On the Weights of Nations: Assigning Voting Weights in a Heterogeneous Union

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Abstract

Consider a voting procedure where countries, states, or districts comprising a union each elect representatives who then participate in later votes at the union level on their behalf. The countries, provinces, and states may vary in their populations and composition. If we wish to maximize the total expected utility of all agents in the union, how to weight the votes of the representatives of the different countries, states or districts at the union level? We provide a simple characterization of the efficient voting rule in terms of the weights assigned to different districts and the voting threshold (how large a qualified majority is needed to induce change versus the status quo). Next, in the context of a model of the correlation structure of agents preferences, we analyze how voting weights relate to the population size of a country. We then analyze the voting weights in Council of the European Union under the Nice Treaty and the recently proposed constitution, and contrast them under different versions of our model.

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1 Introduction

Unions of nations, states, or districts, make large numbers of decisions by votes. These votes are generally cast by representatives of the nations, states or districts, who in turn have been elected by their local populations. Such two-stage procedures are sensible due to the costs of involving full populations in innumerable decisions. Nevertheless, such two-stage procedures introduce distortions in the decision process, due to the fact that a single vote by a representative does not always adequately represent the heterogeneity of votes that would be cast by the population itself.

Understanding such two-stage voting procedures and characterizing the voting rules in this context that maximize welfare is especially important given the numerous applications. A particularly important and timely application is to the Council of the European Union, which we discuss in more detail below.

Given that the countries, states, or districts comprising the union may be of different sizes and have different compositions in terms of distributions of citizens' preferences, it makes sense to weight the votes of the representatives.[1] For instance, some obvious difficulties can result if countries differ in population and their voting power is not weighted. Then, small countries might impose decisions that a majority of the affected people are against.

But what are the "right" weights for each of the countries? Should they be proportional to population, or are there additional considerations to be made? In fact, many political arrangements fitting our model (exactly or approximately) tend to assign weights in a way that gives less to large countries or districts and more to small countries or districts, than what they would get under weights that are proportional to population.

We provide an analysis under the objective of maximizing the total expected utility of the population of the union. We characterize the voting rules that achieve this maximum, which we call the "efficient" voting rules. To be specific, we refer to the case where votes are of a "yes" or "no" variety. That is, there are two alternatives and so there is no issue of strategic voting or cycles. Representatives vote on behalf of their constituencies, and the union makes decisions weighting the votes of the representatives and applying some threshold in order to choose "yes" over "no.". The characterization of voting rules is then in terms of both the weights assigned to various countries or districts, and the threshold required to select "yes" over "no".

We show that the selection of optimal weights and thresholds can be treated separately,

[1] Alternatively, one can think of adjusting the number of representatives that each country, state, or district has - and we shall come back to discuss this.

with the weights depending on the differing compositions of countries, and the threshold depending on the general bias in favor of "no" over "yes".

In particular, the efficient weights can be described as follows. Look at the vote by a given representative of a country. Suppose that he or she has voted "yes" on a given issue. We can ask the following question. Given the vote of "yes", what is the surplus of people in the country who favor "yes" over "no". For instance if 62% of the people favor "yes" and 38% favor "no", then 24% more of the population favor "yes" versus "no". This is a measure of how much this country would benefit if we choose "yes" versus "no". The efficient weight is exactly the *expectation* of this surplus conditional on seeing a "yes" vote by the representative. This follows fairly directly from the fact that our objective is to please the most people possible across countries.

We emphasize that this perspective is very different from the rhetoric that often underlies political discussions, where the vote by representatives are taken to coincide with the wishes of the whole of their country. Most importantly, this measure of the surplus of the population favoring one alternative over another is not always directly related to the population size; and so treating representatives' votes in proportion to their population size can be inefficient.

In particular, beyond a general characterization of efficient voting rules, we also provide a model of population behavior (that we refer to as the "block model") that allows us to pinpoint the efficient voting weights and thresholds under two focal scenarios. This allows us to understand when weights that are proportional to population would be appropriate, and when a rescaling that is less than proportional to population would be in order. Our model thus offers some simple tests of the extent to which, by calculus or accident, the weights attributed to nations in a given union are efficient.

After the development of our theoretical model, rest of the paper is devoted to the analysis of the voting systems of the Council of European Union, as suggested under both the Nice treaty of 2000 and the Constitutional Convention of 2003. The voting rules are quite different, with the Nice Treaty assigning weights that are less than proportional to a country's population and the proposed Constitution assigning weights that are directly proportional to a country's population. We show that these two conflicting proposals coincide with the two polar cases of our "block model" of population behavior. Which set of weights is more efficient then boils down to understanding which one of these two models is a better fit of reality. The two proposals also differ in the voting thresholds they suggest. We emphasize that the optimality of weights and thresholds can be completely disassociated from each other. Thus, we separately discuss how the different thresholds correspond to different hypotheses about the bias of voters in favor of the status-quo over change.

Relation to the Literature

There is a rich literature in cooperative game theory that examines weighted majority games. One main thread in that literature has been to produce power indices measuring things such as the relative probabilities that different voters are pivotal. These include the Banzhaf (1965) and Shapley-Shubik (1954) indices, among others. One central way in which our analysis differs from most of that literature is that we are interested in total satisfaction in terms of expected utilities rather than a measure of pivots or what is often called decisiveness.[2]

While there are some power measures have analyzed satisfaction (expected utility) and contrasted it with decisiveness (see for instance Dubey and Shapley (1979), Barry (1980) and Laruelle and Valenciano (2003)); our perspective is still quite different. Most importantly, our aim is not to produce some measure of power or satisfaction or to compare rules under such measures, but instead to study the optimal design of voting rules. We provide a full characterization of the voting rules that maximize total expected utility and show how these relate to the underlying distributions of agents' preferences, among other things.

To the extent that the previous literature has thought about designing rules, it has focussed on equating the power of agents, rather than maximizing the total expected utilities of agents. This dates to the seminal work of Penrose (1946). Depending on the distribution of preferences, these two objectives can lead to quite different voting rules. And, interestingly, maximizing total expected utility can result in large inequalities in the treatment of individuals across countries. We provide some results outlining how the asymmetric treatment of agents depends on the situation.

Perhaps the closest predecessor to the theoretical part of our work is that of Felsenthal and Machover (1999), who also study the design of two-stage voting rules from an optimization perspective. Their objective is to minimize the expected difference between the size of the majority and the number of supporters of the chosen alternative.[3] Their objective differs from maximizing total expected utility in that it does not account for the surplus of voters in favor of an alternative when the majoritarian alternative is selected, but only accounts for the deficit when the majoritarian alternative is not selected. While these two perspectives differ, they lead to the same weights in the particular case of large countries of i.i.d. voters,

[2] Rae (1969) was the first to analyze voting rules under this utilitarian perspective of maximizing expected utility or satisfaction rather than decisiveness (see also Badger (1972) and Curtis (1972)). He was interested in the normative foundations of simple majority rule as a form of direct democracy.

[3] See Felsenthal and Machover for an illuminating discussion of their objective, and some of the imprecisions in the previous literature.

where the weights are proportional to the square root of a country's population size, as originally suggested by Penrose (1946) from an even different perspective.

The setting with a large number of i.i.d. voters is special and not so realistic, and the coincidence of the weights is peculiar to that case. Our analysis applies to a more general model, where the weights that maximize total expected utility usually differ from the square root of population size. In particular, we show how the efficient voting rules vary in interesting ways according to the correlation structure of agents' preferences, as well as the bias for one alternative over another (for instance for the status-quo as opposed to change), and the behavior of countries' representatives. This is the first analysis that accounts for such correlations and other factors that we are aware of.

Finally, there is also a literature that is related to our analysis of the structure of the European Union's decision-making. Some of that literature has brought ideas from the literature on weighted games to assess the relative power of different countries under the Nice Treaty (e.g., see Laruelle (1998), Laruelle and Widgrén (1998), Sutter (2000), Baldwin, Berglöf, Giavazzi, and Widgrén (2001), Bräuninger and König (2001), Galloway (2001), Leech (2002), and some of the references cited there). As the foundations of our analysis of voting rules differs from the previous literature and power indices, so does our analysis of the Nice Treaty and the new Constitution. Among other things, we identify conditions on the correlation structure of citizens' preferences that would justify the various rules that have been proposed, something which does not appear previously.

2 A Simple Example

We begin by presenting a simple example that gives a preview of some of the issues that arise in designing an efficient voting rule. The example shows why in some cases it will be efficient to use weights that are not proportional to population.

Example **1** *Non-Proportional versus Proportional Weights*

Consider a world with three countries. Countries 1 and 2 have populations of one agent each. Country 3 has a population of three agents.

Each agent has an equal probability of supporting alternative a as alternative b. An agent gets a payoff of 1 if their preferred alternative is chosen, and -1 if the other alternative is chosen. Thus, total utility can be deduced simply by keeping track of the number of agents who support each alternative.

First, let us consider a situation where we weight countries in proportion to their populations and then use a threshold of 50% of the total weight. That would result in weights of $w = (1, 1, 3)$ and a threshold of 2.5. This reduces to letting country 3 choose the alternative.

Here it is possible for a minority of agents to prefer an alternative and still have that be the outcome. For instance, if two agents in country 3 prefer a, and all other agents prefer b, then a is still chosen.

Let us compare this to the efficient weights - that is, those that maximize the total expected utility. Here those weights turn out to be (1,1,1.5), and the threshold is 1.75. Thus, this voting rule is equivalent to one vote per country. The proof that this is the efficient rule comes from our characterization theorem below, but we can see the improvement in utility directly.

First, note that it is still possible for a minority of agents to prefer a and a majority to prefer b, but to still have a selected. For instance, this happens if agents in countries 1 and 2 prefer a, but agents in country 3 all prefer b. Despite the fact that the rule is not always making the correct choice in terms of maximizing the total utility, there is an important distinction between the efficient rule and the proportional rule here. Fewer configurations of preferences under the efficient weights lead to incorrect (minority-preferred) decisions.

Let us list configurations that are problematic in terms of agents preferences, where the last three agents are the agents in country 3.

The only way that a can be the outcome and only be preferred by a minority under the efficient weights is when preferences are (a;a;b,b,b).

However, under the weights that are proportional to population there are three preference configurations that can lead to a being chosen when preferred by a minority. These are (b;b;a,a,b), (b;b;a,b,a) and (b;b;b,a,a).

When we compute the total expected utility (summed across all agents) it is 1.75 under the efficient weights compared to 1.5 under the population weights, which reflects this difference in potential incorrect decisions.

This example is clearly a very stark one. It illustrates some of the ideas that we will run across in what follows. More generally, the characterization of the efficient rule will depend on many considerations including the distribution of agents' preferences, the way in which representatives of a country act, and the configuration of countries. In some cases weights that are proportional to population are efficient, while in other cases non-proportional weights are efficient. We now turn to that more general analysis.

3 The Model

Decisions and Agents

A population of agents is divided into m countries.

Country i consists of n_i agents and we denote this set by C_i. The total number of agents is $n = \sum_i n_i$.

Although we use the language of a union of countries, the model equivalently applies to any voting procedure where different groups elect representatives who then vote on their behalf.

These agents must make a decision between two alternatives that we label a and b.

A state of the world s will be a description of agents' preferences over the two alternatives. In a given state of the world, each agent is either a supporter of alternative a or a supporter of alternative b. We need only keep track of the difference in utility that a agent has for alternatives a and b. Thus, without loss of generality we normalize things so that agent j gets a utility of s_j if a is chosen and a utility of 0 if b is chosen.

So, a state of the world is a vector $s \in I\!R^n$, with element s_j being the difference between agent j's valuations for a and b.

A Two Stage Voting Procedure

The decision making process is described as follows.

The First Stage

In the first stage, a country's representative decides whether to vote for a or b. This decision will generally depend on the state of agents' preferences.

We use $r_i = a$ to denote that the representative of country i will vote for a, and $r_i = b$ to denote that the representative will vote for b.

At this point we remain agnostic on how the decision of a representative's vote relates to the state of agents' preferences.

Possibilities are that the representative is elected with a mandate, or that the representative is an existing politician who polls the population, or that the representative is a dictator, bureaucrat, etc., who might decide on how to vote quite differently. Later in the paper we will consider a situation where the "representative" is in fact that, namely he or she votes in accordance with a majority of the population.

The Second Stage

In the second stage, the representatives from each country meet and vote according to a weighted voting rule with a qualified majority. In particular, each representative casts a vote for either a or b. The vote of the representative of country i is given a weight $w_i \in I\!R_+$. The tally of votes for a is simply the sum of the w_i's of the representatives who cast votes for a, and similarly for b. Alternative a is selected if its tally of weights exceeds the qualified majority threshold (denoted $\beta \in [0, \sum_i w_i]$), alternative b is selected if the tally of weights for a is less than the qualified majority threshold, and ties are broken by the flip of a fair coin.

Let $v : I\!R^n \to \{-1, 0, 1\}$ denote the outcome of this two stage voting procedure as a function of the state. Here $v(s) = 1$ is interpreted as meaning that alternative a is chosen, $v(s) = -1$ means that alternative b is chosen, and $v(s) = 0$ denotes that a tie has occurred and a coin is flipped.

We let V denote the set of all such weighted voting rules with qualified majorities.

The reason that we code $v(s)$ in this way is that the utility of a agent j in state s can now be written as $v(s) \times s_j$.[4] Thus the total utility summed across all agents in all countries is

$$v(s) \sum_j s_j,$$

and the total expected utility of the union using a voting rule v is denoted

$$E\left[\sum_j v(s) s_j\right].$$

Equivalent Voting Rules

We must recognize that different weights and thresholds can lead to the same voting rule, and so voting rules will only be defined up to an equivalence class of weights and thresholds.

Beyond defining two different pairs of weights and thresholds to be equivalent if their induced voting rules always make the same choices, we need a coarser requirement for our main results due to the fact that tie-breaking is not completely tied down under efficient voting rules.

Let us say that a profile of voting weights and threshold w, β with induced voting rule v is *equivalent up to ties* to a profile of voting weights and threshold w', β' with induced voting rule v' if $v(s) = v'(s)$ for all s such that $v'(s) \neq 0$.

[4] To be careful, this denotes twice the utilities in the sense that s_j is the difference between the utility for a and b, and this difference is now doubled in our accounting. We do this to accommodate ties in voting.

This is not quite an equivalence relationship, as it allows v to break ties in a different way from v'.[5]

To see why we define equivalence only up to ties consider a simple example. There are two countries and each consists of a single agent whose utilities take on values in $\{-1, 1\}$. Let w' be (1,1) and the threshold be 1. Note that the induced voting rule v' would be efficient for this example. When things are unanimous, v' picks the unanimous choice, but when s_1 and s_2 are of opposite signs, the rule flips a coin and so $v'(s) = 0$. Alternative weights $w = (1 + \varepsilon, 1)$ with a threshold of $1 + \frac{\varepsilon}{2}$ would also be efficient, but would favor the first agent in the case of a tie. Thus, its induced voting rule v would be more resolute than v', but would make the same choices in any case where efficiency was at stake.

Equivalent voting weights and thresholds can be rescalings of each other, but also might not be. For instance with three countries, $w = (3, 2, 2)$ with a threshold of 3.5 is equivalent to $w' = (1, 1, 1)$ with a threshold of 1.5 - they both select the alternative that at least two countries to voted for.

4 Efficient Voting Rules

Let us consider the problem of assigning the weights and setting the threshold of the qualified majority in a manner so that the resulting voting rule maximizes the expected sum of the utilities of all agents in the union.

In this regard, the best one could hope for would be to choose a when $\sum_j s_j > 0$ and b when $\sum_j s_j < 0$. With the two-stage procedure this optimum cannot be achieved. The reason is that we are losing information in a two stage procedure. In the second stage we see only the votes of the representatives. This comes only in the form of a vote for a or b, which includes only indirect information about the preferences of agents.

Efficient Voting Rules

Efficient voting rules are those designed to capture as much information as possible. In particular, we can still ask which $v \in V$ maximizes

$$E\left[\sum_j v(s) s_j\right].$$

We call such a voting rule an *efficient* voting rule.

[5] This is an asymmetric relationship: v can be equivalent up to ties with v' while the reverse might not hold.

We now introduce the only assumption needed for our first result.

Unbiased Countries

Let us say that a country is *unbiased* if

$$E\left[\sum_{k\in C_i} s_k \,|r_i = b\right] = -E\left[\sum_{k\in C_i} s_k \,|r_i = a\right].$$

An unbiased country is one where what we learn about how much a country cares about a from the fact that the country supports a is the same as what we learn about how much a country cares about b from the fact that the country supports b.

THEOREM **1** *Suppose that s_j is independent of s_k when j and k are in different countries, and that each country is unbiased. A profile of voting weights and a threshold is efficient if and only if it is equivalent up to ties to the weights*

$$w_i^* = E\left[\sum_{k\in C_i} s_k \middle| r_i = a\right]$$

and the 50% threshold of $\frac{\sum_i w_i^*}{2}$.

The theorem tells us that efficient weights are those which are proportional to what a vote tells us about the total utility for a (or b if negative) in a country.

Note that expression for the weight of a country will generally increase as the size of a country increases. So, all else held equal, increasing the size of a given country will increase the voting weight it will get. However, this weight is not proportional to a country's size. We return later to discuss how these weights behave in the context of a setting that will allow us to develop some of these calculations explicitly.

Before turning to such calculations, let us examine another aspect of the voting rules.

4.1 Biased Countries and Threshold Voting

The efficient voting rules identified by Theorem 1 had 50% majority thresholds, and thus are ones where a and b receive neutral treatment in counting votes. In many contexts, especially where b is interpreted as a status quo, there might be reason to consider other thresholds.

In particular, in contexts where countries are biased rather than unbiased, thresholds other than 50% turn out to be the efficient ones.

Unbiased Countries

Let us say that country i is *biased* with bias $\gamma_i > 0$ if

$$E\left[\sum_{k\in C_i} s_k \,|r_i = b\right] = -\gamma_i E\left[\sum_{k\in C_i} s_k \,|r_i = a\right].$$

A biased country is one where we have different expectations about how much the country's voters care about a over b when their representative votes for a, compared to our expectations about how much the country's voters care about b over a when their representative votes for b.

THEOREM **2** *Suppose that s_j is independent of s_k when j and k are in different countries, and that each country has the same bias factor γ. A weighted voting rule is efficient if and only if it is a weighted voting rule with qualified majority threshold and weights that are equivalent up to ties to the threshold $\frac{\gamma\sum_i w_i^*}{\gamma+1}$ and weights*

$$w_i^* = E\left[\sum_{k\in C_i} s_k \middle| r_i = a\right].$$

It is important to note that the threshold depends on the bias γ, while the weights are determined by the expectations that come from each country. Thus one can judge whether a rule's weights are optimal independently of the threshold, and vice versa.

We emphasize that there are no assumptions other than the common bias behind this theorem, and yet we obtain an essentially unique characterization of efficient voting rules and a strong form of separability of weights and thresholds.

Because of its generality, the model has a variety of interesting implications. Let us discuss a few key implications.

First, the extent to which a country's representative's vote is tied to the utilities of the agents in the country has important consequences. For example, if the representative's vote was purely random and uncorrelated with the utilities of his constituency, then that country's weight would be 0. More generally, the closer the tie between a representative's vote and the population's utilities, the larger the weight that a country receives.

Second, the weights are affected by the distribution of opinions inside a country. In particular, the correlation structure within a country is an important determinant of the expected size of the surplus of utilities for one alternative or the other. For instance, if a country's agents had perfectly correlated opinions (and the representative voted in accordance with them), then a vote for an alternative would indicate a strong surplus of utility in favor of that alternative. The more independent the population's opinions the lower the expected surplus of utility in any given situation. Thus, higher correlation among agents' utilities will generally lead to higher weights.

Third, the efficient weights take into account the intensity of preferences. So, relatively larger utilities lead to relatively larger weights. Thus, a country that cares more intensely about issues is weighted more heavily than a country that cares less, all else held equal. Due to practical and philosophical difficulties with the appraisals of utilities, one might want to be agnostic on this dimension and just treat all s_j's equally in the sense of only assigning them values of +1 or -1. We do this in the following section. Then accounting for utilities amounts to counting supporters.

Fourth, because of all the things that lie behind the calculations of the weights, the relation between the size of countries and their relative weights is ambiguous. For example, a large country with a representative who is a dictator whose vote is uncorrelated with his population's preferences receives a smaller weight than a smaller country with a representative whose vote is very responsive to his population's preferences.

In order to apply the theory and calculate weights as a function of a country's population, we no introduce a model that is more specific about the distribution of agents' preferences and how representatives vote.

5 A Block Model

We now specialize to what we call a "block model" which works as follows.

First, we treat agents' utilities equally, in the sense that we only account for them as +1 or -1, and will disregard personal intensities. This may be defended on grounds of practicality, but also more philosophically as an equal treatment condition.

Second, we assume that representatives vote for the alternative that has a majority of support in their country.

Third, we make the following specific assumptions about the distribution of the utilities of agents. We consider a world where each country is made up of some number of blocks of constituents, where agents within each constituency think alike - that is have perfectly correlated preferences, and where agents across constituencies think independently. We take the blocks within a country to be of the same size.

These assumptions are a stylized version of what we generally see. They reflect the fact that countries are often made up of some variety of constituencies, within which agents tend to have very highly correlated preferences. For instance, the farmers in a country might have similar opinions on a wide variety of issues, as will union members, intellectuals, etc.

By adjusting the size and number of blocks in a country we obtain varying expressions for the efficient weights of that country.

Efficient Weights in the Block Model

In the block model, we let N_i be the number of blocks in country i. In most applications the numbers N_i are likely to be relatively small. Then letting p_i be the size of each block, then we obtain the following expression for the efficient weight of country i.

$$w_i^b = p_i 2^{-N_i} \sum_{x > \frac{N_i}{2}} (2x - N_i) \frac{N_i!}{x!(N_i - x)!}. \tag{1}$$

There are two prominent variations on the block model that we consider in what follows.

We call the first variation the *absolute size block model*. In this variation, blocks are of a fixed size across all countries. In this case, a country's population can be measured in blocks, and a larger country has more blocks than a smaller one. Here the p_i's are the same across all countries.

We call the second variation the *relative size block model*. In this variation, all countries have the same number of blocks, and the size of the blocks in a given country adjust according the country's population size. Here the N_i's are the same across all countries.

Thus, we get the following expressions for the efficient weights in the two specializations of the block model.

Efficient Weights in the Absolute Size Block Model

Given that the population size of a block (p_i) is the same across all countries, these can be cancelled out, and the weights in the absolute size block model, w_i^a, reduce to:

$$w_i^a = 2^{-N_i} \sum_{x > \frac{N_i}{2}} (2x - N_i) \frac{N_i!}{x!(N_i - x)!}. \tag{2}$$

Efficient Weights in the Relative Size Block Model

In the relative size block model, as the number of blocks (N_i) are the same in all countries, the difference in the weights then comes only in how many agents are represented in a block. When calculating the weights, the weights turn out to be directly proportional to the population size of the countries. Thus,

$$w_i^r = p_i. \tag{3}$$

The efficient weights for various sizes of countries are given in the following table. The country size refers to number of blocks for the absolute block model and to some number of population units (say millions of people) in the relative block model.

Country Size in Units	Weight in the Absolute Block Model	Weight in the Relative Block Model
1	1	1
2	1	2
3	1.5	3
4	1.5	4
5	1.875	5
6	1.875	6
7	2.186	7
8	2.186	8
9	2.461	9
10	2.461	10
11	2.707	11
12	2.707	12
13	2.933	13
14	2.933	14
15	3.142	15
16	3.142	16
17	3.338	17
18	3.338	18
19	3.524	19
20	3.524	20

While the weights in the relative size block model are directly proportional to a country's population, they are less than proportional in the absolute block model. In that model they are graphed as follows.

Figure 1:

We note that for large numbers of blocks, the weights in the absolute block model vary with the square root of the number of blocks, which is consistent with weights originally proposed by Penrose (1946),[6] while for small numbers of blocks they diverge from this.

Asymmetries and Non-Monotonicities in Expected Utilities

Our perspective has been to maximize the sum of expected utilities, and in the block model as we have only looked at the sign of utilities, this amounts to maximizing the expected number of agents who are in agreement with the alternative chosen. What we emphasize here is that this is quite different from trying to equalize expected utilities across agents. In particular, efficient rules can necessarily treat agents asymmetrically, depending on the size of the country they live. Let us examine this in more detail for the two variations on the block model.

Let us compare the expected utilities of agents living in two countries of different population size, under the efficient voting rule in the two variations of the block model.

PROPOSITION **1** *In the relative size block model, agents living in the larger country have expected utilities which are at least as large as agents living in the smaller country; and whenever the two countries weights are not equivalent*[7] *then the agents in the larger country have a strictly higher expected utility. In the absolute size block model, the comparison of expected utilities of agents across countries can go either way depending on the specifics of the context.*

The proof of the proposition is straightforward. We offer a simple argument for the relative size block model, and an example showing ambiguity for the absolute size block model.

In the relative size block model, any agent's block in any country has exactly the same probability of agreeing with the agent's representative's vote. Thus, the expected utilities of agents in different countries differ only to the extent that their representatives receive different weights. As larger countries have larger weights, the claim in the proposition follows directly.

[6] See also Felsenthal and Machover (1999), as discussed in the introduction. Here we end up with similar expressions, but only in one specific version of the block model, and only for large populations with relatively small blocks, and for quite different reasons. More generally, the weights we obtain will differ from the square root, especially when the number of blocks is small or when we leave the absolute size block model.

[7] Two countries weights are equivalent if there exists a set of weights that lead to the same voting rule where these two countries weights are identical.

To see the ambiguity in the absolute block size model let us examine an example. Consider a union of three countries. Let us examine the expected utilities of the agents as we vary the number of blocks in the various countries.[8]

Populations of Countries in Blocks	Efficient Voting Weights	Expected Utility of a Agent in Country 1 or 2	Expected Utility of a Agent in Country 3
(1,1,1)	(1,1,1)	.5	.5
(1,1,3)	(1,1,1.5)∼(1,1,1)	.5	.25
(1,1,5)	(1,1,1.875)∼(1,1,1)	.5	.1875
(1,1,7)	(1,1,2.186)∼(0,0,1)	0	.3125
(2,2,7)	(1,1,2.186)∼(0,0,1)	0	.3125
(3,3,7)	(1.5,1.5,2.186)∼(1,1,1)	.25	.15625

There are some interesting things to note here. The changes in voting weights result in non-monotonicities in expected utilities in several ways. In the cases of (1,1,3) and (1,1,5), a agent in country 1 or 2 has a higher utility than a agent in country 3. However, once country 3 hits a population of 7, then its weight is such that the votes from countries 1 and 2 are irrelevant. Thus, a agent would rather be in the larger country when the configuration is (1,1,7), while a agent would prefer to be in a smaller country when the configuration is (1,1,3) or (1,1,5). Also, we se that as we increase country 3's population for 3 to 5, its agents' utilities fall, but then increasing the population from 5 to 7 leads to an increase in its agents' utilities. This contrasts with decreases in utilities of agents in the other countries.

This example shows us that there are no regularities that we can state concerning agents' utilities in the absolute size block model. The difficulty is that changes in population might

[8]The calculations are as follows. A agent gets a 1 when his or her preferred outcome is chosen and a -1 if it is not. For a agent in country 1 in the (1,1,1), (1,1,3), and (1,1,5) cases, there is a 3/4 chance at least one of the other countries will prefer the agent's preferred alternative and a 1/4 chance that the other two countries will both favor the other alternative. This leads to 3/4 chance of utility of 1 and 1/4 chance of utility of -1. For a agent in country 3 in the (1,1,3) case, there is a 3/4 chance his or her preferred alternative will match the country's vote and a 1/4 chance it will not. In the first case, there is then a 3/4 chance this will receive a vote from at least one of the other two countries and a 1/4 chance it does not. In the second case, there is a 1/4 chance that the agent's preferred alternative will still be passed by the other two countries and a 3/4 chance it will not. More generally, it is easy to check that the agent's ex ante expected utility conditional on his or her country's vote being in the winning majority is simply $\frac{w_i^*}{n_i}$, and conditional on his or her country's vote being on the losing side is $-\frac{w_i^*}{n_i}$. Then we can just calculate the probability that a given country's vote will be in the winning majority, given the weights.

dilute a given agents' impact within a country, but might also lead to a relative increase of that country's voting weight. As these two factors move against each other, changes can lead to varying effects.

Another issue that we might consider in addition to comparing agents utilities across countries, is to examine how the overall expected utility varies under efficient voting rules as we change the division of a given population into different districts or countries. This issue is also generally ambiguous, regardless of which version of the block model one considers. For instance, one might conjecture that if we start with one division of a population into districts, and then further subdivide the population into finer districts, we would enhance efficiency since agents would become closer to their representatives. However, this is not always the case. To see this note that if we start with a union of just one district or country, then we essentially have direct democracy. This is the most efficient possible. But then dividing this into several districts or countries would lead to a lower total expected utility under the efficient rule, than having just one district. Now, if we continue to further subdivide the districts, we eventually reach a point where each agent resides in a district of one, which brings us back to direct democracy and full efficiency! Generally, subdivisions lead to conflicting changes: on the one hand having a smaller number of agents within a district gives them a better say in the determination of their representative's vote, but on the other hand their representative is now just one among many. This leads to non-monotonicities and ambiguities of the types discussed above.

6 The European Union

Let us now examine the voting rule to be used in the Council of Ministers of the European Union under the Nice Treaty (December 2000) and compare it to the efficient voting rules under the variations of the block model.

The following are the voting weights for the European Council of Ministers under the Nice Treaty for the expansion of the EU from 15 to 27 members.[9] The vote is by qualified majority. At least 255 of the 345 votes (73.9%) must be cast in approval of a proposal for it

[9]The previous weights for the 15 members were 10 for Germany, France, Italy and the U.K.; 8 for Spain; 5 for Belgium, Greece, the Netherlands, and Portugal; 4 for Austria and Sweden; 3 for Denmark, Ireland and Finland; and 2 for Luxembourg, with 62 of 87 votes (71%) required for approval of a proposal.

to pass.[10,11]

[10]There are two other qualifications as well: (i) that the votes represent at least 14 of the 27 countries and (ii) that the votes represent at least 62% of the total population. Calculations by Bräuninger and König (2001) suggest that there are relatively few scenarios in which the weighted vote threshold of 255 votes would be met while one of the other two criteria would fail. It appears that the only impact will be from the population threshold and that this will only involve a few configurations of votes providing a very slight boost in power to Germany and slight decrease in power to Malta. Thus, for practical purposes, these additional considerations are relatively unimportant and the voting weights themselves are the main component of the voting procedure.

[11]There are discrepancies in the Nice Treaty in that some statements imply a threshold of 258 votes and others a threshold of 255 votes. It appears that the correct number is the 255.

Country	Population	Votes (i.e., weights)
Germany	82.8	29
U.K.	59.5	29
France	59.3	29
Italy	57.6	29
Spain	40	27
Poland	38.7	27
Romania	22.4	14
Netherlands	15.9	13
Greece	10.6	12
Czech	10.3	12
Belgium	10.2	12
Hungary	10.1	12
Portugal	10	12
Sweden	8.9	10
Bulgaria	7.8	10
Austria	8.1	10
Slovakia	5.4	7
Denmark	5.3	7
Finland	5.2	7
Ireland	3.8	7
Lithuania	3.6	7
Latvia	2.4	4
Slovenia	1.9	4
Estonia	1.4	4
Cyprus	0.8	4
Luxembourg	0.5	4
Malta	0.4	3

Let us examine the efficient voting weights and compare those to the actual weights. The following table provides the actual weights and the efficient weights based on two different sizes of voting blocks.

The efficient weights in the absolute size block model are calculated for two different block sizes: 1 million and 2 million. So for instance, in the case of 1 million sized blocks, Germany is seen as having 83 blocks, France as 59, and Italy as 58, etc. This leads to efficient voting weights of 7.3, 6.2 and 6.1 for these countries, respectively.[12] Recall that voting weights are not affected by rescaling. So, we need to rescale the efficient weights to the scale of the actual weights. We find the scaling factor by regressing the actual weights on the efficient weights (with no intercept). This leads to a scaling factor of 4.58 for the case of 1 million sized blocks and 9.01 for the case of 2 million sized blocks. The efficient weights reported below are those directly from (2) multiplied by the scaling factor.

The efficient weights in the relative size block model are calculated directly by rescaling the population sizes to best fit the actual weights (recall that weights are completely equivalent under rescalings). The scaling factor here is .58.

[12]Countries with a faction of a block are simply scaled to a corresponding fraction of the efficient weight of 1 for a one block country.

Country	Population	Nice Treaty Weights	Absolute Block Efficient Weights: 1M Sized Blocks	Absolute Block Efficient Weights: 2M Sized Blocks	Relative Block Efficient and Constitution Weights
Germany	82.8	29	33.4	33.4	48.3
U.K.	59.5	29	28.4	27.9	34.7
France	59.3	29	28.4	27.9	34.6
Italy	57.6	29	27.9	27.9	33.6
Spain	40	27	22.9	22.7	23.3
Poland	38.7	27	22.9	22.7	22.6
Romania	22.4	14	16.9	17.5	13.1
Netherlands	15.9	13	14.2	14.3	9.3
Greece	10.6	12	12.4	12.3	6.2
Czech	10.3	12	11.4	12.3	6.0
Belgium	10.2	12	11.4	12.3	5.9
Hungary	10.1	12	11.4	12.3	5.9
Portugal	10	12	11.4	12.3	5.8
Sweden	8.9	10	11.4	9.7	5.2
Bulgaria	7.8	10	10.1	9.7	4.6
Austria	8.1	10	10.1	9.7	4.7
Slovakia	5.4	7	8.7	8.1	3.1
Denmark	5.3	7	8.7	8.1	3.1
Finland	5.2	7	8.7	8.1	3.0
Ireland	3.8	7	6.9	6.5	2.2
Lithuania	3.6	7	6.9	6.5	2.1
Latvia	2.4	4	4.6	6.5	1.4
Slovenia	1.9	4	4.6	6.2	1.1
Estonia	1.4	4	4.6	4.5	.8
Cyprus	0.8	4	3.7	2.6	.5
Luxembourg	0.5	4	2.3	1.6	.3
Malta	0.4	3	1.8	1.3	.2

The Nice Treaty weights compared to the efficient weights are pictured as follows. A regression of the Nice Treaty weights on the efficient weights under the absolute size block model provides an R^2 of 96% for the case of 1 million sized blocks and 95% for the case of

Figure 2:

2 million sized blocks (with F-statistics in each case over 600).[13] A regression of the Nice Treaty weights on the efficient weights under the relative size block model provides an R^2 of .80 and an (F-statistic of 102).

The relationship between the different weights is pictured as follows.

Discussion

The above analysis suggests that the voting weights under the Nice Treaty are very close to being efficient, if the world is well-approximated by the absolute size block model. If the world is better approximated by the relative size block model, then the weights are not so close to being efficient.

[13] As a comparison, the fit using weights directly proportional to population is only 81%, and so the efficient weights provide a much closer match to the Nice Treaty weights.

It is interesting to compare the voting rule under the Nice Treaty to that under the draft of the Constitution produced by the Constitutional Convention in June of 2003, which are proposed to take affect in November of 2009 (see Article 24). Under the proposed voting rule there, weights will be proportional to population and the threshold will be 60% of the total population.[14] Those weights would not be very efficient if the world is well approximated by the absolute size block model, but would be a perfect fit under the relative size block model.

Thus, we are left with an empirical question. If the world is a good match to the absolute size block model then the Nice Treaty weights are almost perfectly efficient, while if the world is a good match to the relative size block model then the new Constitution's weights are the efficient ones. Of course, if the world lies somewhere between these two variations, then so will the shape of the weights. While it seems clear that countries such as Luxembourg and Malta consist of more than one block, it also seems clear that the smallest countries have fewer voting blocks than the largest ones. This suggests that the weights should be nonlinear, although perhaps not quite to the level suggested by the absolute size block model.

Let us also discuss the voting thresholds. The threshold under the Nice treaty is 73.9% of the weights - which would be efficient if countries have a bias of roughly $\gamma = 3$. This indicates a strong bias for the status quo. In contrast, the threshold of 60% under the Constitution would be efficient if countries have a bias of roughly $\gamma = 1.5$. This is also a bias for the status quo, but a less pronounced one.

At least two other considerations might lie behind the selection of a voting rule, both in terms of weights and thresholds. One is its stability. As the rules can be amended, considerations other than efficiency enter the long-run picture, as only certain rules will survive. [15] Another is the issue of fairness or equality. As we have shown, efficient weights do not necessarily lead to the same expected utilities for agents in different countries. For instance Proposition 1 showed that larger countries are favored under proportional weights in the relative size block model.

In conclusion, in this paper, we have provided a framework for designing and analyzing efficient voting rules in the context of votes by representatives of countries, districts, etc. We have shown that the model can be directly applied to analyzing voting rules such as those of the European Union, and that the relative merits of different rules reduce to readily

[14]The rule is more complicated than this, as it requires at least 50% of member states (at least 14 of the 27 countries) to vote yes as well as 60% in terms of the weighted voting. Thus, there could arise instances where 60% of the weights come from fewer than 50% of the countries, in which case the vote will not pass. While this is an important consideration, as a first approximation we take the 60% weight to be the binding constraint.

[15]See Barbera and Jackson (2000) and Sosnowska (2002) for an examination of the stability of voting rules.

identifiable hypotheses that are amenable to empirical testing.

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Appendix

Proof of Theorem 1: This is a special case of Theorem 2. ▮

Proof of Theorem 2: Given that countries are biased with common factor γ, it follows that for any country i

$$E\left[\sum_{k\in C_i} s_k | r_i = a_i\right] = -\gamma E\left[\sum_{k\in C_i} s_k | r_i = b_i\right]. \tag{4}$$

An efficient voting rule maximizes

$$E\left[\sum_k v(s)s_k\right].$$

We can rewrite this as

$$\sum_{r_1,\ldots,r_m} E\left[\sum_k v(s)s_k \,|r_1,\ldots,r_m\right] P(r_1,\ldots,r_m),$$

where $(r_1,\ldots,r_m)$ is the event where the realization of representatives (i.e., votes of the countries) is $(r_1,\ldots,r_m)$. Note that we can write $v(s)$ as a function of $(r_1,\ldots,r_m)$ instead of s. Hence, the total expected utility is

$$\sum_{r_1,\ldots,r_m} E\left[\sum_k v(r_1,\ldots,r_m)s_k \,|r_1,\ldots,r_m\right] P(r_1,\ldots,r_m),$$

Given the independence across countries, we can write this as

$$\sum_{r_1,\ldots,r_m} v(r_1,\ldots,r_m)\sum_i \left(E\left[\sum_{k\in C_i} s_k \,|r_i\right] P(r_i)\right).$$

It then follows that if we can find voting weights w and a threshold that maximize

$$v(r_1,\ldots,r_m)\sum_i E\left[\sum_{k\in C_i} s_k \,|r_i\right] \tag{5}$$

pointwise for each $(r_1,\ldots,r_m)$, then these must be an efficient weights and threshold pair. Moreover, if we find one that leads to a 0 whenever there is indifference between a and b, then and all efficient weight-threshold pairs must be equivalent to such a weight-threshold pair.

Note that for any given $(r_1, \ldots, r_m)$, maximizing expression (5) requires setting $v(r_1, \ldots, r_m) = 1$ when

$$\sum_i E\left[\sum_{k \in C_i} s_k \,|r_i\right] > 0 \tag{6}$$

and $v(r_1, \ldots, r_m) = -1$ when

$$\sum_i E\left[\sum_{k \in C_i} s_k \,|r_i\right] < 0, \tag{7}$$

and does not have any requirement in the case that this expression is equal to 0.

With an abuse of notation, let us write $r_i = 1$ when $r_i = a$ and $r_i = -\gamma$ when $r_i = b$. We do this based on equation (4), as we can then rewrite (6) and (7) as $v(r_1, \ldots, r_m) = 1$ when

$$\sum_i r_i w_i^* > 0 \tag{8}$$

and $v(r_1, \ldots, r_m) = -1$ when

$$\sum_i r_i \gamma w_i^* < 0, \tag{9}$$

where w_i^* is as defined in Theorem 2.

So, one efficient voting rule is sums the weights w_i^*, but adjusting them to have a factor of 1 when the representative chooses a and a factor of $-\gamma$ when the representative chooses b. This is the same as using the efficient weights and then having a threshold of $\frac{\gamma}{\gamma+1}(\sum_i w_i^*)$. Then we flip a coin in the case of a tie. Any efficient voting rule must agree with this one except in the case where this rule results in an expression equal to 0. This concludes the proof of the theorem. ▮