Implementation of the Ordinal Shapley Value for a three-agent economy[1]

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Abstract

We propose a simple mechanism that implements the Ordinal Shapley Value (Pérez-Castrillo and Wettstein [2005]) for economies with three or less agents.

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1 Introduction

The Ordinal Shapley Value (OSV) is a way to allocate gains realized by cooperation in general economic environments. It is invariant with respect to the utility representation of the agents' preferences and enjoys several desirable properties such as efficiency, monotonicity, anonymity, and individual rationality (see Pérez-Castrillo and Wettstein [2005]). It provides a reasonable outcome for a large class of environments even where competitive equilibria or core allocations may fail to exist.

The OSV is a normative solution concept. An alternative approach to the problem of sharing gains from cooperation consists of proposing mechanisms whose equilibria yield "good" outcomes.[1] In this paper, we propose the use of a *bidding mechanism*, which combines and adapts to exchange economies proposals suggested in Pérez-Castrillo and Wettstein [2001] and [2002]. Informally, the mechanism proceeds as follows: In stage 1 the agents bid to choose the proposer. Each agent bids by submitting an n-tuple of real numbers, one number for each agent (including himself). The number submitted by agent i for an agent j, is a commitment to forego a commodity bundle in case j is chosen as the proposer. The bids submitted by an agent must sum up to zero. The agent for whom the aggregate bid (sum of bids submitted for him by all agents including himself) is the highest is chosen as the proposer. Before moving to stage 2, all the agents (including the proposer) pay the "bid" (i.e., the promised commodity bundles) they submitted for the proposer. In stage 2 the proposer offers a feasible allocation of the total initial resources. The offer is accepted if all the other agents agree. In case of acceptance each agent receives the bundle suggested for him in this allocation. In the case of rejection all the agents other than the proposer play the same game again where the new initial endowments incorporate the allocations paid and received by the end of stage 1.

We prove that the proposed bidding mechanism implements in Subgame Perfect equilibrium the OSV correspondence for economies with at most three agents.

[1] See Moore and Repullo [1988] and Maniquet [2003] for papers dealing with implementation in general environments. Winter [1994], Dasgupta and Chiu [1996] and Vidal-Puga and Bergantiños [2003] deal with the implementation of the Shapley value in Transferable Utility games.

2 The Ordinal Shapley Value and the Bidding Mechanism

We consider a pure exchange economy E consisting of a set $N = \{1, 2, ..., n\}$ of agents and $k \geq 2$ commodities. Agent $i \in N$ is described by $\{\succeq^i, w^i\}$, where $w^i \in \mathbb{R}^k$ is the vector of initial endowments and $\succeq^i$ is a continuous and monotonic preference relation defined over $\mathbb{R}^k$. We denote by $\succ^i$ and $\sim^i$ the strict preference and indifference relationships associated with $\succeq^i$, and $e \equiv (1, ..., 1) \in \mathbb{R}^k$.

For this economy, Pérez-Castrillo and Wettstein [2005] propose and prove the existence of a solution concept, called the *Ordinal Shapley Value* (OSV), the construction of which relies on the notion of *concessions*. The use of concessions allows to "measure" the benefits from cooperation. Concessions are made in terms of the reference bundle e.

Definition 1 *The Ordinal Shapley Value is defined recursively.*

($n = 1$) *In the case of an economy with one agent with preferences* $\succeq^1$ *and initial endowments* $a^1 \in R^k$, *the OSV is given by the initial endowment:* $OSV(\succeq^1, a^1) = \{a^1\}$.

For $n \geq 2$, *suppose that the solution has been defined for any economy with* $(n-1)$ *or less agents.*

(n) *In the case of an economy* $(\succeq^i, a^i)_{i \in N}$ *with a set* N *of* n *agents, the* OSV $((\succeq^i, a^i)_{i \in N})$ *is the set of efficient allocations* $(x^i)_{i \in N}$ *for which there exists an* $n-$*tuple of concession vectors* $(c^i)_{i \in N}, c^i \in R^{n-1}$ *for all* $i \in N$ *that satisfy:*

$n.1$) *for all* $j \in N$, *there exists* $y(j) \in OSV\left((\succeq^i, a^i + c^j_i e)_{i \in N \backslash j}\right)$ *such that* $x^i \sim^i y(j)^i$ *for all* $i \in N \backslash j$, *and*

$n.2$) $\sum\limits_{i \in N \backslash j} c^j_i = \sum\limits_{i \in N \backslash j} c^i_j$ *for all* $j \in N$.

By definition, the OSV is *efficient*. It is also *consistent* in the sense that any set of $(n-1)$ agents is indifferent between keeping their allocation or taking the concessions made by the remaining agent and reapplying the solution concept to the $(n-1)$ economy (property $n.1$). Moreover, to ensure *fairness*, the concessions balance out (property $n.2$). In fact, Pérez-Castrillo and Wettstein [2005] show that the concessions associated with OSV allocations satisfy the stronger condition that they are symmetric, that is, the concession c^i_j of agent i to agent j is equal to the concession c^j_i of agent j to i. Also, the matrix of concessions associated with any OSV allocation is unique.

The bidding mechanism to implement the OSV is recursively defined as follows:

If there is only one agent $\{i\}$, he receives his initial endowments, so he obtains utility $u^i(w^i)$. (If only one player plays, there is no bidding stage.)

Given the rules of the mechanism for at most $n-1$ agents, the mechanism for $N = \{1, \ldots, n\}$ proceeds as follows:

$t = 1$: Each agent $i \in N$ makes bids $b^i_j \in \Re$, one for every $j \in N$, with $\sum_{j\in N} b^i_j = 0$. Hence, at this stage, a strategy for player i is a vector $(b^i_j)_{j\in N} \in \mathcal{H}^n$, where $\mathcal{H}^n = \left\{z \in \Re^n | \sum_{j\in N} z_j = 0\right\}$.

For each $i \in N$, define the *aggregate bid* to player i by $B_i = \sum_{j\in N} b^j_i$. Let $\alpha = \text{argmax}_i(B_i)$ where an arbitrary tie-breaking rule is used in the case of a non-unique maximizer. Once the proposer α has been chosen, every player $i \in N$ pays $b^i_\alpha e$ and receives $(B_\alpha/n) \cdot e$.

$t = 2$: The proposer α offers a feasible allocation $(x^1, ..., x^n) \in \mathbb{R}^{kn}$ given the initial resources $(w^i)_{i\in N}$.

$t = 3$: The agents other than α, sequentially, either accept or reject the offer. If an agent rejects it, then the offer is rejected. Otherwise, the offer is accepted.

If the offer is accepted, each agent i receives x^i. Therefore, the final payoff to an agent i is $u^i(x^i)$. On the other hand, if the offer is rejected, all players other than α proceed to play the same game where the set of agents is $N\backslash\{\alpha\}$ and the initial resources for these players are $(w^i - (b^i_\alpha - B_\alpha/n)\, e)_{i\in N\backslash\{\alpha\}}$; while player α is on his own with resources $w^\alpha - (b^\alpha_\alpha - B_\alpha/n)e$. The final payoff to α is $u^\alpha(w^\alpha - b^\alpha_\alpha e + (B_\alpha/n)e)$. The final payoff to any agent $i \neq \alpha$ is the payoff he obtains in the game played by $N\backslash\{\alpha\}$.

3 The implementation for economies with at most three agents

We start by proving several properties of the OSV allocations for economies with two agents.

Lemma 1 *(a) For a two-agent economy, both agents are indifferent among the OSV allocations.*

(b) The concession c is the same for every OSV allocation and is a continuous function of the initial endowments.

(c) Let $x \in OSV\left((\succeq^i, w^i)_{i=1,2}\right)$ *and* $x' \in OSV\left((\succeq^i, w^i + \lambda e)_{i=1,2}\right)$*,* $\lambda > 0$*. Then,* $x^{i\prime} \succ^i x^i$ *for* $i = 1, 2$*.*

Proof. *(a)* As shown in Pérez-Castrillo and Wettstein [2004], the OSV for a two-agent economy consists of the efficient allocations (x^1, x^2) such that $x^1 \sim w^1 + ce$ and $x^2 \sim w^2 + ce$ for some $c \geq 0$. Consider now $x, y \in OSV\left((\succeq^i, a^i)_{i=1,2}\right)$ and denote by $c = c_2^1 = c_1^2$ and $d = d_2^1 = d_1^2$ the concessions associated respectively with x and y. Then, $x^1 \sim^1 a^1 + ce$, $x^2 \sim^2 a^2 + ce$, $y^1 \sim^1 a^1 + de$, and $y^2 \sim^2 a^2 + de$. It is immediate that $x^1 \prec^1 y^1$ if and only if $x^2 \prec^2 y^2$. The efficiency of both allocations x and y implies $x^1 \sim^1 y^1$ and $x^2 \sim^2 y^2$.

(b) The previous argument also shows that $c = d$, while the continuity of preferences implies that the concession varies continuously with the initial endowments.

(c) There exist c and c' such that: $x^1 \sim w^1 + ce$, $x^2 \sim w^2 + ce$, $x'^1 \sim w^1 + (\lambda + c')e$ $x'^2 \sim w^2 + (\lambda + c')e$. The allocation x' is Pareto efficient in $(\succeq^i, w^i + \lambda e)_{i=1,2}$ whereas x is feasible, yet not Pareto efficient for the economy $(\succeq^i, w^i + \lambda e)_{i=1,2}$. Hence, it must be the case that $\lambda + c' > c$ and $x'^i \succ^i x^i$ for $i = 1, 2$. ∎

The next theorem shows that the set of Subgame Perfect equilibrium outcomes (SPE) of the bidding mechanism coincides with the OSV for economies with at most three agents.

Theorem 1 *The bidding mechanism implements the Ordinal Shapley Value in Subgame Perfect Equilibrium in economies with* $n \leq 3$*.*

Proof. The proof proceeds by induction.

(a) We first prove that every SPE outcome of the bidding mechanism is in the $OSV\left((\succeq^i, w^i)_{i \in N}\right)$.

For $n = 1$ the proof is trivial. Note also that for economies with one agent, there is only one OSV allocation.

Claim 1. In any SPE, any agent i different than the proposer α accepts the proposal x at $t = 3$ if $x^i \succ y^i$ for every $i \in N \backslash \{\alpha\}$, where $y \in OSV\left((\succeq^j, w^j - (b_\alpha^j - B_\alpha/n)e)_{j \in N \backslash \{\alpha\}}\right)$.

Proof: First, by induction, in case of rejection the agents expect to obtain an allocation in the OSV in the economy without the proposer (and where the concessions have been added to or substracted from their initial endowment); second, by Lemma 1 *(a)*, agents in a two-agent economy are indifferent among OSV allocations (as is the case for a one-agent economy).

Claim 2. In any SPE of the game that starts at $t = 2$, the proposer α proposes an allocation x that is Pareto efficient and satisfies $x^i \sim y^i$ for every $i \in N/\{\alpha\}$, where $y = OSV\left((\succeq^j, w^j - (b^j_\alpha - B_\alpha/n)e)_{j\in N\setminus\{\alpha\}}\right)$. Moreover, every agent $i \in N\setminus\{\alpha\}$ accepts any offer x such that $x^i \succsim y^i$ for every $i \in N/\{\alpha\}$.

Proof: These are clearly equilibrium strategies for the agents other than the proposer. As regards the proposer, he cannot gain by switching to another offer that is accepted. If he makes an unacceptable offer he obtains the bundle $w^\alpha - (b^\alpha_\alpha - B_\alpha/n)e)$ which if preferred to x^α would violate the Pareto efficiency of the proposal x.

Claim 3. In any SPE, $B_i = 0$ for $i \in N$. Moreover, each agent is indifferent about the identity of the agent who is chosen as the proposer.

Proof: Denote by M the set of agents for whom the aggregate bid is the largest, that is, $M \equiv \{i \in N | B_i = \max_{j\in N} B_j\}$. We first claim that any agent $j \in N$ is indifferent between any agent in M being chosen as the proposer. Indeed, if j would strictly prefer some particular agent, say $i \in M$ to win, agent j would slightly increase his bid to agent i and decrease his bid to the other agents in M so that agent i is chosen as the proposer for sure. Following the change, by Lemma 1 *(b)*, agent j would be better off.

If $M = N$, Claim 3 is proven. Suppose, by way of contradiction, that $M \neq N$ and denote by m $(< n)$ the cardinality of M. Assume, for convenience, $1 \in M$. We now show that agent 1 can achieve a better outcome by changing the bids he submitted in stage 1. Consider the following change in the bids by agent 1: $b^{1\prime}_1 = b^1_1 - \epsilon$, $b^{1\prime}_i = b^1_i - 2\epsilon$ for any $i \in M\setminus\{1\}$, $b^{1\prime}_j = b^1_j + (2m-1)\epsilon$ for a particular $j \notin M$, and $b^{1\prime}_j = b^1_j$ otherwise, with $\epsilon > 0$ and small enough. Then, $B_1 > B'_1 > B'_i$ for all $i \neq 1$. In particular, 1 is chosen as the proposer for sure. We claim that 1 is strictly better following this change in bids. To see this, note first that b^i_1 did not change for any $i \neq 1$ and $B'_1 < B_1$. Hence, in the economy with agents $N\setminus\{1\}$, after the change in the bids the "initial endowments" change from $(w^j - (b^j_1 - B_1/n)e)_{j\in N\setminus\{1\}}$ to $(w^j - (b^j_1 - B'_1/n)e)_{j\in N\setminus\{1\}}$. Given $B_1 > B'_1$, by Lemma 1 *(c)*, all agents in $N\backslash\{1\}$ are worse off in the OSV of the second economy than in the OSV of the former. Hence, at stage 2, agent 1 can offer an allocation that is worse off for all $j \in N\backslash\{1\}$ and, by Pareto efficiency, better off for himself. Therefore, agent 1 is better off after bidding according to $b^{1\prime}$ than after bidding according to b^1.

Claim 4. In any SPE, the offer x made by the proposer at $t=2$ always belongs to $OSV\left((\succeq^j, w^j)_{j\in N}\right)$. Moreover, the agents' bids at $t=1$ are $b^i_j = c^i_j$ for all $i, j \in N$, $i \neq j$, where c is the matrix of concessions of x, and $b^i_i = -\sum_{j\in N\backslash\{i\}} c^i_j$.

To prove Claim 4, denote by $x = x(i) \in OSV\left((\succeq^j, w^j - b^j_i e)_{j\in N\backslash\{i\}}\right)$ the proposal that agent $i \in N$ makes if he is chosen as the proposer (we notice that Claim 3 states that $B_i = 0$). We are going to prove that $x \in OSV\left((\succeq^j, w^j)_{j\in N}\right)$. First, according to Claim 2, x is a Pareto efficient allocation. Moreover, the $n-$tuple of vectors of bids $(b^i)_{i\in N}$ satisfies:

i) By Claim 3, $x^k \sim^k x(j)^k$ for $j \in N, k \neq j$, where $x(j) \in OSV\left((\succeq^k, a^k - b^k_j e)_{k\in N\backslash\{j\}}\right)$,

ii) $\sum\limits_{i\in N\backslash\{j\}} b^j_i = \sum\limits_{i\in N\backslash\{j\}} b^i_j$ for all $j \in N$ (by Claim 3, $B_j = 0$, i.e., $\sum\limits_{i\in N\backslash\{j\}} b^i_j = -b^j_j$; moreover, the rules of the mechanism impose that $\sum\limits_{i\in N} b^j_i = 0$, i.e., $-b^j_j = \sum\limits_{i\in N\backslash\{j\}} b^j_i$).

Therefore, the allocation x is in the set $OSV\left((\succeq^j, w^j)_{j\in N}\right)$ taking the matrix of concessions $c^i_j = b^i_j$ for all $i, j \in N$, $i \neq j$.

(b) We now prove every allocation x in the set $OSV\left((\succeq^i, w^i)_{i\in N}\right)$ is an SPE outcome of the bidding mechanism. We denote c the matrix of concessions of x. We propose the following strategies for the case of n agents:

At $t=1$, each agent i, $i \in N$, announces $b^i_j = c^i_j$ for every $j \in N\backslash\{i\}$ and $b^i_i = -\sum_{j\in N\backslash\{i\}} c^i_j$.

At $t=2$, agent i, if he is the proposer, proposes an allocation z that is Pareto efficient and satisfies that $z^j \sim y^j$ for every $j \in N/\{i\}$, where $y \in OSV\left((\succeq^j, w^j - (b^j_i - B_i/n)e)_{j\in N\backslash\{i\}}\right)$. (We recall that, according to Lemma 1 *(a)*, in economies with one or two agents either there is only one OSV allocation or agents are indifferent among the several OSV allocations.)

At $t=3$, agent i, if agent $j \in N/\{i\}$ is the proposer, accepts any offer z such that $z^i \succsim y^i$, where $y \in OSV\left((\succeq^k, w^k - (b^k_j - B_j/n)e)_{k\in N\backslash\{j\}}\right)$ and rejects it otherwise.

First of all, we notice that if the agents make the previous bids, then the aggregate bid to each one is zero. This is a direct consequence of the symmetry of the concessions.

Second, if they make these bids, the proposal at $t = 2$ will certainly be x, given that x is efficient and guaranties the rest of the agents their OSV of the game without the proposer and with the proposer's concessions added to their initial endowment. Hence, if the agents follow the previous strategies, the final outcome is always x.

We prove that the strategies are indeed SPE strategies. By the induction argument, what the agents other than the proposer, say agent j, expect after the bids is some allocation in $OSV\left((\succeq^k, w^k - (b_j^k - B_j/n)e)_{k\in N\setminus\{j\}}\right)$. Therefore, it is easy to check that the previous strategies are SPE strategies from $t = 2$ on. Consider now the strategies at $t = 1$. Remember that we have shown that $B_i = 0$ for all $i \in N$. If agent i changes his bid, the proposer will be the agent (or one of the set of agents) to whom i increases his bid. Denote by α the proposer, and $B'_\alpha > 0$ the new aggregate bid. If $\alpha = i$, then the other agents will face a situation where all their initial endowments increase by the same amount B'_i/n. By Lemma 1 *(c)*, all these agents are better off in the new situation, hence agent i is worse off. If the new proposer is $\alpha \neq i$, then the outside option for agent i will be a situation where all the agents other than i and α will see their initial endowment increased by B'_α/n while agent i's initial endowment will decrease by $(n-1)B'_\alpha/n$. An argument similar to that of Lemma 1 *(c)* shows that agent i's situation is worse off after the change. Therefore, deviating is not profitable. ■

The major difficulty with extending this result for any number of agents is intimately related to the transfer paradox (Safra [1984]). We briefly explain here this difficulty.

It is crucial for our result that (as stated in Claim 3) the equilibrium aggregate bids are zero for every agent. For this result, it must be the case that a proposer can not gain by increasing the bid for himself and facing at the proposal stage agents with larger endowments. However, similar to the transfer paradox, an agent can be worse off in an OSV allocation when the initial endowments of all agents (including himself) increase. If this happens, the proposer may find it "easier" (less costly in terms of his own welfare) to make an acceptable proposal to the set of agents with larger endowments.

The mechanism constructed provides a non-cooperative foundation for the OSV for environments with a small number of agents. It also shows that the concessions underlying the OSV concept can be interpreted in terms of bids.

4 References

Dasgupta, A. and Y. S. Chiu (1996), "On implementation via demand commitment games," *International Journal of Game Theory* 27, 161-189.

Maniquet, F., 2003. Implementation under perfect information in economic environments. *Soc. Choice Welfare* 21, 323-346.

Moore, J., Repullo, R., 1988. Subgame perfect implementation. *Econometrica* 46, 1191–1220.

Pérez-Castrillo, D. and D. Wettstein (2001): "Bidding for the Surplus: A Non-Cooperative Approach to the Shapley Value," *Journal of Economic Theory*, 100, 274-294.

Pérez-Castrillo, D. and D. Wettstein (2002), "Choosing wisely: A multi-bidding approach," *American Economic Review* 92, 1577-1587.

Pérez-Castrillo, D. and D. Wettstein (2004) "An Ordinal Shapley Value for Economic Environments (Revised Version)," Universitat Autonoma de Barcelona W. P. 634.

Pérez-Castrillo, D. and D. Wettstein (2005): "An Ordinal Shapley Value for Economic Environments," *Journal of Economic Theory*, forthcoming.

Safra, Z. (1984) "On the Frequency of the Transfer Paradox," *Economics Letters* 15, 209-212.

Vidal-Puga, J. and G. Bergantiños (2003), "An implementation of the Owen value," *Games and Economic Behavior* 44, 412-427.

Winter, E. (1994), "The demand commitment bargaining game and snowballing cooperation," *Economic Theory* 4, 255-273.