Sharing the surplus: A just and efficient proposal for environments with externalities[*]

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Abstract

Economic activities, both on the macro and micro level, often entail wide-spread externalities. This in turn leads to disputes regarding the compensation levels to the various parties affected. We propose a general, yet simple, method of deciding upon the distribution of the gains (costs) of cooperation in the presence of externalities. This method is shown to be the unique one satisfying several desirable properties. Furthermore, we illustrate the use of this method to resolve the sharing of benefits generated by international climate control agreements.

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1 Introduction

Achieving cooperation and sharing the resulting benefits in the presence of externalities is a central question in many economic environments. These issues are often decided by international agreements, when countries are the players involved. Take for example the Kyoto protocol drafted in 1997 to address climate control. It was further elaborated upon in the Buenos Aires plan of action put together in 1998. The Bonn agreements in 2001 resolved several outstanding issues and paved the way to the Marrakesh accords that contained a clearer picture of the Kyoto protocol. Of concern were the commitments undertaken by the various parties and the enforcement as well as compensation mechanisms set in place.

The GATT (General Agreement on Tariffs and Trade) was signed in 1947 in Geneva and focused on trading arrangements. After eight rounds of multilateral trade negotiations, the Uruguay round concluded with the signing in 1994 of the Marrakesh agreement, closing down the GATT which was replaced in 1995 by the World Trade Organization (WTO). The WTO stated goals are to promote world trade. It examines among other things the effect of regional trade agreements on the world wide trading system and the ways to compensate (or reward) the parties adhering to its policy recommendations.

Another instance of such an agreement is The Treaty on the Non-Proliferation of Nuclear Weapons signed at Washington, London, and Moscow in 1968, which dealt with the nuclear arms race. The purpose was to prevent the spread of nuclear weapons and eventually lead to nuclear disarmament. The implementation of the treaty objectives had to overcome two major problems. The first one is assuring countries agreeing to it of adequate protection should a non-adhering country develop a nuclear arsenal. The second is the (economic) sanctions that should be imposed on violating countries, as well as the (economic) compensation for countries adhering to it.

On the industry level, there are mergers and agreements between firms coordinating their market behavior or research activity. Two major issues are the payoffs expected by the parties involved, and the reorganization of activities which determines to a large degree the sharing of these payoffs.

A common denominator to all of the above scenarios is that they entail cooperation

in the presence of externalities. Each one of those can be broadly described as a situation where, what a group of players, taking a joint action, may expect to get, depends both on the action taken, as well as on the organization and actions of players outside this group.

Recently there has been a surge of literature that deals with the question of what coalitions would arise in such cooperative environments and how would the gains of cooperation be shared among the players. This "strategic" approach has been taken by Bloch (1996) which studied the sequential formation of coalitions in environments with externalities through the analysis of extensive form games. Ray and Vohra (1999) allowed for more general environments while studying a similar problem. They defined an extensive form bargaining game, and studied its stationary subgame perfect equilibria outcomes with emphasis on the resulting coalition structure. In Ray and Vohra (2001) the provision of public goods was analyzed and the resulting equilibrium coalition structure was characterized. Bloch (2002) offered a survey of problems and results in the industrial organization literature.

These works, while discussing in part the efficiency of the outcome reached, did not address other properties of the resulting allocations. They refrained from taking the axiomatic point of view asking what should the coalition structure and sharing of the surplus look like. This might in part be due to the fact that in contrast to cooperative environments with no externalities for which there exist focal solutions such as the Shapley value or the core, there is no "focal" solution for cooperative environments with externalities.

In this work we propose a general, yet simple, method of dividing the gains (costs) of cooperation in the presence of externalities. The solution offered can be applied to environments where the externalities are positive (that is, what a group of players expects to get is larger the more grouped the rest of players are) as well as to situations with negative externalities. It satisfies the desirable properties (axioms) of efficiency, anonymity (symmetry), linearity, and the further reasonable property that players which have no effect whatsoever on the outcome ("dummy" players) should not receive any part of the surplus. In contrast to the case of no externalities, where these conditions are sufficient to generate a unique sharing method (Shapley, 1953), there are several ways to satisfy them in the presence of externalities. We proceed to ask more stringent, yet reasonable, conditions leading to a unique way of surplus sharing.

First we study the implications of a stronger symmetry axiom, capturing the idea that all players with "identical power" should receive the same outcome. We prove that this leads to a natural method of constructing a solution, that is proceeding via averages. This method associates to each group of players a value that is some average of what they can obtain in the different scenarios, and then it allocates to each player the Shapley value of this average game.

There are still several ways to share the surplus from cooperation that satisfy all the properties required so far. This allows to ask for one more desirable property, namely, that when a pair of players has exactly the same power acting separately or together, the outcome received as a pair coincides with the outcome received as singletons. We construct a (simple) sharing method that satisfies all the axioms and show it is unique.

We elaborate further on the proposed sharing method by providing additional properties it satisfies. We present a "marginalistic view" of the method, similar to the popular marginalistic expression for the Shapley value for games with no externalities. We also prove the method satisfies a "strong dummy property", in that the addition of a "dummy" player leaves the outcomes of all other players intact.

A distinct advantage of such an approach setting forward a set of requirements the sharing method should satisfy is that it enables one to focus on principles rather than particulars. If these requirements seem reasonable then their prediction should be accepted as reasonable too. This approach moves the discussion from considering particular examples to considering general guidelines.

Two previous attempts to provide sharing methods in the presence of externalities were Myerson (1977) and Bolger (1989). Our method is simpler and uses a more intuitive average approach. We also offer a more suitable definition of a "dummy" player than Myerson (1977) and avoid the Bolger (1989) problem whereby the "strong dummy property" is violated. The issue of coalition formation and value in environments with externalities has recently been raised by Maskin (2003). He considered a sequential process of coalition formation, where offers made and decisions to accept are required to satisfy a set of reasonable requirements, and characterized the resulting sharing method. The efficiency of the solution depends in part on the type of the externality present. We note that the value and coalitions structure predictions of Aumann and Dreze (1974) and Hart

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and Kurz (1983) dealt with environments with no externalities.

Finally, we emphasize the applicability of our proposal by applying our method to data provided by Eyckmans and Tulkens (2003) on international environmental agreements.[1] The environment consists of six players: the U.S., Japan, European Union, China, Former Soviet Union, and Rest of the World. The data describes the coalitional payoffs for any partition of those players. We calculate and comment on the country payoffs prescribed by our value.

The paper proceeds as follows: Section 2 introduces the environment; Section 3 presents the three basic requirements of symmetry, "dummy" player, and linearity and the class of efficient sharing methods that satisfy them. Section 4 presents the new strong symmetry axiom as well as the average approach and shows the two are equivalent. Section 5 introduces the final similar influence axiom. It constructs a sharing method satisfying all axioms, shows it is unique, and discusses several properties of the new value. Section 6 offers a detailed comparison of our value with those of Myerson and Bolger. Section 7 applies our value to data on environmental agreements, generating a proposal regarding the distribution of gains across the various countries. Section 8, concludes and offers further directions of research. Finally, an Appendix includes all the proofs.

2　The environment

The economic environment we study can be described as follows. We denote by $N = \{1, ..., n\}$ the set of players. A coalition S is a group of players, that is, a non-empty subset of N, $S \subseteq N$. An embedded coalition is a pair (S, P), where S is a coalition and $P \ni S$ is a partition of N. An embedded coalition hence, specifies the coalition as well as the structure of coalitions formed by the other players. Let $\mathcal{P}$ denote the set of all partitions of N. It represents all the possible ways in which the society can be organized. The set of embedded coalitions is denoted by ECL and defined by:

$$ECL = \{(S, P) \mid S \in P, \ P \in \mathcal{P}\}.$$

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[1]See Eyckmans and Tulkens (2003) for a description of the computational model generating this data set.

We denote by (N, v) a game in partition function form (or a *Partition Function Game*), where $v : ECL \rightarrow \mathbb{R}$ is a characteristic function that associates a real number with each embedded coalition. Hence, $v(S, P)$ with $S \in P$, $P \in \mathcal{P}$, is the worth of coalition S when the players are organized according to the partition P. In our environment, players can make transfers among them. For technical convenience, we use the convention that the empty set $\varnothing$ is in P for every $P \in \mathcal{P}$, and assume that the characteristic function satisfies $v(\varnothing, P) = 0$.

A game is *with no externalities* if and only if the payoff that the players in a coalition S can jointly obtain if this coalition is formed is independent of the way the other players are organized. This means that in a game with no externalities, the characteristic function satisfies $v(S, P) = v(S, P')$ for any two partitions of the set of players $P, P' \in \mathcal{P}$ and any coalition S which belongs both to P and P'. Hence, the worth of a coalition S can be written without reference to the organization of the remaining players, $\hat{v}(S) \equiv v(S, P)$ for all $P \ni S$, $P \in \mathcal{P}$.

A game is *with externalities* if and only if the worth of some coalitions depend on the way the other players are organized, that is, there is at least one coalition $S \subseteq N$, and two partitions P and P' containing S, such that $v(S, P) \neq v(S, P')$. In this case, it is necessary to specify not only the coalition whose worth we are interested in but also the organization of the other players.

In this paper we make a proposal for the division of the surplus in such Partition Function Games. By a solution concept, or *a value*, we mean a mapping φ which associates with every game (N, v) a vector in R^n that satisfies $\sum_{i \in N} \varphi_i(N, v) = v(N, (N, \varnothing))$. A value determines the payoffs for every player in the game and, by definition, it is always *efficient* since the value of the grand coalition is shared among the players. Note that we are assuming that all the players end up together. Hence, we have in mind economic environments where forming the grand coalition is the most efficient way of organizing the society, that is, $v(N, (N, \varnothing)) \geq \sum_{S \in P} v(S, P)$ for every partition $P \in \mathcal{P}$. All international negotiations highlighted in the Introduction (as well as many other interesting economic environments) clearly satisfy that the players maximize total surplus when they take decisions jointly, because they can internalize the externalities.

To illustrate some properties, we will use very simple examples. In particular, we will

refer to games that we will denote by $(N, w_{S', P'})$, that satisfy

$$w_{S', P'}(S', P') = w_{S', P'}(N, (N, \varnothing)) = 1, \text{ and } w_{S', P'}(S.P) = 0 \text{ otherwise.}$$

In the game $(N, w_{S', P'})$ there are only two cases where a coalition has a positive worth, the first is for the coalition S' when the players are organized according to the partition P', and the second is for the grand coalition.

3 The "basic" axioms

The most natural requirements to impose on a value are those underlying the construction of the Shapley value in games without externalities, namely the axioms of linearity, symmetry, and finally the "dummy" player axiom. We first define the notion of a dummy player and the operations of addition, multiplication by a scalar, and permutation of games.

A player $i \in N$ is called a *dummy player* in the game (N, v) if and only if for every $(S, P) \in ECL$, then $v(S, P) = v(S', P')$ for any embedded coalition (S', P') that can be obtained from (S, P) by changing the affiliation of player i. Hence, for a player i to be a dummy player it must be the case that he alone receives zero for any organization of the other players. Also a dummy player has no effect on the worth of any coalition S. In games in partition function form, this also means that if player i is not a member of S, changing the organization of players outside S by moving player i around will not affect the worth of S.[2]

The *addition* of two games (N, v) and (N, v') is defined as the game $(N, v + v')$ where $(v + v')(S, P) \equiv v(S, P) + v'(S, P)$ for all $(S, P) \in ECL$. Similarly, given the game (N, v) and the scalar $\lambda \in \mathbb{R}$, the game $(N, \lambda v)$ is defined by $(\lambda v)(S, P) \equiv \lambda v(S, P)$ for all $(S, P) \in ECL$.

Let σ be a permutation of N. Then the σ *permutation* of the game (N, v) denoted by $(N, \sigma v)$ is defined by $(\sigma v)(S, P) \equiv v(\sigma S, \sigma P)$ for all $(S, P) \in ECL$.

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[2] This definition of a dummy player agrees with the Bolger (1989) definition and it is different than the Myerson (1977) definition. A dummy player in Myerson (1977) is not necessarily a dummy player according to our definition. However a dummy player in our setup is a dummy player according to Myerson (1977) as well. See Section 6 for more details.

The three basic axioms a value φ should satisfy are immediately derived from the original Shapley (1953) value axioms and are:

1. Linearity: A value φ satisfies the linearity axiom if:

    1.1. For any two games (N, v) and (N, v'), $\varphi(N, v + v') = \varphi(N, v) + \varphi(N, v')$.

    1.2. For any game (N, v) and any scalar $\lambda \in \mathbb{R}$, $\varphi(N, \lambda v) = \lambda \varphi(N, v)$.[3]

2. Symmetry: A value φ satisfies the symmetry axiom if for any permutation σ of N, $\varphi(N, \sigma v) = \sigma \varphi(N, v)$.

3. Dummy player: A value φ satisfies the dummy player axiom if for any player i which is a dummy player in the game (N, v), $\varphi_i(N, v) = 0$.

The axiom of linearity means that when a group of players shares the benefits (or the costs) stemming from two different issues, how much each player obtains does not depend on whether they consider the two issues together or one by one. Hence, the agenda does not affect the final outcome. Also, the sharing does not depend on the unit used to measure the benefits. Symmetry is a property of anonymity: the payoff of a player is only derived from his influence on the worth of the coalitions, it does not depend on his "name". Finally, the dummy player axiom only makes sure that a player with absolutely no influence on the gains that any coalition can obtain, should not receive nor pay anything.

Shapley (1953) proved that these three basic axioms characterize a unique value in the class of games with no externalities. Let us denote by $(N, \widehat{v})$ a game with no externalities, where $\widehat{v} : 2^N \to \mathbb{R}$ is a function that gives the worth of each coalition (independently of the partition structure). The Shapley value ϕ can be written as:

$$\phi_i(N, \widehat{v}) = \sum_{S \subseteq N} \beta_i(S, n) \widehat{v}(S) = \sum_{\substack{S \subseteq N \\ S \ni i}} \beta_i(S, n) MC_i(S) \text{ for all } i \in N, \quad (1)$$

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[3]Note that in games with no externalities, the axiom of Linearity can be reduced to part 1.1. In fact, in that class of games, property 1.2 is implied by part 1.1 together with the other axioms. In games with externalities this is not the case.

where $MC_i(S)$ is the marginal contribution of player $i \in S$ to the coalition S, $MC_i(S) \equiv \widehat{v}(S) - \widehat{v}(S \backslash \{i\})$, and we have denoted by $\beta_i(S, n)$ the following numbers:

$$\beta_i(S, n) = \begin{cases} \frac{(|S|-1)!(n-|S|)!}{n!} & \text{for all } S \subseteq N, \text{ if } i \in S \\ -\frac{|S|!(n-|S|-1)!}{n!} & \text{for all } S \subseteq N, \text{ if } i \in N \backslash S. \end{cases}$$

These three basic axioms impose some structure on a value for Partition Function Games, as can be seen by the results in the Appendix. However, they still leave a considerable amount of leeway as regarding the question of how one should distribute $v(N, (N, \varnothing))$ among the players. As will become clear later, the two values of Myerson (1977) and Bolger (1989) indeed satisfy these basic axioms, as do many other possible values one could define.

In the next section we describe an alternative (and stronger) symmetry axiom leading to a very natural method of constructing a value for Partition Function Games, namely "taking averages".

4 The strong symmetry axiom and the average approach

The symmetry axiom imposes much more structure on a value for games with no externalities, than it does on a value when there are externalities. Consider for example the game with no externalities $(N, \widehat{v})$, where $\widehat{v}(S') = \widehat{v}(N) = 1$ and all other coalitions receive zero. In such a game the symmetry axiom implies that all the players who do not belong to S' should obtain the same payoff. Take now the game with externalities $(N = \{1, 2, 3, 4, 5\}, w_{S', P'})$ where only the embedded coalition $(S', P') = (\{1, 2\}, (\{1, 2\}, \{3\}, \{4, 5\}, \varnothing))$ and the grand coalition have a worth of 1, and all other embedded coalitions have zero worth. Symmetry implies that players 4 and 5 should receive the same payoff. But symmetry does not tell anything about the payoff of player 3 as compared with them. However, the role of the three players in this game is, in some sense, similar: it is only when they form the partition $(\{1, 2\}, \{3\}, \{4, 5\}, \varnothing)$ that the coalition $\{1, 2\}$ generates value. If the position of any of them changes, $\{1, 2\}$ gets zero. Our strong symmetry axiom will propose that player 3 should receive the same as 4 and

5. It captures the intuitive notion that individuals with "identical power" should receive the same payoff.

The strong symmetry axiom strengthens the symmetry axiom by requiring that the payoff of a player should not change after permutations in the set of players in $N\backslash S$, for any embedded coalition structure (S, P). For simplicity, we illustrate this new condition through the game $(N, w_{S',P'})$ introduced before. We consider a permutation of the set $N\backslash S'$ so that we obtain a new $(S', P'') \in ECL$, we denote such a permutation by $\sigma_{S',P'}$. For example, a permutation $\sigma_{S',P'}$ can generate

$$(S', P'') = (\{1, 2\}, (\{1, 2\}, \{4\}, \{3, 5\}, \varnothing)).$$

Strong symmetry requires that player 3 receive the same payoff in the games $(N, w_{S',P'})$ and $(N, w_{S',P''})$. Note that both P' and P'' are of equal sizes.

Formally, given a coalition S and a partition P containing that coalition the $\sigma_{S,P}$ permutation of the game (N, v) denoted by $(N, \sigma_{S,P} v)$ is defined by $(\sigma_{S,P} v)(S, P) = v(S, \sigma_{S,P} P)$, $(\sigma_{S,P} v)(S, \sigma_{S,P} P) = v(S, P)$, and $(\sigma_{S,P} v)(R, Q) = v(R, Q)$ for all $(R, Q) \in ECL\backslash \{(S, P), (S, \sigma_{S,P} P)\}$.

2'. A value φ satisfies the strong symmetry axiom if:

(a) for any permutation σ of N, $\varphi(N, \sigma v) = \sigma \varphi(N, v)$,

(b) for any $(S, P) \in ECL$ and for any permutation $\sigma_{S,P}$, $\varphi(N, \sigma_{S,P} v) = \varphi(N, v)$.

The strong symmetry axiom, naturally, implies symmetry and reduces to the Shapley symmetry for games with no externalities. It imposes in addition to symmetric treatment of individual players, the symmetric treatment of "externalities" generated by players in a given embedded coalition structure. Exchanging the names of the players inducing the same externality does not affect the payoff of any player.

When we add the strong symmetry axiom to the two basic axioms of linearity and dummy player, we can look for values for games with externalities in a different and very appealing way. We will refer to this way as the "average approach", that we now describe.

In an environment with externalities, the worth of a group of players is influenced by the way the outside players are organized. What should then be the worth "assigned" to

that group of players? The most natural and obvious candidate is to take an average of the different worths of this group for all the possible organizations of the other players. Repeating this process for all groups leads to an "average" game with no externalities. A focal candidate now for a value for the original game with externalities, is the Shapley value for the average game.

More formally, the "*average approach*" consists of, first constructing an average game $(N, \widetilde{v})$ associated to the Partition Function Game (N, v) by assigning to each coalition $S \subseteq N$ the average worth $\widetilde{v}(S) \equiv \sum_{P \ni S, P \in \mathcal{P}} \alpha(S, P) v(S, P)$, with $\sum_{P \ni S, P \in \mathcal{P}} \alpha(S, P) = 1$. We refer to $\alpha(S, P)$ as the "weight" of the partition P in the computation of the value of coalition $S \in P$. Second, the average approach constructs a value φ for the Partition Function Game (N, v) by taking the Shapley value of the game $(N, \widetilde{v})$. Therefore, if a value φ is obtained through the average approach then, for all $i \in N$,

$$\varphi_i(N, v) = \sum_{S \subseteq N} \beta_i(S, n) \widetilde{v}(S) = \sum_{S \subseteq N} \left[\beta_i(S, n) \sum_{P \ni S, P \in \mathcal{P}} \alpha(S, P) v(S, P) \right].$$

We say a value is constructed through the average approach, if it can be derived in the two stage procedure described above of constructing an average game and calculating its Shapley value.

The following theorem shows the relationship between the average approach and the strong symmetry axiom:

Theorem 1 *Assume the value φ satisfies linearity and dummy player. Then, φ can be constructed through the average approach if and only if it satisfies the strong symmetry axiom.*

Theorem 1 provides additional intuition and support for the strong symmetry axiom: under the two basic axioms of linearity and dummy player, it is equivalent to the possibility of using the average approach. Similarly, it clearly states which is the additional property we are assuming if we use the (natural) average approach to construct a value.

The average approach as such does not imply any restrictions regarding the different weights. However, to fulfill the symmetry and the dummy player axioms, the weights must satisfy several constraints. First, the weights must be symmetric, that is, they must only

depend on the size distribution of the partition. Second, the dummy player axiom imposes a certain link between the weight of partition P for the coalition S and the weights of the partitions that result from moving any player in S to the coalitions in P other than S. The next corollary describes the precise restrictions stemming from the dummy player axiom.

Corollary 1 *A value φ satisfies linearity, strong symmetry, and dummy player if and only if it can be constructed through the average approach with symmetric weights satisfying the following condition:*

$$\alpha(S, P) = \sum_{R \in P \setminus S} \alpha(S \setminus \{i\}, P \cup (R \cup \{i\}) \setminus R) \tag{2}$$

for all $i \in S$ and for all $(S, P) \in ECL$ with $|S| > 1$.[4]

The three requirements of linearity, strong symmetry, and dummy player do not yield a unique value for games with externalities. To illustrate this statement, we provide in the following tables the parametrized family of values that satisfy the three axioms for games with three and four players. We write in the table the weight $\alpha(S, P)$ of each embedded coalition structure (S, P).

To illustrate how the values share the benefits of cooperation in some examples, we also include at the end of each row in the tables (a row corresponds to an embedded coalition structure (S, P)) the payoff that each player in the coalition S obtains in the game $(N, w_{S,P})$ (remember that $w_{S,P}(S, P) = w_{S,P}(N, (N, \varnothing)) = 1$, and $w_{S,P}(S', P') = 0$ otherwise). Note that, by strong symmetry, what each player in $N \setminus S$ receives is equal, and augments the payment to members of S to one. For example, when $(S, P) = (\{1\}, (\{1\}, \{2\}, \{3\}))$, the value gives the following payoff profile: $\varphi(N, w_{S,P}) = \left(\frac{2-a}{3}, \frac{1+a}{6}, \frac{1+a}{6}\right)$.

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[4]When $R = \varnothing$, we slightly abuse notation (to keep it simple) by assuming that the partition $P \cup (\varnothing \cup \{i\}) \setminus \varnothing$ also includes the empty set.

(S, P) for $n = 3$	$\alpha(S, P)$	$\varphi_i(N, w_{S,P})$ for $i \in S$
$(\{i\}, (\{i\}, \{j\}, \{k\}, \varnothing))$	$1 - a$	$\frac{2-a}{3}$
$(\{i\}, (\{i\}, \{j, k\}, \varnothing))$	a	$\frac{1+a}{3}$
$(\{ij\}, (\{i, j\}, \{k\}, \varnothing))$	1	$\frac{1}{2}$
$(\{N\}, (N, \varnothing))$	1	$\frac{1}{3}$

<div align="center">Table 1</div>

The three player case serves to clearly demonstrate why strong symmetry is not sufficient to guarantee uniqueness of the value. For this case, strong symmetry and symmetry are equivalent and fail to provide a unique value since any real number a generates a different value satisfying for $n = 3$ all three axioms.

This table allows us to informally discuss some features of the family of solutions proposed so far. For $a > 1$ player i in the game $(\{1, 2, 3\}, w_{\{i\}, (\{i\}, \{j\}, \{k\}, \varnothing)})$ would receive less than $\frac{1}{3}$. The same would happen for $a < 0$ in the game $(\{1, 2, 3\}, w_{\{i\}, (\{i\}, \{j, k\}, \varnothing)})$. One may argue that this is not a convincing feature in these games. Indeed, the only coalition other than the grand coalition that may generate some profits is $\{i\}$, hence it does not seem sensible that player i ends up enjoying less than a third of the whole profit. It may hence, be more sensible to consider values such that $a \in [0, 1]$. In fact, since the issue we are interested in is related to the existence of externalities one may argue that $a = 0$ and $a = 1$ may not be appropriate since they are ignoring the effect of the coalition structure.

The table for four players is given as follows:

(S, P) for $n = 4$	$\alpha(S, P)$	$\varphi_i(N, w_{S,P})$ for $i \in S$
$(\{i\}, (\{i\}, \{j\}, \{k\}, \{l\}, \varnothing))$	$1 - b - 2c$	$\frac{1}{4}(2 - b - 2c)$
$(\{i\}, (\{i\}, \{j\}, \{k, l\}, \varnothing))$	c	$\frac{1}{4}(1 + c)$
$(\{i\}, (\{i\}, \{j, k, l\}, \varnothing))$	$b - c$	$\frac{1}{4}(1 + b - c)$
$(\{i, j\}, (\{i, j\}, \{k\}, \{l\}, \varnothing))$	$1 - b$	$\frac{1}{12}(4 - b)$
$(\{i, j\}, (\{i, j\}, \{k, l\}, \varnothing))$	b	$\frac{1}{12}(3 + b)$
$(\{i, j, k\}, (\{i, j, k\}, \{l\}, \varnothing))$	1	$\frac{1}{3}$
$(N, (N, \varnothing))$	1	$\frac{1}{4}$

<div align="center">Table 2</div>

The parameters b and c can be any real numbers. Again, as discussed before one may impose constraints on these parameters so as to yield a relatively larger payoff to the player(s) generating positive profits on their own. This would imply $b \in (0,1)$, $c \in \left(0, \frac{1}{3}\right)$ and $b \in (c, 1 - 2c)$. Still many values remain possible.

In the next section we introduce the final axiom and obtain a unique value.

5   The similar influence axiom and the value

The fourth axiom that we propose addresses the issue that similar environments should lead to similar payoffs for the players. To understand the motivation for this axiom take $N = \{1,2,3\}$ and consider the games $(N, w_{S',P'})$ and $(N, w_{S',P''})$, where $S' = \{1\}$, $P' = (\{1\}, \{2,3\}, \varnothing)$ and $P'' = (\{1\}, \{2\}, \{3\}, \varnothing)$. The two games are very similar. In both only player 1 can produce some benefits alone. The only difference is, that in the first game players 2 and 3 should be together for the benefits to player 1 to be realized, while in the second game players 2 and 3 should be separated. The payoffs for the three players in these games according to any value φ satisfying the three previous axioms are:
$\varphi(N, w_{S',P'}) = (\frac{1+a}{3}, \frac{2-a}{6}, \frac{2-a}{6})$ and $\varphi(N, w_{S',P''}) = (\frac{2-a}{3}, \frac{1+a}{6}, \frac{1+a}{6})$.

The payoff of players 2 and 3 (hence, the payoff of player 1 as well) can differ very much depending on whether they influence the worth of player 1 by staying together or separated. However, we think that this influence is very similar and therefore it is sensible that players 2 and 3 should receive the same payoff in both games. This idea leads to the next axiom.

To introduce the similar influence axiom, we first define the notion of "similar influence". We say that a pair of players $\{i,j\} \subseteq N, i \neq j$, has *similar influence* in games (N,v) and (N,v') if $v(T,Q) = v'(T,Q)$ for all $(T,Q) \in ECL\backslash\{(S,P),(S,P')\}$, $v(S,P) = v'(S,P')$, and $v(S,P') = v'(S,P)$, where the only difference between the partitions P and P' is that $\{i\}, \{j\} \in P\backslash S$ while $\{i,j\} \in P'\backslash S$.

4. Similar influence: A value φ satisfies the similar influence axiom if for any two games (N,v) and (N,v') and for any pair of players $\{i,j\}$ that has similar influence in those games, we have $\varphi_i(N,v) = \varphi_i(N,v')$ and $\varphi_j(N,v) = \varphi_j(N,v')$.

Note that when applied to the following simple class of games $(N, w_{S,P})$, the similar influence axiom reduces to the requirement that for any two such games with $(S, P), (S, P') \in ECL$, where the only difference between P and P' is that a pair of players $i, j \in N \backslash S, i \neq j$, are singletons in P and are a pair in P' (or the other way around), we have $\varphi_i(S, P) = \varphi_i(S, P')$ and $\varphi_j(S, P) = \varphi_j(S, P')$.

To see the restrictions of this axiom for games with small number of players, notice that it implies $a = 1/2$ for the games with three players that we introduced at the end of last section. Similarly, for games with four players, the similar influence axiom implies that the parameters defining the value are $b = 1/2$ and $c = 1/6$.

In the next theorem we show there is a unique value satisfying the four axioms, and provide an explicit and simple formula to calculate it.

Theorem 2 *There is a unique value φ^* satisfying linearity, strong symmetry, dummy player, and similar influence. The value φ^* is given by:*

$$\varphi_i^*(N, v) = \sum_{(S,P) \in ECL} \frac{\prod_{T \in P \backslash S} (|T| - 1)!}{(n - |S|)!} \beta_i(S, n) v(S, P)$$

for all game (N, v) and for all player $i \in N$.

For simplicity, we will refer from now on to the value φ^* identified in Theorem 2 as *the value*. We now give a first interpretation of it. Remember that $\beta_i(S, n)$ is the coefficient of $v(S)$ in the expression of the Shapley value in a game with no externalities. It seems reasonable that the coefficient multiplying $v(S, P)$ should be smaller, since $v(S, P)$ is the worth of coalition S only if the partition P forms. The factor multiplying $\beta_i(S, n)$ measures how to "discount" the outcome for the players, depending on the partition P. This factor is nothing but the weight associated with the partition P in the average approach, as stated in the following corollary:

Corollary 2 *The value φ^* can be constructed through the average approach by using, for all $(S, P) \in ECL$, the following weights:*

$$\alpha^*(S, P) = \frac{\prod_{T \in P \backslash S} (|T| - 1)!}{(n - |S|)!}.$$

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According to Corollary 2, the weights are all strictly positive and more weight is given to those partitions with large coalitions than to partitions with a large number of small coalitions.

To gain more intuition about the value, and to see how it relates to the original Shapley value for games with no externalities, we now provide another way of writing and computing the value as an average of marginal contributions. We take the convention that $|\varnothing| = 1$ and we write:

$$MC_i(S, P) \equiv v(S, P) - \sum_{R \in P \setminus S} \frac{|R|}{(n - |S| + 1)} v(S \setminus \{i\}, P \cup (R \cup \{i\}) \setminus R).$$

That is, $MC_i(S, P)$ is a marginal contribution of player $i \in S$ to the coalition S, given the coalition structure P, where the worth of the coalition $S \setminus \{i\}$ is some average of the worth of this coalition in all the possible coalition structures that can emerge by moving i in P. Then we can write φ^* as follows:

$$
\begin{aligned}
\varphi_i^*(N, v) &= \sum_{\substack{(S,P) \in ECL \\ S \ni i}} \frac{\prod_{T \in P} (|T| - 1)!}{n!} MC_i(S, P) \qquad (3) \\
&= \sum_{\substack{S \subseteq N \\ S \ni i}} \left[\beta_i(S, n) \sum_{\substack{P \ni S \\ P \in \mathcal{P}}} \alpha^*(S, P) MC_i(S, P) \right].
\end{aligned}
$$

The expression (3) is similar to the formula (1) for the Shapley value, once we interpret $\sum_{\substack{P \ni S \\ P \in \mathcal{P}}} \alpha^*(S, P) MC_i(S, P)$ as the (average) marginal contribution of player $i \in S$ to the coalition S. Hence, the payoff of player i, according to the value φ^*, is an average of his marginal contribution to the different groups of players he can join, taking into account all the ways the whole society can be organized.

6 Comparison with previous values and further properties

Two previous solutions for the problem of sharing surplus with externalities were proposed by Myerson (1977) and Bolger (1989). Myerson (1977) adapts the Shapley value axioms to

environments with externalities and derives an extension, that we will denote φ^M, of the Shapley value for this class of environments. The three axioms that uniquely characterize the Myerson's extension are linearity, symmetry, and a carrier axiom requiring that the surplus is shared only among the members of the carrier. The Myerson value of a player is given by:

$$\varphi_i^M(N,v) = \sum_{(S,P)\in ECL} (-1)^{|P|-1}\,(|P|-1)! \left[\frac{1}{n} - \sum_{\substack{T\in P\backslash S \\ i\notin T}} \frac{1}{(|P|-1)\,(n-|T|)} \right] v(S,P),$$

where $|P|$ is the number of non-empty coalitions in P.

The carrier axiom implies both efficiency and a dummy player concept much stronger than the one assumed in our analysis. A set S of players is a *carrier* if: $v(\widetilde{S},P) = v(\widetilde{S}\cap S, P\wedge\{S,N\backslash S\})$ for all $(\widetilde{S},P)$ where $P\wedge Q = \{S\cap T | S\in P, T\in Q, S\cap T\neq\varnothing\}$. The *carrier axiom* states that if S is a carrier in the game (N,v), the sum of payoffs assigned to the members of S equals $v(N,(N,\varnothing))$. We can say that, in the game (N,v), a player $i\in N$ is a dummy player, in the Myerson sense if there exists a carrier set S with $i\notin S$. Given that all the dummy players are symmetric, they all get zero according to the Myerson value.

A problematic aspect of the carrier axiom is that in many cases a dummy player (in the Myerson's sense) might, through changes in his position in the partition, affect the outcome reached. Take for example the game with three players $(\{1,2,3\},w_{S',P'})$, where $(S',P') = (\{1\},(\{1\},\{2,3\},\varnothing))$. In this game, player 1 is a carrier and hence players 2 and 3 are dummy players. Therefore, $\varphi_1^M(N,w_{S',P'}) = 1$ and $\varphi_2^M(N,w_{S',P'}) = \varphi_3^M(N,w_{S',P'}) = 0$. However, player 2 can affect the outcome since player 1 will get zero rather than one if player 2 does not join player 3. Thus we feel player 2 is not "really" a dummy player.

Also note that, due to the carrier axiom, the Myerson value yields very different outcomes to games that are quite similar. Consider the game $(\{1,2,3\},w_{S',P''})$, where $(S',P'') = (\{1\},(\{1\},\{2\},\{3\},\varnothing))$. This game is similar to the game $(N,w_{S',P'})$ proposed previously in the sense that the "worth" of the players is similar. However, players 2 and 3 are not dummy now and the Myerson value for these players is surprising since it does not allocate any payoff to player 1 : $\varphi_1^M(N,w_{S',P''}) = 0$ and $\varphi_2^M(N,w_{S',P''}) = \varphi_3^M(N,w_{S',P''}) = 1/2$.

Bolger (1989) obtains a unique value φ^B characterized by our properties of linearity, symmetry, and dummy player, and an additional requirement based on the behavior of the value in simple games (the worth of any coalition is either one or zero). He says that the coalition S is winning with respect to (S, P) if $v(S, P) = 1$. Now, consider an embedded coalition $(S\backslash\{i\}, P \cup (R \cup \{i\})\backslash R)$ obtained from (S, P) by moving player $i \in S$ from S to $R \in P\backslash S$. In Bolger's terminology this is called a move for player i. Such a move is called a pivot move if S wins with respect to (S, P) and $S\backslash\{i\}$ loses with respect to $(S\backslash\{i\}, P \cup (R \cup \{i\})\backslash R)$. The additional property Bolger (1989) introduces states that for simple games, a player i obtains the same payoff in two games (N, v) and (N, v') if he has the same number of pivot moves in both games. There is no closed form expression for φ^B.

The values φ^M and φ^B while satisfying our basic properties, cannot be constructed through the average approach, as shown in the next proposition.

Proposition 1 *The values φ^M and φ^B fail to satisfy the strong symmetry axiom.*

Straightforward calculations show that the values proposed by Bolger and Myerson do not satisfy the similar influence property as well.

Finally, we would like to comment on the behavior of the various values with respect to an alternative axiom concerning the dummy player. Our dummy player (as well as Bolger's) axiom imposes that a dummy player should obtain zero, but it does not require that this player does not influence the payoffs obtained by the other players. In games with no externalities, the basic axioms do indeed imply this additional property. However, this is not necessarily so in games with externalities. We call this property the "strong dummy player" property:

3'. Strong Dummy Player: A value φ satisfies the strong dummy player axiom if for any dummy player j in the game (N, v), $\varphi_i(N, v) = \varphi_i(N\backslash\{j\}, v)$ for all i in $N\backslash\{j\}$.

We note that, given that a value is efficient, the strong dummy player axiom implies the dummy player axiom. The next proposition shows that even though the strong dummy property was not imposed as a requirement, our value does satisfy it.

Proposition 2 *The value φ^* satisfies the strong dummy player property.*

Bolger's value (as pointed out in Bolger, 1989) violates the strong dummy player axiom. On the other hand, the value proposed by Myerson (1977) satisfies the requirement; his dummy player property (implied by his carrier axiom) is still much more demanding than the strong dummy player property. Hence, the strong dummy player axiom is not sufficient to characterize, together with symmetry and linearity, a unique value. In fact, the class of values satisfying these three axioms is still large. Even if we substitute symmetry by strong symmetry, or if we add the similar influence axiom, a value is still not singled out.

7 The value allocation for a world climate change program

The purpose of our approach is to provide a simple method to share the gains from co-operation in economic environments. In this section, we illustrate the approach in a particularly interesting environment where we can use the characteristic function estimated in Eyckmans and Tulkens (2003).

These authors consider a question that has received a lot of attention both in policy debates and in the economic literature: how to reach international agreements on climate control and how to design the compensation mechanisms. This is a game with positive externalities, since outside countries benefit whenever other countries reach an agreement. Eyckmans and Tulkens (2003) analyze global climate negotiations from a game-cooperative point of view. They consider six regions: USA (player 1), Japan (player 2), European Union (EU, player 3), China (player 4), Former Soviet Union (FSU, player 5) and the Rest of the World (ROW, player 6). They determine the agreements reached and the payoffs accruing to the various regions under alternative modes of cooperation, that is, they estimate the value function v for any ECL. They focus their analysis on the stability properties of alternate transfer systems.

We approach the problem from an axiomatic point of view and calculate the payoff configuration generated by our value for this "climate change game". The game is given by the payoffs received by each coalition for any of the 203 possible partitions of the six

player set.[5] This game is characterized by wide-spread externalities. To illustrate this feature, we report a few payoff configurations.

When the partition consists of six singletons, the payoffs to the six players respectively are: 78.352, 42.909, 102.729, 9.140, 23.793, and 81.134 yielding a total world welfare of 338.057. If all six regions form one coalition they receive 339.831, which is the largest payoff the world can receive. When the USA, Japan and the EU form one coalition whereas China, the FSU and ROW stay as singletons the payoffs are given by 224.005 to the three member coalition and 9.142, 23.798 and 81.154 to players 4, 5 and 6 respectively. If players 4, 5 and 6 form a coalition as well then the payoff to the coalition $\{1, 2, 3\}$ is 224.913 and it is 114.394 to the coalition $\{4, 5, 6\}$. The previous number illustrates that this is an environment with positive externalities.

Our value takes into account these and all other externalities and yields the following payoff configuration to the six players respectively:[6]

$$\varphi^*(N, v) = (78.692,\ 43.070,\ 103.203,\ 9.293,\ 23.933,\ 81.639).$$

Our value recommendation can be attained by having the players carry out efficient climate change policy, and a transfer formula assuring that each region receives its payoff. This suggestion naturally ignores the possibility of manipulation and strategic behavior on part of the regions. It also takes for granted the complete information assumption. We think of it as a desirable outcome, that an international court or body of nations can justify as an arbitration offer to settle international disputes.

8 Conclusion

We set out to provide an axiomatic solution concept for environments with externalities. The construction proceeded in stages. We first took the natural extensions of the Shapley axioms to our environment and studied their implications. They generated a large family of possible values. We then strengthened the symmetry axiom and showed it is equivalent to an average approach for resolving the value problem.

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[5]We are grateful to Johan Eyckmans for providing the data set to illustrate our value.

[6]We thank Nina Kirma for computational assistance.

The average approach amounts to calculating a value for a game with externalities by associating with it a game with no externalities, where each coalition is assigned a payoff which is an average of its payoff over all possible partitions containing it. The Shapley value of the average game is then taken to be the value of the original game. There are several restrictions on the weighting method, but still many values remain as possible solutions.

The final axiom we added regarded the behavior of the value in very similar games. This was called the similar influence axiom, since the only difference between the games was the pairing of two singletons in one partition into a pair in the other game. We wanted the value assigned to each of the two concerned players to be the same in both games. We showed there is a unique value that satisfies all these axioms. This value, given by a simple formula, can be easily calculated and generates a payoff vector for any environment with externalities.

There have been two previous attempts to provide a normative solution to the sharing problem in environments with externalities, by Myerson (1977) and Bolger (1986). The main advantages of our value are the plausible nature of the axioms characterizing it and its ease of calculation. It uses a reasonable dummy player property, and furthermore satisfies the strong dummy player property whereby the removal of a dummy player has no effect on the values of the other players.

Our value can be used to resolve distributional problems in very general settings. It can determine a benchmark result arbitrators might consider as a good compromise. We applied it to a climate change game based on actual data and generated a payoff configuration that can serve as a basis for a system of transfers across regions.

There are several open questions regarding the axioms characterizing the value. It is not clear which, if any, of the basic axioms can be relaxed by the introduction of the strong dummy player axiom. It is also of interest to study whether or not there exist axioms different than the similar influence axiom, which are sensible in certain economic environments and which lead to a unique value. In actual applications it might also be that certain suggestions generated by only a subset of the axioms form an appropriate solution. Also, the value may be characterized through properties different from the ones we use in this paper.

The approach in this paper has been axiomatic and it would be interesting to construct game forms that implement the value constructed. Pérez-Castrillo and Wettstein (2001) provided a deterministic mechanism that implements the Shapley value in pure strategy Subgame Perfect Equilibrium.

The analysis throughout the paper proceeded under the assumption of transferable utility. The extension to environments without side payments remains an interesting topic of further research.

9   Appendix

Prior to providing the proofs of the results stated in the paper, we start by deriving the properties of solutions satisfying the basic axioms of linearity, symmetry, and dummy player. In order to study the implications of these as well as further axioms, it is convenient to introduce a natural basis for the space of Partition Function Games. Consider the following simple class of games, a generic element of which is denoted by $(N, v_{S,P})$, with $(S, P) \in ECL$. The game $(N, v_{S,P})$ is defined by $v_{S,P}(S', P') = 1$ if $(S', P') = (S, P)$ and zero otherwise. This set of games forms a *basis* for the set of Partition Function Games, since for all (N, v) we can write:

$$v = \sum_{(S,P) \in ECL} v(S, P) \cdot v_{S,P}.$$

We will use in this Appendix the basis formed by the games $v_{S,P}$ instead of the alternative (and possibly more natural) basis $w_{S,P}$ for computational simplicity.

The following proposition establishes several properties of the value φ implied by the basic axioms. Properties (a), (b), and (c) state immediate implications from, respectively, linearity, efficiency (implicitly included in the definition of a value), and symmetry. Property (d) highlights the implication of the fact that if a value satisfies the three basic axioms, then it must coincide with the Shapley value for games with no externalities.

Lemma 1 *If the value φ satisfies linearity, symmetry, and dummy player, then:*

(a) $\varphi_i(N, v) = \sum_{(S,P) \in ECL} \varphi_i(N, v_{S,P}) v(S, P)$ *for all $i \in N$.*

(b) $\sum_{i \in N} \varphi_i(N, v_{S,P}) = 0$ *for all $(S, P) \in ECL$, $(S, P) \neq (N, (N, \varnothing))$.*

(c) $\varphi_i(N, v_{S,P}) = \varphi_j(N, v_{S,P})$ *for all $i, j \in S$, for all $(S, P) \in ECL$.*

(d) $\sum_{\substack{P \ni S \\ P \in \mathcal{P}}} \varphi_i(N, v_{S,P}) = \beta_i(S, n)$ *for all $S \subseteq N$, for all $i \in N$.*

Proof. Properties (a), (b), and (c) are immediate. To prove (d), for any $S \subseteq N$ denote by $(N, \widehat{v}_S)$ the game with no externalities defined by $\widehat{v}_S(S, P) = 1$ for any $P \ni S$ and zero otherwise. The Shapley value of player $i \in S$ in $\widehat{v}_S$ is $\phi_i(N, \widehat{v}_S) = \beta_i(S, n)$ and it should coincide with $\varphi_i(N, \widehat{v}_S)$. Since $\widehat{v}_S = \sum_{P \ni S, P \in \mathcal{P}} v_{S,P}$, linearity of φ implies that $\varphi_i(N, \widehat{v}_S) = \sum_{P \ni S, P \in \mathcal{P}} (N, v_{S,P})$, thus proving the proposition. ∎

The dummy property also implies important restrictions (stemming from the structural properties of Partition Function Games) on the behavior of the value over basis games.

Lemma 2 *If the value φ satisfies linearity, symmetry, and dummy player, then*

$$\varphi_i(N, v_{S,P}) + \sum_{R \in P \setminus S} \varphi_i(N, v_{S \setminus \{i\}, P \cup (R \cup \{i\}) \setminus R}) = 0 \tag{4}$$

for all $i \in S$ and for all $(S, P) \in ECL$ with $|S| > 1$.

Proof. Consider $(S, P) \in ECL$ with $|S| > 1$ and $i \in S$. Define the game $(N, v_{S,P}^i)$ as $v_{S,P}^i(S', P') = 1$ for $(S', P') = (S, P)$ and for all $(S', P') = (S \setminus \{i\}, P \cup (R \cup \{i\}) \setminus R)$, for some $R \in P \setminus S$, otherwise $v_{S,P}^i(S', P') = 0$; that is, $v_{S,P}^i = v_{S,P} + \sum_{R \in P \setminus S} v_{S \setminus \{i\}, P \cup (R \cup \{i\}) \setminus R}$. The proposition follows immediately from the fact that player $i \in S$ is a dummy player in $(N, v_{S,P}^i)$, hence his value in $(N, v_{S,P}^i)$ must be zero. ∎

We now proceed to provide the proofs of the other propositions stated in the paper.

Proof. of Theorem 1. Linearity implies that the value φ satisfies the strong symmetry axiom for all games if and only if it satisfies the axiom for the games $(N, v_{S,P})$ for $(S, P) \in ECL$. If φ can be constructed through the average approach, then for all $(S, P) \in ECL$, $\varphi_i(N, v_{S,P}) = \alpha(S, P) \beta_i(S, n)$ for all $i \in N$, for some vector of weights $(\alpha(S, P))_{(S,P) \in ECL}$. The expression $\beta_i(S, n)$ is the same for all players $i \in S$, and it is also

the same for all players $i \in N \backslash S$. Therefore, $\varphi_i(N, v_{S,P}) = \varphi_j(N, v_{S,P})$ for all $i, j \in S$ and for all $i, j \in N \backslash S$, which is equivalent to the requirement of the strong symmetry axiom for the basic games $(N, v_{S,P})$. Hence, the value φ satisfies the strong symmetry axiom for all games.

Now assume φ satisfies the linearity, dummy player, and strong symmetry axioms. We first show that for all $(S, P) \in ECL$, the ratio $\varphi_i(N, v_{S,P})/\beta_i(S, n)$ is the same for any $i \in N$. By strong symmetry, the property holds for $(S, P) = (N, \{N, \varnothing\})$. For $(S, P) \in ECL$, $(S, P) \neq (N, \{N, \varnothing\})$, both $\varphi_i(N, v_{S,P})$ and $\beta_i(S, n)$ are the same for all players in S, and they are also the same for all players in $N \backslash S$, because of the strong symmetry axiom. Moreover, by Proposition 1(b) (that holds because strong symmetry implies symmetry), $\sum_{i \in N} \varphi_i(N, v_{S,P}) = 0$, i.e., $|S| \, \varphi_i(N, v_{S,P}) + (n - |S|) \, \varphi_j(N, v_{S,P}) = 0$, for all $i \in S, j \in N \backslash S$. Given that $|S| \, \beta_i(S, n) + (n - |S|) \, \beta_j(S, n) = 0$, for all $i \in S, j \in N \backslash S$, it also holds that $\varphi_i(N, v_{S,P})/\beta_i(S, n) = \varphi_j(N, v_{S,P})/\beta_j(S, n)$, for all $i \in S, j \in N \backslash S$.

Second, define the weights as follows: $\alpha(S, P) = \varphi_i(N, v_{S,P})/\beta_i(S, n)$, for any $i \in N$. By Proposition 1(d), $\sum_{P \ni S, P \in \mathcal{P}} \varphi_i(N, v_{S,P}) = \sum_{P \ni S, P \in \mathcal{P}} \alpha(S, P)\beta_i(S, n) = \beta_i(S, n)$ for all $S \subseteq N$. Hence, $\sum_{P \ni S, P \in \mathcal{P}} \alpha(S, P) = 1$, for all $S \subseteq N$.

Finally, we claim that the value φ can be constructed through the average approach, using the weights $\alpha(S, P)$. Indeed,

$$
\varphi_i(N, v) = \sum_{(S,P) \in ECL} \varphi_i(N, v_{S,P})v(S, P) = \sum_{(S,P) \in ECL} \alpha(S, P)\beta_i(S, n)v(S, P)
$$
$$
= \sum_{S \subseteq N} \beta_i(S, n) \sum_{P \ni S, P \in \mathcal{P}} \alpha(S, P)v(S, P).
$$

∎

Proof. of Corollary 1. The necessary condition is immediately implied by Theorem 1 and Proposition 2. For the sufficient part, note that if the value is constructed through the average approach, then linearity and efficiency are immediate. Also, strong symmetry is direct consequence of the fact that (a) $\beta_i(S, n)$ only depends on the players being in or outside the set S and (b) both $\beta_i(S, n)$ and $\alpha(S, P)$ only depend on the size of S and on the size distribution of P. Finally, it is also easy to check that, given the other axioms, the dummy axiom holds if and only if condition (4) holds, which corresponds to (2) when it is written in terms of the weights $\alpha(S, P)$. ∎

Proof. of Theorem 2. We start by showing that the value φ^* satisfies the four axioms. It obviously satisfies linearity, strong symmetry, and similar influence. To show the dummy player axiom, note that if $i \in N$ is a dummy player in a game (N, v), then

$$
\begin{aligned}
\varphi_i^*(N, v) &= \sum_{(S,P)\in ECL} \varphi_i^*(N, v_{S,P})v(S, P) \\
&= \sum_{\substack{(S,P)\in ECL \\ S\ni i}} \left[\varphi_i^*(N, v_{S,P})v(S, P) + \sum_{P'\in\mathcal{P}(i,P)} \varphi_i^*(N, v_{S\setminus\{i\},P'})v(S\setminus\{i\}, P') \right] \\
&= \sum_{\substack{(S,P)\in ECL \\ S\ni i}} \left[\varphi_i^*(N, v_{S,P}) + \sum_{P'\in\mathcal{P}(i,P)} \varphi_i^*(N, v_{S\setminus\{i\},P'}) \right] v(S, P).
\end{aligned}
$$

Substituting in the expressions for φ_i^*, we get $\varphi_i^*(N, v_{S,P}) + \sum_{P'\in\mathcal{P}(i,P)} \varphi_i^*(N, v_{S\setminus\{i\},P'}) = 0$ and hence $\varphi_i^*(N, v) = 0$ whenever player i is a dummy player in (N, v).

We now prove that if a value φ satisfies the four axioms, then $\varphi = \varphi^*$. Since φ can be constructed through the average approach, let us denote by $\alpha(S, P)$ the weights associated to φ. Proving that $\varphi = \varphi^*$ is equivalent to proving that the weights $\alpha(S, P)$ are the same as the weights $\alpha^*(S, P)$ associated to φ^*, that is,

$$
\alpha(S, P) = \alpha^*(S, P) = \frac{\prod_{T\in P\setminus S}(|T| - 1)!}{(n - |S|)!} \quad \text{for all } (S, P) \in ECL. \tag{5}
$$

By symmetry, $\alpha(S, P)$ only depends on the sizes of the coalitions in P. Hence, denoting $s = |S|$, we can write $\alpha(S, P) = \rho(s; t)$, where $t = (t_1, ..., t_h)$ with $\sum_{k=1}^h t_k = n - s$, is the vector of sizes of the coalitions in P different from S. We prove (5) if we show that:

$$
\rho(s; t) = \frac{\prod_{k=1}^h (t_k - 1)!}{(n - s)!} \quad \text{for all } s \le n, \text{ for all } t = (t_1, ..., t_h), \text{ with } \sum_{k=1}^h t_k = n - s. \tag{6}
$$

We prove that the expression (6) holds through an induction argument on the size of the coalition S, going from $s = n$ to $s = 1$.

($s = n$) If $s = n$, equation (6) holds since the only embedded coalition structure for $S = N$ is $(N, (N, \varnothing))$ and $\alpha(S, P) = \rho(n; 0) = 1$.

(s) We make the induction argument that (6) holds for all $(s'; t')$ with $s' > s$. We then prove that it also holds for every $(s; t)$. Denote

$$\mathcal{T}' = \left\{ t' = (t_1', ..., t_h') \mid t_1' \geq ... \geq t_h' \text{ and } \sum_{k=1}^{h} t_k' = n - s - 1 \right\}.$$

By the induction argument, we know that $\rho(s+1; t')$ satisfies (6) for all $t' \in \mathcal{T}'$. We endow the set $\mathcal{T}'$ with the following complete lexicographic order: $t' = (t_1', ..., t_h') \succeq \widehat{t'} = (\widehat{t_1'}, ..., \widehat{t_h'})$ if $t_h' > \widehat{t_h'}$, or $t_h' = \widehat{t_h'}$ and $t_{h-1}' > \widehat{t_{h-1}'}$, and so on, or $t' = \widehat{t'}$.. Also, we say $t' \succ \widehat{t'}$ if $t' \succeq \widehat{t'}$ and $t' \neq \widehat{t'}$. We rename the vectors in $\mathcal{T}'$ as t'^1, t'^2, etc., so that $t'^1 \prec t'^2 \prec ...$

For all $t' \in \mathcal{T}'$, we denote by $\mathcal{T}(t')$ the set of vectors t of the form $t = (t_{-k}', t_k' + 1)$ or $t = (t', 1)$. Notice that the sum of the components of the vectors in $\mathcal{T}(t')$ is $n - s$. Moreover, all vector t with $\sum_{k=1}^{h} t_k = n - s$ belongs to $\mathcal{T}(t')$ for some $t' \in \mathcal{T}'$. Hence, we prove property (6) for s if and only if we prove the following property:

$$\rho(s; t) = \frac{\prod_{k=1}^{h} (t_k - 1)!}{(n - s)!} \text{ for all } t = (t_1, ..., t_h) \in \mathcal{T}(t'^m), \text{ for all } t'^m \in \mathcal{T}'. \tag{7}$$

We prove (7) through a second induction argument. This time we do the induction on the number of elements of $\mathcal{T}'$. We will use Corollary 1 that we rewrite as

$$\rho(s + 1; t') - \sum_{k=1}^{h} \rho(s; t_{-k}', t_k' + 1) - \rho(s; t', 1) = 0 \text{ for all } t' \in \mathcal{T}'. \tag{8}$$

($s, m = 1$) It is clear that $t'^1 = (1, ..., 1)$, that is, t'^1 is a vector with $n - s - 1$ unitary coordinates. The set $\mathcal{T}(t'^1)$ is composed of $(n - s - 1)$ vectors of the form $(2, 1, ..., 1)$ and a vector $(1, ..., 1)$. By the similar influence axiom, $\rho(s; 2, 1, ..., 1) = \rho(s; 1, ..., 1)$.[7] Hence, (8) implies that $\rho(s + 1; t'^1) = (n - s)\rho(s; t^1)$, i.e.,

$$\rho(s; 1, ..., 1) = \rho(s; 2, 1, ..., 1) = \frac{1}{(n - s)}\rho(s + 1; t'^1) = \frac{1}{(n - s)}\frac{\prod_{k=1}^{h} (t_k'^1 - 1)!}{(n - s - 1)!} = \frac{1}{(n - s)!},$$

which corresponds to (7) for the vectors $(1, ..., 1)$ and $(2, 1, ..., 1)$, which are the two types of vector in $\mathcal{T}(t'^1)$. Hence, the induction property holds for $m = 1$.

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[7] Indeed, the similar influence axiom implies that $\alpha(S, P) = \alpha(S, P')$ whenever the only difference between P and P' is that $\{i\}, \{j\} \in P$ whereas $\{i, j\} \in P'$, with $i, j \in N \backslash S$.

(s, m) We now suppose that (7) holds for up to $t'^{(m-1)}$, and prove that it also holds for t'^m. By the strong influence axiom, $\rho(s; t) = \rho(s; \widehat{t})$ if $\widehat{t}$ is the vector equal to t except that each component $t_k = 2$ has been substituted in $\widehat{t}$ by two components with value 1. Let us write all the vectors in $\mathcal{T}(t'^m)$ as vectors where each component with value 2 is substituted with two components with value 1, and let us denote the set of such vectors as $\mathcal{T}_1(t'^m)$.

By using the induction hypothesis, we prove that $\rho(s; t)$ is known for all $t \in \mathcal{T}_1(t'^m)$, except possibly the highest vector in $\mathcal{T}_1(t'^m)$ (when we apply the lexicographic order $\succeq$ defined above). To do the proof, let us write $t'^m = (t'^{m2}, t'^{m1})$, where $t'^{m1} = (1, ..., 1)$ and $t_k'^{m2} \geq 2$ for all $k = 1, ..., r$, where $t'^{m2} = (t_1'^{m2}, ..., t_r'^{m2})$ (t'^{m1} may not exist for $m > 1$). It is clear that the highest vector in $\mathcal{T}_1(t'^m)$ is $(t_{-r}'^{m2}, t_r'^{m2} + 1, t'^{m1})$. Consider now any other vector in $\mathcal{T}_1(t'^m)$. We distinguish between two possibilities:

1. If $t = (t_{-z}'^{m2}, t_z'^{m2} + 1, t'^{m1})$ for some $z < r$, then let $t'^p \equiv (t_{-\{z,r\}}'^m, t_z'^m + 1, t_r'^m - 1)$. Note that $t'^p \prec t'^m$, hence $p < m$, and also note that $t = (t_{-r}'^p, t_r'^p + 1)$. Hence, by induction hypothesis we already know $\rho(s; t)$.

2. If $t = (t'^{m2}, t_{-z}'^{m1}, t_z'^{m1} + 1)$, or $t = (t'^m, 1)$, then $\rho(s; t) = \rho(s; t'^m, 1)$. Let $t'^p \equiv (t_{-r}'^m, t_r'^m - 1, 1)$. Again, since $(t'^m, 1) = (t_{-r}'^p, t_r'^p + 1)$, $\rho(s; t)$ is already known.

Hence, we have proven that $\rho(s; t)$ is known for all $t \in \mathcal{T}_1(t'^m)$, except possibly the highest vector in $\mathcal{T}_1(t'^m)$. Therefore, in the formulae (8) (for $t' = t'^m$) we know all the values except possibly the one corresponding to the highest vector in $\mathcal{T}_1(t'^m)$. Moreover, all the other values are different from zero, hence the solution for the remaining value is unique. Since $\rho(s; t) = \prod_{k=1}^{h} (t_k - 1)! / (n - |S|)!$ is an expression that does satisfy the equation, it is the unique solution.

This concludes the proof of Theorem 2. ■

Proof. of Proposition 1. Consider the basis game $(N, v_{S,P})$ where $S = \{1\}$ and $P = (\{1\}, \{2, 3\}, \{4\}, \varnothing)$. The Myerson and Bolger values assign to for this game the following payoffs: $\varphi_1^M = -1/3$, $\varphi_2^M = \varphi_3^M = 1/6$, and $\varphi_4^M = 0$; whereas $\varphi_1^B = 7/144$, $\varphi_2^B = \varphi_3^B = -1/72$, and $\varphi_4^B = -1/48$. Hence both values violate the strong symmetry axiom and thus, by our Theorem 1 cannot be constructed through the average approach. ■

Proof. of Proposition 2. We first introduce the following notation: Consider two sets N and M, with $N \subset M$, and $P \in \mathcal{P}(N)$ and $Q \in \mathcal{P}(M)$. We say $Q \sqsupset P$, or $P \sqsubset Q$ if the partition Q is equal to P when we take out of Q the players in $M \backslash N$. To show the strong dummy property, let $\alpha \in N$ be a dummy player in the game (N, v). For every player $i \in N$ we have

$$
\begin{aligned}
\varphi_i(N, v) &= \sum_{(S', P') \in ECL(N)} \varphi_i(S', P') v(S', P') \\
&= \sum_{(S, P) \in ECL(N \backslash \{\alpha\})} \sum_{\substack{S' \supseteq S \\ P' \sqsupset P \\ (S', P') \in ECL(N)}} \varphi_i(S', P') v(S', P') \\
&= \sum_{(S, P) \in ECL(N \backslash \{\alpha\})} v(S, P) \sum_{\substack{S' \supseteq S \\ P' \sqsupset P \\ (S', P') \in ECL(N)}} \varphi_i(S', P').
\end{aligned}
$$

Substituting in the expressions for φ_i, we get $\sum_{\substack{S' \supseteq S, P' \sqsupset P \\ (S', P') \in ECL(N)}} \varphi_i(S', P') = \varphi_i(S, P)$ and hence $\varphi_i(N, v) = \sum_{(S, P) \in ECL(N \backslash \{\alpha\})} v(S, P) \varphi_i(S, P) = \varphi_i(N \backslash \{\alpha\}, v)$ for every $i \in N$. $\blacksquare$

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