Incremental Risk Vulnerability[1]

Guenter Franke[2], Richard C. Stapleton [3] and Marti G Subrahmanyam [4]

September 23, 2005

[2]Fakultät für Wirtschaftswissenschaften und Statistik, University of Konstanz, email: guenter.franke@uni-konstanz.de

[3]University of Manchester and University of Melbourne, email Richard.Stapleton1@btinternet.com

[4]Stern School of Business, New York University, email: msubrahm@stern.nyu.edu

Abstract

We present a necessary and sufficient condition on an agent's utility function for a simple mean preserving spread in an independent background risk to increase the agent's risk aversion (incremental risk vulnerability). Gollier and Pratt (1996) have shown that declining and convex risk aversion as well as standard risk aversion are sufficient for risk vulnerability. We show that these conditions are also sufficient for incremental risk vulnerability.

In addition, we present sufficient conditions for a restricted set of stochastic increases in an independent background risk to increase risk aversion.

Journal of Economic Literature Classification Numbers:
D 52, D 81

1   new bit for introduction

In this paper, we consider a particular set of increases in an independent background risk. In reality all investors face some level of background risk, so a relevant question is how they would react to an increase in such a risk. We look for conditions on utility functions such that an agent becomes more averse to a market risk given an increase in background risk.

This question has been considered previously by Kimball (1993) and by Eeckhoudt, Gollier and Schlesinger (1996). Kimball considers the set of 'patent increases' in background risk[1]. He shows that standard risk aversion is a sufficient condition for 'incremental risk vulnerability' for this set of increases. Eeckhoudt, Gollier and Schlesinger (1996), on the other hand, consider the larger sets of first order and second order stochastic dominance increases in background risk. They find that the conditions on utility functions are quite restrictive and exclude many commonly assumed utility functions.

In this paper, we follow Kimball's general approach by looking at the effect of particular sets of increases in background risk. We begin with a set of non-stochastic mean-preserving spread increases that we term 'simple increases'. For this set, we derive a necessary and sufficient condition for incremental risk vulnerability. We find in this case, a condition on utility which is weaker than standard risk aversion.

An interesting subset of our simple increases is the set of monotonic increases in background risk. This set is interesting, first, because it reflects the property that background risks increase in scale. Second, this case generalizes naturally to the set of stochastic increases that 'improve' according to nth order stochastic dominance. Kimball (1993) has shown that, in the case of stochastic increases that improve according to third-order stochastic dominance, standard risk aversion is a sufficient condition for incremental risk vulnerability. Considering the case of n-th order improvements, we find a sufficient condition for incremental risk vulnerability, which is less restrictive than standard risk aversion.

2   Introduction

Many economic decisions are made in a context where some of the risks are tradable, while others are not. These non-tradable or background risks are not controllable by the decision-maker and yet influence the agent's risk-taking behavior with respect to the tradable claims.

---

[1]Patent increases are those such that an agent who is more risk averse than another, always requires a larger risk premium to bear the increased risk than the other. See Kimball (1993), p 603.

Eeckhoudt and Kimball (1992) and Meyer and Meyer (1998) demonstrate this for the demand for insurance, Franke, Stapleton and Subrahmanyam (1998) for portfolio choice. A central question, in this context, is whether an additive background risk makes the agent more risk averse.

Gollier and Pratt (1996) answer this question by considering an agent who starts without background risk and then faces an independent background risk. They introduce the concept of risk vulnerability and show that risk vulnerability is equivalent to the notion that an undesirable risk can never be made desirable by the presence of an independent, unfair risk. Furthermore, the background risk makes the agent more risk averse. Hence, such a background risk reduces the agent's demand for a risky asset, given a choice between a risky and a risk-free asset. Gollier and Pratt derive a necessary and sufficient condition for risk vulnerability. They show that a sufficient condition for risk vulnerability is either that the absolute risk aversion of the agent is declining and convex or that the agent is standard risk averse in the sense of Kimball (1993). In a recent paper Keenan and Snow (2003) relate Gollier and Pratt's condition of local risk vulnerability to compensated increases in risk, introduced by Diamond and Stiglitz (1974). They show that the introduction of a small fair background risk increases risk aversion of agents more, the higher is their index of local risk vulnerability.

Usually, agents have to bear some background risk, but the level of this risk may change. Therefore the relevant question is not so much whether the *presence* of background risk makes the agent more risk averse, but whether an *increase* in this background risk makes the agent more risk averse. Kimball (1993) analyzes patent increases in background risk. He shows that such an increase raises the risk aversion of an agent if it raises the expected marginal utility conditional on his tradable income and if the agent is standard risk averse. Kimball argues that the background risk X is patently more risky than the background risk x if X can be obtained from x by adding a random variable v such that the distribution of v conditional on x improves for increasing x according to third-order stochastic dominance. Eeckhoudt, Gollier and Schlesinger (1996) consider this issue in the context of increases in an independent background risk that exhibit second order stochastic dominance. Given this broad set of increases in background risk they derive necessary and sufficient conditions, which leave room only for a small set of utility functions. Finally, Eichner and Wagener (2003) discuss the conditions on two-parameter, mean-variance preferences such that the agent is variance vulnerable, i.e. an increase in the variance of an independent background risk induces the agent to take less tradable risk.

Intuitively, there must be an inverse relation between the set of admissible increases in background risk considered and the set of utility functions that exhibit the characteristic of increased risk aversion. Therefore, in this article we consider a smaller, but plausible set of

increases in background risk, the benefit being obtaining a broader set of utility functions that have the desired attribute.

Rothschild and Stiglitz (1970) define a mean preserving spread of an existing risk as a shift in the probability mass from the center to the tails of the distribution. As pointed out by Eeckhoudt, Gollier and Schlesinger (1996), this is equivalent to a second degree stochastic dominance shift, provided the mean is fixed. To this definition we add the restriction that the increase in background risk raises the non-tradable income in some states above a threshold level and lowers it in some states below the threshold. We call this increase a *simple mean preserving spread.*

Let y be the independent background risk with $E(y) = 0$, then a simple mean preserving spread is a deterministic change in y, $\Delta(y)$, such that $\Delta(y) \leq [=] [\geq] 0$ for $y < [=] [>] y_0$ for a given a threshold level y_0, and $E[\Delta(y)] = 0$. In this case, note that the rank order of outcomes below y_0 may change, as well as the rank order of outcomes above y_0.

We introduce the concept of *incremental risk vulnerability.* An agent is incremental risk vulnerable if a simple mean preserving spread in background risk makes the agent more risk averse. In section 2 we derive a necessary and sufficient condition for incremental risk vulnerability. It turns out that the sufficient conditions for risk vulnerability given by Gollier and Pratt are also sufficient for incremental risk vulnerability. However, declining risk aversion is not required. All utility functions with a negative third and a negative fourth derivative are also incremental risk vulnerable.

In section 3, we further consider a restricted set of stochastic increases in background risk and derive sufficient conditions for risk aversion to increase. These conditions are illustrated by examples.

3   Characterization of Incremental Risk Vulnerability

In this section we present a necessary and sufficient condition for the utility function to exhibit incremental risk vulnerability. The agent's income, W, is composed of the tradable income w and the non-tradable income y, i.e. $W = w + y$. The non-tradable income represents an additive background risk. y is assumed to be distributed independently of w and to have a zero mean. Moreover, y is assumed to be bounded from below and above, i.e. $y \in (\underline{y}, \bar{y})$. Finally, $W = w + y \in (\underline{W}, \bar{W})$ is assumed. Let $(\Omega, \mathcal{F}, \mathcal{P})$ be the probability space on which the random variables are defined.

Definition 1 *(A Simple Mean Preserving Spread in Background Risk)*
Let y be a background risk with $E(y) = 0$. Then a simple mean preserving spread in the background risk changes y to $y + s\Delta(y)$, with $E(\Delta(y)) = 0$, where $\Delta(y) \leq [=] [\geq] 0$ for $y < [=] [>] y_0$, and $s \geq 0$ denotes the scale of the increase.

The agent's utility function is $u(W)$. We assume that the utility function is state-independent, strictly increasing, strictly concave, and four times differentiable on $W \in (\underline{W}, \bar{W})$. We assume that there exist integrable functions on $\omega \in \Omega$, u_0 and u_1 such that

$$u_0(\omega) \leq u(W) \leq u_1(\omega)$$

We also assume that similar conditions hold for the derivatives $u'(W), u''(W)$ and $u'''(W)$. The agent's expected utility, conditional on w, is given by the derived utility function, as defined by Kihlstrom et al. (1981) and Nachman (1982):

$$\nu(w) = E_y[u(W)] \equiv E[u(w + y)|w] \tag{1}$$

where E_y indicates an expectation taken over different outcomes of y. Thus, the agent with background risk and a von Neumann-Morgenstern concave utility function $u(W)$ acts like an individual without background risk and a concave utility function $\nu(w)$. The coefficient of absolute risk aversion is defined as $r(W) = -u''(W)/u'(W)$ and the coefficient of absolute prudence as $p(W) = -u'''(W)/u''(W)$. The absolute risk aversion of the agents derived utility function is defined as the negative of the ratio of the second derivative to the first derivative of the derived utility function with respect to w, i.e.,

$$\hat{r}(w) = -\frac{\nu''(w)}{\nu'(w)} = -\frac{E_y[u''(W)]}{E_y[u'(W)]} \tag{2}$$

It is worth noting that, in the absence of background risk, $\hat{r}(w)$ is equal to $r(w)$, the coefficient of absolute risk aversion of the original utility function.
We are now in a position to define incremental risk vulnerability.

Definition 2 *(Incremental Risk Vulnerability)*
An agent is incremental risk vulnerable if a simple mean preserving spread in background risk increases the agent's derived risk aversion for all w.

This definition also includes the case in which the agent initially has no background risk. This case is analyzed by Gollier and Pratt (1996). Hence incremental risk vulnerability implies risk vulnerability subject to $E[\Delta(y)] = E[y] = 0$. Gollier and Pratt allow also for

a non-random negative y which then necessitates declining risk aversion. Since we only consider fair background risks, declining risk aversion is not implied by incremental risk vulnerability.

The main result of this paper is the following proposition which presents a necessary and sufficient condition for a marginal simple mean preserving spread in background risk to raise derived risk aversion, i.e. $\partial \hat{r}(w)/\partial s > 0$.

Proposition 1 *(Derived Risk Aversion and Simple Mean Preserving Spreads in Background Risk)*

If $u'(W) > 0$ and $u''(W) < 0$, then for any simple mean preserving spread in background risk,

$$\partial \hat{r}(w)/\partial s > [=][<]\ 0,\ \forall (w, y, s) \Longleftrightarrow$$
$$u'''(W_2) - u'''(W_1) < [=][>] - r(W)[u''(W_2) - u''(W_1)],$$
$$\forall\ (W, W_1, W_2),\ \underline{W} < W_1 \le W \le W_2 < \bar{W}, W_2 - W_1 < \bar{y} - \underline{y}\ .$$

<u>Proof</u>: See Appendix 1.

Proposition 1 allows us to analyze the effect of any simple mean preserving spread in an independent background risk. Since a finite increase in background risk is the sum of marginal increases, the sufficiency condition in Proposition 1 also holds for finite increases in background risk.

In order to interpret the necessary and sufficient condition under which a simple mean preserving spread in a background risk will raise the risk aversion of the derived utility function, first consider the special case in which background risk changes from zero to a small positive level. This is the case analyzed previously by Gollier and Pratt (1996) and by Keenan and Snow (2003). In this case, we have

Corollary 1 *Starting with no background risk, for any marginal increase in background risk,*
$$\hat{r}(w) > [=][<]\ r(w) \quad \textit{if and only if} \quad \frac{\partial \theta}{\partial W} < [=][>]\ 0,\ \forall\ W$$
where $\theta(W) \equiv u'''(W)/u'(W)$.

<u>Proof</u>: Let $W_2 - W_1 \to dW$. In this case, $u'''(W_2) - u'''(W_1) \to u''''(W)dW$. Similarly $u''(W_2) - u''(W_1) \to u'''(W)dW$.

Hence, the condition in the Proposition yields, in this case, $u''''(W) < [=][>] -r(W)u'''(W)$. This is equivalent to $\partial\theta/\partial W < [=][>] 0, \ \forall \ W \ \square$

In Corollary 1, $\theta(W) = u'''(W)/u'(W)$ is a *combined* prudence/risk aversion measure. This measure is defined by the product of the coefficient of absolute prudence and the coefficient of absolute risk aversion. The corollary says that for a small background risk derived risk aversion exceeds [is equal to] [is smaller than] risk aversion if and only if $\theta(W)$ decreases [stays constant] [increases] with W. Hence, it is significant that *neither* decreasing prudence *nor* decreasing absolute risk aversion is necessary for derived risk aversion to exceed risk aversion. However, the combination of these conditions is sufficient for the result to hold, since the requirement is that the product of the two must be decreasing. The condition in corollary 1 is thus weaker than standard risk aversion, which is characterized by *both* absolute risk aversion and absolute prudence being positive and decreasing. Note that the condition in this case is the same as the 'local risk vulnerability' condition derived by Gollier and Pratt (1996). Local risk vulnerability is $r'' > 2rr'$, which is equivalent to $\theta' < 0$. Keenan and Snow (2003) define $-\theta'$ as the local risk vulnerability index. They show for a small background risk that the difference between derived risk aversion and risk aversion increases in this index.

Since an interior maximum of $r(w)$ implies $r'(w) = 0$ and $r''(w) < 0$, it rules out local risk vulnerability. Therefore, we have

Corollary 2 *Risk vulnerability and incremental risk vulnerability rule out all utility functions with an interior maximum of absolute risk aversion.*

An alternative way to interpret Corollary 1 and Proposition 1 is to assume $u''' > 0$. In this case, Corollary 1 states that a marginal increase in background risk, starting with no background risk, makes the agent more risk averse if and only if temperance $t(W) = -u''''(W)/u'''(W)$ exceeds risk aversion $r(W)$ everywhere. Proposition 1 states that a simple mean preserving spread in background risk makes an agent more risk averse if and only if $-[u'''(W_2) - u'''(W_1)]/[u''(W_2) - u''(W_1)] > r(W)$, for $W_1 \leq W \leq W_2$. The left hand side of this inequality can be interpreted as an average temperance over the range $[W_1, W_2]$. In their analysis of second order stochastic dominance shifts in background risk, Eeckoudt, Gollier and Schlesinger (1996) find the much stronger condition $t(W) \geq r(W'), \forall (W, W')$.

We now apply Proposition 1 to show that standard risk aversion is sufficient for incremental risk vulnerability.

Corollary 3 *Standard risk aversion is a sufficient condition for derived risk aversion to increase with a simple mean preserving spread in background risk.*

Proof: Standard risk aversion requires both positive, decreasing absolute risk aversion and positive, decreasing absolute prudence. Further, $r'(W) < 0 \Rightarrow p(W) > r(W)$ and hence $u'''(W) > 0$. It follows that the condition in the Proposition for an *increase* in the derived risk aversion can be written as [2]

$$\frac{u'''(W_2) - u'''(W_1)}{u''(W_2) - u''(W_1)} < -r(W_1)$$

or, alternatively,

$$p(W_1) \left[1 - \frac{u'''(W_2)}{u'''(W_1)}\right] / \left[1 - \frac{u''(W_2)}{u''(W_1)}\right] > r(W_1)$$

Since $p(W_1) > r(W_1)$, a sufficient condition is that the ratio of the square brackets exceeds 1. This, in turn, follows from decreasing absolute prudence, $p'(W) < 0$. Hence, standard risk aversion is a sufficient condition □

Gollier and Pratt (1996) showed not only that standard risk aversion is sufficient for risk vulnerability, but so also is declining and convex absolute risk aversion $r(w)$. The next corollary shows that the latter condition is also sufficient for incremental risk vulnerability.

Corollary 4 *Declining and convex absolute risk aversion is a sufficient condition for derived risk aversion to increase with a simple mean preserving spread in background risk.*

Proof: From

$$\hat{r}(w) = E_y \left[\frac{u'(W)}{E_y[u'(W)]} r(W)\right],$$

$$\partial \hat{r}(w)/\partial s = E_y \left[\frac{u'(W)}{E_y[u'(W)]} r'(W)\Delta(y)\right] + E_y \left[r(W)\frac{\partial}{\partial y}\left[\frac{u'(W)}{E_y[u'(W)]}\right]\Delta(y)\right] \quad (3)$$

As shown in the appendix, it suffices to consider a three-point distribution of background risk (y_1, y_0, y_2) with $y_1 < 0, y_2 > 0, y_1 < y_0 < y_2$ and $\Delta(y_0) = 0, \Delta(y_1) < 0, \Delta(y_2) > 0$. The first term in equation (3) is positive whenever r is declining and convex. This follows since $E(\Delta(y)) = 0$ and $\Delta(y_2) > \Delta(y_1)$ implies that $E[r'(W)\Delta(y)] \geq 0$. Since $u'(W)$ is

---

[2]Note that whenever $r'(W)$ has the same sign for all W, the three-state condition in the Proposition (i.e. the condition on W, W_1, and W_2) can be replaced by a two-state condition (a condition on W_1 and W_2).

declining, it follows that the first term in (3) is positive. Now consider the second term: $\partial[u'(W)/E_y[u'(W)]]/\partial y \, \Delta(y)$ is positive for y_1 and negative for y_2 and has zero expectation. Therefore a declining r implies that the second term is positive. Hence a sufficient condition for $\partial \hat{r}(w)/\partial s > 0$ is a declining and convex r $\square$

Although corollaries 3 and 4 use the property of declining risk aversion, this property is clearly not required for incremental risk vulnerability, as already noted by Gollier and Pratt.

Corollary 5 : *For every utility function with $u'''(W) < 0$ and $u''''(W) \leq 0$ a simple mean preserving spread in background risk raises derived risk aversion.*

<u>Proof</u>: $u''''(W) \leq 0$ implies that the left hand side of the condition in Proposition 1 is non-positive. $u'''(W) < 0$ implies that the right hand side is positive $\square$

A utility function with $u'''(W) < 0$ exhibits negative prudence and increasing risk aversion. Yet this utility function has the property of incremental risk vulnerability if the fourth derivative is also negative. In terms of equation (3), the second term is now negative, but it is overcompensated by a strongly positive first term due to strong convexity of r.

An example of a utility function with the properties stated in corollary 5 is the HARA-function

$$u(W) = \frac{1-\gamma}{\gamma} \left[A + \frac{W}{1-\gamma} \right]^{\gamma}, \text{where } \gamma \in (1,2), W < A(\gamma - 1) \ .$$

4 Stochastic Increases in Background Risk and Risk Aversion

A simple mean preserving spread in background risk is a deterministic change relating $\Delta(y)$ to y. A natural generalization is to consider a stochastic change e such that y is replaced by $(y + e)$ with e being distributed independently of w, but perhaps dependently on y. In the case of dependence, the distribution of e is assumed to improve with increasing y according to second-order stochastic dominance, i.e. the distribution of e conditional on y second-order stochastically dominates the distribution conditional on a smaller y. It will be assumed throughout that this improvement can be captured by the differential $\partial e/\partial y$. This differential is zero in the case of independence. We, first, derive sufficient conditions on e and on absolute risk aversion to ensure an increase in derived risk aversion and, second, illustrate these conditions.

We analyse the agent's derived risk aversion $\hat{r}(w)$ in the presence of only the y-risk and the derived risk aversion $\hat{\hat{r}}(w)$ in the presence of the $(y + e)$-risk. For this purpose we define $r_e(w + y)$ as the derived risk aversion over the e-risk, given the income $(w + y)$.

$$r_e(w + y) \equiv \frac{E_e[-u''(w + y + e)]}{E_e[u'(w + y + e)]}; \qquad \forall (w + y).$$

Proposition 2 provides sufficient conditions for the e-risk to raise the agent's risk aversion.

Proposition 2 *Let e be a random variable which is distributed independently of w, but perhaps dependently on y. In case of dependence, the distribution of e improves with increasing y according to second-order stochastic dominance.*
Then

$$\hat{\hat{r}}(w) \geq \hat{r}(w), \quad \forall\, w,$$

if

$$r_e(w + y) \geq r(w + y), \quad \forall\, (w + y), \tag{4}$$

and

$$dr_e(w + y)/dy \leq 0, \quad \forall\, (w, y). \tag{5}$$

This proposition is proved in Appendix 2. Condition (4) requires the risk aversion of an agent with income $w + y$ to be higher in the presence of the background risk, e. Condition (4) rules out a subset of the second-order stochastic dominance increases in background risk as analysed by Eeckhoudt, Gollier and Schlesinger (1996). It also rules out a simple mean preserving spread since $y_2 > y_1$ does not imply $y_2 + \Delta(y_2) > y_1 + \Delta(y_1)$. Condition (5) requires the derived risk aversion $r_e(w + y)$ to decline. For a small e-risk, condition (5) implies declining risk aversion of u. Hence condition (5) requires this property to be preserved under the e-risk.

Both conditions are quite natural given a utility function with declining risk aversion. The following corollaries illustrate Proposition 2.

Corollary 6 *The increase in background risk from y to $(y + e)$ raises the derived risk aversion if e is a random variable, distributed independently of y, with nonpositive expectation and if the agent is risk vulnerable.*

<u>Proof</u>: Risk vulnerability and nonpositive expectation of e imply condition (4). Since e is independent of y and declining risk aversion is preserved under background risk, condition (5) holds □

Next, consider the case in which the distribution of e depends on y such that the distribution of e improves with increasing y according to a second-order stochastic dominance shift.

Corollary 7 *Assume $r' < 0$, $r'' > 0$ and $E(e|y) \leq 0$ $\forall y$. Moreover, the distribution of e may improve with increasing y according to second-order stochastic dominance. Then the increase in background risk replacing y by $(y + e)$ raises the derived risk aversion.*

<u>Proof</u>: From Gollier and Pratt(1996), $r' < 0$, $r'' > 0$ and $E(e|y) \leq 0$ imply risk vulnerability and, hence, condition (4). In Appendix 2 condition (5) is shown to hold, too □

5 Conclusion

This paper considers the effect on derived risk aversion of increases in background risk. We first take the case of deterministic increases which are simple mean preserving spreads. We present a necessary and sufficient condition for such an increase to raise the derived risk aversion of an agent. Standard risk aversion and declining, convex risk aversion are shown to be sufficient conditions.

We then analyse the effect of stochastic increases in background risk. If such an increase is independent of the existing background risk and has a non-positive expectation, it raises derived risk aversion if the agent is risk vulnerable. If the distribution of the increase improves with increasing realisations of the existing background risk according to second-order stochastic dominance and the conditional expectation of the increase is non-positive, then the derived risk aversion of an agent with declining, convex risk aversion increases.

Appendix 1

Proof of Proposition 1

From the definition of $\hat{r}(w)$,

$$\hat{r}(w) = \frac{E_y[-u''(W)]}{E_y[u'(W)]} \qquad (6)$$

we have the following condition. For any distribution of y and for any $s \geq 0$,

$$\partial\hat{r}(w)/\partial s > [=][<] \ 0 \iff f(w,y,s) > [=][<] 0, \qquad (7)$$

where $f(w,y,s)$ is defined as

$$f(w,y,s) \equiv E_y\left[\Delta(y)\left\{-u'''(W) - u''(W)\hat{r}(w)\right\}\right]. \qquad (8)$$

Necessity

We now show that

$$\begin{aligned} f(w,y,s) \ \ &> [=][<] \ \ 0 \Longrightarrow \\ u'''(W_2) - u'''(W_1) \ \ &< [=][>] \ \ -r(W)\left[u''(W_2) - u''(W_1)\right], \forall \ W_1 \leq W \leq W_2 \end{aligned}$$

Consider a background risk with three possible outcomes, y_0, y_1, and y_2, such that $y_1 < y_0 < y_2$ and $\Delta(y_1) < \Delta(y_0) = 0 < \Delta(y_2)$. Define

$$W_i \ \ = \ \ w + y_i + s\Delta(y_i), \quad i = 0, 1, 2,$$

and let q_i denote the probability of the outcome y_i. For the special case of such a risk, equation (8) can be written as

$$f(w,y,s) = q_1|\Delta(y_1)|\left\{-u'''(W_2) + u'''(W_1) - [u''(W_2) - u''(W_1)]\hat{r}(w)\right\} \qquad (9)$$

since

$$E[\Delta(y)] = \sum_{i=0}^{2} q_i\Delta(y_i) = 0$$

so that

$$q_1|\Delta(y_1)| = q_2\Delta(y_2)$$

Now $\hat{r}(w)$ can be rewritten from (6) as

$$
\begin{aligned}
\hat{r}(w) &= E_y\left\{\frac{u'(W)}{E_y[u'(W)]}\frac{-u''(W)}{u'(W)}\right\} \\
&= E_y\left\{\frac{u'(W)}{E_y[u'(W)]}r(W)\right\}
\end{aligned}
\tag{10}
$$

Hence, $\hat{r}(w)$ is the expected value of the coefficient of absolute risk aversion, using the risk-neutral probabilities given by the respective probabilities multiplied by the ratio of the marginal utility to the expected marginal utility. Thus, $\hat{r}(w)$ is a convex combination of the coefficients of absolute risk aversion at the different values of y. For the three-point distribution being considered, $\hat{r}(w)$ is a convex combination of $r(W_0)$, $r(W_1)$, and $r(W_2)$. Suppose that $y_0 = 0$. Then $q_0 \to 1$ is feasible. Hence, as $q_0 \to 1$, $\hat{r}(w) \to r(W_0)$. Therefore, in condition (9) we replace $\hat{r}(w)$ by $r(W_0)$. Since W_0 can take any value in the range $[W_1, W_2]$, $f(w, y, s)$ must have the required sign for *every* value of $r(W_0)$, where $W_1 \le W_0 \le W_2$. Thus, since $q_1|\Delta(y_1)| > 0$, the condition as stated in Proposition 1 must hold. As $y \in (\underline{y}, \bar{y})$, $W_2 - W_1 < \bar{y} - \underline{y}$.

Sufficiency

To establish sufficiency we use a method similar to that used by Pratt and Zeckhauser (1987) and Gollier and Pratt (1996).

a) We first show

$$
\begin{aligned}
u'''(W_2) - u'''(W_1) &< -r(W)\left[u''(W_2) - u''(W_1)\right], \ \forall \ W_1 \le W \le W_2 \\
\implies f(w, y, s) &> 0, \ \forall \ (w, y, s)
\end{aligned}
$$

We need to show that $f(w, y, s) > 0$, for all non-degenerate probability distributions of y. Hence, we need to prove that the minimum value of $f(w, y, s)$ over *all* possible probability distributions $\{q_i\}$, with $E(\Delta(y)) = 0$, must be positive. In a manner similar to Gollier and Pratt (1996), this can be formulated as a mathematical programming problem, where $f(w, y, s)$ is minimized, subject to the constraints that all q_i are non-negative and sum

to one, and $E(\Delta(y)) = 0$. Equivalently, this can be reformulated as a parametric linear program where the non-linearity is eliminated by writing $\bar{r}$ as a parameter

$$\min_{\{q_i\}} f(w, y, s) = \sum_i q_i \left[\Delta(y_i) \left\{ -u'''(W_i) - u''(W_i)\bar{r} \right\} \right] \tag{11}$$

s.t.

$$\sum_i q_i \Delta(y_i) = 0 \tag{12}$$

$$\sum_i q_i = 1, \tag{13}$$

the definitional constraint for the parameter $\bar{r}$

$$\bar{r} \sum_i q_i u'(W_i) = - \sum_i q_i u''(W_i) \tag{14}$$

and the non-negativity constraints

$$q_i \geq 0, \quad \forall i. \tag{15}$$

Consider the optimal solution. Since this optimization problem has three constraints, there are three variables in the basis. Number these as $i = 1, 2, a$, with $\Delta(y_1) < 0 < \Delta(y_2)$ and $y_1 + s\Delta(y_1) < y_2 + s\Delta(y_2)$. The associated probabilities are q_1, q_2, q_a, such that $q_1\Delta(y_1) + q_a\Delta(y_a) + q_2\Delta(y_2) = 0$. There are two possibilities with respect to the state a.

Either:
$a = 0$. Then $\Delta(y_a) = \Delta(y_0) = 0$. Hence, we immediately obtain equation (16).

or:
$a \neq 0$. In this case we drop the constraint on $q_0 \geq 0$ (with all the other q_is staying non-negative). Hence the probability associated with y_0 can be negative. Dropping this constraint will lead to a condition that is too demanding. However, since we are searching for a sufficient condition, this is fine. In the original optimisation, all the non-basis variables had nonnegative coefficients in the objective function in the final simplex tableau. Allowing $q_0 < 0$ must result therefore in q_0 replacing either q_1, q_2 or q_a in the optimal basis. Also, the new f-value is either lower or the same as before.

Suppose, first, that q_0 replaces q_a in the optimal basis. Then the new basis variables are q_1, q_2 and q_0. Since $\Delta(y_0) = 0$, we can write the objective function (11) as

$$f^*(w, y, s) = q_1 \Delta(y_1) \left[-u'''(W_1) - u''(W_1)\bar{r} \right] + q_2 \Delta(y_2) \left[-u'''(W_2) - u''(W_2)\bar{r} \right] \qquad (16)$$

Since $q_1 \Delta(y_1) + q_2 \Delta(y_2) = 0$, it follows that (14) can be rewritten as

$$f^*(w, y, s) = q_1 \Delta(y_1) \left[(-u'''(W_1) - u''(W_1)\bar{r}) - (-u'''(W_2) - u''(W_2)\bar{r}) \right] \qquad (17)$$

Hence

$$u'''(W_2) - u'''(W_1) < -\bar{r} \left[u''(W_2) - u''(W_1) \right] \qquad (18)$$

is a sufficient condition for $f^* > 0$, given $\bar{r}$.

As shown in equation (10), $\bar{r}$ is a convex combination of $r(W_a)$, $r(W_1)$ and $r(W_2)$ with $W_1 < W_a < W_2$, hence $\bar{r} \in \{r(W) | W \in [W_1, W_2]\}$. Hence, a sufficient condition for (18) is that

$$u'''(W_2) - u'''(W_1) < -r(W) \left[u''(W_2) - u''(W_1) \right] \qquad (19)$$

for all $\{W_1 \leq W \leq W_2\}$ as given by the condition of Proposition 1.

Alternatively, suppose that q_0 replaces either q_1 or q_2 in the optimal basis. In this case the above argument remains the same with q_a instead of either q_1 or q_2, in equation (16).

b) By an analogous argument, it can be shown that $\partial \hat{r}(w)/\partial s < [=] \, 0$ is equivalent to $u'''(W_2) - u'''(W_1) > [=] \, -r(W)[u''(W_2) - u''(W_1)] \; \forall \; \{W_1 \leq W \leq W_2\}$ $\square$

Appendix 2

Proof of Proposition 2

We need to show that conditions (4) and (5) are sufficient for $\hat{\hat{r}}(w) - \hat{r}(w) \geq 0$. $E_v[\cdot]$ denotes expectations over v. From the definition of the twice derived risk aversion, $\hat{\hat{r}}$,

$$
\begin{aligned}
\hat{\hat{r}}(w) &= E_{y+e} \left[\frac{u'(w + y + e)}{E_{y+e} u'(w + y + e)} r(w + y + e) \right] \\
&= E_y \left[\frac{E_e u'(w + y + e)}{E_{y+e} u'(w + y + e)} E_e \left\{ \frac{u'(w + y + e)}{E_e u'(w + y + e)} r(w + y + e) \right\} \right]
\end{aligned}
$$

$$= E_y \left[\frac{E_e u'(w + y + e)}{E_{y+e} u'(w + y + e)} r_e(w + y) \right],$$

where $r_e(w + y)$ is as defined on page 9. Hence

$$
\begin{aligned}
\hat{\hat{r}}(w) - \hat{r}(w) &= E_y \left[\left(\frac{E_e u'(w + y + e)}{E_{y+e} u'(w + y + e)} - \frac{u'(w + y)}{E_y u'(w + y)} \right) r_e(w + y) \right] \\
&+ E_y \left[\frac{u'(w + y)}{E_y u'(w + y)} (r_e(w + y) - r(w + y)) \right]
\end{aligned}
$$

Condition (4) implies that the second term is positive or zero. The first term is similar to a covariance term since the term in ( ) has zero expectation. Hence the first term is nonnegative if the term in ( ) is single crossing downwards and $r_e(w + y)$ is declining in y. The latter is implied by condition (5). Therefore, to complete the proof we have to establish the single crossing downward property. For notational simplicity, let $Z(w + y)$ denote the term in ( ),

$$Z(w + y) = \frac{E_e u'(w + y + e)}{a} - \frac{u'(w + y)}{b},$$

with a and b being appropriately defined constants.

Differentiating with respect to y yields

$$
\begin{aligned}
Z'(w + y) &= \frac{E_e u''(w + y + e)(1 + \frac{\partial e}{\partial y})}{a} - \frac{u''(w + y)}{b} \\
&= -\frac{E_e u'(w + y + e)}{a} r_e(w + y) + \frac{u'(w + y)}{b} r(w + y) + \frac{E_e u''(w + y + e)\frac{\partial e}{\partial y}}{a}.
\end{aligned}
$$

For $Z = 0$ it follows that $sgn Z'(w + y) = sgn[r(w + y) - r_e(w + y) + [E_e u'(w + y + e)]^{-1} E_e u''(w+y+e)(\partial e/\partial y)]$. Hence condition (4) implies $Z'(w+y) \leq 0$ at a crossing point if e is distributed independently of y, i.e. $\partial e/\partial y \equiv 0$. Then only one crossing point exists, therefore $Z(w + y)$ is downward sloping. If the distribution of e improves with increasing y according to second-order stochastic dominance, then $E_e u''(w+y+e)(de/dy) < 0$ if $u''' > 0$. $u''' > 0$ follows from condition (5) because $dr_e(w + y)/dy \leq 0$ holds for a small risk only if $r' < 0$. Hence, at a crossing point, $Z'(w + y) \leq 0$. $\square$

Proof of Corollary 7

We need to show that condition (5) holds if the distribution of e improves with increasing y according to second-order stochastic dominance. Since

$$
\begin{aligned}
r_e(w+y) &= E_e\left[\frac{u'(w+y+e)}{E_e u'(w+y+e)} r(w+y+e)\right], \\
\frac{dr_e(w+y)}{dy} &= E_e\left[\frac{u'(w+y+e)}{E_e u'(w+y+e)} \frac{dr(w+y+e)}{dy}\right] \\
&+ E_e\left[\frac{d}{dy}\left(\frac{u'(w+y+e)}{E_e u'(w+y+e)}\right) r(w+y+e)\right].
\end{aligned}
$$

The first term is a "risk-adjusted" expectation of $dr(w+y+e)/dy$. If e were distributed independently of y, then $r' < 0$ would imply a negative expectation. This is reinforced for $r' < 0$ and $r'' > 0$ if the distribution of e improves according to second-order stochastic dominance.

Now consider the second term. Using the proof technique of Gollier and Pratt (1996, p. 1122) it follows that this term is negative if it is for every binomial distribution of e. Suppose that e is distributed independently of y. Then $u'' < 0$ and $u''' > 0$ imply that $u'(w+y+e)/E_e u'(w+y+e)$ declines [increases] in y for the lower [higher] realization of e. Hence $r' < 0$ implies that the second term is negative. This is reinforced if the distribution of e improves according to second-order stochastic dominance. Hence $dr_e(w+y)/d(w+y) \le 0$
□

References

Diamond, P.A. and Stiglitz, J.E. (1974), "Increases in Risk and Risk Aversion", *Journal of Economic Theory*, 8, 337-360.

Eeckhoudt, L., Gollier, C. and Schlesinger, H. (1996), "Changes in Background Risk and Risk Taking Behaviour", *Econometrica*, 64, 683-689.

Eeckhoudt, L., and Kimball, M.S. (1992), "Background Risk, Prudence, and the Demand for Insurance", in *Contributions to Insurance Economics*, G. Dionne, ed., Kluwer Academic Publishers, Boston, 239-254.

Eichner, Th. and Wagener, A. (2003), "Variance Vulnerability, Background Risks, and Mean-Variance Preferences", *The Geneva Papers on Risk and Insurance Theory*, 28, 173-184.

Franke, G., Stapleton, R.C. and Subrahmanyam, M.G. (1998), "Who Buys and Who Sells Options: The Role of Options in an Economy with Background Risk", *Journal of Economic Theory*, 82, 89-109.

Gollier, C. and Pratt, J. W. (1996), "Weak Proper Risk Aversion and the Tempering Effect of Background Risk", *Econometrica*, 64, 1109-1123.

Keenan, D.C. and Snow, A. (2003), "Locally Greater Vulnerability to Background Risk", *The Geneva Papers on Risk and Insurance Theory*, 28, 161-172.

Kihlstrom R.E., Romer D. and Williams S. (1981), "Risk Aversion with Random Initial Wealth", *Econometrica*, 49, 911-920.

Kimball, M.S. (1993), "Standard Risk Aversion", *Econometrica*, 61, 589-64.

Meyer, P.J. and Meyer, J., (1998), "Changes in Background Risk and the Demand for Insurance", *The Geneva Papers on Risk and Insurance Theory*, 23, 29-40.

Nachman, D.C. (1982), "Preservation of 'More Risk Averse' under Expectations", *Journal of Economic Theory*, 28, 361-368.

Pratt, J.W. (1964), "Risk Aversion in the Small and in the Large", *Econometrica*, 32, 122-136.

Pratt, J.W. and Zeckhauser, R.J. (1987), "Proper Risk Aversion", *Econometrica*, 55, 143-154.

Rothschild, M. and Stiglitz J.E. (1970), "Increasing Risk: I. A definition", *Journal of Economic Theory*, 2, 225-243.