Search and bargaining in large markets with homogeneous traders

Clara Ponsatí*
Institut d'Analisi Economica - CSIC
and
CODE - Universitat Autònoma de Barcelona

September 2001

Abstract

We study pair-wise decentralized trade in dynamic markets with homogeneous, non-atomic, buyers and sellers that wish to exchange one unit. Pairs of traders are randomly matched and bargain a price under rules that offer the freedom to quit the match at any time. Market equilibria, prices and trades over time, are characterized. The asymptotic behavior of prices and trades as frictions (search costs and impatience) vanish, and the conditions for (non)convergence to walrasian prices are explored.

As a side product of independent interest, we present a self-contained theory of non-cooperative bargaining with two-sided, time-varying, outside options.

JEL Numbers: $C7$, $D4$, $D5$.
Keywords:Bargaining, Market Equilibrium, Outside Options.

*I am indebted to N. Burani, A. Rubinstein, J. Sákovics, J. Watson and an anonymous referee for useful comments. Financial support from the Spanish and Catalan Governments (grants PB98-870 and SGR98-0062 respectively) and the Spain-USA Cooperation Committee is gratefully acknowledged. The usual disclaimer applies. e-mail: clara.ponsati@uab.es

1 Introduction

This paper is a contribution to the theory of price formation in markets with *decentralized trade and bargaining,* focussing on the distinctive effect of *voluntary search*. We consider an economy à la Rubinstein and Wolinsky [1985] where homogneous buyers and sellers that wish to exchange one unit are randomly matched in pairs, and each pair bargains over the price at which to trade. Each period, agents that trade leave the market, agents that fail to trade stay and new agents enter. Markets that remain stationary as the flow of entry exactly matches the measure of consumated trades are only a particular case; generally non-stationary market equilibria must be considered.

Buyer-seller pairs play an alternating offer bargaining game. Upon rejection of an offer, both players may quit the match. If neither *decides* to search for an alternative partner, they remain matched and continue bargaining. Thus, bargaining is an infinite horizon game with two-sided outside options, the *market options*, whose value over time is determined by the strategies of traders throughout the market.

When a market is non-stationary, the value of the market options varies over time. Bargaining games with two-sided outside options that change over time have not been studied in the literature[1]. Consequently, the first step of our analysis is to characterize subgame perfect equilibria (spe) in such bargaining games. We provide such a characterization and subsequently build on it as we focus our attention on market equilibria.

A fundamental insight that we obtain in the analysis of the bargaining game is that (although the bargaining protocol permits an infinite sequence of alternating proposals) the set of spe always contains a profile of *ultimatum strategies*, namely one in which the proposer offers just the outside option, and the responder accepts.[2] This observation about individual bargaining pairs has important implications as we consider market equilibria: The use of ultimatum strategies by all traders in a market results in a unique sequence of market options. Under this sequence of market options,

[1]Models of bargaining in non-stationary environments, as Merlo and Wilson [1995], and Coles and Muthoo [2000], do not explicitely consider the possibility that bargainers opt out.

[2]The utimatum profile is generaly not the unique spe, though; other spe can prevail when the combined values of the anticipated outside options of both bargainers are not too great.

playing ultimatum strategies is a spe for all bargaining pairs. Therefore a *market equilibrium in ultimatum strategies* (meu) always exists.

For an important subset of environments the meu is the unique market equilibrium. In fact, in spite of the (potentially) large set of spe of the bargaining game, the endogenous nature of the market options drastically limits the range of market equilibria. In stationary markets, for instance, the meu prevails uniquely when search and bargaining cost are not too high (and coincides with the unique equilibrium in Rubinstein and Wolinsky [1985] as long as search costs are independent of bargaining cost). In non-stationary environments, if search costs are low and players are impatient, then the meu prevails as the unique market equilibrium as well. In general, however, if players are not too impatient a continuum of market equilibria exists and a non-degenerate interval of equilibrum prices can be supported.

In non-stationary markets, as frictions vanish, prices along the meu approach the walrasian price, an asymptotic prediction consistent with the results of Binmore and Herrero [1988] and Gale [1987]. Nevertheless, multiple equilibria can persist in the limit provided that frictions vanish along a path where search costs stay higher than the bargaining cost. Consequently, there exist sequences of market equilibria where, as frictions vanish, prices fail to converge to the walrasian price.

The non-cooperative foundations of walrasian and/or efficient outcomes in markets where trade takes place in decentralized pair-wise meetings is a fundamental issue that has been the object of important contributions and heated controversy.[3] Non-steady states in markets with non-atomic and homogeneous agents were studied by Binmore and Herrero [1988] under the assumption that all active agents participate in the matching process each period. Since search is not voluntary in their model, their bargaining pairs effectively play an ultimatum game, and thus their unique equilibrium is closely related to our meu. Gale [1987] , Wolinsky [1987] , Bester [1988] , and Muthoo [1993] present a variety of models in which players exit or take outside options voluntarily. In all these models, players must reject a proposal before they can search for outside options. As weak as such a requirement might appear, it is not: Shaked [1994] sharply makes this point in a model where a unique seller confronts many buyers. Because the seller can take outside options when a buyer rejects her proposal, a continuum of

[3]See Osborne and Rubinstein [1990] and Gale [2000] for excellent surveys of the literature.

market equilibria can be supported (provided that the search costs of the seller are neither too small nor too large); and the multiplicity remains as frictions vanish. Our results bridge the gap between these two strands of literature. We confirm that bargaining procedures matter, and that their relevance does not vanish as frictions do. At the same time, however, we show that in the absence of market power the scope of multiplicity is greatly reduced.

In as much as frictionless markets are identified with the walrasian outcome, we must conclude that this identification is tantamount to the assertion that agents play ultimatum strategies. While our findings support ultimatum games as a reduced form of more complex bargaining games, they also point out that this simplifying approach overlooks interesting (and asymptotically persistent) equilibria.

The remainder of the paper is organized as follows. Section 2 presents the model. Section 3 characterizes spe of alternating offer bargaining games with two-sided time-varying outside options, Section 4 characterizes market equilibria in non-stationary environments. Steady states are characterized in section 5. Section 6 concludes.

2 The model

Consider a market where trade is carried out by decentralized agreements of buyer-seller pairs that meet randomly over time and negotiate the price to trade one unit of an indivisible good. Time is measured in discrete equally spaced periods $t = 1, 2, ...$ and players discount the future with a common discount parameter δ, $0 < \delta < 1$. If a pair trades the good at price p in period t, the seller obtains payoff $p\delta^t$ and the buyer obtains $(1-p)\,\delta^t$. There is a continuum of buyers and sellers, with initial measures b_0 and s_0 and a constant flow of entry of buyers and sellers, $z = (b, s)\,, b \geq 0$ and $s \geq 0$. The state of the market at t, $z_t = (b_t,\, s_t)$, is the pair of measures of buyers and sellers searching for a partner at the beginning of each date t.

Without loss of generality, let us assume that sellers are initially the short side of the market, $b_0 = 1$ and $s_0 \leq 1$, and $b \geq s \geq 0$.[4] Given the initial populations, as traders leave the market in buyer-seller pairs, sellers

[4]Symmetric results hold in reverse case. Moreover, assuming that the initial short side is also the short side in the entry process is almost without loss of generality. If entry is higher in the initial short side, it remains the short side for only finitely many periods.

remain the short side of the market. Thus as $t \to \infty$, unless $s = b$ and this measure coincides with the measure of realized trades (i.e. as in Rubinstein and Wolinsky [1985]), the market cannot remain stationary and $\frac{s_t}{b_t} \to 0$ as $t \to 1$.

At each period $t \geq 0$, unmatched sellers and buyers meet at random. The probabilities of finding a trading counterpart at t are denoted π_{bt} and π_{st}, $0 < \pi_{bt} < \pi_{st} < 1$, respectively for buyers and sellers. For the sake of simplicity, when discussing non-stationary environments we assume that sellers find a match with a constant probability, $\pi_{st} = \gamma, 0 < \gamma < 0$, and that buyers are meet a seller with probability $\pi_{bt} = \gamma \frac{s_t}{b_t}$. Search frictions are said to vanish if $\pi_{st} \to 1$ and $\pi_{bt} \to \frac{s_t}{b_t}$ as $t \to \infty$.

When a buyer and a seller meet, they play a bargaining game to set a price for the good. Each bargaining game is characterized by a protocol that establishes the order and timing of proposals and the possibilities of exit to return to the market. Sellers are selected to act as first proposer with probability $\frac{1}{2}$. The bargaining games have two-sided outside options which are (potentially) time varying, endogenously determined as the present value of reentering the market.

Since there is a continuum of buyers and sellers, agents that return to the market find their previous partners with probability 0, so that, without loss of generality, we can assume that each player's strategy does not depend on the history of her previous meetings. Moreover, the value of the outside options to a player in a given bargaining pair is unaffected by her own (or the opponent's) bargaining strategy in that bargaining pair. Thus, from the point of view of each bargaining pair, market outside options can be taken as exogenous. Strategies and subgame perfect equilibria for the bargaining games are defined in the usual way.

A *market profile of strategies* specifies a bargaining strategy for each trader in the market. Given a market profile of strategies, an initial state z_0 and an entry flow z, the measure of agents that trade at t and a distribution of prices at which these trades occur are uniquely determined. For each trader, the *market option at t* at a market strategy profile is the expected value of opting out from a bargaining pair at t. We say that a *market profile of strategies sustains a market equilibrium* if the profile of (sequences of) market options that it generates is such that, for all buyer-seller pairs, the prescribed strategies are a spe of the bargaining game with outside options.

Addressing situations in which buyer-seller pairs play an alternating offers bargaining game with endogenous and time varying outside options demands a characterization of spe outcomes with exogenous time varying outside options. A detailed discussion of bargaining games with exogenous, time invariant, and certain options that can be taken at all times is in Ponsatí and Sákovics [1998][5]. In what follows we consider the more general case of two-sided outside options that are given by infinite sequences of known values.

3 Bargaining with two-sided, time varying, outside options

Consider an alternating proposal bargaining game à la Rubinstein, where upon any rejection both players can exit the game[6] and obtain payments given by a pair of infinite sequences $\{x_{1t}, x_{2t}\}$, where x_{1t} (x_{2t}) denotes the outside option at t of the player chosen to be the first proposer (responder).

We assume that the sequences of outside options satisfy $x_{it} \geq \delta x_{it+1}$ for both players and all t. This is a most natural assumption: a player always has the "option at t" of waiting for the option that comes at $t+1$, and this option has value δx_{it+1} at t, thus δx_{it+1} must be a lower bound of the outside options of player i for all t.

Observe that if $x_{1t} + x_{2t} > 1$ at some t, equilibrium requires that players take their options at that t. Then the game turns effectively into one with a finite horizon, and the (unique) spe strategies are obtained by a straightforward backward induction. Consequently, in what follows we restrict attention to the case that $x_{1t} + x_{2t} \leq 1$ for all t.

The key observation that drives our results is the following simple but powerful lemma.

Lemma 1 *If player 1 proposes at t, immediate agreement on $(1 - x_{2t}, x_{2t})$ is an outcome that can be supported by an spe.*

Proof: Consider the following strategies: if Player i is the proposer he always asks for $1 - x_{jt}$; the responder accepts any proposal that is not worse than the (candidate) equilibrium proposal; if the proposer asks for more, then the responder rejects and takes her outside option; if the responder

[5]The case of uncertain outside options is discussed in Ponsatí and Sákovics [1999] and Ponsatí and Sákovics [2001].

[6]It does not matter whether the decisions to exit are sequential of simultaneous.

does not accept a proposal the proposer opts out. It is straightforward to verify that $x_{it} \geq \delta x_{it+1}$ implies that these strategies constitute an spe. ■

The intuition behind Lemma 1 is the following. Given that next period's proposer is assumed to have a large bargaining power, the current proposer expects little if bargaining continues and thus her threat to take the outside option upon rejection is credible. If options were available for one player only, her threat to opt out would not be credible unless options were sufficiently valuable because she would obtain a reasonable payoff even as a responder. In contrast, when both players can opt out, they both can commit to opting out if their proposals are rejected, thus depriving their opponent of all their bargaining power as responders - except for the one given to them by the outside option - turning each other's commitment to opt out credible.

If the outside options are sufficiently valuable, the unique best response upon an hypothetical rejection is to opt out and thus Lemma 1 describes the unique spe outcome. Alternatively, for options of low value, opting out may not be the best response and the set of spe outcomes may contain strategy profiles in which responders attain a share above their outside option.

Consider an spe and denote the expected value of acting as the responder at t by v_{it}^r. It is necessary that, for all t, $v_{it}^r \leq \delta \min\left\{(1 - x_{jt+1}), (1 - \frac{x_{jt}}{\delta})\right\}$ since i, as a responder, cannot expect more than the discounted value of the maximum share that can be obtained acting as the proposer in the following period: A share $(1 - x_{jt+1})$ is an upper bound to the payoffs that i can expect, as a proposer, in period $t+1$. Still, this upper bound cannot be attained if it yields a share for j at $t+1$ that is below $\frac{x_{jt}}{\delta}$ (otherwise, j would take the outside option upon i's rejection at t). Now, $x_{jt} \geq \delta x_{jt+1}$ implies that $\min\left\{(1 - x_{jt+1}), (1 - \frac{x_{jt}}{\delta})\right\} = (1 - \frac{x_{jt}}{\delta})$. On the other hand observe that when $x_{it} > \delta v_{it+1}^r$ opting out at t is the unique best response of player i upon rejection of her proposal at t. Hence, $v_{it}^r \leq \delta(1 - \frac{x_{jt}}{\delta})$ for all t and $x_{it} \leq \delta v_{it+1}^r$ for some t are necessary to sustain spe outcomes different than that of Lemma 1.

Note that $x_{it} \leq \delta v_{it+1}^r$ and $v_{it+1}^r \leq \delta(1 - \frac{x_{jt+1}}{\delta})$ require that $x_{it} + \delta x_{jt+1} \leq \delta^2$. We say that the *sequence of outside options is δ-bounded* when $\{x_{1t}, x_{2t}\}$ satisfies $x_{it} \leq \delta(\delta - x_{jt+1})$ for $i, j = 1, 2$ and for all t. Our next result, Proposition 2, is a characerization of spe outcomes that crucially relies on δ-boundedness.

Proposition 2 *If the sequence of outside options is $\delta-$bounded, the out-*

comes that can be supported by an spe in a subgame starting at t are either immediate efficient agreements that give Player i, the proposer, a payoff in $[(1-\delta)+x_{it}, 1-x_{jt}]$ *or, for any period* $\tau > 0$*, a delayed agreement that gives Player i a share in* $\left[((1-\delta)+x_{1\tau})\,\delta^{-\tau}, (\delta+x_{1\tau})\delta^{(1-\tau)}\right]$*, whenever this interval is non-empty. If the sequence of outside options is not* $\delta-$*bounded the outcome characterized by Lemma 1 is the unique spe outcome.*

Proof: We prove our statements for spe with immediate agreements. Supporting delayed spe when there is a continuum of efficient spe outcomes is standard.

If there are equilibria different from that of Lemma 1, they must be such that players remain in the match upon rejection of an spe proposal. We have argued above that in a spe if i, acting as a proposer, stays in the match (upon an hypothetical rejection) the outside options must satisfy x_{it} $+\delta x_{jt+1} \leq \delta^2$.

Observe that if x_{it} $+\delta x_{jt+1} \leq \delta^2$ holds for all i, j and t, $t+1$, for all $s \in [(1-\delta)+x_{it}, 1-x_{jt}]$ there is an spe that yields a share s to i when the game starts at t and i acts as the proposer. The following strategies support one such SPE: as proposer, player i offers $(1-s)$ to j, as responder player j accepts only shares greater or equal to $(1-s)$; if player j rejects, she stays in the game and, acting as a proposer the following period, she plays the extreme spe of Lemma 1. If player j rejects proposal $(1-s)$ or greater, player i takes the outside option.

On the other hand the outcome of Lemma 1 prevails uniquely at t when i is proposing if $x_{it}+\delta x_{jt+1} > \delta^2$. ■

Our characterization is summarized by stating that, in the present environments, the outside option principle[7] - that outside options of small value are irrelevant and thus the outcome is the (unique) outcome of the game without options - fails: Although the bargaining game is not an ultimatum game an spe in ultimatum strategies always exists. We now turn attention to the implications of our results for market equilibria.

[7]See Shaked and Sutton [1984].

4 Market Equilibria

Consider a market profile where all traders use ultimatum strategies: as proposers, agents offer a share equal to their partners' market option, as responders they accept any offer of a share at least as great as their own market option, and in case of rejection all opt out. At this profile all matched pairs trade and the market options x_{st} and x_{bt} must satisfy[8]

$$x_{bt} = \delta\pi_{t+1}v_{bt+1} + \delta\left(1 - \pi_{t+1}\right)x_{bt+1},$$

and

$$x_{st} = \delta\gamma v_{st+1} + \delta\left(1 - \gamma\right)x_{st+1},$$

where v_{bt} and v_{st} denote the expected values of being in a bargaining pair at t, respectively for buyers and sellers. Under ultimatum strategies these expected values must solve

$$v_{bt} = \frac{1 + (x_{bt} - x_{st})}{2}$$

and

$$v_{st} = \frac{1 + (x_{st} - x_{bt})}{2},$$

and consequently the market options must solve the following system of difference equations:

$$\begin{array}{c} x_{bt} = \delta\left(\frac{\pi_{t+1}}{2}(1 - x_{st+1}) + (1 - \frac{\pi_{t+1}}{2})x_{bt+1}\right) \\ x_{st} = \delta\left(\frac{\gamma}{2}\left(1 - x_{bt+1}\right) + \left(1 - \frac{\gamma}{2}\right)x_{st+1}\right). \end{array} \tag{1}$$

The following is proved in the Appendix:

Lemma 3 *There is a unique sequence $\{x_{bt}, x_{st}\}$ solving (1). This unique solution can be written*

$$\begin{array}{c} x_{st} = \frac{\gamma}{2}\sum\limits_{\tau=0}^{\infty}\delta^{\tau+1}(1 - a_{t+1+\tau}) \\ x_{bt} = \frac{1}{2}\sum\limits_{\tau=0}^{\infty}\delta^{\tau+1}\pi_{t+1+\tau}(1 - a_{t+1+\tau}) \end{array} \tag{2}$$

[8] The subindex b in π_{bt} is omitted.

where $a_t = x_{bt} + x_{st}$ *uniquely solves*

$$a_t = \delta\left((1 - \Delta_{t+1}) + \Delta_{t+1} a_{t+1})\right), \tag{3}$$

with

$$\Delta_t = 1 - \frac{\pi_t + \gamma}{2}. \tag{4}$$

Remark 1 *The outside options at* t*,* x_{st} *and* x_{bt}*, are simply the present value of the surplus exceeding the market options that each agent expects to appropriate from* $t+1$ *onwards.*

Remark 2 *As* $t \to \infty$*,* $a_t \to \frac{\delta\gamma}{2(1-\delta)+\delta\gamma}$*,* $x_{st} \to \frac{\delta\gamma}{2-\delta\gamma}$ *and* $x_{bt} \to \frac{\delta 2(1-\gamma)}{2-\delta\gamma}$.

With a unique solution to (1) we are ready to prove that a unique market equilibrium in ultimatum strategies exists. With the value of the market options at hand, the result is an immediate consequence of Lemma 1.

Proposition 4 *There is a market equilibrium in ultimatum strategies.*

Proof: Assume that all agents in the market play ultimatum strategies, so that the market options are uniquely given as in Lemma 3. Observe that since $a_t < 1$ (which is shown to hold in the proof of Lemma 3), equation (1) implies that $x_{st} \geq \delta x_{st+1}$ and $x_{bt} \geq \delta x_{bt+1}$. Thus, by Lemma 1, the pair of ultimatum strategies is an spe of the bargaining game for each match and thus the market profile constitutes a market equilibrium. ■

Are there other market equilibria? That is, are there market equilibria in which some traders use strategies that yield market options other than those of Lemma 3? In principle, since bargaining games with two-sided outside options may have other spe, it is possible that, in addition to the meu, other market profiles can be sustained as market equilibria. To address these questions we set upper and lower bounds to the payoffs that traders can attain in a market equilibrium.

Let $\overline{v}_{st}$ and $\overline{v}_{bt}$ ($\underline{v}_{st}$ and $\underline{v}_{bt}$) denote the supremum (infimum) of the payoffs that sellers and buyers can obtain, respectively, at any market equilibrium. At a market equilibrium, for each buyer-seller pair the bargaining game starting in period t yields expected spe values v_{st} and v_{bt}, in $[\underline{v}_{st}, \overline{v}_{st}]$ and $[\underline{v}_{bt}, \overline{v}_{bt}]$.

Observe that in order to evaluate the extreme values that sellers and buyers can attain at each t in a market equilibrium there is no loss of generality if we restrict attention to uniform strategy profiles. For uniform profiles, at each t, all buyers (and similarly all sellers) obtain the same market option if they abandon a bargaining match, and these must lie in intervals $[\underline{x}_{st}, \overline{x}_{st}]$ and $[\underline{x}_{bt}, \overline{x}_{bt}]$ such that

$$\begin{aligned} \overline{x}_{st} &= \delta\gamma\overline{v}_{st+1} + \delta(1-\gamma)\overline{x}_{st+1} \\ \underline{x}_{st} &= \delta\gamma\underline{v}_{st+1} + \delta(1-\gamma)\underline{x}_{st+1} \\ \overline{x}_{bt} &= \delta\pi_{t+1}\overline{v}_{bt+1} + \delta(1-\pi_{t+1})\overline{x}_{bt+1} \\ \underline{x}_{bt} &= \delta\pi_{t+1}\underline{v}_{bt+1} + \delta(1-\pi_{t+1})\underline{x}_{bt+1}. \end{aligned} \tag{5}$$

If the meu is not the unique market equilibrium, then at the market equilibrium where sellers (buyers) obtain their best market option $\overline{x}_{st}$ ($\overline{x}_{bt}$), buyers (sellers) must attain their worst possible market option $\underline{x}_{bt}(\underline{x}_{st})$. Consider a market profile that gives sellers' market options $\overline{x}_{st} > \underline{x}_{st}$ while the buyers' options are $\underline{x}_{bt} < \overline{x}_{bt}$. At such such profile buyers cannot be playing ultimatum strategies: By Proposition 2, market options $\overline{x}_{st}$ and $\underline{x}_{bt}$, different from the meu market options, can be equilibrium options only if the following are spe strategies for each bargaining pair: as proposer, the seller gives the buyer only $\underline{x}_{bt}$ and, credibly, threatens to take the market option upon rejection; on the other hand, when buyers act as first proposers they offer sellers their continuation value within the pair because, given that their market option is just $\underline{x}_{bt}$, they cannot credibly threaten to opt out. That is, depending on whether the seller or the buyer is selected as first proposer, one extreme or the other of the interval of spes is played. Using Proposition 2, and observing that symmetric conditions must hold in a market equilibrium where buyers obtain $\overline{x}_{bt}$ while sellers obtain $\underline{x}_{st}$, the extreme expected values can be written in terms of the extreme options as

$$\begin{aligned} \overline{v}_{st} &= \tfrac{1+\delta}{2} - \underline{x}_{bt} \\ \underline{v}_{bt} &= \tfrac{1-\delta}{2} + \underline{x}_{bt} \\ \underline{v}_{st} &= \tfrac{1-\delta}{2} + \underline{x}_{st} \\ \overline{v}_{bt} &= \tfrac{1+\delta}{2} - \underline{x}_{st}, \end{aligned} \tag{6}$$

and substituting (6) in (5) we obtain

$$\begin{array}{c}\overline{x}_{st} = \gamma\delta[\frac{1+\delta}{2} - \underline{x}_{bt+1}] + (1-\gamma)\delta\overline{x}_{st+1} \\ \underline{x}_{st} = (1-\delta)\frac{\delta\gamma}{2} + \delta\underline{x}_{st+1} \\ \overline{x}_{bt} = \pi_{t+1}\delta[\frac{1+\delta}{2} - \underline{x}_{st+1}] + (1-\pi_{t+1})\delta\overline{x}_{bt+1} \\ \underline{x}_{bt} = (1-\delta)\frac{\delta\pi_{t+1}}{2} + \delta\underline{x}_{bt+1}.\end{array} \tag{7}$$

It is straightforward to verify that equation (7) yields a unique sequence taking values in $[0,1]^4$. Furthermore, recursive forward induction yields

$$\begin{array}{c}\underline{s}_t = \frac{\delta\gamma}{2} \\ \underline{b}_t = \frac{\delta(1-\delta)}{2}B_t\end{array} \tag{8}$$

where

$$B_t \equiv \sum_{i=0}^{\infty} \pi_{t+i}\delta^i. \tag{9}$$

Substituting (8) yields

$$\begin{array}{c}\overline{x}_{st} = \gamma\delta S_t + (1-\gamma)\delta\overline{x}_{st+1} \\ \overline{x}_{bt} = \pi_{t+1}\delta[\frac{1+\delta(1-\gamma)}{2}] + (1-\pi_{t+1})\delta\overline{x}_{bt+1}\end{array} \tag{10}$$

where

$$S_t \equiv \frac{1}{2}[1 - \delta^2(1-\delta)\sum_{i=0}^{\infty}\pi_{t+i}\delta^i]. \tag{11}$$

Substitution in (10) yields

$$\begin{array}{c}\overline{s}_t = \gamma\delta\sum_{k=0}^{\infty}\delta^k(1-\gamma)^k S_{t+k}, \\ \overline{b}_t = \delta\frac{1+\delta(1-\gamma)}{2}\sum_{k=1}^{\infty}\pi_{t+k}\delta^{k-1}\prod_{j=1}^{k-1}(1-\pi_{t+k})^k.\end{array} \tag{12}$$

Observe moreover, that $\underline{s}_t < \overline{s}_t$ and $\underline{b}_t < \overline{b}_t$. The first inequality is immediate. To check the second it suffices to note that

$$\underline{b}_t = \frac{\delta(1-\delta)}{2}\sum_{i=0}^{\infty}\pi_{t+i}\delta^i < \frac{\delta(1-\delta)}{2}\pi_{t+1}\sum_{i=0}^{\infty}\delta^i = \frac{\pi_{t+1}\delta}{2}$$

and

$$\overline{b}_t = \pi_{t+1}\delta[\frac{1+\delta(1-\gamma)}{2}] + (1-\pi_{t+1})\delta\overline{x}_{bt+1} > \frac{\pi_{t+1}\delta}{2}.$$

The proof of the following propositions is now complete.

Proposition 5 *At any market equilibrium, the market options at each t must lie in the intervals $[\underline{s}_t, \overline{s}_t]$ and $[\underline{b}_t, \overline{b}_t]$, where $\{\underline{s}_t, \overline{s}_{t.}\underline{b}_t, \overline{b}_t\}$ is the unique solution to (7).*

Our next result, Proposition 6, establishes a necessary and sufficient condition to sustain all the payoffs within the bounds set in Proposition 5 in a market equilibrium. This condition demands that the sequences of pairs of market options that combine the upper bound for one side of the market and the lower bound for the other side are δ-bounded. Furthermore, Proposition 7 establishes the existence of boundary values separating markets with a unique market equilibrium, the meu, from markets with a continuum of market equilibria.

Proposition 6 *If the sequences $\{\overline{s}_t, \underline{b}_t\}$ and $\{\overline{b}_t, \underline{s}_t\}$ are δ-bounded, then for each sequence $\{x^{\epsilon}_{st}\}$, defined by $x^{\epsilon}_{st} = \epsilon\underline{s}_t + (1-\epsilon)\overline{s}_t$ for each $0 \leq \epsilon \leq 1$, there is a sequence $\{x_{bt}\}$, with $x_{bt} \in [\underline{b}_t, \overline{b}_t]$, such that $\{x^{\epsilon}_{st}, x_{bt}\}$ are market equilibrium options. Otherwise the meu is the unique market equilibrium.*

Proof: Consider bargaining games with outside options $\overline{s}_t$ and $\underline{b}_t$. Observe that $\overline{s}_t \geq \delta\overline{s}_{t+1}$, $\underline{b}_t \geq \delta\ \underline{b}_{t+1}$. By Proposition 2, with δ-bounded outside options $\{\overline{s}_t, \underline{b}_t\}$, all $v_{st} \in [\frac{1-\delta}{2} + \overline{s}_t, \frac{1+\delta}{2} - \underline{b}_t]$ and $v_{bt} \in [\frac{1-\delta}{2} + \underline{b}_t, \frac{1+\delta}{2} - \overline{s}_t]$ are expected gains attainable in spe. Pick a profile where $v_{st} = \frac{1+\delta}{2} - \underline{b}_t$ and $v_{bt} = \frac{1-\delta}{2} + \underline{b}_t$, and observe that the market options that arise under this profile are $\overline{s}_t$ and $\underline{b}_t$. Symmetrically we may select market profiles that yield options $\overline{b}_t$ to the buyers and $\underline{s}_t$ to the sellers and that are spe of games with these outside options iff $\{\overline{b}_t, \underline{s}_t\}$ is δ-bounded. Hence, if $\{\overline{s}_t, \underline{b}_t\}$ and $\{\overline{b}_t, \underline{s}_t\}$ are δ-bounded they are market equilibrium sequences of options.

On the other hand, assume that $\{\overline{s}_t, \underline{b}_t\}$ or $\{\overline{b}_t, \underline{s}_t\}$ is not δ-bounded, then the bargaining game with outside options has a unique spe, namely the ultimatum strategies spe, which cannot yield $\{\overline{s}_t, \underline{b}_t\}$ nor $\{\overline{b}_t, \underline{s}_t\}$ as a market equilibrium sequence of options.

Let us now explore how to sustain other market equilibria with sequences of outside options $\{x_{st}, x_{bt}\}$, $x_{st} \in [\underline{s}_t, \overline{s}_t]$ and $x_{bt} \in [\underline{b}_t, \overline{b}_t]$, when $\{\overline{s}_t, \underline{b}_t\}$

and $\{\overline{b}_t, \underline{s}_t\}$ are δ-bounded. By Proposition 2, in a bargaining game with δ-bounded outside options $\{x_{st}, x_{bt}\}$, all $v_{st} \in [\frac{1-\delta}{2} + x_{st}, \frac{1+\delta}{2} - x_{bt}]$ and $v_{bt} \in [\frac{1-\delta}{2} + x_{bt}, \frac{1+\delta}{2} - x_{st}]$ are expected gains attainable in spe. On the other hand, if $\{x_{st}, x_{bt}\}$ is a market equilibrium sequence of options sustained with a profile that yields expected gains v_{st} to all the sellers and v_{bt} to all the buyers, then the outside options are uniquely given by

$$x_{st} = \delta\gamma v_{st+1} + \delta(1-\gamma)x_{st+1} = \delta\gamma \sum_{i=0}^{\infty} v_{st+i}(1-\gamma)^i \delta^i,$$

$$x_{bt} = \delta\pi_{t+1}v_{bt+1} + \delta(1-\pi_{t+1})x_{bt+1} = \delta \sum_{i=1}^{\infty} v_{bt+i}\pi_{t+i} \prod_{k=1}^{i}(1-\pi_{t+i-k})\delta^i.$$

Consider a sequence of expected values $v_{st}^{\epsilon} = \underline{v}_{st} + \epsilon(\overline{v}_{st} - \underline{v}_{st})$ and $v_{bt}^{\epsilon} = 1 - v_{st}^{\epsilon}$, and write x_{st}^{ϵ} and x_{bt}^{ϵ} to denote the associated market options. Since $x_{st}^{\epsilon} = \epsilon\underline{x}_{st} + (1-\epsilon)\overline{x}_{st}$ is a linear combination of $\underline{x}_{st}$ and $\overline{x}_{st}$, and since x_{bt}^{ϵ} and x_{st}^{ϵ} are bounded above by $\overline{x}_{bt}$ and $\overline{x}_{st}$, the necessary inequalities for δ-boundedness of $\{x_{st}^{\epsilon}, x_{bt}^{\epsilon}\}$ hold. By Proposition 2 all values $v_{st} \in [\frac{1-\delta}{2} + x_{st}^{\epsilon}, \frac{1+\delta}{2} - x_{bt}^{\epsilon}]$ and $v_{bt} \in [\frac{1-\delta}{2} + x_{bt}^{\epsilon}, \frac{1+\delta}{2} - x_{st}^{\epsilon}]$ are attainable as spe payoffs in a bargaining game with outside options $\{x_{st}^{\epsilon}, x_{bt}^{\epsilon}\}$. Observe that

$$v_{st}^{\epsilon} + x_{bt}^{\epsilon} \leq \overline{v}_{st} + \overline{x}_{bt} \leq \frac{1+\delta}{2}$$

and

$$v_{bt}^{\epsilon} + x_{st}^{\epsilon} \leq \overline{v}_{bt} + \overline{x}_{st} \leq \frac{1+\delta}{2}.$$

Hence

$$v_{st}^{\epsilon} \in [\frac{1-\delta}{2} + x_{st}^{\epsilon}, \frac{1+\delta}{2} - x_{bt}^{\epsilon}]$$

and

$$v_{bt}^{\epsilon} \in [\frac{1-\delta}{2} + x_{bt}^{\epsilon}, \frac{1+\delta}{2} - x_{st}^{\epsilon}].$$

Therefore $\{x_{st}^{\epsilon}, x_{bt}^{\epsilon}\}$ is a market equilibrium sequence of options. ∎

Remark 3 *Market equilibria that yield intermediate sequences of options* $\{x_{st}^{\epsilon}\}$ *to the sellers have a direct interpretation: they are the result of a uniform profile of intermediate strategies. An alternative interpretation is that sellers obtain* $\{x_{st}^{\epsilon}\}$ *because buyers play a non uniform profile: a proportion* ϵ *of the buyers have market options* $\overline{b}_t$*, and the remainder* $(1-\epsilon)$ *that have market options* $\underline{b}_t$*; and sellers play a uniform profile that responds optimally to the strategies (and the options) of the type of buyer that they confront in the bargaining game.*

Our next result characterizes parameter configurations for which the meu is the unique market equilibrium.

Proposition 7

$$\left.\begin{array}{r} i)\textit{For each } (z_0,z,\gamma) \textit{ there is a } \delta^*, 0<\delta^*<1, \\ ii)\textit{For each } (z_0,z,\delta) \textit{ there is a } \gamma^*, \gamma^*<1, \end{array}\right\} \textit{such that,}$$

$$\textit{iff} \left\{\begin{array}{c} \delta \leq \delta^*, \\ \gamma \geq \gamma^*, \end{array}\right\} \textit{the meu is the unique market equilibrium.}$$

Proof: i) Observe that $\underline{s}_t$, $\overline{s}_t$, $\underline{b}_t$, and $\overline{b}_t$ are continuous in all parameters. For each configuration $(z_{0,}z,\gamma)$, consider

$$\begin{aligned} \lim_{\delta\to 1} \underline{s}_t &= \frac{\gamma}{2} \\ \lim_{\delta\to 1} \underline{b}_t &= 0 \\ \lim_{\delta\to 1} \overline{s}_t &= \gamma \\ \lim_{\delta\to 1} \overline{b}_t &= \frac{2-\gamma}{2}\sum_{i=1}^{\infty}\pi_{t+i}\prod_{k=1}^{i}(1-\pi_{t+i-k}), \end{aligned}$$

and observe that all the inequalities for δ-boundedness hold strictly in the limit. By continuity, for all parameter configurations, there is $\delta<1$ such that all inequalities are preserved. Hence, for each parameter configuration there is a $\delta^*<1$ such that $[\underline{s}_t,\overline{s}_t]$ and $\left[\underline{b}_t,\overline{b}_t\right]$ are the intervals of market equilibria outside options iff $\delta>\delta^*$.

ii) Simply observe that for each parameter configuration with $\delta<1$, there is a $\gamma^*<1$, (possibly $\gamma^*>\delta$!), such that $\{\overline{s}_t,\underline{b}_t\}$ is not $\delta-$ bounded. Recall that

$$\overline{s}_t = \gamma\delta S_t + (1-\gamma)\delta\overline{s}_{t+1}$$

and

$$\underline{b}_t = \frac{\delta(1-\delta)}{2} B_t.$$

Observe that

$$\overline{s}_t > \frac{\delta\gamma}{2}\left[1 + \delta\left(1 - B_t\right) + \delta^2 B_t\right]$$

implying that

$$\overline{s}_t + \delta \underline{b}_{t+1} > \frac{\delta\gamma}{2}\left[1 + \delta\left(1 - B_t\right) + \delta^2 B_t\right] + \frac{\delta^2(1-\delta)}{2} B_{t+1}$$

and

$$\frac{\delta\gamma}{2}\left[1 + \delta\left(1 - B_t\right) + \delta^2 B_t\right] + \frac{\delta^2(1-\delta)}{2} B_{t+1} \geq \frac{\delta\gamma(1+\delta)}{2}.$$

For each $\delta < 1$, if $\gamma \geq \frac{2\delta}{1+\delta}$ the previous inequalities imply that

$$\overline{s}_t + \delta \underline{b}_{t+1} > \frac{\delta\gamma(1+\delta)}{2} \geq \delta^2.$$

■

It is now immediate that as players become arbitrarily patient and search frictions vanish, expected prices in a market equilibrium may not approach 1, the walrasian price.

Corollary 8 *There exist sequences $\{\delta_n, \gamma_n\} \rightarrow \{1, 1\}$ such that for each term in the sequence a market equilibrium where sellers attain outside options*

$$\underline{s}_{nt} = \frac{\delta_n \gamma_n}{2}$$

exist, and moreover

$$\lim_{(\delta_n, \gamma_n) \rightarrow (1,1)} E(p_n) \leq \frac{3}{4} < 1.$$

5 Steady States

In this section we look at the special case of markets that remain at a steady state $(1, s_0)$. In a *stationary market equilibria* all relevant information about the market at each date t is summarized by δ, $0 < \delta < 1$, and the constant matching probabilities of buyers and sellers π_b and π_s, $0 < \pi_b \leq \pi_s < 1$.

In a stationary market equilibrium, the (endogenously given) outside options of the sellers and the buyers at each t, x_{st} and x_{bt}, must lie in time independent intervals $[\underline{x}_s, \overline{x}_s]$ and $[\underline{x}_b, \overline{x}_b]$.

Proposition 9 characterizes market equilibria in the special case of stationary markets. Here the extent of multiplicity is greatly reduced: if frictions are not too great, a unique steady state market equilibrium à la Rubinstein and Wolinsky [1985] prevails. The basic intuition is that the shares that players obtain are determined by the value of the outside opportunities and in a market where trades and entry keep the measure of buyers and sellers constant these outside opportunities offer positive prospects even to the short side of the market.

Proposition 9 *Assume that matches, trades and entry keep the state of the market constant at* $(s_0, 1)$*. For each* δ *,* π_s *and* π_b *there is* $\pi_\delta < 1$*, such that the following holds.*
I) *If* $\pi_s \leq \pi_\delta$ *there is a continuum of spe for each bargaining pair. The market options of the sellers and the buyers lie in the intervals* $[\underline{x}_s, \overline{x}_s]$ *and* $[\underline{x}_b, \overline{x}_b]$ *at each t, where:*

$$\begin{aligned} \underline{x}_s &= \frac{\delta\pi_s}{2} \\ \overline{x}_s &= \frac{\delta\pi_s}{2}\cdot\frac{1+\delta(1-\pi_b)}{(1-\delta)+\delta\pi_s} \\ \underline{x}_b &= \frac{\delta\pi_b}{2} \\ \overline{x}_b &= \frac{\delta\pi_b}{2}\cdot\frac{1+\delta(1-\pi_s)}{(1-\delta)+\delta\pi_b}. \end{aligned} \tag{13}$$

II) *Otherwise each buyer-seller pair bargaining game has a unique spe in which the proposer gives the responder the value of her market option. These options are:*

$$\begin{aligned} x_s &= \frac{\frac{\delta}{2}\pi_s}{(1-\delta)+\frac{\delta}{2}(\pi_s+\pi_b)} \\ x_b &= \frac{\frac{\delta}{2}\pi_b}{(1-\delta)+\frac{\delta}{2}(\pi_s+\pi_b)}. \end{aligned} \tag{14}$$

Proof: See Appendix ■

The present characterization does not rely on a specific functional form for the matching technology, and consequently a greater variety of asymptotic outcomes can be explored.

Observe that

$$\lim_{\delta \to 1} x_s = \frac{\pi_s}{\pi_b + \pi_s}.$$

If we let $\pi_b = \pi_s s_0$, and assume that for each state of the market the probability that sellers find a match is unrelated to the cost of bargaining δ, then we obtain that $\lim_{\delta \to 1} x_s = \frac{s_0}{s_0+1}$. That is the outcome highlighted by Rubinstein and Wolinsky [1985]: each side of the market appropriates a surplus share proportional to the relative measure of the other.

Other asymptotic outcomes are possible, though, if the matching probabilities are related to the delay caused by searching for a new partner in the market. For instance, agents might be constrained to contacting at most one potential partner per period, and only a portion of potential partners are available. In this case, δ and π_s are no longer independent, and π_s may be a function of δ. As an example, let $\pi_b = \delta^{\frac{k}{s_0}}$ and $\pi_s = \delta^k$. Now

$$x_s = \frac{\frac{\delta^{k+1}}{2}}{(1-\delta) + \frac{\delta^{k+1}}{2}\left(\delta^{\frac{k+1}{1+s_0}}\right)},$$

$$x_b = x_s \delta^{\frac{k}{1+s_0}}.$$

The market approaches a frictionless market in the limit as $\delta \to 1$ and we obtain $\lim_{\delta \to 1} x_s = \lim_{\delta \to 1} x_b = \frac{1}{2}$. In this example, the relative scarcity of sellers in the market is offset by the fact that search costs are always higher than the cost of delaying the agreement in bargaining. Therefore, the relative bargaining power of both buyers and sellers is symmetric and the 50-50 split prevails (uniquely) regardless of the relative sizes of the two populations.

6 Conclusions

Bargaining games with two-sided, varying, outside options are more complicated than standard bargaining games, and our predictions about isolated bargaining pairs are substantially less sharp. In the context of a market, however, the added complexity of the bargaining game is not an obstacle for a complete and explicit characterization of equilibria, even in non-stationary markets. Here, the outside option principle is no longer at work, all outside options are relevant and the range of prices depends fundamentally on the state of the market. When non-stationary market equilibria are unique, they are sustained by profiles of ultimatum strategies and they converge to the walrasian outcome as frictions vanish. However, since the bargaining mechanism admits a continuum of spe, a continuum of market equilibria may prevail and, as frictions vanish, convergence to the walrasian price is not assured. Therefore, in markets where buyer-seller pairs play alternating offer games with freedom to search, the convergence to walrasian outcomes when frictions vanish cannot be assured unless the restriction to ultimatum strategies is exogenously imposed. We must conclude that bargaining procedures matter, and that their relevance does not vanish as frictions do.

Further research is needed to extend the present work to markets with heterogeneous buyers and sellers. In homogeneous markets, even at non-walrasian prices, efficiency (constrained to the feasibility imposed by the matching process) is ensured as long as all realized matches reach immediate agreement. As this no longer holds in non-homogeneous markets, it is an open question whether inefficient market equilibria can be supported.

Exploration of heterogeneous markets should also illuminate our intuition about stationary markets. As long as entry is exogenous, homogeneous markets approach a stationary state only if equal numbers of buyers and sellers enter. A meaningful model of steady states in markets will have to address endogenous entry as well as endogenous exit, demanding that heterogeneity plays an explicit role.

References

[1] Bester, H. [1988]: Bargaining, Search Costs and Equilibrium Price Distributions, Review of Economic Studies, 55, 201-214.

[2] Binmore, K. G. and M. J. Herrero [1988]: Matching and Bargaining in Dynamic Markets, Review of Economic Studies, 55, 17-31.

[3] Coles, M. and A. Muthoo [2000]: Bargaining Equilibrium in a Non-Stationay Environment, University of Essex discussion paper.

[4] Gale, D. [1987]: Limit Theorems for Markets with Sequential Bargaining, Journal of Economic Theory, 43, 20-54.

[5] Gale, D. [2000]: Strategic Foundations of General Equilibrium:Dynamic Matching and Bargaining Games, Cambridge University Press.

[6] Merlo, A. and C. Wilson [1995]: A Stochastic Model of Sequential Bargaining with Complete Information, Econometrica, 63, 371-99.

[7] Muthoo, A. [1993]: Sequential Bargaining and Competition, Economic Theory, 3, 353-63

[8] Osborne, M. and A. Rubinstein [1990]: Bargaining and Markets, Academic Press, San Diego.

[9] Ponsatí, C. and J. Sákovics [1998]: Rubinstein Bargaining with Two-Sided Outside Options, Economic Theory, 11, 667-672.

[10] Ponsatí, C. and J. Sákovics [1999]: Bargaining in a Changing Environment, in Alkan, A. , C.D. Aliprantis and N.C. Yannelis, Current Trends in Economics: Theory and Applications (Studies in Economic Theory, vol 9) Springer, Berlin Heidelberg New York, 398-404.

[11] Ponsatí, C. and J. Sákovics [2001]: Randomly Available Outside Options in Bargaining, Spanish Economc Review, 3, forthcoming.

[12] Rubinstein, A. and A. Wolinsky [1985]: Equilibrium in a Market with Sequential Bargaining, Econometrica, 53, 1133-1150.

[13] Rubinstein, A. and A. Wolinsky [1990]: Decentralized Trading, Strategic Behavior and the Walrasian Outcome, Review of Economic Studies, 57, 63-79.

[14] Shaked, A.[1994]: Opting Out: Bazaars versus Hi-Tech Markets, Investigaciones Economicas,18, 421-432.

[15] Shaked, A. and J. Sutton [1984]: Involuntary Unemployment as Perfect Equilibrium in a Bargaining Model, Econometrica, 52, 1351-1364.

[16] Wolinsky, A. [1987]: Matching, Search and Bargaining, Journal of Economic Theory, 42, 311-333.

A Appendix

Proof of Lemma 3:

Let $a_t = x_{bt} + x_{st}$ it is imediate that $\{a_t\}$ must solve 3.

Since all matches reach trade the measures of traders active at each date and the probabilities that searchers find a partner are uniquely determined from the exogenous parameters of the market (z_0, z, γ), and so is the sequence $\{\Delta_{t+1}\}$. By forward substitution of the sequence $\{\Delta_{t+1}\}$ we obtain that a_t must satisfy

$$a_t = \sum_{\tau=0}^{\infty} \delta^{\tau+1}(1 - \Delta_{t+1+\tau})\Psi_{t+\tau} \tag{15}$$

where

$$\Psi_t = 1, \quad \text{and for } k \geq 1, \quad \Psi_{t+k} = \Psi_{t+k-1}\Delta_{t+k}.$$

Since $\Delta_t < 1 - \frac{\gamma}{2}$, and $\Psi_{t+k} < \gamma^{k-1}$, the right hand side of (15) converges and

$$a_t < \sum_{\tau=0}^{\infty} \delta^{\tau+1}(1 - \frac{\gamma}{2})\gamma^{\tau-1} \equiv A_0$$

for all t. Moreover,

$$a_t < \delta\left((1 - \Delta_{t+1}) + \Delta_{t+1}A_0\right)$$

and hence

$$a_t < \frac{\delta}{2}\left(\gamma + (2 - \gamma)\, A_0\right) \equiv A_1.$$

Observe that the sequence

$$A_k = \frac{\delta}{2}\left(\gamma + (2 - \gamma)\, A_{k-1}\right)$$

is such that for all t, and k, $a_t < A_{k+1}$, and

$$a_t \leq \lim_{k\to\infty} A_k = \frac{\delta\gamma}{2\,(1-\delta) + \delta\gamma} < \delta.$$

Therefore (15) yields a unique solution to (3) and $a_t \in (0, \delta)$ for all t. It is now straightforward that the unique solution to (1) must satisfy (2). ■

Proof of Proposition 9:

For each buyer-seller pair the bargaining game starting in period t yields spe values v_{st} and v_{bt}, in $[\underline{v}_{st}, \overline{v}_{st}]$ and $[\underline{v}_{bt}, \overline{v}_{bt}]$. Observe that, since the matching probabilities are constant for buyers and sellers, this intervals do not depend on t. The market options must lie in $[\underline{x}_{st}, \overline{x}_{st}]$ and $[\underline{x}_{bt}, \overline{x}_{bt}]$ that must also be independent of t. Therefore

$$\begin{aligned}
\overline{x}_s &= \delta\pi_s\overline{v}_s + \delta(1-\pi_s)\overline{x}_s \\
\underline{x}_s &= \delta\pi_s\underline{v}_s + \delta(1-\pi_s)\underline{x}_s \\
\overline{x}_b &= \delta\pi_b\overline{v}_b + \delta(1-\pi_b)\overline{x}_b \\
\underline{x}_b &= \delta\pi_b\underline{v}_b + \delta(1-\pi_b)\underline{x}_b.
\end{aligned}$$

In order to characterize the equilibria that yield the supremum (infimum) value to sellers (buyers) there is no loss of generality if we restrict attention to uniform profiles. For these class of strategy profiles, all buyers and sellers expect the same value if they return to the market, that is, they have the same market option. By Lemma 1 there is always an equilibrium in which the proposer gives the responder just the outside option. Since both players act as first proposer (responder) with probability $\frac{1}{2}$, we have that:

$$\begin{aligned}
\overline{v}_s &= \tfrac{1}{2}[(1+\delta) - 2\underline{x}_b] \\
\underline{v}_s &= \tfrac{1}{2}[(1-\delta) + 2\underline{x}_s] \\
\overline{v}_b &= \tfrac{1}{2}[(1+\delta) - 2\underline{x}_s] \\
\underline{v}_b &= \tfrac{1}{2}[(1-\delta) + 2\underline{x}_b].
\end{aligned}$$

Substition yields (13).

The bounds set in (13) characterize the set of spe provided that: $\overline{x}_s + \delta\underline{x}_b \leq \delta^2$, $\overline{x}_b + \delta\ \underline{x}_s \leq \delta^2$, $\underline{x}_s + \delta\overline{x}_b \leq \delta^2$ and $\underline{x}_b + \delta\overline{x}_s \leq \delta^2$. It is easy to check that $\overline{x}_s + \delta\underline{x}_b \leq \delta^2$ is the most stringent of these inequalities. Thus, as soon as $\overline{x}_s + \delta\underline{x}_b > \delta^2$ we have a unique spe in which proposers offer the responder just the (unique!) outside option. Observe that $\overline{x}_s + \delta\underline{x}_b > \delta^2$ if and only if

$$\frac{\delta\pi_s}{(1-\delta)+\delta\pi_s} \geq \frac{\delta\,(2-\pi_b)}{1+\delta\,(1-\pi_b)}$$

and note that $\frac{\delta\pi_\delta}{(1-\delta)+\delta\pi_\delta} = \frac{\delta(2-\pi_b)}{1+\delta(1-\pi_b)}$ yields $\pi_\delta < 1$ for all π_b and all $\delta \leq 1$. Hence $\overline{x}_s + \delta\ \underline{x}_b > \delta^2$ if and only if $\pi_s > \pi_\delta$.

If each buyer-seller pair plays a bargaining game that yields unique spe values v_s and v_b, then x_s and x_b must be given by

$$x_s = \delta \pi_s v_s \sum_{t=0}^{\infty} (1 - \pi_s)^t \delta^t = \frac{\delta \pi_s v_s}{(1-\delta) + \delta \pi_s}$$

$$x_b = \delta \pi_b v_b \sum_{t=0}^{\infty} (1 - \pi_b)^t \delta^t = \frac{\delta \pi_b v_b}{(1-\delta) + \delta \pi_b}.$$

Now if the endogenously given x_s and x_b are such that $x_s + \delta x_b > \delta^2$ and $x_b + \delta x_s > \delta^2$, then by Proposition 2 ii) we must have $v_s = \frac{1}{2}(1 - x_b) + \frac{1}{2}x_s$ and $v_b = \frac{1}{2}(1 - x_s) + \frac{1}{2}x_b$.

Thus, x_s and x_b must solve $x_s = \alpha_s(1 - x_b)$, $x_b = \alpha_b(1 - x_s)$,with

$$\alpha_s = \frac{\frac{\delta}{2}\pi_s}{(1-\delta) + \delta \pi_s}$$

and

$$\alpha_b = \frac{\frac{\delta}{2}\pi_b}{(1-\delta) + \delta \pi_b}.$$

That yields (14). ■