Tacit Collusion and Capacity Withholding in Repeated Uniform Price Auctions*

Emmanuel Dechenaux† Kent State University

Dan Kovenock Purdue University

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Abstract

This paper contributes to the study of tacit collusion by analyzing infinitely repeated multiunit uniform price auctions in a symmetric oligopoly with capacity constrained firms. Under both the Market Clearing and Maximum Accepted Price rules of determining the uniform price, we show that when each firm sets a price-quantity pair specifying the firm's minimum acceptable price and the maximum quantity the firm is willing to sell at this price, there exists a range of discount factors for which the monopoly outcome with equal sharing is sustainable in the uniform price auction, but not in the corresponding discriminatory auction. Moreover, capacity withholding may be necessary to sustain this outcome. We extend these results to the case where firms may set bids that are arbitrary step functions of price-quantity pairs with any finite number of price steps. Surprisingly, under the Maximum Accepted Price rule, firms need employ no more than two price steps to minimize the value of the discount factor above which the perfectly collusive outcome with equal sharing is sustainable on a stationary path. Under the Market Clearing Price rule, only one step is required. That is, within the class of step bidding functions with a finite number of steps, maximal collusion is attained with simple price-quantity strategies exhibiting capacity withholding.

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†Corresponding author. Kent State University, Department of Economics, Kent, OH 44242-0001. Phone: (330)-672-1146. Fax: (330)-672-9808. E-mail: edechena@kent.edu.

1 Introduction

This paper contributes to the study of tacit collusion by analyzing infinitely repeated multiunit uniform price auctions with capacity constrained firms. As in our earlier work on discriminatory auctions, we modify the Bertrand-Edgeworth approach by allowing each firm to simultaneously set a price-quantity pair specifying the firm's minimum acceptable price and the maximum quantity the firm is willing to sell at this price.[1] Using this game, we analyze the feasibility of perfect collusion using two different rules for determining the uniform price. Under the first rule, which we call the Market Clearing Price rule, the uniform price is equal to the minimum price at which the quantity offered by the firms is greater than or equal to demand. Under the second rule, called the Maximum Accepted Price rule, the uniform price is equal to the highest submitted price at which the residual demand left over from supply provided at strictly lower prices is strictly greater than zero. Both definitions have been used extensively in the literature (see, for example, Green and Newbery, 1992 and von der Fehr and Harbord, 1993).

When each firm sets a price-quantity pair there exists a range of discount factors for which the monopoly outcome with equal sharing is sustainable in either of the uniform price auctions, but not in the corresponding discriminatory auction. Moreover, capacity withholding may be necessary to sustain this outcome.

We extend these results to the case where firms may set bids that are arbitrary step functions of price-quantity pairs with any finite number of price steps. Surprisingly, under the Maximum Accepted Price rule, firms need employ no more than two price steps to minimize the value of the discount factor above which the perfectly collusive outcome with equal sharing is sustainable on a stationary path. Under the Market Clearing Price rule, only one step is required. That is, within the class of step bidding functions with a finite number of steps, maximal collusion is attained with simple price-quantity strategies exhibiting capacity withholding.

These results are particularly relevant for markets such as electricity markets in which uniform price and discriminatory auctions govern exchange. Our simple model captures some of the basic features of operating electricity markets, such as the UK spot market, the Spanish wholesale market or the Victoria Power Exchange. In these markets, capacity constrained firms compete by offering step bidding functions that vary in their complexity depending on the market.

The theoretical literature on capacity constrained uniform price auctions applied to electricity markets can be traced back to Green and Newbery (1992) and von der Fehr and Harbord (1993).[2] The former assumes that capacity constrained firms offer continuous supply functions, while the latter assumes that firms submit discrete step functions similar to those in this paper. In both papers, the analysis is static, and thus

[1]See Dechenaux and Kovenock (2003). Fabra (2003) provides a comparison of infinitely repeated uniform price and discriminatory auctions based on the Bertrand-Edgeworth approach.

[2]More recent theoretical work related to this study includes Baldick and Hogan (2001), Boom (2003), Borenstein *et al.* (2000), Crampes and Créti (2003), Crawford *et al.* (2003), Fabra *et al.* (2004), García-Díaz and Marín (2003), Gutiérrez-Hita and Ciarreta (2003), Lave and Perekhodstev (2001), Le Coq (2003) and Ubéda (2004).

ignores the strategic implications of repeated interaction. Although, as Borenstein, Bushnell and Wolak (2002) note, most electricity markets provide favorable conditions for firms to collude, surprisingly, little attention has been paid to the theoretical modeling of collusion in electricity markets. An exception is Fabra's (2003) comparison of the uniform price and discriminatory auctions in Bertrand-Edgeworth duopoly supergames.

Fabra (2003) has shown that under Bertrand-Edgeworth (B-E) duopoly, divisions of the monopoly profit can be supported in the infinitely repeated uniform price auction for strictly lower discount factors than in the infinitely repeated discriminatory auction. However, this result is only valid for a subset of symmetric capacities for which non-stationary paths with bid rotation can be sustained as perfect equilibria of the uniform price auction. For example, in the duopoly, if each firm's capacity is large enough to supply the monopoly output, incentives to deviate from perfectly collusive paths in the uniform price auction are no less than in the discriminatory auction. Furthermore, on the non-stationary paths with bid rotation that minimize incentives to deviate in the uniform price auction, firms do not equally share monopoly profit. Expanding the strategy space to price-quantity pairs, thereby allowing for physical withholding, has important implications for the sustainability of perfect collusion in the uniform price auction. A direct implication of capacity withholding is that, in contrast to B-E competition, when capacity is such that $n-1$ firms can supply the monopoly output, the monopoly outcome can be supported for a strictly wider range of discount factors in the uniform price auction than in the discriminatory auction. Moreover, this result holds even if we restrict attention to stationary paths on which each firm obtains an equal share of the monopoly profit.

In the discriminatory auction the incentive to deviate from perfect collusion is minimized on a stationary path on which each firm sets the monopoly price and offers its whole capacity. On the other hand, in the uniform price auction, if the uniform price is given by the Market Clearing Price rule, the stationary path on which each firm withholds capacity to offer its share of monopoly output at a price below some critical level (strictly lower than the monopoly price) minimizes firms' incentives to deviate in the class of stationary paths with equal sharing of the monopoly profit. If the uniform price is given by the Maximum Accepted Price Rule, then incentives to deviate from perfect collusion are minimized when $n-1$ firms withhold capacity to offer their share of the monopoly output. The remaining firm acts as the price setter and offers capacity at the monopoly price. Together these two results provide a conclusive theoretical link between equilibrium capacity withholding and the ability to support tacitly collusive outcomes.

The remainder of the paper is organized as follows. In Section 2, we describe the model and the simultaneous move price-quantity uniform price auction under two alternative definitions of the uniform price and characterize the Nash equilibria of the game. In Section 3, we introduce notation and definitions used in analyzing the price-quantity supergame. In Section 4, we show that under both formulations of the uniform price, capacity withholding relaxes incentives to deviate on perfectly collusive stationary perfect equilibrium paths with equal sharing. On such paths, incentives to deviate are minimized when n firms withhold capacity under the Market Clearing Price rule and when $n-1$ firms withhold capacity under the Maximum Accepted Price rule. Section 5 extends the results in section 4 to L-step bidding functions, $L \geq 1$, and shows that bidding functions with at most two steps are sufficient in order to minimize firms' incentives to deviate from a perfectly collusive path. One step is required under the Market Clearing Price rule and two steps under the Maximum Accepted Price Rule. Section 6 concludes.

2 The simultaneous move price-quantity game

2.1 The model

Consider a market for a homogeneous good. There are n firms in the industry. Let $N = \{1, \ldots, n\}$ denote the set of firms. Firm i's cost function is such that unit cost c_i is constant up to capacity k_i. Firms are symmetric: $k_i = k$ and $c_i = c = 0$ for all i. Let $d(p)$ be market demand and assume that it satisfies the following assumptions.

A1 $d(p)$ is continuous on $[0, \infty)$. $\exists\, \overline{p} > 0$ such that $d(p) = 0$ if $p \geq \overline{p}$ and $d(p) > 0$ if $p < \overline{p}$. $d(p)$ is twice continuously differentiable and $d'(p) < 0$ on $(0, \overline{p})$. Finally, $pd(p)$ is strictly concave on $[0, \overline{p}]$ with maximizer p^m.

These assumptions guarantee that there exists a unique unconstrained monopoly price, p^m. Inverse demand exists and is denoted by $P(y)$, where y is output. To ensure that there exists a unique Cournot equilibrium with a strictly positive price in the quantity-setting game with n symmetric firms (without capacity constraints), demand given by $d(p)$ and zero marginal cost, we further assume[3]

A2 $d'(p) + pd''(p) < 0$.

Under assumptions analogous to A1 for $P(y)$, this is equivalent to assuming that $\log P(y)$ is strictly concave over the relevant range and implies that Cournot quantity best response functions are downward sloping.[4] Denote by $r(z)$ a firm's Cournot best-response to an aggregate quantity z set by other firms. That is, $r(z)$ maximizes $P(x + z)x$ with respect to x. Let y^c be the quantity set by each firm in the Cournot equilibrium with strictly positive price.

In the one-shot simultaneous move price-quantity game, firms simultaneously set price-quantity pairs, (p, q), where $p \in \mathbb{R}_+$ and $q \in [0, k]$. Firm i's strategy space is thus $S_i = \mathbb{R}_+ \times [0, k]$. A strategy profile $(\mathbf{p}, \mathbf{q}) = ((p_1, q_1), \ldots, (p_n, q_n))$ is an element of $\times_{i=1}^{n} S_i$. In this paper, we restrict the analysis to pure strategies.

Define $\hat{q}_i = \min\{q_i, d(0)\}$ to be the effective quantity offered by firm i. Given a

[3] See Deneckere and Kovenock (1999), who also compare and contrast these conditions to inverse-demand based conditions guaranteeing existence and uniqueness of Cournot equilibrium. Note also that in the absence of capacity constraints, if $c_i = 0$ for every i, bootstrap Cournot equilibria exist in which equilibrium price is zero and every group of $n-1$ firms sets their aggregate quantity $q > d(0)$.

[4] See Deneckere and Kovenock (1999), Theorem 6.

strategy profile $(\mathbf{p}, \mathbf{q})$ and a coordinate $p \in \mathbb{R}_+$ of the price vector $\mathbf{p}$, define the set $L(p|\mathbf{p}, \mathbf{q}) \equiv \{i \in N | p_i = p\}$. $L(p|(\mathbf{p}, \mathbf{q})$ is the set of firms setting price p. We have $L(p|(\mathbf{p}, \mathbf{q}) = \emptyset$ if for all i, $p_i \neq p$. Let $L^-(p|\mathbf{p}, \mathbf{q}) \equiv \cup_{z<p} L(z|\mathbf{p}, \mathbf{q})$ be the set of all firms charging a price strictly less than p. To simplify notation, we often drop the argument $(\mathbf{p}, \mathbf{q})$.

We assume efficient rationing. Hence, given a strategy profile $(\mathbf{p}, \mathbf{q})$, the residual demand faced by firms in $L(p)$ is

$$R(p|\mathbf{p}, \mathbf{q}) = \max\{d(p) - \sum_{j \in L^-(p)} q_j, 0\}.$$

If $L^-(p)$ is empty, then we define $R(p|\mathbf{p}, \mathbf{q}) = d(p)$. Note that here, the residual demand is the demand left over from supply provided at strictly lower prices.

If, in case of a tie in price at p, we assume that firms share residual demand in proportion to their effective quantities offered, then for $i \in L(p|\mathbf{p}, \mathbf{q})$, sales are

$$s_i(p|\mathbf{p}, \mathbf{q}) = \min\left\{\hat{q}_i, \frac{\hat{q}_i}{\sum_{l \in L(p)} \hat{q}_l} R(p|\mathbf{p}, \mathbf{q})\right\}.$$

In this context, the literature has defined a uniform price auction in two distinct ways. We will examine each in turn. In the first definition, we follow Green and Newbery (1992) who use a specification in which the uniform price is the price at which the quantity demanded is equal to the quantity supplied.[5] This formulation leaves open the possibility that the uniform price will not be one of the submitted bids. See Figure 1 for an illustration.

Definition 1 (Market Clearing Price) *Given a strategy profile* $(\mathbf{p}, \mathbf{q})$ *in the uniform price auction, the uniform price* $\mathbf{P}^e(\mathbf{p}, \mathbf{q})$ *is the unique price that solves*

$$\min\{p | \sum_{i \in \mathcal{L}(p|\mathbf{p}, \mathbf{q})} \hat{q}_i \geq d(p)\},$$

where $\mathcal{L}(p|\mathbf{p}, \mathbf{q}) = L^-(p|\mathbf{p}, \mathbf{q}) \cup L(p|\mathbf{p}, \mathbf{q})$.

Definition 2 is the approach used by von der Fehr and Harbord (1993).[6] The price each firm receives in the uniform price auction is equal to the maximum accepted price, where the maximum accepted price is the highest submitted price at which the residual demand leftover from supply provided at strictly lower prices is strictly positive. Note that in this definition, the uniform price must be one of the submitted prices, and thus may not clear the market. See Figure 1 for an illustration. For Definition 2, we require slightly more notation. Let $\mathbf{p} = (p_1, \ldots, p_n)$ and define

$$\mathcal{P}(\mathbf{p}, \mathbf{q}) = \{p \in \{p_1, \ldots, p_n\} | R(p|\mathbf{p}, \mathbf{q}) > 0\}.$$

$\mathcal{P}(\mathbf{p}, \mathbf{q})$ is the set of submitted prices with $R(p|\mathbf{p}, \mathbf{q}) > 0$.

[5] See also Boom (2003) and Ubéda (2004).

[6] See also Crampes and Créti (2003), Fabra (2003), von der Fehr *et al.* (2004) and Le Coq (2002).

Definition 2 (Maximum Accepted Price) *Given a strategy profile* $(\mathbf{p}, \mathbf{q})$ *in the uniform price auction, the uniform price* $\mathbf{P}^a(\mathbf{p}, \mathbf{q})$ *is equal to the maximum accepted price, that is*

$$\mathbf{P}^a(\mathbf{p}, \mathbf{q}) = \max \mathcal{P}(\mathbf{p}, \mathbf{q}) \ \ \textit{if} \ \ \mathcal{P}(\mathbf{p}, \mathbf{q}) \neq \emptyset,$$

and

$$\mathbf{P}^a(\mathbf{p}, \mathbf{q}) = \overline{p} \ \ \textit{if} \ \ \mathcal{P}(\mathbf{p}, \mathbf{q}) = \emptyset.$$

For $u \in \{e, a\}$, firm i's payoff, $i = 1, \ldots, n$, under the two alternative definitions is simply:

$$\pi_i(\mathbf{p}, \mathbf{q}) = \mathbf{P}^u(\mathbf{p}, \mathbf{q}) s_i(p_i | \mathbf{p}, \mathbf{q}).$$

2.2 Pure strategy equilibria

We now define critical prices that are useful in characterizing a firm's profit from deviating from a given profile. We also characterize a firm's minmax payoff.

First, for $q < d(0)$, define the residual demand monopoly price for a firm with capacity k, $p^r(k, q)$:

$$p^r(k, q) \equiv \max\{\arg\max_p \{p[d(p) - q]\}, P(k + q)\}.$$

$p^r(k, q)$ is unique for every pair (k, q) given our assumptions on demand. From the strict concavity of $pd(p)$, it is clear that whenever $p^r(k, q)$ is strictly positive, it is strictly decreasing in q. A firm's profit from setting $p^r(k, q)$ after lower-priced firms have sold a quantity q is $\underline{\pi}(k, q) \equiv p^r(q)[d(p^r(q)) - q]$. For $q \geq d(0)$, a firm's residual demand after other firms have sold a quantity q is zero for all p. In this case, we define $p^r(k, q) \equiv 0$ and it follows that $\underline{\pi}(k, q) = 0$ for all $q \geq d(0)$.

Defining $p^r \equiv p^r(k, (n-1)k)$, it is straightforward to show that if $(n-1)k < d(0)$, then a firm's minmax payoff, $\underline{\pi}$, is $\underline{\pi} = p^r[d(p^r) - (n-1)k] > 0$. If $(n-1)k \geq d(0)$, then by definition, $p^r((n-1)k) = 0$ and each firm's minmax payoff is $\underline{\pi} = 0$.

Following Deneckere and Kovenock (1992), let $\underline{p}(k, q)$ be the unique price less than or equal to $p^r(k, q)$ at which a firm is indifferent between being the low-priced firm at $\underline{p}(k, q)$ and being a monopolist on residual demand left after q is sold and earning $\underline{\pi}(k, q)$. $\underline{p}(k, q)$ is equal to the smallest solution to

$$p \times \min\{d(p), k\} = \underline{\pi}(k, q).$$

If $q < d(0)$, then $\underline{p}(k, q) > 0$. If $q \geq d(0)$, by definition $\underline{\pi}(k, q) = 0$, and thus $\underline{p}(k, q) = 0$. In the continuation, we will use the notation $\underline{p}$ to denote $\underline{p}(k, (n-1)k)$. Moreover, since $k_i = k$ for every i, when there is no ambiguity we use $p^r(q)$ to denote $p^r(k, q)$, $\underline{\pi}(q)$ to denote $\underline{\pi}(k, q)$ and $\underline{p}(q)$ to denote $\underline{p}(k, q)$.

We can now state the following lemma describing equilibrium in the one shot price-quantity uniform price auction with common capacities $k_i = k$ and common unit costs $c_i = 0$, for every i, which we denote by $\Gamma^u(k, 0)$, where $u \in \{e, a\}$ indicates the definition of the uniform price that is employed.

Proposition 1 *The sets of pure strategy equilibria, $E^u(k,0)$, of the one shot uniform price auctions $\Gamma^u(k,0)$, $u = e, a$, are completely characterized as follows:*

(i) Suppose $k \leq y^c$. Then $E^a(k,0) = \{(\mathbf{p}^*,\mathbf{q}^*)|p_i^* \leq P(nk)$ and $q_i^* = k,\ \forall i \in N$, with $p_j^* = P(nk)$ for at least one $j \in N\}$ and $E^e(k,0) = \{(\mathbf{p}^*,\mathbf{q}^*)|p_i^* \leq P(nk)$ and $q_i^* = k,\ \forall i \in N\}$.

(ii) Suppose $k \geq \frac{d(0)}{n-1}$. Then $E^a(k,0) = \{(\mathbf{p}^*,\mathbf{q}^*)|\exists\ i,j,\ i \neq j,$ such that $p_i^* = p_j^* = 0$ and, $\forall h \in L(0|\mathbf{p}^*,\mathbf{q}^*)$, $\sum_{l\in L(0|\mathbf{p}^*,\mathbf{q}^*)\setminus\{h\}} \hat{q}_l^* \geq d(0)\}$. Define $C(0) \equiv \{(\mathbf{p}^*,\mathbf{q}^*)|p_i^* \leq \underline{p}((n-1)y^c)$ and $q_i^* = y^c,\ \forall i \in N\}$. Then $E^e(k,0) = E^a(k,0) \cup C(0)$.

(iii) Suppose $k \in (y^c, \frac{d(0)}{n-1})$. Define $\underline{y}$ to be the unique $y \in (-(n-2)k, k)$ such that $\underline{\pi}((n-2)k + y) = p^r k$. Then $E^a(k,0) = \{(\mathbf{p}^*,\mathbf{q}^*)|\exists\ j \in N$ such that $p_j^* = p^r$ and $q_j^* \in [\underline{y}, k]$ and $\forall i \neq j, p_i^* \leq \underline{p}$ and $q_i^* = k\}$. Define the set $D^e(k,0) \equiv \{(\mathbf{p}^*,\mathbf{q}^*)|\exists\ j \in N$ such that $p_j^* \leq p^r = \mathbf{P}(\mathbf{p}^*,\mathbf{q}^*)$ and $q_j^* \in [\underline{y}, k]$ and $\forall i \neq j, p_i^* \leq \underline{p}$ and $q_i^* = k\}$. Then $E^e(k,0) = D^e(k,0) \cup C(0)$, where $C(0)$ is as defined in (ii).[7]

Moreover, for every $k \in \mathbb{R}_+$, $u \in \{e,a\}$ and $i \in N$, there exists a pure strategy equilibrium of $\Gamma^u(k,0)$ in which $\pi_i(\mathbf{p}^*,\mathbf{q}^*) = \underline{\pi}$.

Proof. See the Appendix.

Proposition 1 demonstrates that, under both rules for determining the uniform price, if each firm's capacity k is less than or equal to its n-firm Cournot output, each firm sets price at or below the capacity clearing price $P(nk)$ and sells its capacity. If, for this range of capacities, the Maximum Accepted Price rule is used in determining the market price, at least one of the firms must set price equal to $P(nk)$.

For k in the classical Bertrand region where any $(n-1)$ firms have sufficient capacity to satisfy the whole market demand $d(0)$ at unit cost $c = 0$, the uniform price is always zero in the auction with the Maximum Accepted Price rule. This requires that at least two firms price at zero and set quantities sufficiently large that any unilateral deviation to a higher price yields zero sales. These classical Bertrand equilibia are also contained in the set of equilibria under the Market Clearing Price rule. However, under this rule, the equilibrium set also contains Cournot-like equilibria in which all firms set prices at or below $\underline{p}((n-1)y^c)$ and sell their Cournot quantity y^c. The uniform price in these equilibria is the Cournot price $P(ny^c)$.

For $k \in (y^c, \frac{d(0)}{(n-1)})$, the intermediate range between the Cournot output and the classical Bertrand region, for each $j \in N$, there exists a continuum of equilibria in which firm j sets $p_j^* = p^r$ and all other firms price at or below $\underline{p}$ (except possibly in a special case discussed in footnote 7 where firm j may set its price below p^r). The $(n-1)$

[7] Note that $D^e(k,0) = E^a(k,0)$ unless $\underline{y} = d(p^r) - (n-1)k$, which holds if and only if $k \leq r(d(p^r) - k)$. In the latter case, under the Market Clearing Price rule, strategy profiles $(\mathbf{p}^*,\mathbf{q}^*)$ such that $p_j^* < p^r$, $q_j^* = d(p^r) - (n-1)k$ and $\forall i \neq j$, $p_i^* \leq \underline{p}$ and $q_i^* = k$ are also Nash equilibria with $\mathbf{P}(\mathbf{p}^*,\mathbf{q}^*) = p^r$, $s_j(p_j^*|\mathbf{p}^*,\mathbf{q}^*) = d(p^r) - (n-1)k$ and $s_i(p_i^*|\mathbf{p}^*,\mathbf{q}^*) = k$.

low-priced firms all have sales equal to capacity k and firm j sells to residual demand. In these equilibria the quantity that firm j places on the market must be sufficient to deter a unilateral deviation by a low-priced firm to a price above p^r. The critical supply that achieves this is the quantity $\underline{y}$ defined in Proposition 1, so j must supply at least $\underline{y}$, which we show is greater than or equal to residual demand $d(p^r) - (n-1)k$. Under the Maximum Accepted Price rule these equilibria define the complete set of equilibria. For the Market Clearing Price rule the set must again be augmented by the set of Cournot-like equilibria described in the previous paragraph.

In our analysis of the repeated uniform price auctions that follows the most important aspect of the characterization in Proposition 1 is the fact that, under both uniform price rules, for any common capacity k and any firm i there exists a one-shot Nash equilibrium in which i receives its minmax profit $\underline{\pi}$. This allows the direct construction of credible punishments in the repeated uniform price auctions that force any unilaterally deviating firm down to its minmax per period continuation payoff.

3 The price-quantity supergame

In this section, we examine the supergame $\Gamma^u(k, 0, \delta)$ obtained by infinitely repeating the one shot game $\Gamma^u(k, 0)$ and discounting payoffs with discount factor $\delta < 1$. In the supergame, a path τ is an infinite sequence of action profiles $\{(\mathbf{p}^t, \mathbf{q}^t)\}_{t=0}^{\infty}$. A pure strategy σ_i for firm i is a sequence of functions, $\{\sigma_i(t)\}_{t=0}^{\infty}$, such that for every t, $\sigma_i(t) : H_t \to S_i$, where H_t is the set of possible histories $h_t = (\mathbf{p}^0, \mathbf{q}^0, \ldots, \mathbf{p}^{t-1}, \mathbf{q}^{t-1})$ up to time t and h_0 is the null history. A strategy profile is a vector $\sigma = (\sigma_1, \ldots, \sigma_n)$. Each strategy profile generates an infinite path $\tau(\sigma)$. Firm i's normalized discounted value from period s along a given path $\tau = \{(\mathbf{p}^t, \mathbf{q}^t)\}_{t=0}^{\infty}$ is given by:

$$V_i(\tau, s) = (1-\delta) \sum_{t=s}^{\infty} \delta^{t-s} \pi_i(\mathbf{p}^t, \mathbf{q}^t).$$

We refer to $V_i(\tau, t)$ for $t = 0, 1, 2, 3\ldots$ as firm i's continuation value at t. We let $V_i(\tau) \equiv V_i(\tau, 0)$ denote the payoff associated with the entire path. A security level punishment for firm i is a path on which firm i obtains the discounted sum of its minmax profit, equal to $\underline{\pi}_i$ in normalized terms. The result below establishes that a perfect equilibrium security level punishment in pure strategies exists under both definitions of the uniform price. After any unilateral deviation by firm i, firm i's punishment consists of reverting to a static equilibrium in every period .

Proposition 2 *For every $k \in \mathbb{R}_+$, $\delta \in (0, 1)$, $u \in \{e, a\}$, and $i \in N$, there exists a perfect equilibrium of $\Gamma^u(0, k, \delta)$ which serves as a security level punishment for firm i.*

Proof. From Proposition 1, the simultaneous move game has a pure strategy equilibrium in which firm i obtains its minmax payoff, $\forall i$ and k. Since repeating a minmax one-shot Nash equilibrium forever is a perfect equilibrium security level punishment, the result follows directly. ■

In the continuation, we assume that each firm's capacity is larger than its share of the monopoly output, $k > \frac{d(p^m)}{n}$. If on the other hand $k \leq \frac{d(p^m)}{n}$, all firms are capacity constrained in the one-shot Nash equilibrium and equilibrium payoffs are Pareto optimal. Thus the simultaneous move equilibrium is a collusive outcome immune to deviations for any discount factor.

Consider a stationary path $\tau = \{(p_1, q_1), (p_2, q_2), \ldots, (p_n, q_n)\}$. Let $\pi_i^*(p_{-i}, q_{-i})$ be firm i's optimal deviation profit as a function of the prices and quantities set by the remaining $n-1$ firms. Formally, $\pi_i^*(p_{-i}, q_{-i}) = \sup_{\{p_i, q_i\}} \pi_i(p_i, q_i, p_{-i}, q_{-i})$. Since from Proposition 1, a perfect equilibrium security level punishment exists, the incentive constraints that provide the perfect equilibrium conditions for the stationary path τ are simply:

$$(1-\delta)\pi_i^*(p_{-i}, q_{-i}) + \delta\underline{\pi} \leq \mathbf{P}^u(\mathbf{p}, \mathbf{q}) s_i(p_i|\mathbf{p}, \mathbf{q}). \tag{1}$$

for every $i \in N$ and $u \in \{e, a\}$.

In the next section, we characterize all stationary paths that achieve the monopoly outcome and on which firms share monopoly profits equally. We show that under both uniform pricing rules, there is a range of discount factors for which capacity withholding is necessary for such paths to be supported as perfect equilibrium paths.

4 Capacity withholding and market sharing

In this section, we focus attention on a specific class of paths. We consider paths τ that are stationary and on which the normalized payoffs satisfy $\sum_{i \in N} V_i(\tau) = p^m d(p^m) \equiv \Pi^m$. We say that such paths are *perfectly collusive*. In the remainder of this section, we also impose the condition that $\pi(\mathbf{p}, \mathbf{q}) = \frac{\Pi^m}{n}$; that is, firms share monopoly profits equally. We call a path satisfying the two conditions above a perfectly collusive stationary path with equal sharing.

The following lemma states that if firm i's rivals increase their prices in a way that preserves the ordering of prices between these firms, but do not alter their quantity ceilings, then firm i's payoff from an optimal deviation cannot decrease (and vice-versa). The conditions imposed on the profiles $(\mathbf{p}, \mathbf{q})$ and $(\mathbf{p}', \mathbf{q})$ in the statement of the lemma are sufficient to guarantee that for firm i and for every p, residual demand at p is higher when i's rivals set p'_{-i} than when they set p_{-i}.

Lemma 1 *Suppose $\mathbf{p} = (p_1, \ldots, p_n)$ and $\mathbf{p}' = (p'_1, \ldots, p'_n)$ satisfy the two following conditions: for some $i \in N$, (i) $p'_{-i} \geq p_{-i}$ and (ii) $\forall j, h \in N \setminus \{i\}$, $p_j \geq p_h$ implies $p'_j \geq p'_h$. Then for any vector of quantities $\boldsymbol{q} = (q_1, \ldots, q_n)$, $\pi_i^*(p'_{-i}, q_{-i}) \geq \pi_i^*(p_{-i}, q_{-i})$.*

Proof. Suppose that when the $n-1$ remaining firms set (p_{-i}, q_{-i}), firm i's optimal deviation is to undercut all $n-1$ firms. Since $p'_{-i} > p_{-i}$, firm i's optimal deviation profit under (p_{-i}, q_{-i}) can also be obtained by undercutting all other firms when they set (p'_{-i}, q_{-i}). Thus, its deviation profit cannot be lower in this case. Now suppose that if i's rivals set (p_{-i}, q_{-i}), firm i's optimal deviation consists of setting the residual demand monopoly price after a group of $l \geq 1$ firms have sold a quantity q, $p^r(q)$, to earn $\underline{\pi}(q)$. We show that i cannot obtain less than $\underline{\pi}(q)$ by deviating when its rivals set (p'_{-i}, q_i). First, consider the difference in firm i's residual demand at a given price p when firms set $(\mathbf{p}', \mathbf{q})$ and $(\mathbf{p}, \mathbf{q})$:

$$\max\{d(p) - \sum_{j \in L^-(p|\mathbf{p}',\mathbf{q})\setminus\{i\}} \hat{q}_j, 0\} - \max\{d(p) - \sum_{j \in L^-(p|\mathbf{p},\mathbf{q})\setminus\{i\}} \hat{q}_j, 0\}.$$

From conditions (i) and (ii) in the statement of the lemma, it follows that $L^-(p|\mathbf{p}', \mathbf{q}) \subset L^-(p|\mathbf{p}, \mathbf{q})$, and thus, $\sum_{j \in L^-(p|\mathbf{p}',\mathbf{q})\setminus\{i\}} \hat{q}_j \leq \sum_{j \in L^-(p|\mathbf{p},\mathbf{q})\setminus\{i\}} \hat{q}_j$. Consequently, residual demand at p is weakly larger when other firms set (p'_{-i}, q_i) than when they set (p_{-i}, q_{-i}). It follows that if $\nexists j$ for which $p'_j = p^r(q)$, then firm i obtains a deviation profit $\pi^*_i \geq \underline{\pi}(q)$ from setting exactly $p^r(q)$. If $\exists j$ such that $p'_j = p^r(q)$, then i can obtain a payoff arbitrarily close to π^*_i by infinitesimally undercutting $p'_j = p^r(q)$. Therefore, we have shown that any deviation profit level firm i can guarantee itself under (p_{-i}, q_{-i}), it can also obtain when the other firms set (p'_{-i}, q_{-i}). ■

Lemma 2 characterizes all perfectly collusive stationary paths with equal sharing. Note that the characterization is independent of the definition of the uniform price except for statement (ii).

Lemma 2 *Suppose $k > \frac{d(p^m)}{n}$. In $\Gamma^u(k, 0, \delta)$, on every perfectly collusive stationary path with equal sharing $\tau = ((p_1, q_1), (p_2, q_2), \ldots, (p_n, q_n))$, the following must hold: (i) $\mathbf{P}^u(\mathbf{p}, \mathbf{q}) = p^m$, (ii) $p_i \leq p^m$ with equality for at least one firm if $u = a$, (iii) for every firm $i \in N$, $q_i \geq \frac{d(p^m)}{n}$, with equality if firm i sets $p_i < p^m$.*

Proof. (i) and (ii) follow from the fact that there exists a unique maximizer to $pd(p)$. Thus, if industry profit is Π^m in every period, p^m must be the uniform price in every period as well. It is straightforward to see that if there exists a firm i for which $p_i > p^m$, then, either the uniform price is strictly greater than p^m, or firm i does not have any sales, a contradiction to $V_i(\tau) = \frac{\Pi^m}{n}$. Furthermore, if the uniform price is given by Definition 2, p^m must also be one of the accepted bids, thus at least one firm $i \in N$ must set $p_i = p^m$. To prove (iii), note that it is clear that if $q_i < \frac{d(p^m)}{n}$, then $s_i < \frac{d(p^m)}{n}$, and thus $V_i(\tau) < \frac{\Pi^m}{n}$. Hence, $q_i \geq \frac{d(p^m)}{n}$. To complete the proof of (iii), consider first the Maximum Accepted Price rule. Suppose to the contrary that there exists a firm i for which $p_i < p^m$ and $q_i > \frac{d(p^m)}{n}$. From the definition of s_i, it is clear that the only time a firm's sales are strictly below its quantity ceiling is when it sells to all or a fraction of residual demand. But if this is the case for firm i, then, $R(p|\mathbf{p}, \mathbf{q}) = 0$ for every $p > p_i$, which implies that p^m cannot be the uniform price, thus contradicting (i). Hence, it follows immediately that if $p_i < p^m$, $q_i = \frac{d(p^m)}{n}$. Consider now the Market Clearing Price rule and suppose that there exists a firm i with $p_i < p^m$ and $q_i > \frac{d(p^m)}{n}$. First, note that from Definition 1 and (i), if $p_j < p^m$, $\forall j$, then it must be the case that $\sum_{h \in N} \hat{q}_h = d(p^m)$. Otherwise, at least one firm must be setting p^m. Since we have established above that for every j, $q_j \geq \frac{d(p^m)}{n}$, it follows directly that if additionally, for every j, $p_j < p^m$, then $\sum_{j \in N} \hat{q}_j > d(p^m)$, so that the uniform price is not equal to p^m, a contradiction to (i). Suppose now that l firms, $l \geq 1$, are setting p^m. Since for every firm j setting $p_j < p^m$, $q_j \geq \frac{d(p^m)}{n}$ and there exists i for which $q_i > \frac{d(p^m)}{n}$, it follows that residual demand at p^m is strictly less than $\frac{l}{n}d(p^m)$, so that $s_h < \frac{d(p^m)}{n}$ for at least one firm h setting $p_h = p^m$, a contradiction to the fact that $V_h(\tau) = p^m s_h = \frac{\Pi^m}{n}$. ■

Lemma 2 shows that there are essentially two ways in which firms can achieve a perfectly collusive outcome on a stationary path. Either all firms set the monopoly price and offer the same quantity ceiling. In this case, the sales function prescribes that each firm will obtain an equal share of demand at the monopoly profit; or a group of firms set the monopoly price and share residual demand after another group of (strictly lower-priced) firms offering their share of the monopoly output have sold their quantity. We show in the following sections that the crucial difference between the Market Clearing Price and Maximum Accepted Price approaches is that if the former is used, there can be as many as n low-priced firms, while there must be at least one firm setting p^m under the latter.

4.1 The Market Clearing Price rule

Building on the insight from Lemma 2, we construct a perfect equilibrium path τ^{me} and show that given the imposed stationarity and equal division, such a path minimizes incentives to deviate for all firms among perfectly collusive stationary paths achieving the same payoff per firm. To this effect, let $\underline{q}^m = (n-1)\frac{d(p^m)}{n}$. τ^{me} is characterized as follows:

$$p_i = \underline{p}(\underline{q}^m) \text{ and } q_i = \frac{d(p^m)}{n}, \forall i \in N.$$

Lemma 3 *Suppose $k > \frac{d(p^m)}{n}$. In $\Gamma^e(k, 0, \delta)$, the path τ^{me} minimizes incentives to deviate in the class of perfectly collusive stationary paths with equal sharing.*

Proof. See Appendix.

The path constructed in Lemma 3 is symmetric. Each firm withholds capacity and sets a quantity equal to its share of the monopoly output and a price equal to the maximum price that no firm would want to unilaterally undercut. From Definition 1, it is straightforward to check that the uniform market price is then equal to the monopoly price. The proposition below shows that for some discount factors, withholding is necessary to sustain perfect collusion with equal sharing on a stationary path. The crucial effect of capacity withholding on collusion is that in markets with a large aggregate capacity, it allows firms to set lower prices without affecting the uniform price, thereby reducing incentives to deviate substantially for all firms.

It is important to note that setting $p_i = \underline{p}(\underline{q}^m)$ is not necessary to support this outcome. Every path on which each firm offers exactly its share of monopoly output at a price less than or equal to $\underline{p}(\underline{q}^m)$, for example the marginal cost of zero, yields the same outcome and exactly the same incentives to deviate as τ^{me}.

Proposition 3 *Suppose $k > \frac{d(p^m)}{n}$. There exists a $\overline{\delta}$ such that in $\Gamma^e(k, 0, \delta)$ perfect collusion with equal sharing is sustainable on a stationary path on which no firm withholds* *capacity if and only if $\delta \geq \overline{\delta}$. Furthermore, there exists a $\underline{\delta}^e < \overline{\delta}$ such that if $\delta \in [\underline{\delta}^e, \overline{\delta})$, perfect collusion with equal sharing is sustainable on a stationary path if and only if a subset of the firms withholds capacity. If $\delta = \underline{\delta}^e$, perfect collusion with equal sharing is sustainable on a stationary path if and only if all firms withhold capacity.*

Proof. See Appendix.

4.2 The Maximum Accepted Price rule

If the uniform price is defined as the maximum accepted price, then the result is slightly different. The fact that the uniform price has to be one of the submitted prices adds the extra constraint that one firm must set p^m on the path. This clearly increases incentives to deviate for low-priced firms. Let τ_i^{ma} be characterized as follows, $\forall i \in N$:

$$\begin{aligned} p_i &= p^m \text{ and } q_i = k, \\ p_j &= \underline{p}(q_-^m) \text{ and } q_j = \frac{d(p^m)}{n}, \ \forall j \neq i. \end{aligned}$$

There exist n such paths corresponding to each of the n players with price p^m.

Lemma 4 *Suppose $k > \frac{d(p^m)}{n}$. In $\Gamma^a(k, 0, \delta)$, paths of the form of τ_i^{ma} minimize incentives to deviate in the class of perfectly collusive stationary paths with equal sharing.*

Proof. See Appendix.

Lemma 4 shows that paths on which $n-1$ firms withhold capacity to offer an equal share of the monopoly output at a low price and one firm sets the monopoly price and offers its whole capacity minimize all firms' incentives to deviate on paths in that particular class. The next result shows that if $n > 2$ and capacity is large enough, by withholding output, firms can sustain an equal division of the monopoly outcome for a strictly wider range of discount factors than by offering full capacity.

Again, as for the Market Clearing Price approach, it is important to note that $p_j = \underline{p}(q_-^m)$ is not necessary for the low-priced firms. Every path on which each firm j, $j \neq i$, offers exactly its share of monopoly output at a price less than or equal to $\underline{p}(q_-^m)$ yields the same outcome and exactly the same incentives to deviate as τ_i^{ma}.

Proposition 4 *Suppose $n > 2$ and $k > \frac{2}{n}d(p^m)$. Then there exists a $\underline{\delta}^a < \overline{\delta}$ such that, for every $\delta \in [\underline{\delta}^a, \overline{\delta})$, in the game $\Gamma^a(k, 0, \delta)$ perfect collusion with equal sharing is sustainable on a stationary path if and only if a subset of firms withholds capacity. If $\delta = \underline{\delta}^a$, perfect collusion with equal sharing is sustainable on a stationary path if and only if at least $n-1$ firms withhold capacity.*

Proof. See Appendix.

Proposition 4 shows that for $n > 2$ and for sufficiently large capacity, withholding capacity facilitates collusion. The intuition behind the dependence of the result in

Proposition 4 on the number of firms is as follows. For every n, on a perfectly collusive path with equal sharing that does not feature withholding, an optimal deviation always consists in undercutting the monopoly price to obtain $p^m \min\{k, d(p^m)\}$. On τ_i^{ma}, if there are only two firms in the industry, the single low-priced firm can always undercut the high-priced firm, offer and sell the minimum of demand and its capacity. On the other hand, when there are more than two firms in the industry, an optimal deviation takes one of two forms depending on k. For a range of relatively low capacity values, a low-priced firm offers and sells its capacity at a price below p^m. In this case, since k is low, the deviating firm does not affect the uniform price and receives p^m for its capacity independently of the price it sets as long as this price is below p^m. However, this statement is only valid if undercutting the low-priced firms and offering capacity does not lower the market price. If k is large enough that a deviating firm would have to withhold in order not to affect the uniform price by expanding its quantity up to its capacity, the optimal deviation consists in pricing above the low-priced firms, but below the high price firm. For such values of k, a low-priced firm's deviation profit is strictly below $p^m \min\{k, d(p^m)\}$. Thus, k relatively large is required for withholding to strictly relax incentive constraints as compared to paths on which no firm withholds.

Fabra (2003) analyzes the feasibility of collusion in a Bertrand-Edgeworth duopoly supergame with the Maximum Accepted Price Rule. In Fabra's model the two firms must offer capacity at the price they set. She shows that to support a given level of industry profit, the paths that minimize firms' incentives to deviate are non-stationary. On such paths, firms alternate between being high-priced and low-priced, so that only one firm, the high-priced firm, has an incentive to deviate in any given period. To support perfect collusion, the high-priced firm's incentive to deviate on the non-stationary path is lower than that of the low-priced firm on the stationary path τ_i^{ma} constructed in Lemma 4 in this paper. Thus perfect collusion can be supported for lower discount factors on the non-stationary path with firms switching roles every period than on the stationary path with equal sharing. However on the non-stationary paths, firms do not share industry profit equally so that the question of role selection in the first period arises. Moreover, to support perfect collusion, such paths are feasible only if $k < d(p^m)$, since if $k \geq d(p^m)$, the firms' inability to withhold capacity implies that to obtain industry profit equal to Π^m, each firm must set p^m.[8] The latter observations are all the more relevant in the case $n \geq 3$.[9] Indeed, the range of capacities for which alternating non-stationary paths with one high-priced firm and no withholding are feasible, $k < \frac{d(p^m)}{n-1}$, shrinks as n increases. If $k \geq \frac{d(p^m)}{n-1}$, we conjecture that non-stationary perfect equilibrium paths with more than one high-priced firm setting p^m can be constructed for a range of discount factors, but it is not clear that they minimize incentives to deviate.

We have focused on one possible division of monopoly profits, namely the symmetric allocation, in which each firm receives an equal share of industry profit. Other allocations can also be sustained as perfect equilibria. With symmetric firms and the Market Clearing Price rule, equal sharing is optimal because on the path τ^{me}, all firms

[8]This is because if one firm sets p^m and the other firm $p < p^m$, then the price is equal to p.

[9]Fabra only analyzes the duopoly.

have the same incentive to deviate. However, when the Maximum Accepted Price rule is used, equal sharing may not minimize the critical value of the discount factor below which perfect collusion is sustainable. This is best illustrated by considering the duopoly case. Figure 2 shows the set of sustainable allocations for the low-priced firm as a function of the discount factor on perfectly collusive stationary perfect equilibrium paths similar to τ^{ma}. On such paths, the low-priced firm obtains $p^m s^m$ and the high-priced firm obtains $p^m(d(p^m) - s^m)$. Along the curve $\underline{s}^m$, the low-priced firm's incentive constraint is binding, while along the curve $\overline{s}^m$, the high-priced firm's constraint binds. At every allocation between $\underline{s}^m$ and $\overline{s}^m$, both constraints are slack. It is clear that the critical discount factor obtains at an allocation at which the low-priced-firm receives a greater share of monopoly profits.

To summarize, there are two main insights to be gleaned from the analysis of the price-quantity supergames in this section. First, capacity withholding facilitates collusion on stationary paths no matter which uniform pricing rule is used. This is because it allows firms to set a price that minimizes the other firms' deviation profit without preventing the market price from remaining at p^m. Second, under the Market Clearing Price rule, the reduction in deviation profit achieved on paths with withholding is larger than under the Maximum Accepted Price Rule since under the former, no single firm is required to price at p^m. Using these results, we now compare repeated uniform price auctions to repeated discriminatory auctions.

4.3 Uniform price and discriminatory auctions

Another commonly employed auction mechanism is the discriminatory auction, in which each firm receives the price it bids for its quantity. Propositions 3 and 4 provide a simple way to compare the two institutions. Recall that $\overline{\delta}$ is the critical value of the discount factor above which a path on which each firm sets the monopoly price p^m and offers its capacity in every period can be supported as a perfect equilibrium in the repeated uniform price auction. In both the uniform price and the discriminatory auction, an optimal deviation from such a path consists in undercutting the monopoly price and offering capacity. Therefore, $\overline{\delta}$ also represents the criticial value of the discount factor above which a perfectly collusive path with equal sharing is sutainable in the repeated discriminatory auction.

Theorem 1 *Suppose $k > \frac{d(p^m)}{n}$. Under both definitions of the uniform price, perfect collusion with equal sharing is sustainable on a stationary path in the repeated uniform price auction whenever it is sustainable on a stationary path in the repeated discriminatory auction. Moreover, if either (i) $u = e$ or (ii) $u = a$ and $n > 2$ and $k > \frac{2}{n} d(p^m)$ hold, there exists a nondegenerate interval of discount factors $[\underline{\delta}^u, \overline{\delta})$ for which perfect collusion with equal sharing is sustainable on a stationary path in the repeated uniform price auction, but not in the repeated discriminatory auction.*

Proof. In the discriminatory auction, it follows from a simple extension of Fabra's (2003) Proposition 3 to our price-quantity supergame with n symmetric firms that incentives to deviate are minimized on symmetric paths on which firms set the same price, p^m. Thus, the minimum value of the discount factor for which a perfectly collusive stationary perfect equilibrium path with equal sharing exists is obtained from a firm's incentive constraint on a path on which all firms offer capacity k at a price equal to p^m and is given by $\overline{\delta}$. Hence the result follows from Propositions 3 and 4. ■

The results in Theorem 1 are illustrated for an example with linear demand in Figure 3. Let $d(p) = \max\{0, 1-p\}$. In this case, $p^m = \frac{1}{2}$, $d(p^m) = \frac{1}{2}$ and $y^c = \frac{1}{n+1}$. Suppose that $k > y^c$. Using the Market Clearing Price approach, the optimal deviation on τ^{me} yields profit equal to $\frac{(n+1)^2}{16n^2}$. Using the Maximum Accepted Price rule, it is simple to show that if $n > 2$, for $j \neq i$, the optimal deviation on τ_i^{ma} is for j to undercut i but not $h \neq i, j$. j's deviation profit is then equal to $\frac{(n+2)^2}{16n^2}$. Assuming $k = \frac{1}{2}$ so that each firm's capacity is sufficient to supply the monopoly output, we obtain[10]

$$\overline{\delta} = \begin{cases} \frac{2}{3} & \text{if } n = 2, \\ \frac{n-1}{n} & \text{if } n > 2. \end{cases}$$

$$\underline{\delta}^e = \begin{cases} \frac{1}{5} & \text{if } n = 2, \\ \frac{(n-1)^2}{(n+1)^2} & \text{if } n > 2. \end{cases}$$

$$\underline{\delta}^a = \begin{cases} \frac{2}{3} & \text{if } n = 2, \\ \frac{4+n^2}{(n+2)^2} & \text{if } n > 2. \end{cases}$$

Under the Maximum Accepted Price rule, as in Brock and Scheinkman (1985),[11] the relationship between the lower bound on the discount factor for which perfect collusion is sustainable and the number of firms is always non-monotonic.[12] Therefore, when demand is linear, the result that for sufficiently low capacity (such that minmax payoffs are strictly positive for some n), decreasing the number of firms in the industry makes collusion more difficult continues to hold in the repeated price-quantity uniform price auction. Furthermore, independent of k, the number of firms at which the critical value of the discount factor is minimized is higher in the repeated uniform price auction when the Maximum Accepted Price rule is used. To see this, recall that in the linear demand case, Brock and Scheinkman show that $\overline{\delta}$ is initially decreasing in n. This is because a decrease in punishment payoffs more than offsets an increase in incentives to deviate for such values of n. The same holds in the uniform price auction. However, note that $\frac{d}{dn}\left[\frac{(n+2)^2}{16n^2}\right] < 0$, that is, deviation profit decreases with n in the repeated uniform price auction, while deviation profit is independent of n in the discriminatory auction. Therefore, $\underline{\delta}^a$ must decrease with n for a range of values of n above the minimizer of $\overline{\delta}$.[13] In the uniform price auction with the Market Clearing Price rule, in

[10] All formulas are derived from equations (A1), (A3), (A4) and (A8), which can be found in the Appendix.

[11] See also Davidson and Deneckere (1984) and Lambson (1987) for analyses of discriminatory auctions in a Bertrand-Edgeworth oligopoly supergame.

[12] Brock and Scheinkman's Figure 2 [p.376] is drawn for $\frac{1-\delta}{\delta}$.

[13] In our example the optimal number of firms is between 2 and 3 in both the discriminatory and the uniform price auction. However, the same intuition as that outlined above also works at values of k for which the optimal number of firms is greater than 3.

our example, even at low values of n an increase in incentives to deviate is not offset by a decrease in punishment value. Therefore, $\underline{\delta}^e$ increases in n for all $n \geq 2$. However, this is not generally true. The relationship between $\underline{\delta}^e$ and n depends on capacity, k. If k is sufficiently low, then $\underline{\delta}^e$ decreases with n initially and the relationship between the critical discount factor and the number of firms is again nonmonotonic.

5 Bid functions with an arbitrary number of steps

In this section, we generalize the results obtained in the previous section to a setting in which firms can submit bid functions with an arbitrary finite number of steps. In the one-shot simultaneous move game $\Gamma^u(k, 0, L)$, assume that a firm's strategy is a vector of price-quantity pairs defining an incremental bid function $((p_i^{l_i}, q_i^{l_i}))_{l_i=1}^{L_i}$, $L_i \leq L$ and $p_i^1 < \ldots < p_i^{L_i}$. L is a finite number representing the maximum number of admissible steps in the (non-decreasing) bid function of each firm. For each pair, the price $p_i^{l_i}$ represents the minimum price at which firm i is willing to sell the quantity increment $q_i^{l_i}$. We assume $p_i^{l_i} \in \mathbb{R}_+$ and $\sum_{l_i} q_i^{l_i} \leq k$. A strategy profile is a vector of bid functions $(\mathbf{p}, \mathbf{q}) = ((p_1^{l_1}, q_1^{l_1})_{l_1=1}^{L_1}, \ldots, (p_n^{l_n}, q_n^{l_n})_{l_n=1}^{L_n})$. We denote by $(\mathbf{p}_{-i}, \mathbf{q}_{-i})$ the vector composed of firm i's rivals' bid functions.

Below we define residual demand and a firm's sales at a given price in this setting. To this effect, let $l_i(p)$ be the index of the step associated with price p in firm i's strategy, that is, if there exists l_i such that $p_i^{l_i} = p$, then $l_i(p) = l_i$. Otherwise, define $l_i(p) \equiv \emptyset$. Let p_i^s be the set of prices submitted by firm i for each of its quantity increments, $p_i^s = \{p \in \mathbb{R}_+ | l_i(p) \neq \emptyset\}$ and let $p_i^s(p)$ the set of prices submitted by i that are less than or equal to p. That is, $p_i^s(p) = [0, p] \cap p_i^s$. Define $q_i(p)$ to be the quantity increment associated with price p in firm i's strategy. Formally,

$$q_i(p) = \begin{cases} q_i^{l_i} & \text{if } l_i(p) = l_i, \\ 0 & \text{if } l_i(p) = \emptyset. \end{cases}$$

so that firm i's quantity supplied at p is $\sum_{z \in p_i^s(p)} q_i(z)$.

Define $\hat{q}_i(p) = \min\{q_i(p), d(0)\}$. Assuming efficient rationing and given a strategy profile $(\mathbf{p}, \mathbf{q})$, residual demand at p is then easily defined as follows

$$R(p|\mathbf{p}, \mathbf{q}) = \max\{d(p) - \sum_{i \in N} \sum_{z \in p_i^s(p) \setminus \{p\}} \hat{q}_i(z), 0\}.$$

Note that if the minimum price submitted by any firm is greater than or equal to p, then $R(p|\mathbf{p}, \mathbf{q}) = d(p)$. Given a strategy profile $(\mathbf{p}, \mathbf{q})$, firm i's sales at a price $p \in p_i^s$ are equal to

$$s_i(p|\mathbf{p}, \mathbf{q}) = \min\left\{\hat{q}_i(p), \frac{\hat{q}_i(p)}{\sum_{l \in N} \hat{q}_l(p)} R(p|\mathbf{p}, \mathbf{q})\right\},$$

so that firm i's total sales amount to

$$\mathbf{s}_i(\mathbf{p}, \mathbf{q}) = \sum_{p \in p_i^s} s_i(p|\mathbf{p}, \mathbf{q})$$

and firm i's profit is $\pi_i(\mathbf{p},\mathbf{q}) = \mathbf{P}^u(\mathbf{p},\mathbf{q})\mathbf{s}_i(\mathbf{p},\mathbf{q})$, $\forall u \in \{e,a\}$. Finally, we let $\pi_i^*(\mathbf{p}_{-i},\mathbf{q}_{-i})$ denote firm i's profit from an optimal deviation when its rivals play $(\mathbf{p}_{-i},\mathbf{q}_{-i})$.

The supergames obtained by repeating the component game above are defined in a similar manner as the supergame with $L = 1$. We denote the supergame in which the number of admissible steps is L by $\Gamma^u(k,0,\delta,L)$, for $u \in \{e,a\}$. Lemma 5 below simplifies the analysis of the games with L-step bidding functions by showing that, when compared to the price-quantity approach, expanding the strategy space does not allow firms to obtain greater one-period deviation payoffs.

Lemma 5 *For every $k \in \mathbb{R}_+$, $L \geq 1$, $u \in \{e,a\}$ and $i \in N$, in the game $\Gamma^u(k,0,L)$, $\pi_i^*(\mathbf{p}_{-i},\mathbf{q}_{-i})$, firm i's optimal deviation payoff when its rivals set $(\mathbf{p}_{-i},\mathbf{q}_{-i})$, can be obtained with a bidding function using a single step ($L_i = 1$).*

Proof. See Appendix.

The extension of Lemmas 3 and 4 to the more general setting introduced above is relatively straightforward. Statements similar to those in Proposition 2 and Lemmas 1 and 2 are valid. Beginning with Proposition 1 and making use of Lemma 5, note that the pure strategy equilibria for the simultaneous move game characterized for $L = 1$ are clearly equilibria of the simultaneous move game with $L \geq 2$ admissible steps. We hence have the following result.

Proposition 5 *For every $k \in \mathbb{R}_+$, $L \geq 1$, $u \in \{e,a\}$ and $i \in N$, there exists a pure strategy equilibrium of the game $\Gamma^u(k,0,L)$ in which $\pi_i(\mathbf{p}^*,\mathbf{q}^*) = \underline{\pi}$.*

Allowing for L steps does not change a firm's minmax profit $\underline{\pi}$. This is because the worst that can be imposed on a given firm i is obtained by maximizing i's profit after its rivals have sold their capacity at a price of 0. Therefore, Proposition 2 continues to hold when $L \geq 2$ because perfect equilibrium security level punishments are attained with strategies using only one step. The analog of Lemma 1 for the case $L \geq 2$ states that if for some $i \in N$, $(\mathbf{p},\mathbf{q})$ and $(\mathbf{p}',\mathbf{q}')$ satisfy the two following conditions: first, the total quantity put on the market by i's rivals does not change and second, the residual demand faced by firm i at every p is no less when firms set $(\mathbf{p}',\mathbf{q}')$ than when they set $(\mathbf{p},\mathbf{q})$, then firm i's deviation profit under $(\mathbf{p},\mathbf{q})$ is no greater than under $(\mathbf{p}',\mathbf{q}')$. Formally these two conditions are

$$\sum_{j\in N\setminus\{i\}}\sum_{p\in p_j^s} q_j(p) = \sum_{j\in N\setminus\{i\}}\sum_{p\in p_j^{s\prime}} q_j'(p)$$

and, for every $p \in [0,\overline{p}]$,

$$\sum_{j\in N\setminus\{i\}}\sum_{z\in p_j^s(p)\setminus\{p\}} q_j(z) \geq \sum_{j\in N\setminus\{i\}}\sum_{z\in p_j^{s\prime}(p)\setminus\{p\}} q_j'(z).$$

To generalize Lemma 2, note that the requirements are that the uniform price be p^m and that $s_i(\mathbf{p},\mathbf{q}) = \frac{d(p^m)}{n}$. Moreover, supply of lower-priced units should sum up to a quantity less than $d(p^m)$,[14] while still allowing firms that do not have p^m in the support of their strategy to sell $\frac{d(p^m)}{n}$. Thus on a perfectly collusive stationary path with equal sharing

$$\sum_{j\in N}\sum_{p\in p_j^s(p^m)\setminus\{p^m\}} \hat{q}_j(p) \leq d(p^m) \tag{2}$$

with a strict inequality if the uniform price is given by the Maximum Accepted Price Rule. Furthermore, if $p^m \notin p_h^s$, then

$$\sum_{p\in p_h^s(p^m)} \hat{q}_h(p) = \frac{d(p^m)}{n}.$$

As before, when the uniform price is given by the Market Clearing Price rule, on the perfectly collusive stationary path that minimizes incentives to deviate the price will be set at p^m by having firms withhold their capacity to offer only $\frac{d(p^m)}{n}$ each and set a price that minimizes deviation profits. When the uniform price is given by the Maximum Accepted Price rule, the highest accepted bid has to be p^m so that at least one firm has to offer a positive quantity at p^m.

5.1 The Market Clearing Price rule

For every $L \geq 1$, it follows from the above discussion that a perfectly collusive stationary paths that minimizes incentives to deviate is such that all firms set $p = \underline{p}(q_-^m)$ and $q = \frac{d(p^m)}{n}$. Under the Market Clearing Price definition, one step is necessary and sufficient to minimize incentives to deviate on perfectly collusive stationary paths with equal sharing. Thus, we have Proposition 6.

Proposition 6 *Suppose $k > \frac{d(p^m)}{n}$. For every number of admissible steps $L \geq 1$, in $\Gamma^e(k, 0, \delta, L)$, the path τ^{me} minimizes incentives to deviate in the class of perfectly collusive stationary paths with equal sharing. That is, the incentive to deviate is minimized over all possible finite step functions by employing a simple (one-step) price-quantity strategy.*

Theorem 2 below then follows directly from Propositions 3 and 6. It shows that a path on which firms use a bid function with only one step minimizes the critical value of the discount factor above which a stationary perfect equilibrium path that achieves the monopoly outcome with equal sharing exists.

Theorem 2 *Suppose $k > \frac{d(p^m)}{n}$. For every number of admissible steps $L \geq 1$, in $\Gamma^e(k, 0, \delta, L)$, perfect collusion with equal sharing is sustainable on a stationary path if and only if $\delta \geq \underline{\delta}^e$. That is, if the strategy space is extended to allow for arbitrary finite step functions, whenever perfect collusion with equal sharing is sustainable using strategies involving more than one step, it is sustainable employing simple one-step price-quantity strategies.*

[14]This quantity must be strictly less than $d(p^m)$ if the uniform price is given by the Maximum Accepted Price rule.

5.2 The Maximum Accepted Price rule

The difference between the Market Clearing Price rule and the Maximum Accepted Price rule is that under the latter, on a perfectly collusive path with equal sharing, at least one firm must set $p = p^m$, while this is not required in the former. However, in contrast to the price-quantity supergame, a firm setting p^m may also offer part of its capacity at prices below p^m. The main insight from Lemma 4, namely that incentives to deviate are minimized when as much capacity as possible is offered at a sufficiently low price p^m, applies here as well. However, when $L \geq 2$, we show that all firms offer some quantity at p^m. Since in our model nothing prevents firms from offering infinitesimally small quantities at a given price, we conduct the analysis for a given aggregate quantity ϵ sold at p^m and show that as this quantity goes to zero, the lowest discount factor for which perfect collusion is sustainable on a stationary path with the Maximum Accepted Price approach converges to that in the Market Clearing Price approach, $\underline{\delta}^e$. We may interpret ϵ as being part of the tacitly collusive agreement between the firms.

For a given minimum quantity agreement $\epsilon > 0$, consider the 2-step bid function $(p_i^\epsilon, q_i^\epsilon)$ for firm $i \in N$ where

$$(p_i^\epsilon, q_i^\epsilon) = \left(\left(\underline{p}(q_-^\epsilon), \frac{d(p^m) - \epsilon}{n} \right), \left(p^m, k - \frac{d(p^m) - \epsilon}{n} \right) \right).$$

Let the path τ^ϵ be such that firm i sets $(p_i^\epsilon, q_i^\epsilon)$, $\forall i \in N$. Proposition 7 shows that for a given quantity agreement $\epsilon > 0$, τ^ϵ minimizes incentives to deviate in the class of perfectly collusive paths with equal sharing. Therefore, in characterizing paths that minimize incentives to deviate in the class of perfectly collusive stationary paths with equal sharing, two steps are necessary and sufficient.

Proposition 7 *Suppose $k > \frac{d(p^m)}{n}$. For every number of admissible steps $L \geq 2$ and quantity agreement $\epsilon \in (0, d(p^m)]$, in $\Gamma^a(k, 0, \delta, L)$, the path τ^ϵ minimizes incentives to deviate in the class of perfectly collusive stationary paths with equal sharing. That is, the incentive to deviate is minimized over all possible finite step functions by employing a two-step strategy.*

Proof. See the Appendix.

The intuition behind Proposition 7 is as follows. Suppose that the aggregate quantity offered at a common price $p < p^m$ is $d(p^m) - \epsilon$, for some quantity agreement $\epsilon > 0$, where p is sufficiently low that no firm would ever undercut it. Then, if l firms each offer $\frac{d(p^m)}{n} - \frac{\epsilon}{l}$ at p and $k - \left[\frac{d(p^m)}{n} - \frac{\epsilon}{l}\right]$ at p^m, while the remaining firms simply offer $\frac{d(p^m)}{n}$ at p, the uniform price is indeed p^m and each firm earns $p^m \frac{d(p^m)}{n}$. Furthermore, it is clear that the $n - l$ low-priced firms have greater incentives to deviate than the high-priced firms, since they face a strictly higher residual demand at every price between p and p^m. If k is sufficiently small that the price would remain at p^m if a low-priced firm offered its capacity at p, then an optimal deviation consists in offering capacity at

p to earn $p^m k$. Otherwise, the optimal deviation is to set the residual monopoly price after $n-1$ firms sell their quantity offered at p, where this quantity is equal to

$$(n-l-1)\frac{d(p^m)}{n} + l\left(\frac{d(p^m)}{n} - \frac{\epsilon}{l}\right) = (n-1)\frac{d(p^m)}{n} - \epsilon. \tag{3}$$

Now suppose that $l = n$, so that all firms set $k - \left[\frac{d(p^m)-\epsilon}{n}\right]$ at p^m. Then each firm's optimal deviation is to either offer capacity at p or set the residual monopoly price after $n-1$ firms sell their quantity offered at p, where this quantity is equal to

$$(n-1)\left(\frac{d(p^m)}{n} - \frac{\epsilon}{n}\right) \equiv q_{-}^{\epsilon}. \tag{4}$$

(4) is clearly greater than (3), so that a firm's deviation profit is lowest when all firms offer the same quantity both at the low price and the monopoly price. It follows that on a path that minimizes incentives to deviate, all n firms will offer some quantity at p^m. Thus all firms will have two steps in their bid function. One of the steps must be at a low price in order to prevent rivals from undercutting and the quantity offered at that price must be the highest quantity consistent with p^m being the uniform price. The second step effectively sets the price at p^m. To sustain the perfectly collusive outcome, there is nothing to gain from being allowed to include additional steps in the bid functions. Theorem 3 below follows directly from Proposition 7.

Theorem 3 *Suppose $k > \frac{d(p^m)}{n}$. For every number of admissible steps $L \geq 2$ and quantity agreement $\epsilon \in (0, d(p^m)]$, perfect collusion with equal sharing is sustainable on a stationary path in $\Gamma^\delta(k, 0, a, L)$, if and only if τ^ϵ is sustainable. Moreover, there exists $\underline{\delta}^a(\epsilon)$ such that τ^ϵ is sustainable if and only if $\delta \geq \underline{\delta}^a(\epsilon)$ and $\underline{\delta}^a(\epsilon) \downarrow \underline{\delta}^e$ as $\epsilon \to 0$.*

Apart from directly relaxing firms' incentive constraints on perfectly collusive paths, allowing for more than one step in a firm's bid function generates another interesting difference as compared to the simple price-quantity approach. If $L \geq 2$, on a perfectly collusive stationary path that minimizes incentives to deviate, firms set identical 2-step bid functions. This symmetry in firms' actions is not a property of the most collusive path in the price-quantity supergame since a single firm must play the role of the high-priced firm. This further implies that under the Maximum Accepted Price rule, capacity withholding is effective in relaxing incentive constraints even in a duopoly, while as we have shown in Proposition 4, this is not the case in the price-quantity approach.

6 Conclusion

We have examined the nature of collusive stationary perfect equilibrium paths in an infinitely repeated multiunit uniform price auction with capacity constrained firms. Under two different definitions of the market price in a uniform price auction, each appearing prominently in the literature, we characterize the set of paths that minimize the incentive to deviate while supporting the monopoly price with equal sharing of output. We then show that these paths can be supported as stationary perfect equilibria for a wider range of discount factors than under a repeated discriminatory price auction.

Using the Market Clearing Price rule to determining the uniform price, we show that extending firms' strategy spaces to allow them to place bids involving any finite number of price-quantity pairs neither enhances, nor hinders the firms' ability to collude. Surprisingly, L-step bidding functions, $L \geq 2$, cannot improve upon the ability to collude with 1-step bidding functions, which involve the simple choice of a price-quantity pair, nor can they make collusion more difficult to sustain. Since such step functions are quite common in electricity markets, our analysis extends the analysis of collusion to these more complicated markets with step supply functions.

Using the Maximum Accepted Price rule, the capacity withholding properties stated above continue to hold, although with this rule, bid functions that involve 2 steps ($L \geq 2$) strictly lower the incentive to deviate from the most collusive outcome when compared to the price-quantity game ($L = 1$). Further increases in the number of steps in the bidding function provide no advantage: collusive outcomes are no easier or harder to support when the strategy space is extended to $L = 3$ than with $L = 2$. Hence, under the Maximum Accepted Price rule as well, optimal collusion can be attained with a drastically restricted set of available step supply functions requiring only two steps.

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Appendix

7 Proof of Proposition 1

We first show that all strategy profiles described in (i), (ii) and (iii) of Proposition 1 form Nash equilibria of the respective games $\Gamma^u(k,0)$, $u \in \{e,a\}$. For $k \leq y^c$, we show in (a) that $E^u(k,0)$ is a set of Nash equilibria for $u \in \{e,a\}$. For $k \geq \frac{d(0)}{n-1}$ and $k \in (y^c, \frac{d(0)}{n-1})$, we show in (b) that $E^a(k,0)$ is a set of Nash equilibria for $u \in \{e,a\}$. Next, for $k > y^c$, we show in (c) that $E^a(k,0)$ and $D^e(k,0)$ are sets of Nash equilibria under the Maximum Accepted Price rule and the Market Clearing Price rule, respectively. Finally, we show in (d) that for $k > y^c$, $C(0)$ is an additionnal set of Nash equilibria under the Market Clearing Price rule. We complete the proof of the proposition by demonstrating that for each $u \in \{e,a\}$, the sets of strategy profiles described in (i), (ii) and (iii) characterize the complete set of Nash equilibria of $\Gamma^u(k,0)$.

(a) Suppose $k \leq y^c$. Note that in this case, all firms are capacity constrained at $p^r = \underline{p} = P(nk)$. Furthermore, for all $u \in \{e,a\}$, all strategy profiles in the statement of the proposition yield $\pi_i(\mathbf{p}^*, \mathbf{q}^*) = P(nk)k$, $\forall i \in N$. To prove (i), we examine both definitions of the uniform price separately. Consider first the Market Clearing Price rule or $u = e$. Suppose that there exists a firm i such that, $\forall j \neq i$, $p_j^* \leq P(nk)$ and $q_j^* = k$. We show that firm i's best response to (p_{-i}^*, q_{-i}^*) is the set of price-quantity pairs (p_i^*, q_i^*) such that $p_i^* \leq P(nk)$ and $q_i^* = k$. Let (p_i', q_i') be firm i's strategy and $\mathbf{P}^{e\prime}$ the resulting uniform price. Then we either have $p_i' > \mathbf{P}^{e\prime}$, in which case firm i's profit from (p_i', q_i') is equal to zero, or $p_i' \leq \mathbf{P}^{e\prime}$. In the latter case, one can easily check that $\mathbf{P}^{e\prime} = \max\{p_i', P((n-1)k + q_i')\}$ and that firm i's sales are equal to $\min\{q_i', R(p_i'|p_i', q_i', p_{-i}^*, q_{-i}^*)\}$. Furthermore, firm i's payoff is equal to $\max\{p_i', P((n-1)k + q_i')\} \min\{q_i', R(p_i'|p_i', q_i', p_{-i}^*, q_{-i}^*)\}$, which is maximized by setting $p_i' \leq P(nk)$ and $q_i' = k$, at which firm i obtains a payoff equal to $P(nk)k > 0$.

Consider now the Maximum Accepted Price rule or $u = a$. Suppose that there exists a firm i such that, $\forall j \neq i$, $p_j^* \leq P(nk)$ and $q_j^* = k$. We show that firm i's best response to (p_{-i}^*, q_{-i}^*) is the set of price-quantity pairs (p_i^*, q_i^*) such that $p_i^* \leq P(nk)$ and $q_i^* = k$, unless $p_j^* < P(nk)$, $\forall j \neq i$, in which case, $p_i^* = P(nk)$ and $q_i^* = k$ is firm i's best response. Let (p_i', q_i') be firm i's strategy and $\mathbf{P}^{a\prime}$ the resulting uniform price. Again, if $p_i' > \mathbf{P}^{a\prime}$, firm i obtains a profit of zero. On the other hand, if (p_i', q_i') is such that $p_i' \leq \mathbf{P}^{a\prime}$, then firm i's payoff from (p_i', q_i') is given by $\mathbf{P}^{a\prime} \min\{q_i', R(p_i'|p_i', q_i', p_{-i}^*, q_{-i}^*)\}$. Since $p_j^* \leq P(nk)$, $\forall j \neq i$, if $p_i' > P(nk)$, $P^{a\prime} = p_i'$ and thus firm i's payoff is equal to $p_i' \min\{q_i', R(p_i'|p_i', q_i', p_{-i}^*, q_{-i}^*)\} < P(nk)k$, but firm i could obtain $(Pnk)k$ by decreasing price to $P(nk)$ and offering $q_i' = k$ instead. Furthermore, given any q_i', firm i's payoff from setting $p_i' < P(nk)$ can be no greater than $P(nk)k$ since $k < y^c$ implies $\underline{p} = P(nk)$ and $\underline{p}k = P(nk)k$. So suppose $p_j^* < P(nk)$, $\forall j \neq i$. If $p_i' < P(nk)$, it is clear that $P^{a\prime} < P(nk)$, in which case firm i obtains strictly less than $P(nk)k$, a payoff it could obtain by raising price to $P(nk)$ and offering capacity. On the other hand, if there exists $h \neq i$ for which $p_h^* = P(nk)$, then for any $p_i' \leq P(nk)$, we have $P^{a\prime} = P(nk)$, and thus, by setting $q_i' = k$, firm i obtains exactly $P(nk)k$.

(b) Suppose $k \geq \frac{d(0)}{n-1}$. In this case, $n-1$ firms can serve demand at a price equal to marginal cost. To prove that all strategy profiles $(\mathbf{p}^*, \mathbf{q}^*) \in E^a(k, 0)$ are equilibria, note that $\mathbf{P}^u(\mathbf{p}^*, \mathbf{q}^*) = 0$, for every $(\mathbf{p}^*, \mathbf{q}^*) \in E^a(k, 0)$ and $u \in \{e, a\}$. Hence $\pi_i(\mathbf{p}^*, \mathbf{q}^*) = \underline{\pi} = 0$, $\forall i \in N$. Since $\forall h \in L(0|\mathbf{p}^*, \mathbf{q}^*)$, $\sum_{l \in L(0|\mathbf{p}^*,\mathbf{q}^*)\setminus\{h\}} \hat{q}_l^* \geq d(0)$, residual demand is zero at every $p \in (0, \overline{p}]$. It thus follows that no firm has a profitable deviation in this case.

(c) Suppose $k \in \left(y^c, \frac{d(0)}{n-1}\right)$. Note that the only difference between the sets $E^a(k, 0)$ and $D^e(k, 0)$ is the fact that p_j^* may be less than or equal to p^r in $D^e(k, 0)$. Thus, we first show that strategy profiles in $E^a(k, 0)$ are Nash equilibria under both rules. For $u \in \{e, a\}$, note that all strategy profiles in $E^a(k, 0)$ yield $P^u(\mathbf{p}^*, \mathbf{q}^*) = p^r$, $\pi_j(\mathbf{p}^*, \mathbf{q}^*) = \underline{\pi}$ and for every firm $i \neq j$, $\pi_i(\mathbf{p}^*, \mathbf{q}^*) = p^r k$. By definition of $\underline{p}$, firm j has no incentive to deviate from (p^r, q_j^*). We now show that low-priced firms do not have an incentive to deviate. We first show that a low-priced firm, say i, $i \neq j$, has no incentive to deviate to the residual demand monopoly price after $n-2$ firms have sold k and firm j has sold q_j^*. To this effect, define $\underline{y} \in (-(n-2)k, k)$ as the unique solution in y to:

$$\underline{\pi}((n-2)k + y) = p^r k.$$

Uniqueness of $\underline{y}$ in the interval $(-(n-2)k, k)$ follows from $\underline{\pi}(0) = \max\{p^m d(p^m), P(k)k\} > p^r k$ and $\underline{\pi}((n-1)k) < p^r k$ and the fact that $\underline{\pi}((n-2)k + y)$ is continuous and strictly decreasing as a function of y on the closed interval $[-(n-2)k, k]$. Now, since $\underline{\pi}(q)$ is strictly decreasing in q, if $q_j^* \geq \underline{y}$, then $p^r k \geq \underline{\pi}((n-2)k + q_j^*)$. Therefore, if $q_j^* \geq \max\{d(p^r) - (n-1)k, \underline{y}\}$, firm i prefers to obtain $\pi_i(\mathbf{p}^*, \mathbf{q}^*) = p^r k$ rather than raising price to $p^r((n-2)k + q_j^*)$ and serving residual demand. Finally, firm i can neither increase its payoff by slightly undercutting other low-priced firms, nor by setting a price in the interval $(\underline{p}, p^r)$, since in both cases, it would obtain exactly $p^r k$. Finally, the fact that $\underline{y} \geq d(p^r) - (n-1)k$ follows from the following argument. By definition of $\underline{y}$, if $q_j > \underline{y}$, then $p^r k > \underline{\pi}((n-2)k + q_j) = \max_{q \leq k}\{P(q + (n-2)k + q_j)q\}$. So suppose that $\underline{y} < d(p^r) - (n-1)k$ and let $q_j = d(p^r) - (n-1)k > \underline{y}$. It is clear that in this case $P(k + (n-2)k + q_j)k = p^r k \leq \max_{q \leq k}\{P(q + (n-2)k + q_j)q\}$ (where the first equality follows from $k + (n-2)k + d(p^r) - (n-1)k = d(p^r)$), a contradiction to the above statement. Thus, $\underline{y} \geq d(p^r) - (n-1)k$. Hence, we have shown that all strategy profiles in $E^a(k, 0)$ are Nash equilibria under both rules. To complete the proof for the Market Clearing Price rule, note that if $\underline{y} = d(p^r) - (n-1)k$, then $q_j^* = \underline{y}$ implies $\sum_{h \in N} q_h^* = d(p^r)$. If additionally, for every h, $p_h^* \leq p^r$, which clearly holds for the strategy profiles described in $E^e(k, 0)$, then $\mathbf{P}^e(\mathbf{p}^*, \mathbf{q}^*) = p^r$. This completes the proof that all strategy profiles in $D^e(k, 0)$ are Nash equilibria under the Market Clearing Price rule.

(d) Consider the game $\Gamma^e(k, 0)$. Define $q_-^c \equiv (n-1)y^c$. For $k > y^c$, we now show that any $(\mathbf{p}^*, \mathbf{q}^*)$ where $p_i^* \leq \underline{p}(q_-^c)$ and $q_i^* = y^c$, $\forall i$, is a Nash equilibrium of $\Gamma^e(k, 0)$. For such strategy profiles, the uniform price is equal to $\mathbf{P}^e(\mathbf{p}^*, \mathbf{q}^*) = P(ny^c)$ and each firm's payoff $\pi_i(\mathbf{p}^*, \mathbf{q}^*)$ is $P(ny^c)y^c$ $\forall i$. First, by definition of $\underline{p}(q_-^c)$, no firm has an incentive to undercut any of its rivals' prices and expand output above y^c. Moreover, since y^c is the unique Cournot equilibrium output, $p^r(q_-^c) = P(ny^c)$ and $\underline{\pi}(q_-^c) = P(ny^c)y^c$. Thus no firm has an incentive to deviate from $(\mathbf{p}^*, \mathbf{q}^*)$, from which it follows that $(\mathbf{p}^*, \mathbf{q}^*)$ is an equilibrium.

In the remainder of the proof, we show that for $u \in \{e, a\}$, the equilibria characterized above are the only equilibria of $\Gamma^u(k, 0)$. In the analysis below, let $(\mathbf{p}^*, \mathbf{q}^*)$ be a pure strategy equilibrium. We have shown above that, for all k, $(\mathbf{p}^*, \mathbf{q}^*)$ exists under both uniform price rules. For $i \in N$, let $\overline{\pi}_i$ be firm i's profit in a pure strategy equilibrium.

7.1 Maximum Accepted Price rule

Lemma A1 *Suppose $k < \frac{d(0)}{n-1}$, then, for every $(\mathbf{p}^*, \mathbf{q}^*)$, $\overline{p} > \mathbf{P}^a(\mathbf{p}^*, \mathbf{q}^*) > 0$ and $p_i^* \leq \mathbf{P}^a(\mathbf{p}^*, \mathbf{q}^*)$, $\forall i \in N$.*

Proof. It is clear that $\overline{p} > \mathbf{P}^a(\mathbf{p}^*, \mathbf{q}^*) > 0$ since both $\mathbf{P}^a(\mathbf{p}^*, \mathbf{q}^*) = \overline{p}$ and $\mathbf{P}^a(\mathbf{p}^*, \mathbf{q}^*) = 0$ imply $\overline{\pi}_j = 0$, $\forall j$. But since $k < \frac{d(0)}{n-1}$, we have $\underline{\pi} > 0$, a contradiction to $\overline{\pi}_j \geq \underline{\pi}$ in equilibrium. Finally, suppose there exists a firm i setting $p_i^* > \mathbf{P}^a(\mathbf{p}^*, \mathbf{q}^*)$. Then firm i's profit is equal to zero since its sales are zero. Hence, a similar argument as the above applies. ■

Lemma A2 *If $(\mathbf{p}^*, \mathbf{q}^*)$ is such that there exists firm $i \in N$ for which $0 \leq p_i^* < \mathbf{P}^a(\mathbf{p}^*, \mathbf{q}^*)$, then $\hat{q}_i^* = s_i(p_i^* | \mathbf{p}^*, \mathbf{q}^*) = \hat{k}$.*

Proof. The proof is in two parts. First we show that each firm i setting $p_i^* < \mathbf{P}^a(\mathbf{p}^*, \mathbf{q}^*)$ offers q_i^* such that $\hat{q}_i^* = \hat{k}$. Then, we show that $s_i^* = \hat{k}$ follows. Suppose contrary to the statement of the lemma that there exists a firm i setting $p_i^* < \mathbf{P}^a(\mathbf{p}^*, \mathbf{q}^*)$ and $q_i^* = \hat{q}_i^* < \hat{k}$. It is easy to see that for small enough $\epsilon > 0$, $\mathbf{P}^a(\mathbf{p}^*, q_i^* + \epsilon, q_{-i}^*) = \mathbf{P}^a(\mathbf{p}^*, \mathbf{q}^*)$. But then $\mathbf{P}^a(\mathbf{p}^*, \mathbf{q}^*)(q_i^* + \epsilon) > \overline{\pi}_i = \mathbf{P}^a(\mathbf{p}^*, \mathbf{q}^*) q_i^*$, a contradiction to (p_i^*, q_i^*) being part of an equilibrium. Suppose now that $\hat{q}_i^* = \hat{k} > s_i^*$. This can only be the case if firm i sells to residual demand. But then $p_i^* < \mathbf{P}^a(\mathbf{p}^*, \mathbf{q}^*)$ must be the uniform price, a contradiction. Thus $\hat{q}_i^* = s_i^* = \hat{k}$. ■

Lemma A3 *If $(\mathbf{p}^*, \mathbf{q}^*)$ is such that $\mathbf{P}^a(\mathbf{p}^*, \mathbf{q}^*) > 0$, then either (i) exactly one firm sets $\mathbf{P}^a(\mathbf{p}^*, \mathbf{q}^*)$, or (ii) $s_i(p_i^* | \mathbf{p}^*, \mathbf{q}^*) = k$, $\forall i \in N$.*

Proof. Suppose contrary to the statement of the lemma that a group of l firms, $l \geq 2$, tie at $p = \mathbf{P}^a(\mathbf{p}^*, \mathbf{q}^*)$. It follows immediately that if firm h sets $p_h^* = p$, then firm h can increase its profit by undercutting firms setting p slightly and offering capacity to earn:

$$p \min\{\hat{k}, R(p | \mathbf{p}^*, \mathbf{q}^*)\} \geq p s_h^* = p \min\left\{\hat{q}_h, \frac{\hat{q}_h}{\sum_{j \in L(p)} \hat{q}_j} R(p | \mathbf{p}^*, \mathbf{q}^*)\right\}.$$

where the inequality is strict whenever $s_h^* < \hat{k}$. Using Lemma A2, it follows that either there is only one firm setting $\mathbf{P}^a(\mathbf{p}^*, \mathbf{q}^*)$ or $s_i(p_i^* | \mathbf{p}^*, \mathbf{q}^*) = k$, $\forall i \in N$, which is what we had to prove. ■

Lemma A4 *Suppose $k \leq y^c$, then in every equilibrium, $\mathbf{P}^a(\mathbf{p}^*, \mathbf{q}^*) = P(nk)$.*

Proof. Suppose $\mathbf{P}^a(\mathbf{p}^*, \mathbf{q}^*) < P(nk)$. Then the (possibly multiple) firm(s) setting $\mathbf{P}^a(\mathbf{p}^*, \mathbf{q}^*)$ could increase profit strictly by offering and selling capacity at $P(nk)$ instead. Suppose $\mathbf{P}^a(\mathbf{p}^*, \mathbf{q}^*) > P(nk) > 0$. In this case, from Lemma A3, there can only be one firm, say i, setting the uniform price. Since from Lemma A2, $s_j(p_j^*|\mathbf{p}^*, \mathbf{q}^*) = k$, $\forall j \neq i$, it follows that $\pi_i(\mathbf{p}^*, \mathbf{q}^*) = \mathbf{P}^a(\mathbf{p}^*, \mathbf{q}^*) s_i(p_i^*|\mathbf{p}^*, \mathbf{q}^*) < \underline{\pi}$, a contradiction to equilibrium behavior. ■

Lemma A5 *Suppose $k \geq \frac{d(0)}{n-1}$, then in every equilibrium, $\mathbf{P}^a(\mathbf{p}^*, \mathbf{q}^*) = 0$.*

Proof. Suppose to the contrary that $\mathbf{P}^a(\mathbf{p}^*, \mathbf{q}^*) > 0$. Then from Lemma A3 and $k \geq \frac{d(0)}{n-1}$, it must be the case that there is a unique h setting p_h^*. But from Lemma A2 and $k \geq \frac{d(0)}{n-1}$, $\sum_{j \in L(\mathbf{P}^a)^-} \hat{q}_j = (n-1)\frac{d(0)}{n-1} = d(0)$, contradicting the fact that residual demand is strictly positive at $\mathbf{P}^a(\mathbf{p}^*, \mathbf{q}^*) > 0$ and thus the definition of the uniform price. Thus $\mathbf{P}^a(\mathbf{p}^*, \mathbf{q}^*) = 0$. ■

The fact that the complete set of equilibria of $\Gamma^a(k, 0)$ is given by (i), (ii) and (iii) follows from combining Lemmas A1 to A5 and constructing strategy profiles that satisfy the properties in the lemmas. It is straightforward to show that the only such strategy profiles from which no firm has an incentive to deviate are those characterized in Proposition 1. It is then clear that all Nash equilibria of $\Gamma^a(k, 0)$ are given by (i), (ii) and (iii).

7.2 Market Clearing Price rule

Lemma A6 below identifies the major difference between the two definitions of the uniform price. When the Market Clearing Price rule is used, the uniform price may be a price that no firm submitted. Lemma A6 identifies the properties that an equilibrium in which no firm sets the resulting uniform price should satisfy. It follows from Lemma A6 that all such equilibria are given by strategy profiles we characterized and are contained in either $D^e(k, 0)$ or $C(0)$. In all other equilibria, at least one firm must set the uniform price and it is straightforward to show that in this case, Lemmas A1 to A5 derived for the Maximum Accepted Price rule apply to the Market Clearing Price rule as well. Thus all equilibria of $\Gamma^e(k, 0)$ are characterized by (i), (ii) and (iii) in the statement of Proposition 1.

Lemma A6 *If $(\mathbf{p}^*, \mathbf{q}^*)$ is such that $p_i^* \neq \mathbf{P}^e(\mathbf{p}^*, \mathbf{q}^*)$, $\forall i \in N$, then either*

1. $p_i^* < \mathbf{P}^e(\mathbf{p}^*, \mathbf{q}^*) = P(n \min\{k, y^c\})$ *and* $q_i^* = s_i(p_i^*|\mathbf{p}^*, \mathbf{q}^*) = \min\{k, y^c\}$,

   or $k \in \{\kappa \in (y^c, \frac{d(0)}{n-1})|\underline{y} = d(p^r((n-1)\kappa) - (n-1)\kappa\}$ *and*

2. $(\mathbf{p}^*, \mathbf{q}^*) \in D^e(k, 0)$.

Proof. First, it clear that $\mathbf{P}^e(\mathbf{p}^*, \mathbf{q}^*) > 0$ since otherwise, it must be the case that there exists a firm i setting the uniform price $p_i^* = 0$. It is straightforward to check that $p_i^* < \mathbf{P}^e(\mathbf{p}^*, \mathbf{q}^*)$, $\forall i$, since if this were not the case, a firm setting its price strictly above $\mathbf{P}^e(\mathbf{p}^*, \mathbf{q}^*)$ would obtain a payoff of zero, which is strictly less than what it would obtain by offering a positive quantity at exactly $\mathbf{P}^e(\mathbf{p}^*, \mathbf{q}^*)$. It follows from the above arguments that $p_i^* \neq \mathbf{P}^e(\mathbf{p}^*, \mathbf{q}^*)$, $\forall i$, implies $\mathbf{P}^e(\mathbf{p}^*, \mathbf{q}^*) = P(\sum_{i \in N} q_i^*)$. Now suppose that $k \leq y^c$, then it is clear that in equilibrium, $\mathbf{P}^e(\mathbf{p}^*, \mathbf{q}^*) = P(nk)$. Otherwise it must be the case that some firm, say j, sets $q_j^* < k$, to earn $P(\sum_{i \in N} q_i^*) q_j^*$. But since $k \leq y^c$, such firm could clearly increase its profit by offering k at a price at or below the resulting uniform price to earn $P(k + \sum_{i \in N \setminus \{j\}} q_i^*)k$ instead. Suppose now that $k > y^c$. Then firm i's payoff in this case is equal to $P(\sum_{i \in N} q_i^*) q_i^*$. The unique Cournot equilibrium, $q_i^* = y^c$ $\forall i$, is then clearly an equilibrium of $\Gamma^e(k, 0)$. This completes the proof of 1.

To prove 2, note that in any equilibrium $(\mathbf{p}^*, \mathbf{q}^*)$ with $p_i^* \neq \mathbf{P}^e(\mathbf{p}^*, \mathbf{q}^*)$ and $q_j^* \neq y^c$ for some j, it must be the case that either $q_i^* = k$ or $q_i^* = r(\sum_{h \neq i} q_h^*)$. We now show that at most one firm can have $q_i^* \neq k$. Suppose there exists j for which $q_j^* \neq k$ and $\forall i \neq j$, $q_i^* = k$. Then clearly, $q_j^* = d(p^r) - (n-1)k$, since otherwise, $\overline{\pi}_j < \underline{\pi}$, a contradiction to equilibrium behavior. Moreover, for such ceilings to be part of a Nash equilibrium with $p_h^* \neq \mathbf{P}^e(\mathbf{p}^*, \mathbf{q}^*)$ $\forall h$, it must be the case that $p_j^* < p^r = \mathbf{P}^e(\mathbf{p}^*, \mathbf{q}^*)$ and, $\forall i \neq j$, $p_i^* < \underline{p}$. It is then straightforward to check that this is possible if and only if $\underline{y} = d(p^r) - (n-1)k$. Finally, suppose that there exist i and j setting $q_i^* = r(\sum_{h \neq i} q_h^*) \leq k$ and $q_j^* = r(\sum_{h \neq j} q_h^*) \leq k$. Then, for $l, m \in \{i, j\}$ and $m \neq l$, it follows from $r(\sum_{h \neq m} q_h^*) \leq k$, that $p^r(\sum_{h \neq l} q_h^*) \geq p^r$, which implies $q_l^* \leq d(p^r) - (n-1)k$ $\forall l \in \{i, j\}$. But since $\underline{y} = d(p^r) - (n-1)k$, for $l, m \in \{i, j\}$ $l \neq m$, the following holds: $k \leq r((n-2)k + \underline{y}) \leq r((n-2)k + q_m^*) = r(\sum_{h \neq l} q_h^*)$, with equality if and only $q_m^* = k$. Therefore $r(\sum_{h \neq l} q_h^*) < k$ cannot hold for both $l \in \{i, j\}$. Appealing to the definition of $D^e(k, 0)$ completes the proof of 2 and the proof of the Lemma. ■

Finally, for both uniform price rules, simple computations show that for all values of capacity k and each firm $i \in N$, there exists a pure strategy equilibrium in which firm i obtains its minmax payoff $\underline{\pi}$. ■

8 Proof of Lemma 3

Let τ be a perfectly collusive stationary path with equal sharing. Then τ must satisfy the properties stated in Lemma 2. We first show that incentives to deviate cannot increase for any firm if every firm setting $p_i = p^m$ also sets $q_i = k$. Let $l \geq 1$ be the number of elements of $L(p^m)$. By (iii), for firms in $L(p^m)$, the residual demand is $R(p^m) = d(p^m) - (n-l)\frac{d(p^m)}{n} = l\frac{d(p^m)}{n}$. Furthermore, equal sharing requires that if $i \in L(p^m)$, $s_i = \min\{\hat{q}_i, \frac{\hat{q}_i}{\sum_{j \in L(p^m)} \hat{q}_j} l \frac{d(p^m)}{n}\} = \frac{d(p^m)}{n}$. But then it is easy to see that setting $q_i = k$, $\forall i \in L(p^m)$ implies $s_i = \frac{d(p^m)}{n}$ and cannot increase any firm's incentive to deviate.

We now show that if i and j set $p_i < p^m$ and $p_j < p^m$, then $p_i = p_j$ minimizes incentives to deviate. Suppose this is not the case and, w.l.o.g., that $p_i < p_j$. Then, using (1), i and j's incentive constraints are given by:

$$(1-\delta)\pi_i^*(p_{-i},q_{-i})+\delta\underline{\pi}\leq p^m\frac{d(p^m)}{n}.$$

$$(1-\delta)\pi_j^*(p_{-j},q_{-j})+\delta\underline{\pi}\leq p^m\frac{d(p^m)}{n}.$$

Suppose firm j reduces its price to match p_i. By (ii) in Lemma 2, it must be the case that $q_{-i}=q_{-j}=\frac{d(p^m)}{n}$, thus firms' sales do not change. By Lemma 1 deviation profits $\pi_h^*(p_{-h},q_{-h})$ are non-decreasing in p_{-h} for fixed q_{-h}. Thus deviation profits cannot increase by setting $p_j=p_i$. On the other hand, a firm's profit on the initial path is unaffected since the uniform price does not change and neither do any firm's sales.

Next we show that if i sets $p_i=p^m$, then its incentive to deviate is minimized when the remaining $n-1$ firms set their price equal to $\underline{p}(q_-^m)$ (and thus offer exactly $\frac{d(p^m)}{n}$), from which it follows that all firms will set $\underline{p}(q_-^m)$ and $\frac{d(p^m)}{n}$, which completes the proof of the lemma. By definition of $p^r(q)$, the worst possible deviation profit firm i can guarantee itself when lower-priced firms offer $q<d(0)$ is:

$$p^r(q)[d(p^r(q))-q].$$

The above expression is decreasing in q. Since every firm setting $p<p^m$ must offer $\frac{d(p^m)}{n}$, firm i's worst possible deviation profit is minimized when the number of low-priced firms is the highest, that is, when the remaining firms' aggregate quantity offered is q_-^m. In this case, firm i's deviation profit is given by:

$$p^r(q_-^m)[d(p^r(q_-^m))-q_-^m]=\underline{\pi}(q_-^m). \tag{A1}$$

Note that it follows from $k>\frac{d(p^m)}{n}$ that $\underline{p}(q_-^m)<p^r(q_-^m)<p^m$. Furthermore, if each firm setting $p_h<p^m$ sets $p_h=\underline{p}(q_-^m)$, then firm i's deviation profit is indeed equal to its worst possible deviation profit. It remains to show that the uniform price is actually p^m. This is easily verified using Definition 1 and the fact that if all firms offer $\frac{d(p^m)}{n}$, then $\sum_{i\in N}\hat{q}_i=d(p^m)$ where $N=\mathcal{L}(p^m|\mathbf{p},\mathbf{q})$. ■

9 Proof of Proposition 3

We first characterize $\overline{\delta}$. From Lemma 2, the only perfectly collusive stationary path with equal sharing on which every firm offers its capacity is τ^{sm}, where τ^{sm} is as follows:

$$p_i \;=\; p^m \text{ and } q_i=k,\forall i.$$

From the definition of s_i, sales on τ^{sm} are equal to $\frac{k}{nk}d(p^m)=\frac{d(p^m)}{n}$ and the uniform price is p^m. From (1), on τ^{sm}, the symmetric incentive constraints are:

$$(1-\delta)p^m\min\{k,d(p^m)\}+\delta\underline{\pi}\leq p^m\frac{d(p^m)}{n}. \tag{A2}$$

Thus τ^{sm} is sustainable if and only if (A2) holds. Solving for δ from (A2) satisfied with equality, we obtain:

$$\overline{\delta} = \frac{p^m \min\{k, d(p^m)\} - p^m \frac{d(p^m)}{n}}{p^m \min\{k, d(p^m)\} - \underline{\pi}}. \tag{A3}$$

Therefore if perfect collusion with equal sharing is sustainable for $\delta < \overline{\delta}$, it must be the case that some firm withholds. We now show that there exists $\underline{\delta}^e < \overline{\delta}$ such that if $\delta > \underline{\delta}^e$, perfect collusion with equal sharing is sustainable.

To determine the value of $\underline{\delta}^e$, note that from Lemma 3, on the path that minimizes incentives to deviate, τ^{me}, each firm's deviation profit is given by (A1). Thus, from (1), $\underline{\delta}^e$ is the value of δ that solves

$$(1-\delta)\underline{\pi}(q_-^m) + \delta\underline{\pi} = p^m \frac{d(p^m)}{n}.$$

After rearranging, we obtain:

$$\underline{\delta}^e = \frac{\underline{\pi}(q_-^m) - p^m \frac{d(p^m)}{n}}{\underline{\pi}(q_-^m) - \underline{\pi}} \tag{A4}$$

Since $\underline{\pi}(q_-^m) < p^m \min\{k, d(p^m)\}$ clearly holds for all values of k, $\underline{\delta}^e < \overline{\delta}$ follows. Finally, since we have shown deviation profits are strictly lower on τ^{me} than they are on any other perfectly collusive stationary path with equal sharing, it follows that for $\delta = \underline{\delta}^e$, withholding by all firms is necessary to sustain perfect collusion with equal sharing on a stationary path. ■

10 Proof of Lemma 4

The proof is similar to the proof of Lemma 3 above except for the last two paragraphs, which rely on the particular definition used for the uniform price. When Definition 2 is used, one firm must set p^m on any perfectly collusive stationary path. Thus on the path that minimizes incentives to deviate, deviation profits differ from those under Definition 1.

To complete the proof of Lemma 4, we show that when all but one firm set $p = \underline{p}(q_-^m)$, incentives to deviate are also minimized for the low-priced firms. Suppose $l \geq 1$ firms set $p < p^m$ and offer $\frac{d(p^m)}{n}$ each. Below, we compute such a firm's deviation profit depending on k. First note that incentives to deviate cannot increase if the l low-priced firms set $p = \underline{p}(q_-^m)$ rather than some $p' \in (\underline{p}(q_-^m), p^m)$ (Lemma 1). Suppose then that the l low-priced firms set $p = \underline{p}(q_-^m)$. It is straightforward to check that undercutting p cannot be optimal if it results in the uniform price being reduced to p. If $l = 1$, the optimal deviation clearly consists in undercutting p^m and offering k, for every k. If $l > 1$ and firm j is a low-priced firm, optimal deviations then take one of two possible forms depending on k. Either firm j undercuts p and offers the minimum of k and a quantity infinitesimally less than $\frac{(n-l+1)d(p^m)}{n}$ to obtain:

$$p^m \min\{k, \frac{(n-l+1)d(p^m)}{n}\}; \tag{A5}$$

or firm j maximizes profit on residual demand after $l-1$ low-priced firms have sold their quantity $\frac{d(p^m)}{n}$ to obtain:

$$p^r(q_l^H)[d(p^r(q_l^H)) - q_l^H] \text{ where } q_l^H \equiv \frac{(l-1)d(p^m)}{n}. \tag{A6}$$

If $k \leq \frac{(n-l+1)d(p^m)}{n}$, it is clear that $p^r(q_l^H) \geq p^m$, and thus (A6) is not a possible deviation (since then the deviating firm would be setting a higher price than all $n-1$ remaining firms). In this case, firm j's optimal deviation profit is given by (A5) and equals $p^m k$. If $\frac{(n-l+1)d(p^m)}{n} < k \leq r(q_l^H)$, then $p^r(q_l^H) = P(k + q_l^H) < p^m$. In this case, firm j's optimal deviation consists in selling its capacity at $p^r(q_l^H) < p^m$. Finally, if $k > r(q_l^H)$, $p^r(q_l^H) = P(r(q_l^H) + q_l^H) < p^m$ and firm j's optimal deviation consists in setting the residual demand monopoly price after $l-1$ firms have sold their quantity, and offering $q_j \in [r(q_l^H), k]$ for sale. It is clear that deviation profits in (A5) and (A6) are non-increasing in l. Therefore, they are minimized by setting $l = n-1$. Moreover, inspecting (A5) and (A6), optimal deviations do not depend on p as long as $p \leq \underline{p}(q_-^m)$. Therefore, setting $p = \underline{p}(q_-^m)$ cannot increase incentives to deviate. ■

11 Proof of Proposition 4

Since we have characterized $\overline{\delta}$ in the proof of Proposition 3, we only need to characterize $\underline{\delta}^a$ and compare in order to prove Proposition 4. From Lemma 4, the path that minimizes incentives to deviate in the class of perfectly collusive paths with equal sharing is of the form τ_i^m. On τ_i^m, firm i's optimal deviation yields $p^r(q_-^m)[d(p^r(q_-^m)) - q_-^m]$, which is clearly strictly less than $p^m \max\{k, d(p^m)\}$, a firm's optimal deviation profit on τ^{sm}. On τ_i^m, for $j \neq i$, firm j's optimal deviation is either (A5) or (A6) above after substituting for $l = n-1$. (A5) is then equivalent to $p^m \min\{k, \frac{2d(p^m)}{n}\}$. It is clear that if $n = 2$ or $n > 2$ and $k \leq \frac{2d(p^m)}{n}$, firm j's optimal deviation yields $p^m k$, in which case τ_i^m does not strictly relax incentives to deviate. If $n > 2$ and $k > \frac{2d(p^m)}{n}$, the optimal deviation is given by (A6). Letting $q^H \equiv \frac{(n-2)}{n} d(p^m)$, (A6) is equivalent to:

$$\pi^H \equiv p^r(q^H)[d(p^r(q^H)) - q^H].$$

Since we have $k > \frac{2d(p^m)}{n}$, $P(k + q^H) < p^m$. Moreover, if k is such that $p^r(q^H) \neq P(k + q^H)$, then it is clear that since $q^H > 0$, $p^r(q^H) < p^m$ as well. Hence,

$$\pi^H < p^m \min\{k, d(p^m)\}. \tag{A7}$$

Finally, since $q^H < q_-^m$, $\pi^H > p^r(q_-^m)[d(p^r(q_-^m)) - q_-^m]$, i.e., a low-priced firm's deviation profit is higher than firm i's deviation profit. Therefore, if $k > \frac{2}{n} d(p^m)$, $\underline{\delta}^a$ is given by any low-priced firm's incentive constraint satisfied with equality, which using (1) is given by

$$(1-\underline{\delta}^a)\pi^H + \underline{\delta}^a\ \underline{\pi} = p^m \frac{d(p^m)}{n}.$$

After solving for δ in the above, we obtain:

$$\underline{\delta}^a = \frac{\pi^H - p^m \frac{d(p^m)}{n}}{\pi^H - \underline{\pi}}. \tag{A8}$$

Using (A3) and (A7), it follows from (A8) that $\underline{\delta}^a < \overline{\delta}$.

Finally, since we have shown deviation profits are strictly lower on τ^{ma} than they are on any other perfectly collusive stationary path with equal sharing, it follows that for $\delta = \underline{\delta}^a$, withholding by all but one firms is necessary to sustain perfect collusion with equal sharing on a stationary path. ■

12 Proof of Lemma 5

Let $(\mathbf{p}_{-i}, \mathbf{q}_{-i})$ be a vector firm i's rival's strategies. We show that for both uniform price auction rules, $\pi_i^*(\mathbf{p}_{-i}, \mathbf{q}_{-i}) = \sup_{(p_i,q_i)} \pi_i(p_i, q_i, \mathbf{p}_{-i}, \mathbf{q}_{-i})$ can be obtained with a strategy using a single step. Throughout the proof, we ignore price steps l_i such that $p_i^{l_i}$ is strictly greater than the uniform price. Such price steps are indeed irrelevant since the quantity sold at such steps is equal to zero, so that firm i cannot increase its profit by including them in its strategy. Let $(p_i^*, q_i^*) \in \arg\sup_{(p_i,q_i)} \pi_i(p_i, q_i, \mathbf{p}_{-i}, \mathbf{q}_{-i})$ and p_i^{s*} be the set of prices submitted by firm i. First, $\mathbf{P}^{u*} = \mathbf{P}^u(p_i^*, q_i^*, \mathbf{p}_{-i}, \mathbf{q}_{-i})$ is only possible if firm i's residual demand at strictly positive prices is equal to zero. In this case, $\pi_i^* = 0$ is independent of firm i's strategy and can therefore be obtained by using a single step bidding function (for instance, offering k at $p_i = 0$). Suppose that $\mathbf{P}^{u*} > 0$. There are two cases: either $\mathbf{P}^{u*} \in p_i^{s*}$ or $p_i^{l_i*} < \mathbf{P}^{u*}$, $\forall l_i$. Consider first the case $\mathbf{P}^{u*} \in p_i^{s*}$. It is clear that tying with a group of firms at $\mathbf{P}^{u*}$ cannot be optimal unless $\mathbf{s}_i^* = \mathbf{s}_i(p_i^*, q_i^*, \mathbf{p}_{-i}, \mathbf{q}_{-i}) = \min\{d(\mathbf{P}^{u*}), k\}$, since otherwise, firm i could strictly increase its profit by slightly undercutting $\mathbf{P}^{u*}$. However if $\mathbf{s}_i^* = \min\{d(\mathbf{P}^{u*}), k\}$, then, from Definitions 1 and 2, it is straightforward to check that it must be the case that firm i can obtain $\pi_i^* = \mathbf{P}^{u*}\mathbf{s}_i^*$ by using the one-step strategy $(\mathbf{P}^{u*}, \min\{d(\mathbf{P}^{u*}), k\})$. Suppose then that $\mathbf{s}_i^* < \min\{d(\mathbf{P}^{u*}), k\}$. It follows from the above arguments that firm i must be the only firm with $\mathbf{P}^{u*}$ in its strategy. Thus

$$\pi_i^* = \mathbf{P}^{u*}[\min\{\hat{q}_i(\mathbf{P}^{u*}), d(\mathbf{P}^{u*}) - \sum_{j\in N\setminus\{i\}} \sum_{p\in p_j^s} q_j(p)\} + \sum_{p\in p_j^s\setminus \mathbf{P}^{u*}} \hat{q}_i^{l_i*}]$$

Note that unless $\hat{q}_i(\mathbf{P}^{u*}) = k < d(\mathbf{P}^{u*}) - \sum_{j\in N\setminus\{i\}} \sum_{p\in p_j^s} q_j(p)$ (in which case $\mathbf{s}_i^* = \min\{d(\mathbf{P}^{u*}), k\}$), firm i would never set $q_i^*(\mathbf{P}^{u*}) < d(\mathbf{P}^{u*}) - \sum_{j\in N\setminus\{i\}} \sum_{p\in p_j^s} q_j(p)$. Thus

$$\pi_i^* = \mathbf{P}^{u*}[d(\mathbf{P}^{u*}) - \sum_{j\in N\setminus\{i\}} \sum_{p\in p_j^s} q_j(p) + \sum_{p\in p_j^s\setminus \mathbf{P}^{u*}} \hat{q}_i^{l_i*}]$$

But this is exactly what firm i would obtain if for every $(p_i^{l_i*}, q_i^{l_i*})$ such that $p_i^{l_i*} \in p_i^{s*}(\mathbf{P}^{u*}) \setminus \{\mathbf{P}^{u*}\}$ (the set of steps for which $p_i^{l_i*}$ is strictly below $\mathbf{P}^{u*}$), firm i set the single step $(\mathbf{P}^{u*}, q_i^{l_i*})$ instead. Finally, consider the case $p_i^{l_i*} < \mathbf{P}^{u*}$, $\forall l_i$. Then, from Definitions 1 and 2, it must be the case that $\pi_i^* = \mathbf{P}^{u*}\mathbf{s}_i^*$ is independent of firm i's prices $p_i^{l_i*}$ since $p_i^{l_i*} < \mathbf{P}^{u*}$, $\forall l_i$. Therefore, in this case, any single-step bidding function $(p, \sum_{z \in p_i^{s*}} q_i(z))$ for some $p < \mathbf{P}^{u*}$ achieves a payoff equal to π_i^*. ■

13 Proof of Proposition 7

For $\epsilon \in (0, d(p^m)]$, on τ^ϵ, each firm has the same incentive to deviate in every period. It simple to check that a firm's profit from an optimal unilateral deviation is given by

$$\pi_i^* = p^m k \tag{A9}$$

if $d(p^m) + (n-1)\epsilon \geq k$. It is obtained by expanding output up to k at $\underline{p}(q_-^\epsilon)$. If $k > d(p^m) + (n-1)\epsilon$, then

$$\pi_i^* = \underline{\pi}(q_-^\epsilon), \tag{A10}$$

which is obtained by setting the residual demand monopoly price after firm i's rivals have sold q_-^ϵ. We now show that τ^ϵ minimizes incentives to deviate in the class of stationary perfectly collusive paths with equal sharing. Since on a perfectly collusive path with equal sharing, it must be the case that $\mathbf{P}^a = p^m$, it is clear that incentives to deviate cannot be lowered by moving some quantity offered at $\underline{p}(q_-^\epsilon)$ to some price strictly between $\underline{p}(q_-^\epsilon)$ and p^m. Furthermore, for every i, we can rule out steps such that $p > \mathbf{P}^a = p^m$ and $q_i(p) > 0$ since setting such steps does not affect the outcome (because a firm's sales are equal to zero at such prices), but may increase incentives to deviate. Since on a stationary perfectly collusive paths with equal sharing, for a given $(p_i^{l_i}, q_i^{l_i})$ such that $p_i^{l_i} < p^m$, $p_i^{l_i}$ affects neither firm i's sales nor its profit, it follows that we can, without loss of generality, restrict attention to stationary paths on which firms use at most 2 steps in their bidding function satisfying the following. If for some $i, j \in N$, there exists $p_i \in p_i^s$ and $p_j \in p_j^s$ such that $p_i < p^m$ and $p_j < p^m$, then $p_i = p_j = p$. If q_{-i}^* is the total quantity offered by firm i's rivals at $p < p^m$, other things equal, incentives to deviate are minimized if p is set equal to $p_l = \min_{i \in N}\{\underline{p}(q_{-i}^*)\}$, the highest price that no firm would ever want to undercut. Additionally, for every firm j such that $p^m \in p_j^s$, $q_j(p^m) = k - q_j(p_l)$, minimizes incentives to deviate (since then, when firm j ties with a group of firms at p^m, it cannot deviate by increasing its quantity offered at p^m without decreasing its quantity offered at p_l). Finally, to minimize firm i's deviation profit, the quantity offered by firm i's rivals at a price strictly below p^m $q_{-i}^* = \sum_{j \neq i} q_j(p)$, $p < p^m$, must be as large as possible, while satisfying $\sum_{i \in N} q_i(p) \leq d(p^m) - \epsilon$. thus for every i, this maximum quantity is obtained when $\sum_{i \in N} q_i(p) = d(p^m) - \epsilon \iff q_{-i}^* = d(p^m) - \epsilon + q_i(p)$. It follows that incentives to deviate are minimized at $(q_1(p), \ldots, q_n(p))$ satisfying $q_{-i}^* = d(p^m) - \epsilon + q_i(p) = d(p^m) - \epsilon + q_j(p) = q_{-j}^*$, $\forall i, j$, $i \neq j$. Solving for $q_1(p)$ and substituting in $\sum_{i \in N} q_i(p) = d(p^m) - \epsilon$ yields $q_1(p) = \ldots = q_i(p) = \ldots = q_n(p) = \frac{d(p^m) - \epsilon}{n} = q_-^\epsilon$. It follows that $p_l = \underline{p}(q_-^\epsilon)$ and that each firm offers $k - \left[\frac{d(p^m) - \epsilon}{n}\right]$ at p^m. ■

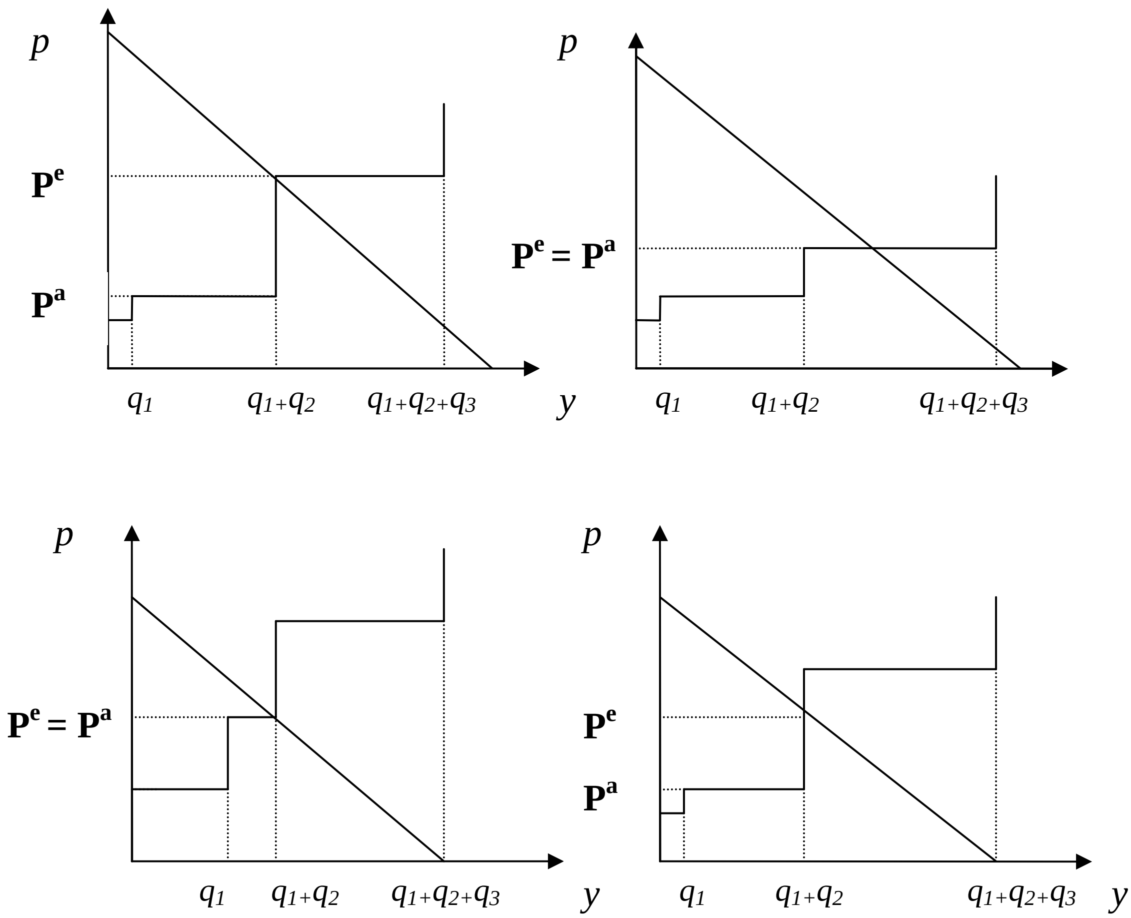

Figure 1: Determining the uniform price under Definition 1 and Definition 2.

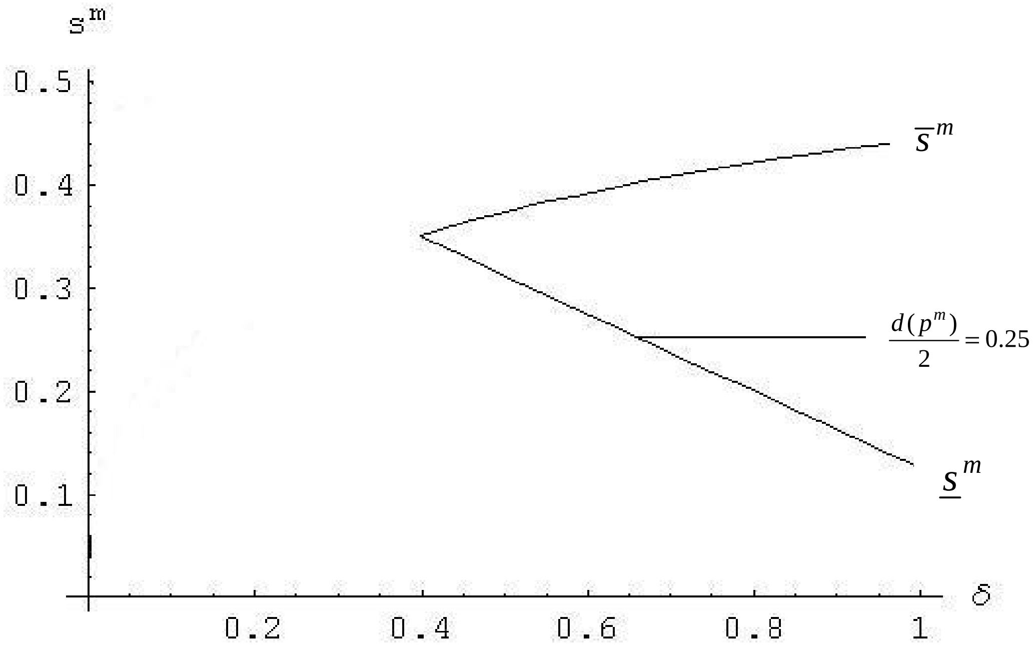

Figure 2: Range of possible divisions of monopoly output for the low-priced firm, firm i, on a stationary perfect equilibrium path for $k = \frac{1}{2}$. $\underline{s}^m$ is the smallest allocation and $\overline{s}^m$ the largest allocation firm i can obtain on such paths.

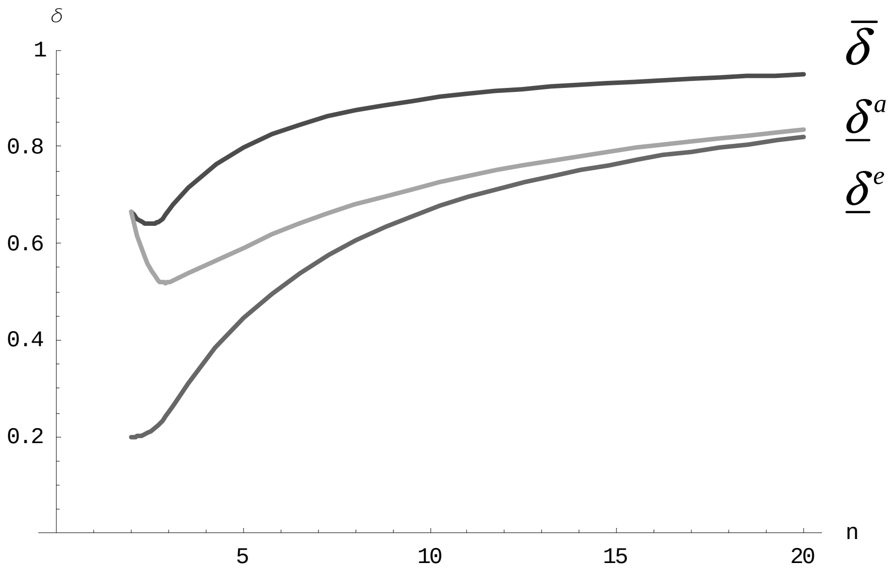

Figure 3: Critical values of the discount factor as a function of the number of firms for $k = \frac{1}{2}$.