and *fairness* requirements.

What is a "fair" and "consistent" sharing? Let us first discuss the rationale of our proposal in the case of two agents. According to our proposal, a sharing is fair if the gains from cooperation are equally distributed among the two agents. A crucial question is how to measure these gains. In our proposal, the benefits from cooperation are measured in terms of e. The gain of each agent is the amount of e units that when added to his initial endowment, yields a bundle indifferent to the bundle received by the sharing. This amount of e assumes the role of the difference in values (in the TU case).

A sharing is consistent if each agent is indifferent between the sharing outcome and what he could get if he were to walk away and keep what remains of the aggregate endowment, after compensating the other agent according to the solution concept. We measure the surplus he can keep by the maximal amount of e units for which, when he receives a bundle indifferent to his initial endowment augmented by that amount of e units, the other agent is left with a bundle equivalent to the bundle he received in the sharing. To state these properties more succinctly we use the notion of a concession just as in the TU case. An efficient sharing is fair and consistent if there exists a pair of

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[5] For a vector $x \in R^n$ and $i \in N$, $x_{-i} \equiv (x_1,...,x_{i-1},x_{i+1},...,x_n\}$.

concessions such that the concession made by agent i to agent j equals the concession made by agent j to agent i, and each agent is indifferent between keeping this allocation or taking the concession proposed by the other (to add to his initial endowment).

Extending this notion to the n-person case, a solution is an efficient allocation for which there exists a matrix of concessions, one from each agent to any other agent, satisfying consistency and fairness. The consistency property now requires that any set of $(n-1)$ agents should be indifferent between keeping their allocation or taking the concessions made by the remaining agent and reapplying the solution concept to the $(n-1)$-agent economy. The recursive nature of the definition implies that this consistency property extends to coalitions of any size. Moreover, to ensure that the allocation reached is "fair", we require the concessions to balance out, in the sense that the sum of concessions one player makes to the others equals the sum of concessions the others make to him. In other words, the surplus generated for any set of $n-1$ agents is the same as the surplus they are willing to concede to the remaining agent.

The formal definition of this solution concept, the OSV, is as follows:

Definition 1 *The Ordinal Shapley Value is defined recursively.*

$(n=1)$ In the case of an economy with one agent with preferences $\succeq^1$ and initial endowments $a^1 \in R^k$, the OSV is given by the initial endowment: $OSV(\succeq^1, a^1) = \{a^1\}$.

For $n \geq 2$, suppose that the solution has been defined for any economy with $(n-1)$ or less agents.

(n) In the case of an economy $(\succeq^i, a^i)_{i\in N}$ with a set N of n agents, the $OSV\left((\succeq^i, a^i)_{i\in N}\right)$ is the set of efficient allocations $(x^i)_{i\in N}$ for which there exists an $n-$tuple of concession vectors $(c^i)_{i\in N}, c^i \in R^{n-1}$ for all $i \in N$ that satisfy:

n.1) for all $j \in N$, there exists $y(j) \in OSV\left((\succeq^i, a^i + c^j_i e)_{i\in N\backslash j}\right)$ such that $x^i \sim^i y(j)^i$ for all $i \in N\backslash j$, and

n.2) $\sum\limits_{i\in N\backslash j} c^j_i = \sum\limits_{i\in N\backslash j} c^i_j$ for all $j \in N$.

It should be noted that the choice of the bundle e to measure the surplus that accrues to each agent is arbitrary. The OSV could be constructed by using any other positive vector.[6] The following analysis is valid regardless of the particular reference bundle chosen.

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[6] Given this fact, it may be more appropriate to use the notation OSV_e instead of OSV. We use

Note also that this solution concept reduces to the Shapley value in economic environments that can be described as a TU environment. In such environments there is a common unit of account which can be thought of as money, and agents' preferences are (normalized) quasi linear of the form $m + v^i(x)$ where m is "money", v^i is a utility function, and x is a commodity vector. If we measure concessions in terms of money (m), our solution yields the Shapley value.

4 The solution in a two-agent economy

For a two-agent economy (E), an OSV is an efficient allocation for which there exists an identical concession for each agent, such that any agent is indifferent between the bundle offered to him in the allocation or taking the concession and staying on his own.

In order to characterize a solution $(x^i)_{i=1,2}$ in the two-agent economy, notice first that, by efficiency, the bundle of player 1, x^1, must be the best for him among all the allocations that leave agent 2 indifferent or better off than the bundle x^2. Moreover, agent 2 is indifferent between x^2 and $w^2 + c^1 e$, and similarly, agent 1 is indifferent between x^1 and $w^1 + c^2 e$. Given that the concessions are the same, $c \equiv c^1 = c^2$, they must satisfy the following equality:

$$\begin{aligned} u^1(w^1 + ce) = max_{(z^1,z^2)}\ & u^1(z^1) \\ \text{s.t. } & u^2(z^2) \geq u^2(w^2 + ce) \\ & z^1 + z^2 \leq w^1 + w^2. \end{aligned}$$

The solution to this equation is given by the maximal real number c (which is non-negative) that satisfies:

$$(u^1(w^1 + ce), u^2(w^2 + ce)) \in A(E)$$

Since preferences are strictly increasing and the sets $A_i(u)$ are bounded, the previous c exists and is *unique*. Note that the concession in the OSV depends on the initial endowments. The OSV for the two-agent economy consists of the efficient allocations (x^1, x^2) such that $u^1(x^1) = u^1(w^1 + ce)$ and $u^2(x^2) = u^2(w^2 + ce)$. When preferences are strictly quasiconcave, the OSV allocation is unique.

OSV for notational simplicity.

For the two-agent economy the *OSV* has a very natural graphical representation. Figure 1 depicts the *OSV* when $n = 2$ and there are two commodities.

[Insert Figure 1]

For two-agent economies, our proposal bears many similarities to two previous solution concepts. First, it is similar to the Pareto-Efficient Egalitarian-Equivalent (*PEEE*) allocation proposed by Pazner and Schmeidler (1978), when addressing the issue of allocating a bundle of goods among a set of agents. The *OSV* allocation when the two agents have the same initial endowments is a *PEEE* allocation as well. Note that by choosing different commodity bundles to concede with, we can generate a family of *OSV* allocations, all of which are *PEEE*.

Nicolò and Perea (2004) also propose an ordinal solution concept for two-person bargaining situations. Their construction yields the *OSV* for the class of exchange economies where aggregate endowments of all the commodities are equal and are shared equally among the two agents. Furthermore, while we require indifference with respect to adding to the two agents initial endowments, multiples of e, they require indifference with respect to adding to each agent's initial endowment a multiple of the other agent's initial endowment.

5 Existence of the *OSV*

It is not obvious there exists an efficient allocation for which one can find concessions satisfying the requirements imposed by the definition of the *OSV*. To show such allocations exist, we invoke in Theorem 2 a fixed point argument. Furthermore we show that allocations in the *OSV* satisfy the desirable property of individual rationality, that is, if $x \in OSV\left((\succeq^i, w^i)_{i\in N}\right)$, then $x^i \succeq^i w^i$, for all $i \in N$.

We first prove the following lemma which plays a crucial role in the proof of Theorem 2 and is used in several propositions and comments throughout the paper.

Lemma 1 *For any economy* $E = (\succeq^i, \beta^i)_{i\in N}$ *and any* $u \in A(E)$*, there exists a unique vector* $a \in R^n$ *that varies continuously with* u*, such that an* OSV *allocation for the* $n-$*agent economy* $(\succeq^i, \beta^i + a_i e)_{i\in N}$ *yields the utility tuple* u*.*

Proof. Lemma 1 is true for $n = 1$ by monotonicity and continuity of the preferences. For $n \geq 1$, we assume it holds for $n-1$ and show it also holds for n. For each $j \in N$, let $(\widehat{a}_i^j)_{i \in N \backslash j}$ be the unique vector such that the economy with $(n-1)$ agents with initial endowments $(\beta^i + \widehat{a}_i^j e)_{i \in N \backslash j}$ has an OSV yielding the $(n-1)-$utility tuple u_{-j} with $(\widehat{a}_i^j)_{i \in N \backslash j}$ being continuous functions of u_{-j}.

To prove the existence of such a vector $a \in R^n$, we propose concessions $\left(c_i^j\right)_{i,j \in N, i \neq j}$ and prove that they support an OSV allocation yielding the utility vector u. The proposal involves the unknowns a_i, for $i \in N$, as follows:

$$c_i^j = -a_i + \widehat{a}_i^j \text{ for } i, j \in N, i \neq j.$$

The proposed concessions must satisfy the "fairness" condition $n.2)$:

$$\sum_{i \in N \backslash j} c_i^j = \sum_{i \in N \backslash j} c_j^i \text{ for } j \in N,$$

yielding, after arrangement, a system of linear equations given by:

$$(n-1)a_j - \sum_{i \in N \backslash j} a_i = \sum_{i \in N \backslash j} \widehat{a}_j^i - \sum_{i \in N \backslash j} \widehat{a}_i^j \equiv \theta^j \text{ for } j \in N.$$

Notice that $\sum\limits_{j \in N} \theta^j = 0$. It is then easy to check that the solutions for this system are all given by the following expressions, where $a_n \in R$:

$$a_i = \frac{1}{n}\left(\theta^i - \theta^n\right) + a_n \text{ for } i \in N.$$

Denote by $\widehat{a}$ the only real number such that u is efficient for an economy where the initial endowments are $\left(\beta^i + \frac{1}{n}\left(\theta^i - \theta^n\right)e + \widehat{a}e\right)_{i \in N}$ and $\widehat{x} \in R^{nk}$ a Pareto efficient allocation in that economy.

We now prove that the allocation $\widehat{x}$, is in $OSV((\succeq^i, \beta^i + a_i e)_{i \in N})$, with $a_i = \frac{1}{n}\left(\theta^i - \theta^n\right) + \widehat{a}$, and the concessions $c_i^j = -a_i + \widehat{a}_i^j$ for $i, j \in N$, $i \neq j$ supporting it (note that since the θ^j's and $\widehat{a}$ are continuous functions of the $u_i's$, so are the $c_i^{j\prime}s.$).

First, take any set of $(n-1)$ agents, say $N \backslash j$. An economy where these agents have initial endowments $\left(\beta^i + a_i e\right)_{i \in N \backslash j}$ and receive concessions $\left(c_i^j\right)_{i \in N \backslash j}$ is identical, by construction, to an economy where agents' initial endowments are $\left(\beta^i + \widehat{a}_i^j e\right)_{i \in N \backslash j}$. Hence, there is an OSV allocation for this $(n-1)-$agent economy where agent i's utility is u^i, for all $i \in N \backslash j$. This corresponds to the $n.1)$ requirement in the definition of an OSV allocation for the $n-$agent economy. Furthermore, by construction, requirement $n.2)$ is satisfied for the concessions $\left(c_i^j\right)_{i,j \in N, i \neq j}$. Finally, note that $\widehat{x}$ is efficient for the $n-$agent economy with initial endowments $\left(\beta^i + a_i e\right)_{i \in N, i \neq j}$ and that it generates utility levels given by u.

To complete the proof of Lemma 1, we show that if an OSV allocation for the economy $(\succeq^i, \beta^i + \overline{a}_i e)_{i \in N}$ yields the utility tuple u, then $\overline{a} = a$. Denote by $(\overline{c}_i^j)_{i,j \in N, i \neq j}$ the concessions associated with this OSV allocation. For any $j \in N$, define now the vector $\widehat{\overline{a}}^j \in R^{n-1}$ by $\widehat{\overline{a}}_i^j \equiv \overline{a}_i + \overline{c}_i^j$, for $i \in N \backslash j$. The economy where agents' initial endowments are $(\beta^i + \widehat{\overline{a}}_i^j e)_{i \in N \backslash j}$ is identical, by construction, to the economy with initial endowments $(\beta^i + \overline{a}_i e)_{i \in N \backslash j}$ when the concessions are $(\overline{c}_i^j)_{i \in N \backslash j}$. Therefore, an OSV allocation for the $(n-1)-$agent economy $(\beta^i + \widehat{\overline{a}}_i^j e)_{i \in N \backslash j}$ yields the utility tuple u_{-j}. The induction argument then implies that $\widehat{\overline{a}}_i^j = \widehat{a}_i^j$ for all $i \in N \backslash j$. Moreover, this argument applies to all $j \in N$. Therefore,

$$\overline{a}_i + \overline{c}_i^j = a_i + c_i^j \text{ for all } i, j \in N, i \neq j.$$

By summing, we obtain:

$$\sum_{j \in N \backslash i} (\overline{a}_i + \overline{c}_i^j) - \sum_{j \in N \backslash i} (\overline{a}_j + \overline{c}_j^i) = \sum_{j \in N \backslash i} (a_i + c_i^j) - \sum_{j \in N \backslash i} (a_j + c_j^i) \text{ for all } i \in N.$$

By the fairness condition of both matrixes $\overline{c}$ and c, and rearranging, we obtain

$$n(\overline{a}_i - a_i) = \sum_{j \in N} \overline{a}_j - \sum_{j \in N} a_j \text{ for all } i \in N.$$

Therefore, the sign of the difference $\overline{a}_i - a_i$ is independent of i. Assume, without loss of generality, that $\overline{a}_i > a_i$ for all $i \in N$. In this case, the n agents have more resources in the economy $(\beta^i + \overline{a}_i e)_{i \in N}$ than in the economy $(\beta^i + a_i e)_{i \in N}$, in contradiction to u being efficient for both economies. ■

In the following theorem we use Lemma 1 to construct a mapping, the fixed points of which, constitute the set of utilities achieved in OSV allocations.

Theorem 2 *The Ordinal Shapley Value is non empty and satisfies individual rationality in economic environments.*

Proof. The proof proceeds by induction. The results hold for $n = 1$. We assume the results hold for any economy with up to $(n-1)$ agents and prove that they hold for any economy with n agents, for $n \geq 2$.

We consider the economy $(\succeq^i, w^i)_{i \in N}$. We proceed to construct a continuous mapping from a suitably set of bounded utility profiles for this economy into itself. Let $\underline{u}^i = u^i(w^i - e)$. The set of utility profiles that constitute the domain (as well as range) of the mapping is denoted by H, and defined by:

$$H \equiv \{u \in R^n \mid \exists \text{ a Pareto efficient allocation } (x^i)_{i \in N} \in R^{nk} \\ \text{with } u^i(x^i) = u^i \text{ and } u^i \geq \underline{u}^i \text{ for } i = 1, ..., n\}.$$

We prove that the set H is homeomorphic to the unit simplex. To show it, we take the following utility representation: $\widehat{u}^i(x) = u^i(x) - u^i(w^i - e)$ and let:

$$\widehat{H} \equiv \{u \in R^n \mid \exists \text{ a Pareto efficient allocation } (x^i)_{i \in N} \in R^{nk} \\ \text{with } \widehat{u}^i(x^i) = u^i \text{ and } \widehat{u}^i \geq 0 \text{ for } i = 1, ..., n\}.$$

H and $\widehat{H}$ are clearly homeomorphic and we show that $\widehat{H}$ is homeomorphic to the $(n-1)-$unit simplex.

In economic environments, $\widehat{H}$ is a compact set (it is bounded by assumption and it is closed since u is continuous, due to the continuity of preferences). Let S be the $(n-1)-$unit simplex in R^n. For each $s \in S$ define the two following sets: $K_s = \{\alpha \in R \mid \alpha s \leq h \text{ for some } h \in \widehat{H}\}$ and $K^s = \{\alpha \in R \mid \alpha s \geq h \text{ for some } h \in \widehat{H}\}$. By the definition of the u^i's, K_s is not empty. Since $\widehat{H}$ is bounded K^s is also not empty. Both sets are closed given that $\widehat{H}$ is compact. Finally, since the union of K_s and K^s is R, their intersection is non-empty, and by the definition of $\widehat{H}$ it must be a singleton which we denote by α_s. Hence, we have shown that for each $s \in S$ there exists a unique α_s such that $\alpha_s s \in \widehat{H}$.

We now consider the function $f : S \longrightarrow \widehat{H}$ defined by: $f(s) = \alpha_s s$ with $\alpha_s \in R$ satisfying $\alpha_s s \in \widehat{H}$. By reversing the arguments used above, the inverse of f exists as well. In fact, $f^{-1}(u) = \frac{u}{\sum_{i=1}^{n} u_i}$ for all $u \in \widehat{H}$. We now show that f is continuous. Let $s_n \rightarrow s$, with $s_n \in S$ for all $n = 1, 2, ...$, and assume by way of contradiction that $f(s_n) \nrightarrow f(s)$. Since $\widehat{H}$ is compact there must in this case be a sequence n_i for which $f(s_{n_i}) \rightarrow \beta \neq f(s)$.

Moreover, $f(s_{n_i}) = \alpha_{n_i} s_{n_i}$ hence $\beta = (\lim_{i\to\infty} \alpha_{n_i})\, s \neq \alpha_s s$. However since both $\alpha_s s$ and $(\lim_{i\to\infty} \alpha_{n_i})\, s$ belong to $\widehat{H}$ (because $\widehat{H}$ is closed) it must be that $(\lim_{i\to\infty} \alpha_{n_i})\, s = \alpha_s s$ (by the definition of $\widehat{H}$) leading to a contradiction. Hence, $f(s_n) \to f(s)$ and f is continuous.

Similarly f^{-1} is continuous as well and we have shown that $\widehat{H}$ is homeomorphic to the $(n-1)-$unit simplex. (See Proposition 4.6.1 in Mas-Colell, 1985, for a similar result).

We now return to the construction of the mapping from H into H. We denote by H^b the "border" of H, the set of all the utility vectors for which the ith component equals $\underline{u}^i$ for some i. Formally,

$$H^b \equiv \{u \in H/\ u^i = \underline{u}^i \text{ for some } i \in N\}.$$

For any vector $u \in H$, we look at $u_{-j} \in R^{n-1}$ for all $j \in N$. Lemma 1 provides for each u_{-j} a unique vector $a^j \in R^{n-1}$ such that an OSV allocation for the $(n-1)-$agent economy $(\succeq^i, w^i + a^j_i e)_{i\in N/j}$ yields the utility tuple u_{-j}. We let $c^j_i(u) \equiv a^j_i$. These are the concessions that agent j "needs" to make in order for the other $n-1$ agents to achieve the utility level u_{-j}.

Using the concessions $\left(c^j_i(u)\right)_{j,i\in N, j\neq i}$ we construct n "net concessions" corresponding to u by:

$$C^i(u) \equiv \sum_{j\in N\setminus i} c^i_j(u) - \sum_{j\in N\setminus i} c^j_i(u), \text{ for all } i \in N.$$

Notice that $\sum\limits_{i\in N} C^i(u) = 0$.

We now define a mapping from H into H. Each utility profile u in H is mapped to a utility profile $\overline{u} \in H$ by increasing (decreasing) the components associated with positive (negative) $C^i(u)$s, making necessary adjustments to preserve feasibility and efficiency. More precisely, we let

$$D(u) \equiv \min_{i\in N,\ C^i(u)<0} \{u^i - \underline{u}^i\} \text{ if } C(u) \neq 0 \in R^n.$$
$$D(u) \equiv 0 \text{ otherwise.}$$

Note that, if u is not in H^b (that is, if u is at the "interior" of H) then $D(u) > 0$ if $C(u) \neq 0$.

Consider the following vector:

$$\widetilde{u}(u) \equiv u + \frac{D(u)}{\max\limits_{i\in N}\{|C^i(u)|\} + 1} C(u) = \begin{pmatrix} u^1 \\ \vdots \\ u^n \end{pmatrix} + \frac{D(u)}{\max\limits_{i\in N}\{|C^i(u)|\} + 1} \begin{pmatrix} C^1(u) \\ \vdots \\ C^n(u) \end{pmatrix}.$$

Denote by $C(u)_+ \in R^n$ the vector defined as follows: $C^i(u)_+ = C^i(u)$ if $C^i(u) > 0$, and $C^i(u)_+ = 0$ if $C^i(u) \leq 0$. Similarly, denote by $C(u)_- \in R^n$ the vector that is defined by $C^i(u)_- = C^i(u)$ if $C^i(u) < 0$, and $C^i(u)_- = 0$ if $C^i(u) \geq 0$.

If $\widetilde{u}(u)$ is feasible and efficient, take $\overline{u}(u) = \widetilde{u}(u)$.

If $\widetilde{u}(u)$ is feasible but not efficient, take

$$\overline{u}(u) = u + \frac{D(u)}{\max_{i \in N}\{|C^i(u)|\} + 1}(C(u) - \delta C(u)_-),$$

where $\delta \in (0,1)$ is the unique real number such that $\overline{u}(u)$ previously defined is feasible and efficient. (The efficiency requirement implies $\delta > 0$, whereas feasibility implies $\delta < 1$.)

If $\widetilde{u}(u)$ is not feasible, take

$$\overline{u}(u) = u + \frac{D(u)}{\max_{i \in N}\{|C^i(u)|\} + 1}(C(u) - \delta C(u)_+),$$

where $\delta \in (0,1)$ is the unique real number such that $\overline{u}(u)$ previously defined is feasible and efficient. (Here, feasibility implies $\delta > 0$, whereas efficiency implies $\delta < 1$.)

To prove that $\overline{u}(u) \in H$, we only need to show that $\overline{u}^i(u) \geq \underline{u}^i$ for all i. If $D(u) = 0$, this property is trivially satisfied. If $D(u) > 0$ then $C(u) \neq 0$. By the definition of $D(u)$ and $\widetilde{u}(u)$, it is easy to check that for i's for which $C^i(u) < 0$ the decrease in coordinate i is small enough so that $\widetilde{u}(u)^i \geq \underline{u}^i$. Second, if $\widetilde{u}(u)^i \geq \underline{u}^i$, then the construction of $\overline{u}(u)$ makes sure that also $\overline{u}(u)^i \geq \underline{u}^i$.

Claim a: The mapping $\overline{u}(u)$ has a interior fixed point.

To prove the claim, notice first that the mapping $\overline{u}(u)$ is continuous. Indeed, the function $D(u)$ is clearly continuous. Also, $C(u)$ is continuous as soon as the "concessions" $c^i_j(u)$ are a continuous function of u. By Lemma 1, the $c^i_j(u)$s are a continuous function of u. Since H is homeomorphic to an $n-$unit simplex, the mapping $\overline{u}(u)$ must have a fixed point. It now remains to show that the fixed point cannot occur on the boundary. We prove it by the way of contradiction.

Suppose that the fixed point u is on the boundary, that is, $\overline{u}(u)^i = u^i = \underline{u}^i$ for some $i \in N$. Assume, without loss of generality that $u^1 = \underline{u}^1$. We claim that $C^1(u) > 0$. First, we prove that $\sum_{i \in N \setminus 1} c^1_i(u) > 0$. Indeed, if $\sum_{i \in N \setminus 1} c^1_i(u) \leq 0$, then after the concessions are made, player 1 obtains at least the utility $u^1(w^1) > \underline{u}^1$ since the aggregate endowment at the disposal of the others is lower or equal to $\sum_{i\in N\setminus 1} w^i$ and the final allocation is efficient.

Second, for u^1 to equal $\underline{u}^1$ it is necessarily the case that $c_1^i(u) < 0$ for all $i = 2, ..., n$. Otherwise, the initial endowment of player 1 when i concedes is at least w^1 and hence, because the OSV is individually rational for any environment with $(n-1)$ agents, his final utility can not be $\underline{u}^1$. Therefore, $C^1(u) > 0$ if $u^1 = \underline{u}^1$.

Third, since the previous reasoning holds for every i with $u^i = \underline{u}^i$, we also know that $D(u) > 0$ since $u^i - \underline{u}^i > 0$ as soon as $C^i(u) < 0$ and $C^i(u) < 0$ for at least one $i \in N$ given that $C^1(u) > 0$.

Therefore, by the construction of our mapping, the utility tuple u is mapped to a point with a strictly larger utility level for agent 1 and cannot constitute a fixed point. This proves Claim a.

Claim b: A utility tuple u is a fixed point of the function $\overline{u}$ if and only if there exists an allocation $x \in OSV\left((\succeq^i, w^i)_{i\in N}\right)$ such that $u(x) = u$.

To prove the claim, let u be a fixed point of the previous mapping, x the feasible allocation that yields the utility level u, and c the matrix constructed using Lemma 1 (for simplicity, we write c, C, and D instead of $c(u)$, $C(u)$, and $D(u)$). We claim that c is the matrix of concessions that support x as an OSV allocation. Given the way we constructed c, each agent is indifferent with respect to the identity of the conceding agent. Requirement $n.1$) of the definition of the OSV is then immediately seen to hold. Also requirement $n.2$) holds since, by interiority of the fixed point, $D > 0$ if $C^j < 0$ for some $j \in N$. In an interior fixed point, $C^j = 0$ for all $j \in N$. Therefore, the concessions satisfy $\sum_{i\in N\setminus j} c_i^j = \sum_{i\in N\setminus j} c_j^i$ for all $j \in N$.

Notice also that the utility corresponding to any OSV allocation is a fixed point of our mapping by construction. Therefore, the set of utilities generated by the OSV allocations coincides with the set of fixed points of the mapping $\overline{u}(u)$.

To complete the proof of the theorem we show that every OSV allocation is individually rational for the economy $(\succeq^i, w^i)_{i\in N}$. Assume by way of contradiction that agent i receives a bundle strictly worse than w^i in an element of $OSV\left((\succeq^i, w^i)_{i\in N}\right)$. It must then be that $\sum_{i\in N\setminus j} c_j^i > 0$, hence $\sum_{i\in N\setminus j} c_i^j > 0$ as well. This however means that there exists a $j \neq i$ for which $c_i^j > 0$. Hence if agent j concedes, agent i is in an environment with $n-1$ agents and initial endowment $w^i + c_i^j e$ which is strictly larger than w^i. By the induction assumption, any OSV allocation for this environment would be preferred to $w^i + c_i^j e$, hence strictly preferred to w^i. This is in contradiction to the original OSV allocation yielding an outcome worse than w^i for agent i.

This concludes the proof that the OSV exists and is individually rational. ■

The proof of Theorem 2 uses a fixed point argument, it does not provide an algorithm to calculate the OSV in a particular economy, and yields no information regarding the possible uniqueness of the solution in particular environments. There is, however, much more information regarding the concessions associated with OSV allocations. First, Lemma 1 implies that the matrix associated with any OSV allocation is unique. Indeed, let $x \in OSV\left((\succeq^i, w^i)_{i\in N}\right)$ and $u^i \equiv u^i(x^i)$ for all $i \in N$. For every $j \in N$, Lemma 1 says that there exists a unique vector $c^j \in R^{n-1}$ such that an allocation in $OSV\left((\succeq^i, w^i + c_i^j e)_{i\in N\setminus j}\right)$ yields the utility tuple u_{-j}. That is, there exists a unique matrix of concessions supporting x. Second, if we identify an allocation in the OSV, then the proof of Lemma 1 indicates how to construct the unique matrix of concessions associated with this allocation.

Finally we consider the following example which has also been analyzed in Hart (1985, example 5.7). The economic environment consists of three agents $(1, 2, 3)$ and three commodities (x_1, x_2, x_3) where preferences for non-negative consumptions and initial endowments are given by:[7]

$$\begin{aligned}
u^1(x_1^1, x_2^1, x_3^1) &= x_1^1 + x_2^1 & w^1 &= (2,2,0)\\
u^2(x_1^2, x_2^2, x_3^2) &= 0.5x_1^2 + x_3^2 & w^2 &= (2,0,2)\\
u^3(x_1^3, x_2^3, x_3^3) &= x_2^3 + x_3^3 & w^3 &= (0,2,2)
\end{aligned}$$

The OSV outcome for this environment (it also happens to be unique) is the allocation:

$$x^1 = (4, 0.3791, 0); x^2 = (0, 0, 3.2745); x^3 = (0, 1.6209, 2.725)$$

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[7] The utility functions, as given in Hart (1985) are defined just over the non-negative orthant. Note that in our set up the utility functions need to be defined over all of R^k. This can be accomplished in several ways without affecting the OSV outcome. One option is to let the utility function equal $-\infty$ for all points outside the non-negative orthant. Alternately (to preserve continuity) the u^i's could be redefined by:

$$u^1(x_1^1,x_2^1,x_3^1) = \min\{x_1^1,2x_1^1\} + \min\{x_2^1,2x_2^1\} + \min\{0,2x_3^1\}$$

and similarly for the other two agents.

and the concessions supporting the outcome are:

$$c_2^1 = c_1^2 = 0.129\,09; c_3^1 = c_1^3 = 0.119\,28; c_3^2 = c_2^3 = 0.112\,74.$$

The associated utility profile is $(u^1, u^2, u^3) = (4.3791, 3.2745, 4.3464)$. Note the Shapley value yields the utility profile $(4.5, 3.5, 4)$ whereas the Harsanyi value yields the utility profile $(13/3, 10/3, 13/3)$.

6 Characteristics of the *OSV*

By definition, the OSV allocations satisfy some fairness and consistency properties. Also, Theorem 2 shows that they are individually rational. The OSV allocations however satisfy several additional appealing properties. The main result of this section provides a characterization of the OSV in terms of coalitional dividends similar to the characterization obtained for the Shapley value. The first step towards this result is to show that the fact that concessions in the previous example are symmetric is not a coincidence. The concessions supporting OSV allocations are always *symmetric* as stated in Proposition 3.

Proposition 3 *If the concession matrix c supports an OSV allocation, then* $c_j^i = c_i^j$ *for all* $i, j \in N, i \neq j$.

Proof. The proof proceeds by induction. It is true for any economy with $n = 2$ agents by the fairness condition. We assume the property is satisfied for every economy with $n-1$, with $n \geq 3$, agents and show it also holds for $(\succeq^i, w^i)_{i \in N}$. Let $x \in OSV\left((\succeq^i, w^i)_{i \in N}\right)$, and let $(c_j^i)_{i,j \in N, i \neq j}$ and $u \in R^n$ be the concessions supporting x and the utility tuple associated with it. For any agent $i \in N$, there must exist some OSV allocation (denoted by $y(i)$) for the $(n-1)-$agent economy $(\succeq^j, w^j + c_j^i e)_{j \in N \backslash i}$ yielding the utility profile u_{-i}. Similarly, for any agent $k \in N \backslash i$ there must exist an OSV allocation (denoted by $y(ki)$) for the $(n-2)-$agent economy $(\succeq^j, w^j + (c_j^i + c_j^{ki})e)_{j \in N \backslash \{i,k\}}$ yielding the utility profile $u_{-\{i,k\}}$, where $(c_j^{ki})_{k,j \in N \backslash i, k \neq j} \in R^{(n-1)(n-2)}$ supports $y(i)$. By Lemma 1 there exists a unique vector $a \in R^{n-2}$ such that an OSV allocation for the $(n-2)-$agent economy $(\succeq^j, w^j + a^j e)_{j \in N \backslash \{i,k\}}$ yields the utility tuple $u_{-\{i,k\}}$. Hence we have $w^j + a^j e = w^j + (c_j^i + c_j^{ki})e$

for any three distinct agents $i, j, k \in N$. By permuting the roles of i and k we obtain:

$$c_j^i + c_j^{ki} = c_j^k + c_j^{ik} \text{ for any three distinct agents } i, j, k \in N. \tag{5}$$

Hence, $c_2^1 + c_2^{31} = c_2^3 + c_2^{13}, c_1^3 + c_1^{23} = c_1^2 + c_1^{32}$, and $c_3^2 + c_3^{12} = c_3^1 + c_3^{21}$. Moreover, by the induction assumption, concessions are symmetric for any economy with $(n-1)$ agents, hence $c_2^{31} = c_3^{21}$, $c_2^{13} = c_1^{23}$, and $c_1^{32} = c_3^{12}$. Using this property and summing the three previous equations, we obtain:

$$(c_1^3 - c_3^1) + (c_2^1 - c_1^2) + (c_3^2 - c_2^3) = 0.$$

We can repeat the same argument with agent 3 replaced by agents $4, ..., n$. Summing up all the equalities, we get:

$$\{(c_1^3 - c_3^1) + ... + (c_1^n - c_n^1)\} + (n-2)(c_2^1 - c_1^2) + \{(c_3^2 - c_2^3) + ... + (c_n^2 - c_2^n)\} = 0.$$

Using the fairness requirement $n.2)$ we get:

$$(c_2^1 - c_1^2) + (n-2)(c_2^1 - c_1^2) + (c_2^1 - c_1^2) = 0.$$

Hence, $c_2^1 = c_1^2$.

Similarly it can be shown that $c_j^i = c_i^j$ for any $i, j \in N, i \neq j$. ■

The following theorem provides a characterization of the OSV analogous to the characterization of the Shapley value in terms of coalitional dividends.

Theorem 3 *Let Φ be a correspondence that associates a set of efficient allocations to every economic environment $(\succeq^i, w^i)_{i \in N}$. Suppose that it satisfies property (Q):*

(Q) For all $x \in \Phi((\succeq^i, w^i)_{i \in N})$ and $u^i \equiv u^i(x^i)$ for all $i \in N$, there exists a vector $(\lambda_S)_{S \subseteq N} \in R^{2^n}$ such that

$$u^i\left(w^i + d_i(T)e + \sum_{\substack{S \ni i \\ S \subseteq T}} \lambda_S e\right) = u^i \text{ for all } T \subseteq N, \text{ for all } i \in T, \tag{6}$$

where $d(T) \in R^{|T|}$ is a vector such that an element of the set $\Phi((\succeq^j, w^j + d_j e)_{j \in T})$ yields the utility tuple u_T.

Then, Φ is a sub-correspondence of the OSV correspondence.

Moreover, the OSV correspondence satisfies property (Q).

Proof. The proof of both claims proceeds by induction. First we show that Φ is a sub-correspondence of the OSV correspondence. When $n = 1$, the result holds trivially. We assume now that the result holds for up to $n-1$ agents and show it holds for n agents, $n \geq 2$.

Take $x \in \Phi((\succeq^i, w^i)_{i \in N})$ and let $(\lambda_S)_{S \subseteq N} \in R^{2^n}$ be the vector associated with x. Consider the matrix $c \in R^n x R^{n-1}$ defined by $c_i^j = \sum_{\substack{S \ni i,j \\ S \subseteq N}} \lambda_S$. We claim that the matrix c supports x as an OSV allocation. First, given that $c_i^j = c_j^i$ for all $i, j \in N, i \neq j$, condition $n.2$) of Definition 1 is satisfied. Second, to prove condition $n.1$), take any $j \in N$ and consider the economy $(\succeq^i, w^i + c_i^j e)_{i \in N \backslash j}$. Notice that since

$$u^i\left(w^i + \sum_{\substack{S \ni i \\ S \subseteq N}} \lambda_S e\right) = u^i = u^i\left(w^i + b_i(N\backslash j)e + \sum_{\substack{S \ni i \\ S \subseteq N \backslash j}} \lambda_S e\right) \text{ for all } i \in N\backslash j,$$

it happens that

$$b_i(N\backslash j) = \sum_{\substack{S \ni i,j \\ S \subseteq N}} \mu \lambda_S = c_i^j \text{ for all } i \in N\backslash j.$$

Therefore, the utility tuple u_{-j} is attainable (and efficient) in the economy $(\succeq^i, w^i + c_i^j e)_{i \in N\backslash j}$ since it is attainable (and efficient) in $(\succeq^i, w^i + b_i(N\backslash j)e)_{i \in N \backslash j}$. Denote by $y(j)$ the efficient allocation that yields u_{-j}. Since for all $T \subseteq N\backslash j$, $\left(b_i(T) - c_i^j\right)_{i \in T}$ is a vector such that an element of the set $\Phi((\succeq^j, w^j + c_i^j e + [b_j(T) - c_i^j]e)_{j \in T})$ yields the utility tuple u_{-j}, the induction hypothesis ensures that $y(j) \in OSV((\succeq^i, w^i + c_i^j e)_{i \in N\backslash j})$. This proves condition $n.1$) and concludes the proof that x is an OSV allocation.

We now show that the OSV allocation satisfies property Q. If $N = \{i\}$, then $\lambda_{\{i\}}$ exists and is unique: $\lambda_{\{i\}} = 0$. Suppose the result holds for any economy with at most $n-1$ agents, for $n \geq 2$. Let $x \in OSV((\succeq^i, w^i)_{i \in N})$ and $u^i \equiv u^i(x^i)$ for all $i \in N$. Denote by $\left(c_j^i\right)_{i,j \in N, i \neq j}$ the concessions supporting x as an OSV allocation and, for all $j \in N$, let $y(j)$ be such that $y(j) \in OSV((\succeq^i, w^i + c_i^j e)_{i \in N\backslash j})$ and $y(j)^i \sim^i x^i$ for all $i \in N\backslash j$.

Applying the induction argument, for all $j \in N$, there exists a unique $(\lambda_S(j))_{S \subseteq N\backslash j} \in R^{2^{n-1}}$ such that:

$$u^i\left(w^i + c_i^j e + d_i(T;j)e + \sum_{\substack{S \ni i \\ S \subseteq T}} \lambda_S(j) e\right) = u^i \text{ for all } T \subseteq N\backslash j, \text{ for all } i \in T,$$

where $d(T;j) \in R^{|T|}$ is the unique vector such that an element of the set $OSV((\succeq^j, w^j + c_i^j e + d_i(T;j)e)_{j\in T})$ yields the utility tuple u_T. We first claim that $\lambda_S(j) = \lambda_S(k)$ for all $S \subseteq N\backslash\{j,k\}$. Indeed, consider the economy $(\succeq^i, w^i)_{i\in S}$ and the unique vector $d(S) \in R^{|S|}$ such that an element of $OSV\left((\succeq^i, w^i + d^i(S)e)_{i\in S}\right)$ yields the utility tuple u_S. By the induction argument, there is a unique vector $(\lambda_B)_{B\subseteq S} \in R^{2^{|S|}}$ such that

$$u^i\left(w^i + d_i(T)e + \sum_{\substack{B\ni i\\ B\subseteq T}} \lambda_B e\right) = u^i \text{ for all } T \subseteq S, \text{ for all } i \in T.$$

Since the vector $d(T)$ is unique, it is immediate that $d_i(T) = c_i^j + d_i(T;j) = c_i^k + d_i(T;k)$ for all $T \subseteq S, i \in T$. And since the vector $(\lambda_B)_{B\subseteq S}$ is unique, it is also immediate that $\lambda_S = \lambda_S(j) = \lambda_S(k)$.

According to the previous claim, we can propose λ_S $(= \lambda_S(j)$ for any $j \notin S)$ for any $S \subset N$. With the vector $(\lambda_S)_{S\subset N}$, the equality $u^i\left(w^i + d_i(T)e + \sum_{\substack{S\ni i\\ S\subseteq T}} \lambda_S e\right) = u^i$ holds for all $T \subset N$ and for all $i \in T$. Moreover, the vector for which the equality happens is unique. The unique value still to be found is λ_N.

For any $i \in N$, consider the value $\lambda_N(i)$ implicitly (and uniquely) defined by:

$$u^i\left(w^i + \sum_{\substack{S\ni i\\ S\subset N}} \lambda_S e + \lambda_N(i)e\right) = u^i.$$

We complete the proof if we show that $\lambda_N(i) = \lambda_N(j)$ for any $i, j \in N$. By induction, for any $i, j \in N$:

$$u^i\left(w^i + c_i^j e + \sum_{\substack{S\ni i\\ S\subseteq N\backslash j}} \lambda_S e\right) = u^i = u^i\left(w^i + \sum_{\substack{S\ni i\\ S\subset N}} \lambda_S e + \lambda_N(i)e\right),$$

hence,

$$\lambda_N(i) = c_i^j + \sum_{\substack{S\ni i\\ S\subseteq N\backslash j}} \lambda_S - \sum_{\substack{S\ni i\\ S\subset N}} \lambda_S = c_i^j - \sum_{\substack{S\supseteq\{i,j\}\\ S\subset N}} \lambda_S.$$

Given the symmetry of the concessions, $c_i^j = c_j^i$, $\lambda_N(i) = \lambda_N(j)$ for all $i, j \in N$, which completes the proof. ■

Therefore, the OSV correspondence is characterized as the union of the correspondences (or as the largest correspondence) that satisfy property *(Q)*. Borrowing the terminology used in TU environments, we refer to the vector $(\lambda_S)_{S\subseteq N}$ as the *coalitional dividends*. Although the coalitional dividends are somewhat more complex to define in our economic environment than they are in TU environments, they reflect the same idea: if $i \in S$, then λ_S is the dividend agent i obtains because he belongs to coalition S. Indeed, given that $d(N) = 0$, the final utility agent i obtains in the OSV allocation characterized by the dividends $(\lambda_S)_{S\subseteq N}$ is $u^i = u^i\left(w^i + \sum_{\substack{S\ni i\\ S\subseteq N}} \lambda_S e\right)$. The added difficulty in our framework is how to measure the value of a coalition, since the additional utility (in terms of e) that agents in a certain coalition S obtain depends upon the level of their initial endowment. Theorem 3 shows that the proper reference to measure the increase in utility is given by the level of utility at the OSV allocation. In TU environments, the reference point is not important since the value of the coalition does not depend on the initial endowment.

It is interesting to point out that the relationship between the coalitional dividends that exists for every OSV allocation and the concessions matrix that supports this allocation, is the same as the one that exists for the Shapley value in TU environments (that was proved in Section 2). It is easy to see that $d(N\backslash j) = (c_i^j)_{i\in N\backslash j}$ for any $j \in N$. Therefore, applying (6) to the sets N and $N\backslash j$, we obtain:

$$u^i\left(w^i + \sum_{\substack{S\ni i\\ S\subseteq N}} \lambda_S e\right) = u^i = u^i\left(w^i + c_i^j e + \sum_{\substack{S\ni i\\ S\subseteq N\backslash j}} \lambda_S e\right) \text{ for any } i \in N\backslash j,$$

hence,

$$c_i^j = \sum_{\substack{S\ni i,j\\ S\subseteq N}} \lambda_S \text{ for all } i, j \in N, i \neq j.$$

We conclude this section with two further properties of the OSV. The next proposition shows that the OSV is *monotonic* in initial endowments.

Proposition 4 *Consider an economic environment* $(\succeq^i, w^i)_{i\in N}$ *where* $\succeq^j \equiv \succeq^k$ *and* $w^j \geq (>) w^k$ *for some* $j \neq k$. *Then,* $x^j \succeq^j (\succ^j) x^k$ *for any* $x \in OSV((\succeq^i, w^i)_{i\in N})$.

Proof. The proof proceeds by induction. Consider first the case of two agents ($n = 2$) and assume $\succeq^1 \equiv \succeq^2$ .Let u represent the preferences of both agents. The unique level of utility that they achieve in the OSV allocations is:

$$\underset{c \in R_+}{Max} \left(u(w^1 + ce), u(w^2 + ce) \right) \left| \left(u(w^1 + ce), u(w^2 + ce) \right) \in A \right\} .$$

It is then immediate that $w^1 \geq w^2$ implies $x^1 \succeq^1 x^2$, for $x = OSV((\succeq^i, w^i)_{i=1,2})$. Moreover, x^1 is strictly preferred to x^2 if w^1 is strictly greater than w^2.

We assume now that the property holds for economies with up to $n - 1$ agents, for $n \geq 3$. We prove, by contradiction, that it also holds for economies with n agents.

Without loss of generality, suppose $\succeq^1 \equiv \succeq^2$, $w^1 \geq w^2$, and $x^1 \prec^1 x^2$ for some $x \in OSV((\succeq^i, w^i)_{i \in N})$. (For notational convenience, we do the proof for the case $w^1 \geq w^2$; the proof is similar when $w^1 > w^2$.) Using property $n.1$) in the definition of an OSV allocation, let $y(1) \in OSV((\succeq^i, w^i + c_i^1 e)_{i \in N \backslash 1})$ be such that $u^i(y(1)^i) = u^i(x^i)$ for all $i \in N \backslash 1$, and $y(2) \in OSV((\succeq^i, w^i + c_i^2 e)_{i \in N \backslash 2})$ be such that $u^i(y(2)^i) = u^i(x^i)$ for all $i \in N \backslash 2$.

Given that $u^i(y(1)^i) = u^i(y(2)^i)$ for all $i \in N \backslash \{1, 2\}$, $u^1(y(2)^1) < u^2(y(1)^2)$, $\succeq^1 \equiv \succeq^2$, and the efficiency of the allocations $y(1)$ and $y(2)$, it must be the case that the total initial resources in the economy $(\succeq^i, w^i + c_i^1 e)_{i \in N \backslash 1}$ are larger than in the economy $(\succeq^i, w^i + c_i^2 e)_{i \in N \backslash 2}$. That is,

$$\sum_{i \in N \backslash 1} c_i^1 > \sum_{i \in N \backslash 2} c_i^2.$$

By symmetry, $c_2^1 = c_1^2$, $c_i^1 = c_1^i$ and $c_i^2 = c_2^i$ for all $i \in N \backslash \{1, 2\}$. Therefore,

$$\sum_{i \in N \backslash \{1,2\}} c_1^i > \sum_{i \in N \backslash \{1,2\}} c_2^i.$$

Let $k \in N \backslash \{1, 2\}$ be such that $c_1^k > c_2^k$, and $y(k) \in OSV((\succeq^i, w^i + c_i^k e)_{i \in N \backslash k})$ be such that $u^i(y(k)^i) = u^i(x^i)$ for all $i \in N \backslash k$. In the $(n - 1)-$agent economy $((\succeq^i, w^i + c_i^k e)_{i \in N \backslash k})$, it happens that $\succeq^1 \equiv \succeq^2$ and $w^1 + c_1^k e > w^2 + c_2^k e$. By the induction hypothesis, $u^1(y(k)^1) \geq u^2(y(k)^2)$, that is, $u^1(x^1) \geq u^2(x^2)$. This is in contradiction to our original hypothesis. ■

The next property, *anonymity* of the OSV is an immediate corollary of the previous proposition.

Corollary 2 *Consider an economic environment* $(\succeq^i, w^i)_{i\in N}$ *where* $\succeq^j \equiv \succeq^k$ *and* $w^j = w^k$ *for some* $j \neq k$*. Then,* $x^j \backsim^j x^k$ *for any* $x \in OSV((\succeq^i, w^i)_{i\in N})$*. Moreover, if the preferences of agents* j *and* k *are strictly quasiconcave, then* $x^j = x^k$ *for any* $x \in OSV((\succeq^i, w^i)_{i\in N})$*.*

7 The weighted OSV

Shapley (1953b) extends the Shapley TU value by considering nonsymmetric divisions of the surplus. He defines the (now called) *weighted Shapley value* by stipulating an exogenously given system of weights $q \in R^n_{++}$, assigning each agent i the share $q_i / \sum_{j\in N} q_j$ of the unit in each unanimity game, and defining the value as the linear extension of this operator to the set of TU games. There exist several characterizations of the weighted Shapley value. The next proposition states, without a proof, a new characterization, similar to the one provided in Corollary 1.[8]

Proposition 5 *A value* ξ *is the* $q-$*weighted Shapley value if and only if for each game* (N, v) *there exists a matrix of concessions* $c(N, v) \equiv (c^i_j(N, v))_{i,j\in N, i\neq j}$*, with* $c^i_j(N, v)$ *in* R *for all* $i, j \in N, i \neq j$*, such that:*

(1) $\xi_i(N, v) = \xi_i(N\backslash j, v) + c^j_i(N, v)$ *for all* $i, j \in N, i \neq j$*, and*

(2) $\sum\limits_{i\in N\backslash j} q^j c^j_i(N, v) = \sum\limits_{i\in N\backslash j} q^i c^i_j(N, v)$ *for all* $j \in N$*.*

Following the same route we took in defining the OSV, we can define a weighted value for economic environments where the weights of the agents are taken into account. We now describe an extension of the OSV which yields the $q-$ weighted OSV $(q - wOSV)$ solution, which reduces to the $q-$ weighted Shapley value in economic environments that can be described as a TU environment. The only difference with respect to the definition of the OSV lies in the "fairness" condition $n.2)$:

Definition 2 *We define the* $q-$ *weighted Ordinal Shapley Value recursively.*

($n = 1$*) In the case of an economy with one agent with preferences* $\succeq^1$ *and initial endowments* $a^1 \in R^k$*, the* $q - wOSV$ *is given by the initial endowment:* $q - wOSV(\succeq^1, a^1) = \{a^1\}$ *.*

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[8] For interpretation, see also Section 4 in Pérez-Castrillo and Wettstein (2001).

Suppose that the solution has been defined for any economy with $(n-1)$ *or less agents.*

(n) In the case of an economy $(\succeq^i, a^i)_{i\in N}$ *with a set* N *of* n *agents, the* $q-wOSV$ $((\succeq^i, a^i)_{i\in N})$ *is the set of efficient allocations* $(x^i)_{i\in N}$ *for which there exists an* $n-$*tuple of concession vectors* $(c^i)_{i\in N}$ *that satisfy*

$n.1)$ *for all* $j \in N$*, there exists* $y(j) \in q - wOSV\left((\succeq^i, a^i + c^j_i e)_{i\in N\backslash j}\right)$ *such that* $x^i \sim^i y(j)^i$ *for all* $i \in N\backslash j$*, and*

$n.2)$ $\sum\limits_{i\in N\backslash j} q^j c^j_i = \sum\limits_{i\in N\backslash j} q^i c^i_j$ *for all* $j \in N$*.*

It is worthwhile to notice that the "weighted fairness" condition $n.2)$, together with the "consistency" requirement $n.1)$ also imply in this case that the concessions that support the $q-wOSV$ are "weighted" symmetric, in that we have $q^j c^j_i = q^i c^i_j$ for all $i, j \in N, i \neq j$. Moreover, very small changes in the proof of Theorem 2 are needed, to establish existence and individual rationality of this value, for any economic environment, which we state as:

Theorem 4 *The* $q-$*weighted Ordinal Shapley Value is non empty and satisfies individual rationality in economic environments, for any* $q \in R^n_{++}$*.*

8 Conclusion

This paper addressed the problem of sharing a joint surplus among the agents creating it. We looked for a solution associating with each economic environment (agents described by preferences and endowments) a set of outcomes (allocations of the aggregate endowment across the agents). We showed there exists such an (ordinal) solution that satisfies efficiency and suitably defined notions of consistency and fairness. This solution being a natural extension of the Shapley value to general environments (NTU games).was called an Ordinal Shapley value.

The OSV provided not just an allocation but also a matrix of concessions "measuring" the gains each agent foregoes in favor of the other agents. Further analysis showed these concessions were symmetric, what agent i concedes to agent j coincides with the concession of agent j to agent i. This symmetry property reduces to the balanced contributions property of the Shapley Value for TU games. The next stage of the analysis characterized the OSV in terms of coalitional dividends. We further showed that the OSV satisfies monotonicity in initial endowments and anonymity. Finally, we constructed a family of $q-$weighted OSV's, which are the ordinal counterparts (in our setting) to the family of $q-$weighted Shapley values for TU games.

The main advantage of this extension compared to previous attempts to extend the value is the fact it is ordinal. It is also defined in the commodity space rather in the "utility" space, whereas several previous ordinal values were defined solely on the utility space (Safra and Samet, 2004).

Since it exists for a large class of environments it can be used to address a variety of distributional issues dispensing of the need to assume quasi-linear preferences or convexity of preferences. Problems of allocating joint costs can be handled as well without restricting the environment through the quasi-linearity in "money" assumption or convexity of the cost function.

The OSV approach allowing for different reference bundles generates a family of outcomes. A similar phenomenon is given by the family of $PEEE$ allocations in Pazner and Schmeidler (1978) where conceivably different allocations are obtained by choosing different rays along which the search for an allocation proceeds.

Further research should clarify the connections between the OSV and other well-known ordinal solution concepts like the core and competitive equilibria outcomes. More research is also needed to determine what happens to the set of OSV allocations as the economy grows and\or more restrictions are imposed on the preferences. The implementability of the OSV remains the topic of further work as well.

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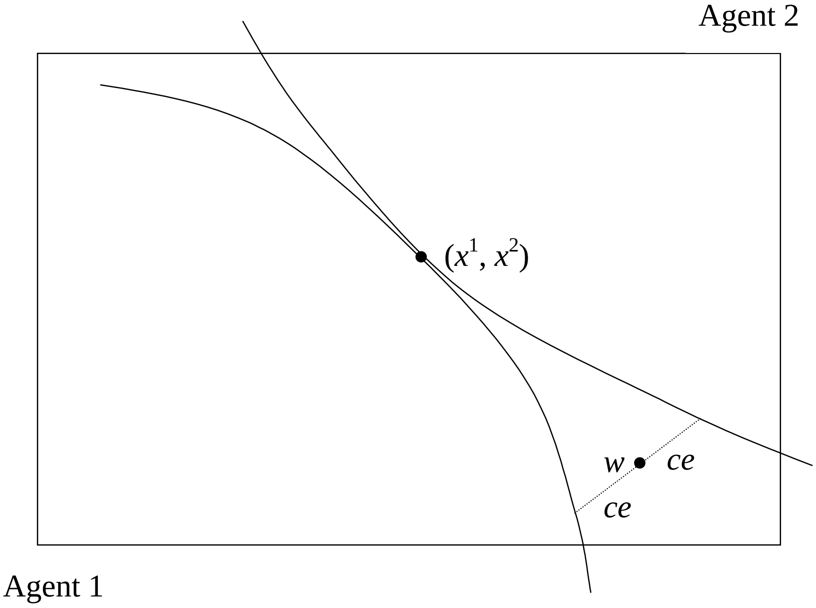

Figure 1: The solution in the two-agent economy.