Stable Condorcet Rules*

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Abstract. We consider the following allocation problem: A fixed number of public facilities must be located on a line. Society is composed of N agents, who must be allocated to one and only one of these facilities. Agents have single peaked preferences over the possible location of the facilities they are assigned to, and do not care about the location of the rest of facilities. There is no congestion. In this context, we observe that if a public decision is a Condorcet winner, then it satisfies nice properties of internal and external stability. Though in many contexts and for some preference profiles there may be no Condorcet winners, we study the extent to which stability can be made compatible with the requirement of choosing Condorcet winners whenever they exist.

Key words: Social Choice Correspondences, Condorcet Rules, Stability, Simpson Rule.

1. Introduction.

Global collective choices can often be described as the union of partial decisions, each of which affects different groups of citizens to a different degree. Take the budget of a government. Some of its items (like the salaries of public servants, if I am one) may be very important to me, while I may be quite indifferent regarding others (like, say, how much of the agricultural subsidies go to olive producers, and how much goes to wheat farmers). But of course, other citizens may feel intensely about what I do not care, and little about my major concerns.

In this paper we explore the connections between the choices over complex decisions, and the partitions of citizens into different interest groups that result from their appraisal of these decisions, based on their particular concerns.

We consider a particular type of allocation problems within this class. A fixed number of public facilities must be located on a line. Society is composed of N agents, who must be allocated to one and only one of these facilities. Agents have single peaked preferences over the possible location of the facilities they are assigned to, and do not care about the location of the rest of facilities. There is no congestion: the location of their facility is the only concern of agents, not the set of other agents with whom they share it. In this particular context, we shall observe that if a public decision is a Condorcet winner (that is, if it would carry the support of a majority over any other possible decision), then it will have very nice stability properties. Specifically, these allocations will be *internally stable (*the location of each facility will be favored over any alternative location by a majority of its users), and also *externally stable* (no user of a facility would prefer to change to another)[1]. Hence, if Social Choice correspondences

[1]This general notion of external stability will be given several rigorous expressions later on. See Section 3.

could always select Condorcet winners, their choices would always share all these stability properties. It is well known, however, that in many contexts and for some preference profiles Condorcet winners may fail to exist. What we do is to formulate a list of desirable properties for allocations and for social choice correspondences, see whether one can design rules that always satisfy some of them, and in addition, choose Condorcet winners whenever they exist.

Our assumption that agents only care about the facility they actually use, approaches our model to a class of coalition formation games, called hedonic games, which have received the attention of different authors (Banerjee, Konishi and Sonmez (1998) and Bogomolnaia and Jackson (1998) among others). In hedonic games, individuals have preferences over the coalitions that they can be part of, presumably because they get some benefit from their association with the rest of agents in the coalition. In our case, agents meet because they share a facility, and in our interpretation they draw no direct advantage from being together. But sharing the facility is tantamount to the wish of being together, and in this respect the two types of models are quite similar.

A recent paper by Milchtaich and Winter (2002) is closely related to ours in several ways. It will be useful to comment on the analogies and on the differences between the two papers, and this will also allows us to complete our previous comments regarding hedonic games. M&W analyze the stability of coalitions in a model of group formation where agents are driven by their attraction to people who are similar to them. They prove the existence of stable partitions in a one-dimensional model where the total number of groups to be formed is bounded above, and agents strive to minimize either the average distance among group members or their distance to the average member.

In our model, group formation is induced by the common use of a facility, rather than by an inherent desire to become associated with similar people. Yet, whenever agents are free to choose their preferred facility, this indirectly results in their association with individuals of similar tastes. Hence, our model is close to that of M&W, and more in general, to the literature on hedonic games. In fact, M&W take a step toward our formulation, when they suggest that a good interpretation of their model comes from assuming that agents choose one location. Indeed, this is what our agents do, once the facilities are located. We add some more richness to the analysis by explicitly allowing agents to play a role in determining the location of the facilities among which they may then choose from.

Both our paper and M&W's depart from the literature on hedonic games by assuming some upper bound in the number of coalitions that can form. This bound is given, in our case, by the possible number of facilities.

The introduction of facilities as explicit objects of analysis allows us to take a significantly distance with respect to M&W's treatment of preferences over sets of agents. In their paper, the average agent plays the role of representative for the group (there are other cases considered, but we retain this one for comparison), but this assumption is rather ad.hoc. In our paper, the median voter emerges as the significant member of the group, since we explicitly model a reason to be part of it: namely, to influence the allocation of the shared facility by means of a majority vote among those who are concerned.

In spite of these differences, their existence proof uses similar techniques to the ones we used to prove our two existence results. For all these rezones, we consider that the two papers are nicely complementary. Notice, however, that since we are considering correspondences, rather than functions, our analysis provides families of partitions, rather than a single one. Our previous assertions apply to any of the partitions in the family.

The rest of the paper is organized as follows. In Section 2, we present our model in detail. In Section 3 we discuss the properties that Condorcet winners have whenever they exist. Section 4 studies the compatibility between internal and external stability. Finally, in Section 5 we present a rule that satisfies the stability properties and always chooses Condorcet winners when they exist.

2. The Model.

The model we present is essentially the one considered in Barberà and Beviá (2001). The major difference is that here we model social decision rules as correspondences, while in the previous paper we considered the case of functions.

We consider problems that involve any finite set of agents. Agents are identified with elements in $\mathbb{N}$, the set of natural numbers. Let $\mathfrak{S}$ be the class of all finite subsets of $\mathbb{N}$. Elements of $\mathfrak{S}$, denoted as $S, S', ...$, stand for particular societies, whose cardinality is denoted by $|S|$, $|S'|$, etc.

We now describe the decisions that societies can face. These are determined by the number and the position of relevant locations, and by the sets of agents who are allocated to each location.

A natural number $k \in \mathbb{N}$ will stand for the number of locations. Then, given $S \in \mathfrak{S}$ and $k \in \mathbb{N}$, an $S/k-$*decision* is a $k-$tuple of pairs $d = (x_h, S_h)_{h=1}^k$, where $x_h \in \mathbb{R}$, and $(S_1, .., S_k)$ is a partition of S. We interpret each x_h as a location and S_h as the set of agents who is assigned to the location $x_{h.}$ Notice that some elements in the partition may be empty. This will be the case, necessarily, if $k > |S|$. We call $d_L = (x_1, .., x_k)$ the vector of *locations*, and $d_A = (S_1, .., S_k)$ the vector of *assignments*.

Let $D(S,k)$ be the set of $S/k-$decisions, $D(k) = \bigcup_{S\in\mathfrak{S}} D(S,k)$ the set of $k-$decisions, and $D = \bigcup\limits_{k\in\mathbb{N}} D(k)$ the set of decisions. For each agent $j \in N$, the set of $k-$decisions which concern j is $D_j(k) = \bigcup_{\{S\in\mathfrak{S}|j\in S\}} D(S,k)$ and the set of decisions that concern j is $D_j = \bigcup\limits_{k\in\mathbb{N}} D_j(k)$.

Agents are assumed to have complete, reflexive, transitive preferences over decisions which concern them. That is, *agent $i's$ preferences* are defined on D_i, and thus, rank any pair of S/k and $S'/k'-$decisions provided that $i \in S \cap S'$. Denote by $\succcurlyeq_i$ the preferences of agent i on D_i.

We shall assume all along that preferences are *singleton-based.* Informally, this means that agents' rankings of decisions only depend on the location they are assigned to, not on the rest of locations or on the assignment of other agents to locations. This assumption is compatible with our interpretation that agents can only use the good provided at one location, and that this is a public good subject to no congestion. Formally, a preference $\succcurlyeq_i$ on D_i is singleton-based if there is a preference $\bar{\succcurlyeq}_i$ on $\mathbb{R}$ such that for all d, $d' \in D_i$, $d \succcurlyeq_i d'$ if and only if $x(i,d)\bar{\succcurlyeq}_i x(i,d')$, where $x(i,d)$ denotes the location to which agent i is assigned under the decision d.

In all that follows, we shall assume that for all $i \in N$, $\succcurlyeq_i$ is singleton-based, and in addition, that the order $\bar{\succcurlyeq}_i$ is *single-peaked.* That is: for each $\bar{\succcurlyeq}_i$, there is an alternative $p(i)$ which is the unique best element for $\bar{\succcurlyeq}_i$; moreover, for all x, y, if $p(i) \geq x > y$, then $x \;\bar{\succ}_i y$, and if $y > x \geq p(i)$, then $x \;\bar{\succ}_i y$. Under the assumption that $\succcurlyeq_i$is singleton-based, there is a one to one relation between preferences on decisions, $\succcurlyeq_i$, and preferences on locations, $\bar{\succcurlyeq}_i$. Thus, from now on we will not make any distinction between the two.

Given $S \in \mathfrak{S}$, *preference profiles for S* are $|S|$-tuples of preferences, and we denote them by $P_S, P'_S, ...$

We denote by $\mathcal{P}$ the set of all preferences described above, and by $\mathcal{P}^S$ the set of preference profiles for S satisfying those requirements.

A *collective choice correspondence* will select a set of $k-$decisions, for each given k, on the basis of the preferences of agents in coalition S, for any coalition $S \subseteq N$. Formally,

Definition 1. *A collective choice correspondence is a correspondence* $\varphi : \bigcup_{S \in \mathfrak{S}} \mathcal{P}^S \times \mathbb{N} \to D$ *such that, for all* $S \in \mathfrak{S}$, $P_S \in \mathcal{P}^S$ *and* $k \in \mathbb{N}$, $\varphi(P_S, k) \subset D(S, k)$.

We rephrase within our model the classical notion of a Condorcet winner.

Definition 2. *An* $S/k-$*decision* $d \in D(S,k)$ *is a Condorcet winner for* P_S *and* k *if*

$$\left|\{i \in S \mid d \succ_i d'\}\right| \geq \left|\{i \in S \mid d' \succ_i d\}\right| \text{ for all } d' \in D(S,k)$$

Given $S \in \mathfrak{S}$, $P_S \in \mathcal{P}^S$ *and* $k \in \mathbb{N}$, *let* $CW(P_S, k)$ *be the set of* $S/k-$*decisions that are Condorcet winners for* P_S *and* k.

3. Condorcet Winners and their Properties.

In our model, Condorcet winners have nice properties whenever they exist. Let us illustrate the class of properties that decisions and rules can be expected to satisfy by considering an example.

Example 1. *A society with 16 agents must locate two facilities and assign each of the agents to one of the two. Agents have Euclidean preferences on the line*[2], *with peaks as follows: For* $i \in \{1, 2\}$, $p(i) = i$, $p(3) = 4.5$, $p(4) = 5$, $p(5) = 6$,

[2]Euclidean preferences have their best element at some point t, and can be represented by the utility function $u(x) = -|t - x|$. They are a special case among the single-peaked.

for all $i \in \{6,..,10\}$, $p(i) = 8$, $p(11) = 10.1$, $p(12) = 11$, $p(13) = p(14) = 13$, $p(15) = p(16) = 14$. *In this society, decisions are expressed by two pairs* $((x_1, S_1), (x_2, S_2))$, *where* x_1 *and* x_2 *are the locations of the public facilities and* S_1, S_2 *are the set of agents who are assigned to these facilities, respectively. Figure 1 provides a sketchy graphical representation of the agents' preferences and it shows a particular decision* $d = ((8, \{1, ..., 11\}), (13, \{12, .., 16\}))$. *With Euclidean preferences it is enough to represent the peaks of the agents on the line. Each circle over a point of the peaks contains a number, indicating what agent has a peak at that point. A decision* $d = ((x_1, S_1), (x_2, S_2))$ *will be represented by the labels* x_1, x_2 *at the appropriate points of the line, and by squares containing the agents in* S_1 *and* S_2 *respectively.*

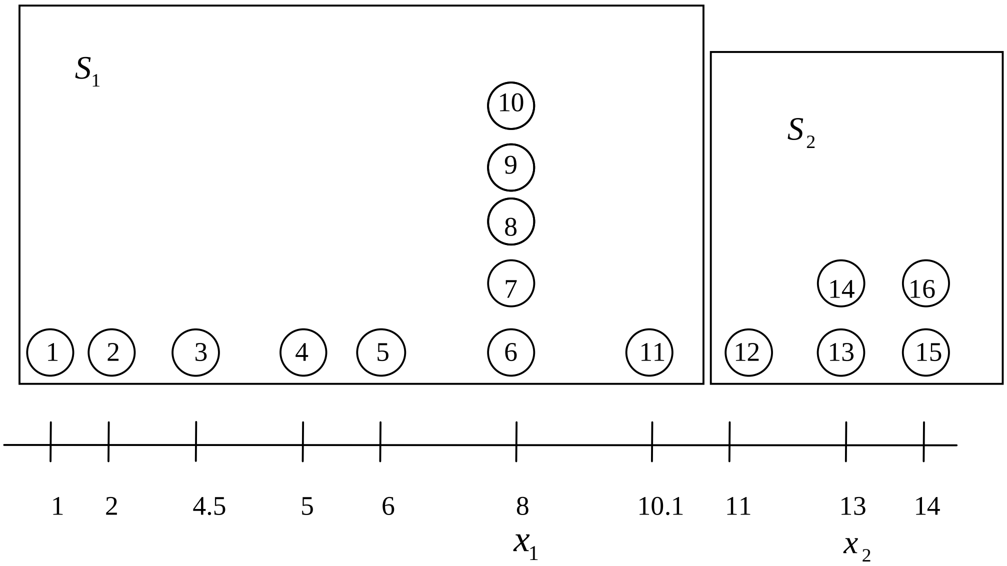

figure 1

If the members of this society were to choose one decision by majority vot-

ing, which one (s) would they select? There may be several possible choices, but they should all be Condorcet winners: that is, decisions which would not loose by majority against any other one. In our example, the decision that we have represented, $d = ((x_1, S_1), (x_2, S_2))$ *is a Condorcet winner.*

The fact that d *is a Condorcet winner has consequences. Specifically, it guarantees that it satisfies a number of nice properties, which we shall here describe informally, and then define precisely.*

(1) The decision d *is Pareto efficient. Indeed, there is no other way to locate the decisions and to assign agents to them which all agents would unanimously support over* d *(with some being possibly indifferent).*

(2) The decision d *satisfies an attractive property of internal stability. Notice that* d *identifies the set* S_i *as that of the users of the facility located at* x_i*. Suppose that agents in* S_i *were in a position to reconsider, just by themselves, whether* x_i *is indeed the best location for the facility that they are the only users of. If these agents were to vote by majority on that issue,* x_i *would indeed be chosen, since it is the median of the peaks of agents in* S_i*, each of which has single peaked preferences. Global decisions give rise to a partition of society, and internal stability means that the members inside each partition agree, by majority, on the locations of their own facility.*

(3) The decision d *satisfies nice properties of external stability. We can attach several meanings to this general notion, all of them related to the question whether agents who are assigned to a certain facility will have incentives to accept this assignment, or else would prefer to join a different set of users. In a first meaning of external stability, an allocation may be envy-free. This means that, in our context, no agent who is assigned to one facility would prefer to use any of the other facilities which are made available to other agents as part of the same decision.*

It is easy to see that, if the initial allocation is efficient, and ours is, then it is envy-free. Indeed, this is due to the absence of congestion in the consumption of the public facilities. Therefore, no agent (or group) will have incentives to change facilities under an efficient decision, unless the agent (or group) can foresee that joining another group may induce a favorable change in the location of the relevant facilities. This leads us to describe other notions of external stability, involving the comparison of any allocation with others that may have resulted if agents had the right to be allocated as they wished. Specifically, we can define Nash and group Nash stability as two properties of external stability, once we clarify the way in which one agent (or set of agents) could modify an initial allocation by joining a group different than the one she (they) were originally assigned to. In our example, we may check that, under the sustained hypothesis of internal stability (or equivalently, if the facility assigned to each group is a majority winner for that group), no agent or group could benefit from joined a different group than the one they are initially assigned to. The fact that decision d *satisfies so many interesting properties is not an accident. Rather, we shall prove that it is a consequence of the fact that this decision is a Condorcet winner among global decisions of the form* $((x_1, S_1), (x_2, S_2))$[3].

Let us now define formally the properties discussed in Example 1, and we prove that whenever a global Condorcet winner exist, it satisfies all those properties.

Definition 3. *An* $S/k-$*decision* $d = ((x_h, S_h))_{h=1}^{k} \in D(S, k)$ *is internally stable*

[3]In this and all other examples, we shall assume that agents have euclidean preferences. One consequence of this is that, in these examples, all groups in partitions satisfying our stability requirements will be connected. But this is just for these special preferences, and it should not be expected in the more general case of single peaked preferences, which are the ones we assume in the paper. The reader may check that our formal proofs never assume connectedness of groups, as it need not hold in the general framework.

if $d_h = (x_h, S_h) \in CW(P_{S_h}, 1)$ *for all* h *such that* $S_h \neq \emptyset$.

As already explained, we may want to consider not one, but several notions related to external stability. If agents envisage the possibility of joining another group, but not of changing the location of any object, then external stability is equal to No-envy. If agents envisage the possibility of joining another group, and consider that the object assigned to this new group may be reallocated accordingly, then external stability is Nash stability. Group Nash stability would be similar, with the added possibility that agents might coordinate with others in the same group when deciding whether or not to change groups.

We now proceed to formally define these notions and to study their compatibility with other desirable features of the collective choice rules.

Definition 4. *An* $S/k-$*decision* $d = ((x_h, S_h))_{h=1}^k \in D(S,k)$ *is envy-free if for all* $i \in S$, $x(i,d) \succcurlyeq_i x_h$ *for all* $x_h \in d_L$.

Both the notion of Nash and group Nash stability are based on the assumption that agents may compare different sets of decisions, resulting from their use of alternative strategies (this is because we are studying social choice correspondences). Since agents' preferences have been defined up to now on single decisions, we must be precise on the kind of set comparisons that we allow for. Rather than introducing very general principles of set comparisons, we propose two specific definitions of Nash and group Nash stability. They are based on the implicit assumption that agents will only engage in strategic changes from one group to another if these changes guarantee an unequivocal gain, whatever pair of allocations we choose to compare out of the choices made by the social choice correspondence.

Definition 5. *An $S/k-$decision $d = ((x_h, S_h))_{h=1}^k \in D(S,k)$ is Nash stable if for all h, j, with $h \neq j$ and for all $i \in S_h$, there is $\bar{x} \in CW(P_{S_j \cup \{i\}}, 1)$ such that $x(i,d) \succcurlyeq_i \bar{x}$.*

Definition 6. *An $S/k-$decision $d = ((x_h, S_h))_{h=1}^k \in D(S,k)$ is group Nash stable if for all h, j, with $h \neq j$ and for all $I \subset S_h$, there is $\bar{x} \in CW(P_{S_j \cup I}, 1)$ such that $x(i,d) \succcurlyeq_i \bar{x}$ for all $i \in I$.*

Let us now explore some of the connections among these stability requirements and two important properties of allocations: efficiency and being a Condorcet winner.

Definition 7. *An $S/k-$decision d is efficient if there is no $S/k-$decision d' such that $d' \succcurlyeq_i d$ for every agent $i \in S$ and $d' \succ_j d$ for some $j \in S$.*

Efficiency in our model requires that each agent should be assigned to the location they most prefer. Thus, an efficient decision is envy-free. Furthermore, any $S/k-$decision that is a Condorcet winner for P_S and k is an efficient decision, consequently, it is envy-free. Moreover, the choice of a global Condorcet winner guarantees internal stability. That is, each one of these locations is a Condorcet winner for the agents who are concerned about it. This last result was shown in Proposition 3 in Barberà and Beviá (2001). For the sake of completeness we formally reproduce it here.

Proposition 1. *Given $S \in \mathfrak{S}$, $P_S \in \mathcal{P}^S$ and $k \in \mathbb{N}$, if an $S/k-$decision $d = ((x_h, S_h))_{h=1}^k \in CW(P_S, k)$, then d is internally stable.*

Finally, our next proposition shows that global Condorcet winners are group Nash stable and, therefore, Nash stable.

Proposition 2. *Given* $S \in \mathfrak{S}$, $P_S \in \mathcal{P}^S$ *and* $k \in \mathbb{N}$ *if an* $S/k-$*decision* $d = ((x_h, S_h))_{h=1}^k \in CW(P_S, k)$, *then* d *is group Nash stable.*

Proof. Suppose that there is a global Condorcet winner which is not group Nash stable. That is, there is $d \in CW(P_S, k)$, h, j, with $h \neq j$ and $I \subset S_h$, such that $\bar{x} \succ_i x(i, d)$ for all $\bar{x} \in CW(P_{S_j \cup I}, 1)$, for all $i \in I$. Without loss of generality suppose that $h = j + 1$. Let $[y_1, y_2] = CW(P_{S_j \cup I}, 1)$. Let d' be such that $d'_l = d_l$ for all $l \notin \{j, h\}$, $d'_j = (y_1, S_j \cup I)$, $d'_h = (x_h, S_h \backslash I)$. We'll prove that d' wins by majority over d, a contradiction to the initial assumption that d is a Condorcet winner. Let us check that $|\{i \in S \mid d' \succ_i d\}| > |\{i \in S \mid d \succ_i d'\}|$. Notice that $\{i \in S \mid d' \succ_i d\} = \{i \in S_j \cup I \mid y_1 \succ_i x_j\}$, and since $y_1 \in CW(P_{S_j \cup I}, 1)$, $|\{i \in S_j \cup I \mid y_1 \succ_i x_j\}| \geq |\{i \in S_j \cup I \mid x_j \succ_i y_1\}|$. Hence, we only need to prove that this inequality is strict. Given that $d \in CW(P_S, k)$, d is envy-free, then $x(i, d) \not\succ_i x_j$ for all $i \in I$, and since all the agents in I prefer any element in $CW(P_{S_j \cup I}, 1)$ to $x(i, d)$, $x_j \notin CW(P_{S_j \cup I}, 1)$, which implies that $x_j < y_1 \leq y$ for all $y \in CW(P_{S_j \cup I}, 1)$. Then $|\{i \in S_j \cup I \mid y_1 \succ_i x_j\}| > |\{i \in S_j \cup I \mid x_j \succ_i y_1\}|$, and trivially, $|\{i \in S_j \cup I \mid x_j \succ_i y_1\}| = |\{i \in S \mid d \succ_i d'\}|$, which contradicts the fact that $d \in CW(P_S, k)$. ■

4. Internal versus External Stability.

We have shown that Condorcet winners are efficient decisions and satisfy both internal and external stability in all its variants. Hence, if a collective choice correspondence that always selects the set of Condorcet winners was well defined, we could guarantee the simultaneous satisfaction of all these desiderata. However, Condorcet winners will not always exist in our setup, as shown in the following example. The inexistence occurs despite our strong restrictions on preferences, which guarantee the existence of Condorcet winners when $k = 1$. Because of that, the design of stable rules will require further attention.

Example 2. ***Condorcet winners may fail to exist.*** *A society with 14 agents must locate two facilities and assign each of the agents to one of the two. Agents have Euclidean preferences on the line, with peaks as follows:* $p(i) = i$, *for all* $i = 1, .., 4$ *and* $p(5) = 32$, $p(6) = 33$, $p(7) = 34$, $p(8) = 67$, $p(9) = 68$, $p(10) = 69$, $p(11) = 97$, $p(12) = 98$, $p(13) = 99$, $p(14) = 100$. *Let us see that* $CW(P_S, 2) = \emptyset$. *We already know that if* $d = (d_1, d_2) \in CW(P_S, 2)$, *then* $d_h \in CW(P_{S_h}, 1)$ *for* $h \in \{1, 2\}$ *and* d *should be an efficient* $S/2-$*decision. We represent in figure 2 the preferences of the agents and the unique decision,* $d = ((x_1, S_1), (x_2, S_2))$ *that satisfies these properties, and thus, the unique potential candidate. However* d *is not a Condorcet winner for* P_S *and 2 since it is defeated by majority by* $d' = ((y_1, S_1'), (y_2, S_2'))$ *(this decision is represented in figure 2 with dotted lines). Indeed,* $\{i \in S \mid d' \succ d\} = \{5, 6, 7, 8, 9, 10, 12, 13, 14\}$, *and* $\{i \in S \mid d \succ d'\} = \{1, 2, 3, 4, 11\}$.

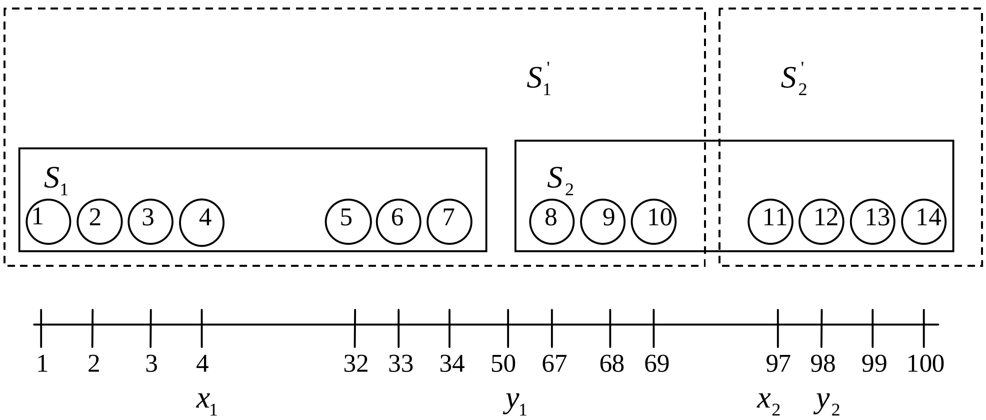

figure 2

In view of this example, we can not expected any collective choice correspondence to always select decisions which are Condorcet winners. We can demand, however, that they always do if such winners exist at all. Our analysis will concentrate on rules that satisfy this requirement, to be called Condorcet rules.

We have seen in Example 2 that for $k > 1$ Condorcet winners may not exist, but when they do, their choice seems quite compelling. Hence, we will demand for a collective choice correspondence to recommend the choice of the Condorcet winners whenever they exist.

Definition 8. *A collective choice correspondence φ is a Condorcet rule if for all $S \in \mathfrak{S}$, $P_S \in \mathcal{P}^S$ and $k \in \mathbb{N}$ such that $CW(P_S, k) \neq \emptyset$, $\varphi(P_S, k) = CW(P_S, k)$.*

The possibility that collective choices may fail to be Condorcet winners at some profiles leads to the following question: are there Condorcet rules which are internally stable and satisfy some form of external stability as well? As a first step toward an answer, we define the correspondence that chooses all decisions satisfying internal stability and efficiency (and thus, the weak form of external stability, envy-freeness).

Definition 9. *Let φ^E be the collective choice correspondence such that for each $S \in \mathfrak{S}$, $P_S \in \mathcal{P}^S$ and $k \in \mathbb{N}$,*

$\varphi^E(P_S, k) = \{d \in D(S, k) \mid d$ *is efficient and* $d_h \in CW(P_{S_h}, 1)$ *for all h s.t.* $S_h \neq \emptyset\}$

The first result proves that internal stability and efficiency can always be met, that is, the image of the correspondence φ^E, as defined, is never empty. To formally state the result, we need to introduce simple functions of the form $f : \bigcup_{S \in \mathfrak{S}} \mathcal{P}^S \to \mathbb{R}$, which can be interpreted to determine a single location for each preference profile. We require these simple functions to satisfy efficiency and participation. Formally,

Definition 10. *A function* $f : \bigcup_{S \in \mathfrak{S}} \mathcal{P}^S \to \mathbb{R}$ *is efficient if for all* $S \in \mathfrak{S}$, *and for all* $P_S \in \mathcal{P}^S$ *there is no location* $x \in \mathbb{R}$ *such that* $x \succcurlyeq_i f(P_S)$ *for all* $i \in S$ *and* $x \succ_j f(P_S)$ *for some* $j \in S$.

Definition 11. *A function* $f : \bigcup_{S \in \mathfrak{S}} \mathcal{P}^S \to \mathbb{R}$ *satisfies participation if for all* $S \in \mathfrak{S}$, *all* $i \notin S$, *and for all* $(P_S, P_{\{i\}}) \in \mathcal{P}^{S \cup \{i\}}$, $f(P_S, P_{\{i\}}) \succcurlyeq_i f(P_S)$.

Participation says that if a new agent joins a group, the outcome can only changes in the direction that benefits the new agent.

Our proof is based on Proposition 1 in Barberà and Beviá (2001), where we show that if a function $f : \bigcup_{S \in \mathfrak{S}} \mathcal{P}^S \to \mathbb{R}$ is efficient and satisfies Participation then the set $\{d = (x_h, S_h)_{h=1}^k \in D(S,k) \mid d$ is efficient and $x_h = f(P_{S_h})$ for all h s.t. $S_h \neq \emptyset\}$ is not empty.

Finally, let us fix some notation. The lower median of a finite collection K of real numbers is denoted by $lmed(K)$. It stands for the median when the cardinality of K is odd, and for the lowest value of the median if the cardinality is even[4].

Proposition 3. *The correspondence* φ^E *is well defined. That is, for each* $S \in \mathfrak{S}$, $P_S \in \mathcal{P}^S$ and $k \in \mathbb{N}$, $\varphi^E(P_S, k) \neq \emptyset$.

Proof. Let $f : \bigcup_{S \in \mathfrak{S}} \mathcal{P}^S \to \mathbb{R}$ be such that for each $S \in \mathfrak{S}$ and $P_S \in \mathcal{P}^S$, $f(P_S) = lmed(p(i))_{i \in S}$. Trivially, f is efficient and satisfies Participation, therefore, by Proposition 1 in Barberà and Beviá (2001), the set $\{d = (x_h, S_h)_{h=1}^k \in D(S,k) \mid d$ is efficient and $x_h = f(P_{S_h})$ for all h s.t. $S_h \neq \emptyset\}$ is not empty, and since for each $S \in \mathfrak{S}$ and $P_S \in \mathcal{P}^S$, $lmed(p(i))_{i \in S} \in CW(P_{S_h}, 1)$, the correspondence φ^E is well defined. ■

[4]We cannot simplify our analysis by assuming an odd number of voters because the nature of our questions require the size of the electorate to be variable and endogenously given.

The rule φ^E, as defined, is not a Condorcet rule, because not all decisions satisfying the two properties of internal stability and efficiency are Condorcet winners. Indeed, a Condorcet rule φ satisfying the internal and the external stability conditions should be a selection from φ^E, but it can not be any selection as shown by the following example.

Example 3. *The society is the same as in Example 1. Consider the decision* $d = ((y_1, S_1), (y_2, S_2))$ *represented in figure 3.*

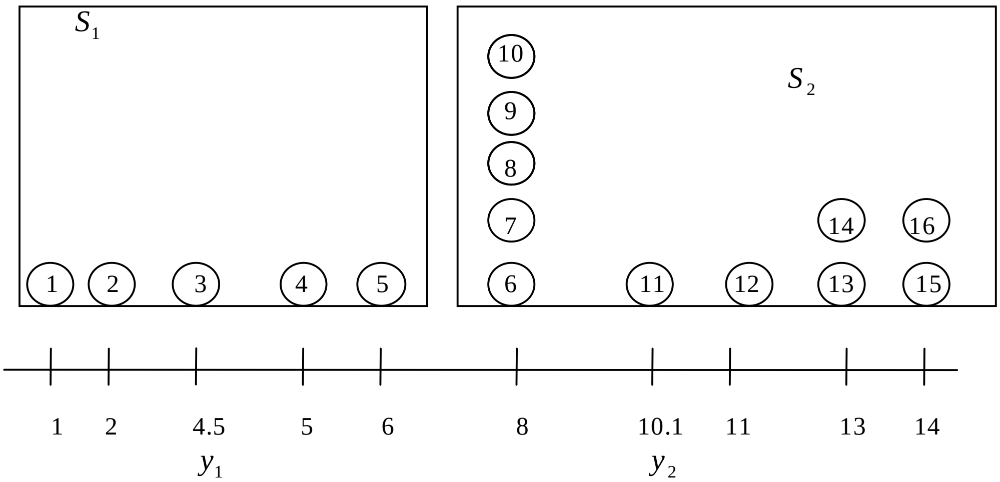

figure 3

The decision d is not a Condorcet winner because it is not group Nash stable. The set of agents {6,7,8,9,10} prefers to join S_1 *taking into account that the location for this new group will be any point between* 6 *and* 8, *which all the agents in this set prefer to the current location they are assigned to.*

Our next remarks refer to the possibility of guaranteeing stronger forms of external stability like Nash or group Nash stability, while also respecting the internal stability requirement. We begin by remarking that it may not always be possible to have internal stability, efficiency and group Nash stability.

Example 4. ***Non existence of efficient, internally stable and group Nash stable rules.*** *Consider a society with 25 agents with euclidean preferences with the following peaks:* $p(1) = 1$, $p(2) = 2$, $p(3) = 3$, $p(i) = 12$ *for all* $i \in \{4, 5\}$, $p(i) = 14$ *for all* $j \in \{6, 7, 8, 9\}$ *and* $p(i) = 15$ *for all* $i \in \{10, ..., 25\}$. *In figure 4, we represent this society and the decision* $d = ((x_1, S_1), (x_2, S_2))$ *which is the unique decision that is internally stable and efficient.*

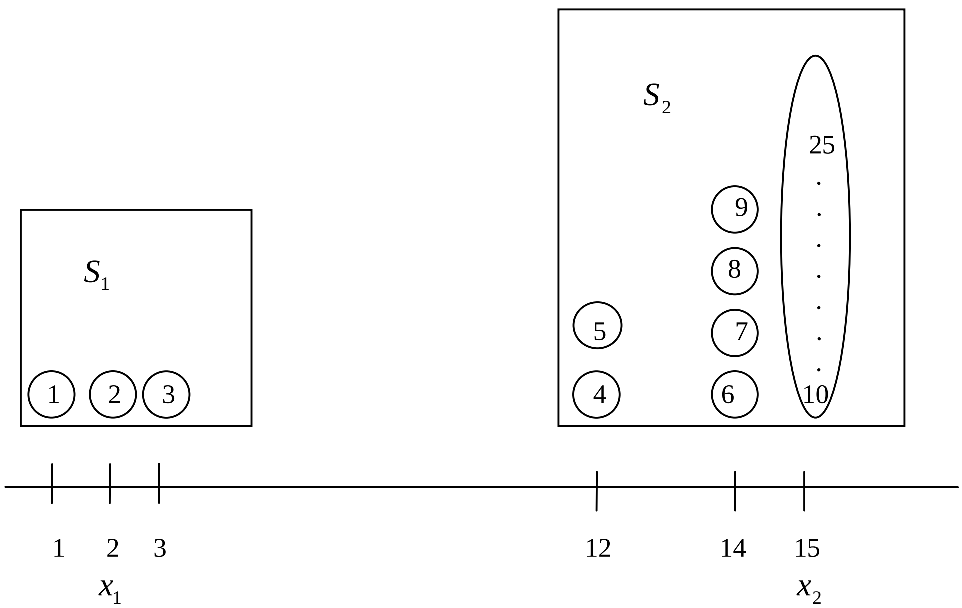

figure 4

No rule satisfying efficiency and internal stability can be group Nash stable. To check that, notice that these rules will always select the median of groups, and thus, $S = \{6, 7, 8, 9\}$ *will prefer to join* S_1 *and then get* $\bar{x}_1 = 14$. *Moreover,* *notice that a circle arises, which also makes this new coalition (and the next) unstable. Indeed, under the assignment* $((14, \{1, 2, 3, 6, 7, 8, 9\}), (15, \{4, 5, 10, .., 25\})$ *the group* $\{4, 5\}$ *would like to join* $\{1, 2, 3, 6, 7, 8, 9\}$ *and get the facility located in* 12*. And the resulting allocation,* $((12, \{1, 2, .., 9\}), (15, \{10, .., 25\})$ *will then induce* $\{6, .., 9\}$ *to join* $\{10, .., 25\}$ *in order to get allocated to 15, which they prefer to 12, then leaving* $\{1, 2, 3, 4, 5\}$ *with a facility in* 3*, for which now* $\{4, 5\}$ *will want to depart, thus closing the cycle.*

We can not combine internal stability with our most demanding notion of external stability. But we can combine it with the satisfaction of Nash stability, as expressed in the following proposition.

Definition 12. *Let* φ^N *be the subcorrespondence of* φ^E *such that for each* $S \in \mathfrak{S}$*,* $P_S \in \mathcal{P}^S$ *and* $k \in \mathbb{N}$*,* $\varphi^N(P_S, k)$ *chooses all Nash stable decisions in* $\varphi^E(P_S, k)$*.*

Proposition 4. *The correspondence* φ^N *is well defined. That is, for each* $S \in \mathfrak{S}$*,* $P_S \in \mathcal{P}^S$ *and* $k \in \mathbb{N}$*,* $\varphi^N(P_S, k) \neq \emptyset$*.*

Proof. Let $S \in \mathfrak{S}$, $P_S \in \mathcal{P}^S$ and $k \in \mathbb{N}$. If there are at most k different peaks, we are done. Hence, suppose that there are at least k different peaks. Let us order the agents by increasing order of their peaks. Let $S^1_h = \{i \in S \mid p(i) = p(h)\}$ for all $h \in \{1, .., k-1\}$, and $S^1_k = \{i \in S \mid p(i) > p(k-1)\}$. Let $x^1 = (x^1_h)^k_{h=1}$ be such that $x^1_h = lmed(p(i))_{i \in S^1_h}$, for $h \in \{1, .., k\}$, and let $d^1 = (x^1_h, S^1_h)^k_{h=1}$. Notice that, $x^1_h = p(h)$ for all $h \in \{1, .., k-1\}$. If there is no $i \in S^1_k$, such that for all $\bar{x} \in CW(P_{S^1_{k-1} \cup \{i\}}, 1)$, $\bar{x} \succ_i x(i, d^1)$ we are done. Otherwise, let $I = \{i \in S^1_k \mid \text{for all } \bar{x} \in CW(P_{S^1_{k-1} \cup \{i\}}, 1),\ \bar{x} \succ_i x^1_k\}$, and let $S_{k-1} = S^1_{k-1} \cup I$. Let $y_{k-1} = lmed(p(l))_{l \in S_{k-1}}$. First of all, notice that for all $i, j \in I$, $lmed(p(l))_{l \in S^1_{k-1} \cup \{i\}} = lmed(p(l))_{l \in S^1_{k-1} \cup \{j\}} \leq y_{k-1} < x^1_k$. Notice that we can have agents in S_k^1 which prefer x_k^1 to any point in $CW(P_{S_{k-1}^1 \cup \{i\}}, 1)$, but they have their peaks between x_{k-1}^1 and y_{k-1}, which implies that they prefer y_{k-1} to x_k^1. Let $S_k' \subset S_k^1$ be the set of those agents. For each $i \in S_k'$, let z_i be such that agent i is indifferent between x_k^1 and z_i. Let us order the agents in S_k' by increasing order of z_i. Then $[z_{i+1}, x_k^1] \subseteq [z_i, x_k^1]$. Take the first agent in this set. Then, $lmed(p(l))_{l \in S_{k-1} \cup \{1\}} \succcurlyeq_1 y_{k-1}$, and since $y_{k-1} \succcurlyeq_1 x_k^1$, $lmed(p(l))_{l \in S_{k-1} \cup \{1\}} \in [z_1, x_k^1]$. If $lmed(p(l))_{l \in S_{k-1} \cup \{1\}} \in [z_2, x_k^1]$, add agent 2 to $S_{k-1} \cup \{1\}$, and we get that $lmed(p(l))_{l \in S_{k-1} \cup \{1,2\}} \succcurlyeq_2 lmed(p(l))_{l \in S_{k-1} \cup \{1\}})$, which will imply that $lmed(p(l))_{l \in S_{k-1} \cup \{1,2\}} \in [z_2, x_k^1]$. We keep adding agents from S_k' in the above defined order whenever $lmed(p(l))_{l \in S_{k-1} \cup \{1,2,..,i\}}) \in [z_{i+1}, x_2^1]$. Let $S_k'^1$ be this subset of agents. Then, for all $i \in S_k' \backslash S_k'^1$, $lmed(p(l))_{l \in S_{k-1} \cup S_k'^1} \notin [z_i, x_k^1]$. Notice that all agents in $S_k' \backslash S_k'^1$ have their peaks between $lmed(p(l))_{l \in S_{k-1} \cup S_k'^1}$ and x_k^1. Once this process is completed, consider the following sets of agents: $S_{k-1}^2 = S_{k-1} \cup S_k'^1$, $S_k^2 = S_k^1 \backslash \{I \cup S_k'^1\}$, and $S_h^2 = S_h^1$ for all $h \in \{1, .., k-2\}$. Let $x_h^2 = lmed((p(l))_{l \in S_h^2}$ for all $h \in \{1, .., k\}$. Notice first that for all $i \in I$, $lmed(p(l))_{l \in S_{k-1}^1 \cup \{i\}} \leq x_{k-1}^2 < x_k^1$ and therefore, $x_{k-1}^2 \succcurlyeq_i x_k^1$. Also, by construction, for all $l \in S_k'^1$, $x_{k-1}^2 \succcurlyeq_l x_k^1$. And since $x_k^2 \geq x_k^1$, for all $l \in S_{k-1}^2$, $x_{k-1}^2 \succcurlyeq_l x_k^2$. Therefore, once the process has finished, no agent that has been moved in order to join a different group, wants to go back to his initial group. If the decision $(x_h^2, S_h^2)_{h=1}^k \in \varphi^N(P_S, k)$ we are done. If not, we repeat the process. Notice that, if in step j we do not get a decision in $\varphi^N(P_S, k)$, it is because for some h, some $i \in S_h^j$ and all $\bar{x} \in CW(P_{S_{h-1}^j \cup \{i\}}, 1)$, $\bar{x} \succ_i x_h^j$. Thus, if this is the case, in each step we add agents from S_h^j to S_{h-1}^j in the way described above. The process will end in a finite number of steps because there are a finite number of agents, at each step $x_h^j \geq x_h^{j-1}$ for all $h \in \{1, ..k\}$, and furthermore, $S_1^{j-1} \subseteq S_1^j$ and $S_k^j \subseteq S_k^{j-1}$. ∎

The correspondence φ^N is efficient, Nash stable and satisfies the internal stability property, but it is still not a Condorcet rule. For example, the decision illustrated in figure 3 is one of the decisions selected by this correspondence but we have seen that is not a Condorcet winner, even though a Condorcet winner exists.

From here on, we face two possible strategies. One is to refine this correspondence in an adequate way so as to guarantee that we get a Condorcet rule. This is the approach we will end up taking. The other would be to start with a Condorcet rule and see whether it is internally and Nash stable. Although we shall not take this route at the end, it is worth exploring one of its possible starting points, as it will be useful in the sequel.

5. Stable Condorcet rules.

The literature on tournaments has proposed many different rules which choose Condorcet winners when they exist, and otherwise make choices based on different types of considerations. One of these rules is the Simpson rule[5], which we now describe.

Definition 13. *Given $S \in \mathfrak{S}$, $P_S \in \mathcal{P}^S$ and $k \in \mathbb{N}$, for any $d, d' \in D(S,k)$, let $N(d,d') = |\{i \in S \mid d \succcurlyeq_i d'\}|$. Given $d \in D(S,k)$, the Simpson score of d, denoted $SC(d)$, is the minimum of $N(d,d')$ over all $d' \in D(S,k)$, $d' \neq d$.*
An $S/$ k-decision d is a Simpson winner for P_S and k if $SC(d) \geq SC(d')$ for all $d' \in D(S,k)$.
The Simpson correspondence, SW, is defined so that for each $S \in \mathfrak{S}$, $P_S \in \mathcal{P}^S$

[5]We have chosen the Simpson rule as it is directly applicable to choice problems involving an infinite set of alternatives. Other criteria (like the Copeland rule) are usually defined for finite sets of alternatives and would need to be adapted.

and $k \in \mathbb{N}$,

$$SW(P_S, k) = \{d \in D(S,k) \mid SC(d) \geq SC(d') \text{ for all } d' \in D(S,k)\}$$

The Simpson rule turns out not to be efficient and internally stable, as the following example shows.

Example 5. *Let us consider Example 2 again. If SW was efficient and internally stable, it should be a selection of* φ^E*. In this example,* $\varphi^E(P_S, 2) = \{((x_1, S_1), (x_2, S_2))\}$*. This decision is represented in figure 5. However, this decision is not a Simpson winner for* P_S *and 2. To see that let* $d' = ((x'_1, S'_1), (x'_2, S'_2))$ *represented in figure 5 with dotted lines. This S/2-decision is such that* $d' \succ_i d$ *for all* $i \in \{5, 6, 7, 8, 9, 10, 12, 13, 14\}$*, and* $d \succ_i d'$ *for all* $i \in \{1, 2, 3, 4, 11\}$*. Therefore the number of voters preferring d to d′ is five, and this is the minimum over all* $\bar{d}$*. So, the Simpson score of d is five. For* $\hat{d} = ((y_1, S_1), (x'_2, S_2))$*, it is easy to see that the Simpson score of* $\hat{d}$ *is six. Then d can not be a Simpson winner. Therefore, SW is not efficient and internally stable.*

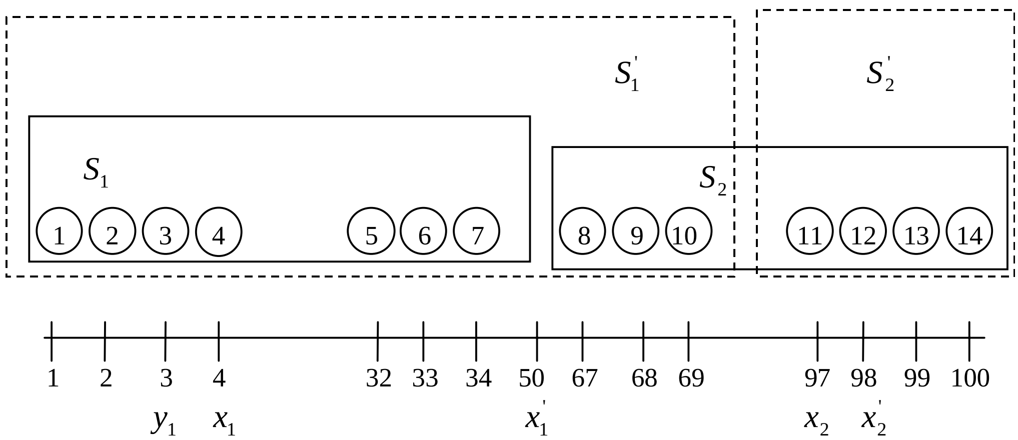

figure 5

In view of the above, we come back to the strategy already described above: start from the correspondence φ^N and select decisions from it in such a way as to preserve the Condorcet criterion. For this purpose, we propose a variation of the Simpson rule.

The Simpson rule, as it is usually defined, computes the Simpson scores on the basis of all pairwise comparisons among the elements of a set. Then it uses these scores to choose among these same elements. Our variation of the rule is based on the idea that one might form Simpson scores on the basis of pairwise comparisons among elements of a certain set D, and then use them to choose from a subset of D. Formally:

Definition 14. *Given a subset of $S/k-$decisions, $H \subset D(S,k)$, the set of $H-$Simpson winners is the set of decisions in H whose Simpson score (still computed on pairwise comparisons over all $D(S,k)$) is maximal on H.*

Hence, the $H-$Simpson winners are those elements in H which have the maximal Simpson scores. But these scores are computed not only in pairwise comparisons among elements in H, but on pairwise comparisons among all elements of the larger set $D(S,k)$.

This method can now be applied to define subcorrespondences of any given correspondence.

Definition 15. *Given a correspondence φ, its Simpson subcorrespondence, φ^*, is defined so that for each $S \in \mathfrak{S}$, $P_S \in \mathcal{P}^S$ and $k \in \mathbb{N}$,*

$$\varphi^*(P_S,k) = \{d \in \varphi(P_S,k) \mid d \text{ is } \varphi(P_S,k) - \textit{Simpson winner}\}$$

Finally, apply this general procedure to φ^N, and define its Simpson subcorrespondence, φ^{*N}. It is straightforward to check that φ^{*N} will satisfy all our requirements.

Proposition 5. *Let φ^{*N} be the Simpson subcorrespondence of φ^N. This correspondence is an internally and Nash stable Condorcet rule.*

Remark 1. *Notice that if we had chosen from the set $\varphi^N(P_S, k)$ on the basis of the Simpson scores computed for the members of this set alone, we could guarantee the choice of Condorcet winners when they exist, but could not always avoid the choice of other non-Condorcet allocations in $\varphi^N(P_S, k)$ (because these may be dominated by allocations outside of $\varphi^N(P_S, k)$).*

6. Final remarks.

We have proven that social choice correspondences satisfying some of our quite demanding requirements do exist. We have also proven that the requirement of Nash stability cannot be strengthened to group Nash stability. We have not provided characterization results, but just examples involving specific rules. Their construction combines a first choice of decisions satisfying basic necessary conditions, followed by a further choice among them (through the Simpson rule) in order to get a Condorcet rule. Our use of the Simpson rule is not essential; other extensions of majority, also respecting the Condorcet criterion, could have been used. However, not all extensions would have been appropriate. The Simpson rule is handy because it is easy to extend to the choice out of an infinity of alternatives (unlike the Copeland rule) and because it is simply based on pairwise comparisons between alternatives.

The above remarks suggest two directions for further research. One would be to find additional social choice correspondences satisfying our properties. The other would involve the axiomatic characterization of the rules we have proposed, or of larger classes containing them.

References

[1] Banerjee, S., Konishi, H., and Sonmez, T. (2001). "Core in a Simple Coalition formation game," *Social Choice and Welfare 18 (1), 135-53.*

[2] Barberà, S., and Beviá, C. (2002). "Self-Selection Consistent Functions," *Journal of Economic Theory 105 (2), 263-277.*

[3] Bogomolnaia, A., and Jackson M. (2002). "The Stability of Hedonic Coalition Structures," *Games and Economic Behavior 38 (2), 201-30.*

[4] Konishi, H., LeBreton, M., and Weber, S. (1998). "Equilibrium in a Finite Local Public Goods Economy," *Journal of Economic Theory* 79 (2), 224-44.

[5] Milchtaich, I., and Winter, E. (2002). "Stability and Segregation in Group Formation," *Games and Economic Behavior* 38, 318-346.