A Characterization of Single-Peaked Preferences*

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Abstract

We identify in this paper two conditions that characterize the domain of single-peaked preferences on the line in the following sense: a preference profile satisfies these two properties if and only if there exists a linear order L over the set of alternatives such that these preferences are single-peaked with respect L. The first property states that for any subset of alternatives the set of alternatives considered as the worst by all agents cannot contains more than 2 elements. The second property states that two agents cannot disagree on the relative ranking of two alternatives with respect to a third alternative but agree on the (relative) ranking of a fourth one.

KEYWORDS: Single-peaked preferences, linear order.

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1 Introduction

Consider a set of alternatives and a set of agents, each of them endowed with a preference relation over these alternatives. For any individual it is easy to construct a linear ordering L of the alternatives such that, as we pick alternatives along the ordering L, their rank in this individual's preference relation is either always increasing, or always decreasing, or first increasing and then decreasing. Such an ordering is called an admissible orientation of this agent. A profile of preference is said to be *single-peaked* if there exists a linear ordering that is an admissible orientation for *all* agents. The class of single-peaked preferences, first scrutinized by Black [3], is perhaps

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one of the most used preference domains when analyzing collective decision problems. These preferences are known for instance to ensure the existence of a Condorcet winner, the transitivity of the majority rule (Inada [7]), or to allow the construction of strategy-proof social choice functions (Moulin [9]). Yet, a major drawback of single-peaked preferences is that this class is, as we have just seen, defined using an underlying linear order of the alternatives. It follows that if we are given a class of preference profiles it is practically impossible to know whether profiles in this class are single-peaked.[1] We identify in this paper a pair of properties that do not mention any specific linear ordering and that any single-peaked preference profile must satisfy, thereby characterizing the domain of single-peaked preferences on the line. Our result reads then as follows: a preference profile satisfies some properties if and only if there exists a linear ordering such that the preferences are single-peaked with respect to this ordering.

Failure for a preference profile to be single-peaked mainly comes from the constraint imposed by the underlying linear-ordering: alternatives must all "fit" in a one-dimensional space.[2] This is so when there are too many individuals with too many different preferences, or when the set of alternatives is "too large". Indeed, whenever there are at most two agents and three alternatives any preference profile is single-peaked.[3] We show that single-peakedness may be lost, however, as soon as we add either another agent or a fourth alternative. Adding a third agent creates a conflict if the set of alternatives that are considered worst by at least one agent is too large. A crucial observation is that a linear ordering L is an admissible orientation for an agent if his worst alternative is not ordered between two more preferred alternatives. Hence, any worst alternative must be either the first or the last in the ordering L. It follows that the set of alternatives that are considered worst by at least one agent must contain at most two alternatives. Otherwise, we say that the preference profile admits an *α-contradiction*. Single-peakedness may also be lost when adding a fourth alternative instead of a third agent. This situation, a bit more technical, can be summarized as follows. There is an alternative called the "pivot" and three alternatives, such that the two individuals agree on the relative ranking of one alternative (with respect to the pivot) but disagree on the relative ranking of the two other alternatives. To illustrate this, consider for instance two individuals, i and j and three alternatives, a, b and c. If the preferences are aP_ibP_ic and cP_jbP_ia, then it suffices to take an order that locates b in

[1]Bartholdi and Trick [2] show that single-peakedness can be identified in polynomial time. While their approach could be suitable when given a specific profile it is of little help if one wants to work with an entire class of profiles.

[2]Single-peakedness can be defined on other underlying structures than that of linear orders. See for instance Demange [5] for single-peaked preferences on a tree.

[3]This assertion will not be proved. It is easily done by simply exhausting all possibilities.

the middle.[4] Things can become more complicate when we introduce d, the fourth alternative. Suppose that $dP_i b$. Hence any linear order must either locate d between a and b (if $aP_i d$) or "beyond" a (if $dP_i a$). If a is indeed j's worst alternative, i.e., i and j both prefer d to b, then the only remaining option is to locate d between a and b. But then we cannot construct any linear order upon which preferences will be single-peaked if $dP_j b$, because in this case d should be located between b and c. If such a configuration occurs we say that the profile admits a β-contradiction. Hence, not having an α and a β-contradiction is necessary for a preference profile to be single-peaked. Our main result states this is also sufficient, thereby providing a characterization of single-peaked preferences on the line. As a byproduct, these properties are shown to also characterize single-caved preferences.

The paper is organized as follows. Notations and the main definitions are provided in Section 2 and our characterization theorem is stated and proved in Section 3and we conclude in Section 4.

2 Preliminaries

2.1 Notations

Let N be the set of agents and $X = \{x_1, \ldots, x_m\}$ be a set of alternatives.[5] For simplicity, we assume that X is finite. For any $A \subset X$, we denote by A^c the complementary set, $A^c = X \backslash A$.

Let L denote any linear order over X, i.e., a complete, transitive, and asymmetric binary relation. We say that two alternatives x and x' are *consecutive* in L if xLx' (resp. $x'Lx$) and there does not exists another alternative x'' such that $xLx''Lx'$ (resp. $x'Lx''Lx$). The fact that k alternatives are consecutive in L when $k \geq 3$ is defined similarly. In the sequel, we shall assume that the order L is *non-directed*, which means that if L and L' are two linear orders such that for any $x, x' \in X$ we have $xLx' \Leftrightarrow x'L'x$ then $L = L'$.[6]

Each agent $i \in N$ is endowed with a complete, transitive, asymmetric preference relation P_i over X. For any set $A \subseteq X$, let $w_i(A, P_i)$ be i's least preferred alternative in A, i.e.,

$$w_i(A, P_i) = \{x \mid \forall\, x' \in A\backslash\{x\},\, x'P_i x\}\,. \tag{1}$$

Let $w(A, P) = \cup_{i \in N} w_i(A, P_i)$. Similarly, let $t_i(A, P_i)$ be i's most preferred alternative in A, i.e, $t_i(A, P_i) = \{x \mid \forall\, x' \in A\backslash\{x\},\, xP_i x'\}$ and let $t(A, P) = \cup_{i \in N} t_i(A, P_i)$. For simplicity, we shall drop the mention to the preferences

[4] The preference relation $aP_i bP_i c$ reads as follows: i's most preferred alternative is a, then b and c is his least preferred alternative.

[5] The sets X and N can be identical

[6] This last assumption is without loss of generality.

and write $w_i(A)$, $w(A)$ and $t_i(A)$ whenever possible. Note also that any preference relation P_i is, in our context, also a linear order. However, to gain in clarity we will not use the terminology "linear ordering" to qualify a preference relation.

2.2 Single-peaked preferences

We first begin by recalling the textbook definition of single-peaked preferences.

Definition 1 A preference profile P is single-peaked if there exists a linear order L such that for each agent $i \in N$, and any two alternatives $x', x'' \in X$, $t_i(X)Lx'Lx''$ or $x''Lx'Lt_i(X)$ imply $x'P_ix''$.

The set of single-peaked preferences is denoted $\mathscr{P}^S$. Obviously, for any agent i it is always possible to find a linear order L such that $t_i(X)Lx'Lx''$ implies $x'P_ix''$. For instance, we can take the trivial order L such that $aLb \Leftrightarrow aP_ib$ for all $a, b \in X$. What is not always possible is the existence of a *common* order L for all players, thereby making the domain of single-peaked preferences a strict sub-domain of the domain of all preference profiles.

Definition 2 Given a preference relation P_i and a set $A \subseteq X$, a linear order L (over A) is an *admissible orientation of A with respect to P_i* if for any triple of alternatives of A, $t_i(A)$, x' and x'' such that $t_i(A)Lx'Lx''$ (resp. $x''Lx'Lt_i(A)$) we have $x'P_ix''$.

We write $\mathscr{L}_i(A, P_i)$ to design the set of all linear orders that are admissible orientations of A with respect to P_i (as we have mentioned, this set is always non-empty), and $\mathscr{L}(A, P)$ stands for the set of all linear orders that are admissible orientations for all individuals, i.e., $\mathscr{L}(A, P) = \cap_{i \in N} \mathscr{L}_i(A, P_i)$. For notational simplicity, we shall sometimes omit the preferences and write $\mathscr{L}_i(A)$ and $\mathscr{L}(A)$. We are now ready to provide an alternative definition of single-peaked preferences.

Definition 3 A profile of preferences P is single peaked if $\mathscr{L}(X, P) \neq \emptyset$.

Observe that a profile of preference is not single-peaked if for any linear ordering of the alternatives there is always an agent for whom there is one alternative ordered between two preferred ones (his preferences are (locally) U-shaped), i.e., $L \notin \mathscr{L}(A, P) \Rightarrow \exists\ i \in N, ; x, x', x'' \in A$ such that x, x', and x'' are consecutive in L (in this order) and $x' = w_i(x, x', x'')$.

3 Results

We begin our characterization by introducing two simple properties and show that they are necessary for single-peakedness.

Definition 4 A preference profile P possesses an α-contradiction if there exists a set of alternatives $A \subseteq X$ such that $\sharp w(A) \geq 3$.

Let $\mathscr{P}^\alpha$ denote the set of all preference profiles that do not possess an α-contradiction. The next Lemma states that belonging to $\mathscr{P}^\alpha$ is a necessary condition for a profile P to be single-peaked.

Lemma 1 $\mathscr{P}^S \subset \mathscr{P}^\alpha$.

Proof Let $P \notin \mathscr{P}^\alpha$, i.e., there exists $A \subseteq X$ such that $\sharp w(A) \geq 3$. Hence, there exist distinct x, x' and x'' such that $\{x, x', x''\} \subseteq w(A)$, which implies that there exist $i, j, k \in N$ such that $w_i(\{x, x', x''\}) = x$, $w_j(\{x, x', x''\}) = x'$ and $w_k(\{x, x', x''\}) = x''$. It follows that if $L \in \mathscr{L}_i(A)$ then L cannot rank x between x' and x'', i.e., $xLx' \Rightarrow xLx''$ and $x'Lx \Rightarrow x''Lx$. Similarly, if $L \in \mathscr{L}_j(A)$ (resp. $L \in \mathscr{L}_k(A)$) then L cannot rank x' between x and x'' (resp. x'' between x and x'). We then have $\cap_{h=i,j,k}\mathscr{L}_h(A) = \emptyset$, and thus $\mathscr{L}(X, P) = \emptyset$, the desired result. ■

Interestingly, the α-contradiction property sheds some light on why free triples correspond to profiles which are not single-peaked. Recall that a free triple occurs when we have three alternatives, a, b and c, and three individuals, i, j and k, such that aP_ibP_ic, bP_jcP_ja and cP_kaP_kb. Clearly, whenever we have a free triple we have an α-contradiction, and thus, according to Lemma 1, single-peakedness does not hold. Now, if we change one agent's preference without changing his worst alternative, then the free triple disappears but, according to Lemma 1, the new profile is still not single-peaked. That is, single-peakedness is a more stringent requirement than the absence of free triples.

The second property we identify concerns the case when the "conflict" between individuals' preferences come from the preference relation of two individuals only. Since there are only two individuals, the conflict cannot come from the worst alternatives (this is captured by the α-contradiction). The second source of conflict we identify deals with the relative ranking of intermediate alternatives.

Definition 5 A profile of preferences P possesses a β-contradiction if there exists two individuals, $i, j \in N$, and a set of 4 alternatives $\{a, b, c, d\}$ such that:

(i) They are opposite in a triple: aP_ibP_id, dP_jbP_ja. Alternative b is called the *pivot*.

(ii) They coincide in ranking cP_ib, cP_jb.

Observe that if there is a β-contradiction between preferences P_i and P_j then agents i and j "agree" to set alternatives a and d as the end points of a common admissible orientation. The disagreement arises however with alternatives b and c. Let $\mathscr{P}^{\beta}$ denote the set of preference profiles that do not possess a β-contradiction. The next Lemma states that if a profile of preferences has a β-contradiction then it it is not single-peaked.

Lemma 2 $\mathscr{P}^S \subset \mathscr{P}^{\beta}$.

Proof Let $P \notin \mathscr{P}^{\beta}$. Then, there exists $i, j \in N$ and $a, b, c, d \in X$ such that aP_ibP_ic and dP_ib, and cP_jbP_ja and dP_jb. Observe that $L \in \mathscr{L}_i$ implies that L cannot rank c between any pair $(x, x') \in \{a, b, d\}$. Similarly, if $L \in \mathscr{L}_j$ then L cannot rank d between any pair $(x, x') \in \{a, b, c\}$. Therefore, $\mathscr{L}_i \cap \mathscr{L}_j \neq \emptyset \Rightarrow \mathscr{L}_i \cap \mathscr{L}_j \subseteq \{L, L'\}$ where $aLbLdLc$ and $aL'dL'bL'c$. Observe that $w_i(a, b, d) = b$, which implies that b cannot be ranked between a and d, and thus $L \notin \mathscr{L}_i$. Similarly, it can be shown that $L' \notin \mathscr{L}_j$, which completes the proof. ■

The previous two properties show two simple structures on a preference profile conducing to non single-peakedness. As our main result shows, they are in fact the only plausible causes. We characterize thus, the nature of preference profiles that are single-peaked on a line. Notice that Theorem 1 do also provide a characterization of the single-peaked domains.

Theorem 1 $\mathscr{P}^S = \mathscr{P}^{\alpha} \cap \mathscr{P}^{\beta}$.

Before proving the theorem let us comment briefly on it. Observe that if $A \subset X$ and $\mathscr{L}(A, P) = \emptyset$, then $\mathscr{L}(X, P) = \emptyset$. Similarly, if for any coalition $S \subset N$ we have $\cap_{i \in S} \mathscr{L}_i(A, P_i) = \emptyset$, then $\cap_{i \in N} \mathscr{L}_i(A, P_i) = \emptyset$ as well. Theorem 1 shows that, to some extent, the converse also holds. More precisely, if $\mathscr{L}(X, P) = \emptyset$ then there exists either a set A of three alternatives and a set S of three players such that $\mathscr{L}(A, (P_i)_{i \in S}) = \emptyset$ or a set B of four alternatives and two players i and j such that $\mathscr{L}(B, (P_i, P_j)) = \emptyset$.

Proof That $\mathscr{P}^S \subseteq \mathscr{P}^{\beta} \cap \mathscr{P}^{\alpha}$ is obtained combining Lemmata 2 and 1. To prove the reverse inclusion, consider any profile $P \in \mathscr{P}^{\beta} \cap \mathscr{P}^{\alpha}$. We show that we can construct a linear order L such that for all $i \in N$, $L \in \mathscr{L}_i(X, P_i)$. To this end, consider the following sequence of sets: $X_1 = w(X)$, and for any $t > 1, X_t = w((\bigcup_{h<t} X_h)^c)$. Note that $P \in \mathscr{P}^{\alpha} \Rightarrow \sharp X_t \leq 2$ for all $t = 1, \ldots$.

We first consider the collection of all sets X_t such that $\sharp X_t = 2$, which we denote $A_1, A_2, \ldots$, ordered as the sets X_t.[7] Consider first A_1, and denote one of its element l_1 and the other r_1, and let L be the linear ordering over X_1 such that $l_1 L r_1$. Clearly, $L \in \mathscr{L}(A_1, P)$. We now proceed by induction to show that we can augment L to include also a relation over

[7]That is, if there are two sets A_h and $A_{h'}$ such that $h < h'$ then there exists two sets X_t and $X_{t'}$ such that $t < t'$.

the elements $A_1 \cup A_2 \cup \dots A_{t+1}$. That is, suppose now that for some $t \geq 1$ we have constructed an ordering L such that $L \in \mathscr{L}(\cup_{h \leq t} A_h, P)$. Let $A_{t+1} = \{x, x'\}$ and the define the sets $R = \{x, x'\} \backslash w(\{l_t, x, x'\})$ and $S = \{x, x'\} \backslash w(\{r_t, x, x'\})$. In words, the set R (resp. S) contains the alternatives that are candidates to be just "after" l_t (resp. "before" r_t). Note that since P has no α-contradiction, R and S are both non-empty. We claim that we can pick two distinct alternatives such that one belongs to R and the other to S. If this were not possible, then we should have $\sharp R = \sharp S = 1$ and $R = S$. Hence, to prove the claim it suffices to show that R and S both containing one identical alternative, say x, yields to a contradiction. Observe that $x' \notin R$ implies that $\exists\ i \in N$ such that $x' \in w(\{l_t, x, x'\})$. It follows that $l_t P_i x'$ and $x P_i x'$. Using the induction hypothesis we obtain $r_t = w_i((X \setminus A \cup \{l_t, r_t\})$, which implies $x' P_i r_t$. Now, since $x' \notin S$ either, $\exists\ j \in N$ such that $x' = w_j(\{x, x', r_t\})$ and thus $x P_j x'$ and $r_t P_j x'$. Like for individual i it can be shown that $x' P_j l_t$. We then have a β-contradiction with P_i and P_j, where the pivot is x'.

Without loss of generality let then $x \in R$ and $x' \in S$ such that $x \neq x'$, and let $l_{t+1} = x$ and $r_{t+1} = x'$. Augment now the ordering L with the following relations $l_t L l_{t+1} L r_{t+1} L r_t$. We claim that $L \in \mathscr{L}(\bigcup_{h \leq t+1} A_h)$. Suppose not. Then, there exists $i \in N$ and three alternatives consecutive in L, say a, b and c, such that $w_i(\{a, b, c\}) = b$. Observe that the induction hypothesis (and how L is constructed) guarantees that $\{a, b, c\} \nsubseteq \{l_1, \dots, l_t\}$ and $\{a, b, c\} \nsubseteq \{r_1, \dots, r_t\}$. It cannot be either that $\{b, c\} = \{l_{t+1}, r_{t+1}\}$, for otherwise we would have $l_{t+1} \notin R$. Similarly (reasoning with S), it can be shown that we must have $\{a, b\} \neq \{l_{t+1}, r_{t+1}\}$. Hence, the only possibility is either $\{a, b, c\} = \{l_{t-1}, l_t, l_{t+1}\}$ or $\{a, b, c\} = \{r_{t+1}, r_t, r_{t-1}\}$. Without loss of generality, suppose that $l_t = w_i(l_{t-1}, l_t, l_{t+1})$. In this case, by the induction hypothesis, we have $l_{t-1} P_i l_t \Rightarrow l_t P_i r_t$. Since l_t and r_t were both selected at stage, say k (i.e., $A_t = X_k$), there must exist another individual j with $l_t = w_j((\bigcup_{h<k} X_h)^c)$. Therefore, $r_t P_j l_t$ and $l_{t+1} P_j l_t$. Since $L \in \mathscr{L}(\bigcup_{h \leq t} A_h)$, $r_t P_j l_t \Rightarrow l_t P_j l_{t-1}$. We then have a β-contradiction with alternatives l_{t-1}, l_t, l_{t+1} and r_t (where l_t is the pivot), contradicting $P \in \mathscr{P}^\beta$.

We now consider the sets X_t such that $\sharp X_t = 1$. Let $A_1, \dots, A_k$ be the sequence of pairs previously obtained and let L be the ordering over $\bigcup_{h \leq k} A_k$ we just constructed. Let $A = \cup_{h=1,\dots,k} A_h$. Denote by $c_0^1, \dots, c_0^{p_0}$ all the alternatives (if any) that appeared before A_1, ordered as in the the sequence $\{X_t\}$. Similarly, for $1 \leq t < k$, let $c_t^1, \dots, c_t^{p_t}$ all singletons (if any) that appeared after A_t and before A_{t+1}. That is, if t is such that $X_t = A_h$ then $X_{t+1} = \{c_h^{p_1}\}$. Finally, let $c_k^1, \dots, c_k^{p_k}$ all singletons (if any) that appeared after A_k. We now show that we can augment the ordering L by including the elements $(c_0^0, \dots, c_k^{p_k})$.

Consider first c_k^1 and augment L with $l_k L c_k^1 L r_k$. We claim that $L \in \mathscr{L}(A \cup \{c_k^1\})$. If not, there exists $i \in N$ and three alternatives consecutive in L, say x, x' and x'', such that $w_i(\{x, x', x''\}) = x'$. Like before, it can be shown that these three alternatives must be either l_{k-1}, l_k, c_k^1 or c_k^1, r_k, r_{k-1}. W.l.o.g. suppose it is the first triple. Hence, $\exists\ j \in N$ such that $l_{k-1} P_j l_k$ and $c_k^1 P_j l_k$. Note that by construction we have indeed $c_k^1 P_h l_k$, $\forall\ h \in N$. Since $L \in \mathscr{L}(A)$, $l_{k-1} P_j l_k \Rightarrow l_k P_j r_k$. Observe now that $\exists\ i \in N$ such that $r_k P_i l_k$ (for otherwise l_k and r_k would not be a pair selected in the first step of the proof) and since $l_{k-1} \in A_{k-1}$ and $l_k \in A_k$, we then have $r_k P_i l_k P_i l_{k-1}$. We then have a $\beta-$contradiction between P_i and P_j where c_k^1 is the pivot.

Consider now c_k^2. We claim that we can augment L either with $l_k L c_k^1 L c_k^2 r_k$ or with $l_k L c_k^2 L c_k^1 r_k$. Suppose we cannot. The same analysis as for c_k^1 rejects the cases in which $c_k^2 \in w(l_{k-1}, c_k^2, l_k))$ and $c_k^2 \in w(r_k, c_k^2, r_{k-1}))$. Like before, this implies that there exists $i \in N$ such that $w_i(\{l_k, c_k^1, c_k^2\}) = c_k^1$ (and a different j such that $w_j(\{c_k^2, c_k^1, r_k\}) = c_k^1$ from which we can deduce that there is a $\beta-$contradiction between P_i and P_j with l_k, c_k^1, c_k^2, r_k where c_k^1 is the pivot. Once we have set $l_k c_k^1 L c_k^2 L r_k$ (or $l_k L c_k^2 L c_k^1 L r_k$), L can be augmented with the remaining alternatives sufficient to augment L in the *same direction*, i.e., setting $l_k L c_k^1 L c_k^2 \ldots L c_k^{s-1} L c_k^s r_k$ (resp. $l_k L c_k^s L c_k^{s-1} \ldots L c_k^2 L c_k^1 L r_k$). The proof that L augmented this way remains an admissible orientation for all individuals is as before and thus is omitted.

For $1 \leq h < k$ a similar reasoning can be used to show that we can augment L with either $l_h c_h^1 \ldots c_h^{p_h} l_{h+1}$ or with $r_{h+1} c_h^{p_h} \ldots c_h^1 r_h$.[8] This completes the proof. ∎

The absence of the basic contradictions is therefore, the fundamental of any single-peaked profile. Since the α-contradiction requires the participation of at least 3 individuals, the following corollary is straightforward.

Corollary 1 *If* $\sharp N = 2$ *then* $\mathscr{P}^S = \mathscr{P}^\beta$.

Similarly, if there are only three alternatives single-peakedness boils down to the absence of an α contradiction:

Corollary 2 *If* $\sharp X = 3$ *then* $\mathscr{P}^S = \mathscr{P}^\alpha$.

Another preference domain, similar to single-peakedness, is the class of *single-caved* preferences.[9] A profile P is single-caved if there exists a linear order L such that for each agent $i \in N$, and any two alternatives $x', x'' \in X \backslash w_i(X)$, $w_i(X) L x' L x''$ or $x'' L x' L w_i(X)$ imply $x'' P_i x'$. For a preference relation P_i, the *reverse* relation, $\overline{P}_i$ is defined as follows: $x' \overline{P}_i x$ if and only if $x P_i x'$, and we denote $\overline{P}$ the profile $(\overline{P}_i)_{i \in N}$. Similarly, let $\overline{\mathscr{P}}$ be the set of

[8]The string $c_0^1, \ldots, c_0^{p_0}$ can be included indifferently to the left of l_1 (in increasing order) or to the right of r_1 (in decreasing order).

[9]Some authors call such preferences *single dipped* (e.g., Klaus *et al.* [8]).

profiles that are reversed of profiles of $\mathscr{P}$. It is easy to see that a profile P is single-caved if and only if $\overline{P}$ is single-peaked.

We now show that Theorem 1 also provides a characterization of the domain of single-caved preferences. This characterization can be made either using the reverse of a single-caved profile (which is single-peaked) or using the reverse versions of the α and β-contradictions.

Definition 6 A preference profile P possesses an α'-contradiction if there exists a set of alternatives $A \subseteq X$ such that $\sharp t(A) \geq 3$.

Definition 7 A profile of preferences P possesses a β'-contradiction if there exists two individuals, $i, j \in N$, and a set of 4 alternatives $\{a, b, c, d\}$ such that *(i)* They are opposite in a triple: aP_ibP_ic, cP_jbP_ja; *(ii)* They coincide in ranking dP_ib, dP_jb.

Denote by $\mathscr{P}^{SC}$, set of preference profiles that are single-caved, and by $\mathscr{P}^{\alpha'}$, and $\mathscr{P}^{\beta'}$ those without an α'-contradiction and a β'-contradiction, respectively.

Proposition 1 $\mathscr{P}^{SC} = \mathscr{P}^{\alpha'} \cap \mathscr{P}^{\beta'} = \overline{\mathscr{P}}^{\alpha} \cap \overline{\mathscr{P}}^{\beta}$.

Proof The proof of the first equality follows the same steps as that of Lemmata 1 and 2 and Theorem 1 and is left to the reader. The second equality follows from the fact that P is single-caved if and only if $\overline{P}$ is single-peaked. ∎

We now apply our characterization result to the class of *acyclical* preferences defined by Ergin [6]. He showed that these preferences are necessary and sufficient to obtain Pareto efficiency, group strategy-proofness and consistency of a wide range of allocation problems including, among others, college admission, Gale and Shapley's marriage matching model, or house allocation. Ergin defined acyclicity by means of two properties. One of these properties considers capacity constraints, which we shall not consider here. As Ergin pointed out, this property is often suppressed as it has no bite in assignment or matching models with quota equal to one as in the marriage model. The second property is a condition on orderings, requiring that there is no two orderings P_i and P_j and three alternatives x, x' and x'' such that the following holds,

$$xP_ix'P_ix'' \quad \text{and} \quad x''P_jx\,. \tag{2}$$

Let $\mathscr{P}^{\gamma}$ be the set of profiles that does not satisfy Eq. (2).[10] We then have the following result,

[10]Since Egin worked with a model encompassing house allocation or college admission he called colleges' or houses' preferences *priority structures*. Note however, that in this context priority structures are, like preferences, linear orderings. To be consistent with the rest of the paper we adapted Ergin's terminology to our framework.

Proposition 2 $\mathscr{P}^\gamma \subset \mathscr{P}^S$.

Proof Let $P \notin \mathscr{P}^S$. From, Theorem 1 is follows that P has either an α or a β-contradiction (or both). Suppose first that $P \notin \mathscr{P}^\alpha$. Hence, there exists $i, j, k \in N$ and $x, x', x'' \in X$ such that, say $x = w_i(x, x', x'')$, $x' = w_j(x, x', x'')$ and $x'' = w_k(x, x', x'')$. W.l.o.g. suppose that $xP_kx'P''_x$. Since $w_i(x, x', x'') = x$ we then have $x''P_ix$.[11] This satisfies Eq. (2), which implies that $P \notin \mathscr{P}^\gamma$. Suppose now that $P \notin \mathscr{P}^\beta$. Thus there exist $i, j \in N$ and x, x', x'' such that $xP_ix'P_ix''$ and $x''P_jx'P_jx$. (We can disregard the relation with respect to the fourth alternative). This obviously satisfies Eq. (2), which completes the proof. ■

Using Proposition 2 and Theorem 1 in Ergin [6] we can then deduce for instance that in the marriage model the deferred acceptance can only be group-strategyproof and Pareto efficient when preferences of the side not proposing are single-peaked.

4 Conclusion

Our results first show that two agents and four alternatives or three agents and three alternatives are enough to make a preference profile not single-peaked. What is perhaps more surprising is that we also obtain the reverse statement: if a profile is not single-peaked then there must be two or three agents without a common admissible orientation (over four or three alternatives, respectively). That is, if single-peakedness breaks down because of, say, ten agents, then it must also breaks down because of two or three of these agents.

As far as we know, Bartholdi and Trick [2] is the only paper that identifies single-peakedness without using an a priori linear order of the alternatives to be ranked. Their approach differs from ours, however, since they do not provide a set of properties that characterize single-peakedness, as we do. Instead, they provide an algorithm to find admissible orientations and show that given a preference profile single-peakedness can be stated in polynomial time.[12]

The understanding of preferences that allow this kind of representation constitutes one of the main problems of, among other fields, Strategy-proof Social Choice. Inspired by classical results (Moulin [9]), a huge amount of literature has discussed two of its main features. First, the generalization of the notion of single-peakedness to non-linear structures, in the search of new non-manipulable rules. Second, the deep analysis of voting by

[11]If $x'P_kxP''_x$ then it suffices to consider individual j instead of i.

[12]To proceed they transform the problem of finding a linear ordering into a matrix problem and apply an algorithm originally proposed by Booth and Lueker [4]. See also Trick [11] for an algorithm to check single-peakedness on a tree.

committees (see Barberà et al. [1]) show that single-peakedness is the *seed* of non-manipulability in several domains. In a recent paper, Nehring and Puppe [10] show the relevance of this approach.

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