HETEROGENEITY OF INVESTORS AND ASSET PRICING IN A RISK-VALUE WORLD*

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Abstract

Portfolio choice and the implied asset pricing are usually derived assuming maximization of expected utility. In this paper, they are derived from risk-value models which generalize the Markowitz-model. We use a behaviorally based risk measure with an endogenous or exogenous benchmark. A richer set of sharing rules is obtained than in an expected utility world. If the risk measure is modelled by a negative HARA-function, then sharing rules are convex or concave relative to each other. More importantly, the pricing kernel convexity increases with heterogeneity of investors. Therefore an increase in heterogeneity raises investors' needs for trading options and makes all European options more expensive relative to the price of the underlying asset.

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1 Introduction

Recently, traditional financial theories of asset pricing have come under intensive discussion. Quite a number of market phenomena have been discovered which cannot or at least cannot easily be explained by traditional theories. The equity premium puzzle [*Mehra* and *Prescot* 1985], short- and long term predictability of stock returns [*Jegadeesh* and *Titman* 1993, *De Bondt* and *Thaler* 1985], high volatility of stock prices [*Shiller* 1981, 1989] and the success of strategies based on value and size [*Vuolteenaho* 2000] are just examples for those phenomena. Interestingly enough these puzzles are not only found at the New York Stock Exchange but they are also present in markets world wide, see, e.g., *Rouwenhorst* (1998) and *Schiereck*, *De Bondt* and *Weber* (2000) for the persistence of short-term predictability. These results are complemented by the analysis of individual trading behavior [*Odean* 1998 a] which reflects biases known from psychological research as well as from experimental work which clearly shows that expected utility does not adequately describe peoples' behavior [*Gneezy* and *Potters* 1997, *Sarin* and *Weber* 1993, *Thaler et al.* 1997, *Weber* and *Camerer* 1998].

The question remains if these puzzles can be solved within the framework of expected utility (EU) or if new theories incorporating behavioral ideas are needed. Ultimately those theories will be chosen which can explain the (some) puzzles and which are based on more meaningful assumptions. In the following, we will present a behavioral asset pricing theory which is based on more realistic behavioral assumptions. This theory allows us to address the important question which impact heterogeneity of investors has on asset pricing, in particular, on option pricing.

Quite a number of different behavioral theories have been developed lately. Most of them focus on specific behavioral aspects of individual decision making. There are papers that build on overconfidence to explain the amount of trading [*Daniel, Hirshleifer* and *Subrahmanyam* 1998, *Odean* 1998 b]. Another stream of research is based on the value function part of prospect theory [*Barberis, Huang* and *Santos* 2000, *Barberis* and *Huang* 2000, *Hong* and *Stein* 1999, *Zuchel* and *Weber* 2000] and tries to understand the observed correlation in stock returns. We will base our work on yet another body of knowledge from psychology, i.e., the understanding of what people perceive when they talk about the riskiness of an asset. This enables us to come up with a risk-value framework which is behaviorally based and better describes peoples' risk judgments than EU.

Risk is at the center of most theories about asset valuation. It is therefore essential to understand what investors perceive the risk of an asset to be. Investors' risk judgments should serve as a basis of asset valuation. Therefore results of psychological research on risk perception and risk judgment are essential for asset valuation.

This paper starts from risk-value models. In a risk-value model, risk and value are taken as primitives. To evaluate an alternative, the decision maker first separately derives value and risk of this alternative. Then value and risk are combined into an overall preference. While value is always defined as the expected outcome, the essential question relates to the measure of risk. Markowitz also used a risk-value model with risk being measured by the variance of the outcome.

Quite a number of attempts have been made to come to a more realistic concept of risk.[1]

New results suggest that risk perception can be modelled as the expected value of the transformed deviation of the outcome from a benchmark. The

[1] The inclusion of third moments of the return distributions, see *Kraus* and *Litzenberger* (1976), or taking semi-variance as a measure of risk, see *Bawa* and *Lindenberg* (1977) are examples of early attempts which, however, are not based on psychological research.

benchmark can be exogenous (any target or aspiration level) or endogenous, e.g. the expected outcome. The transformation is achieved by a risk function. Empirical work suggests that the risk function can be a monotonically decreasing convex function, with positive deviations from the benchmark reducing risk and negative deviations increasing risk. Such a risk function reflects the practitioners' view that negative deviations constitute "risk" and positive deviations "chances". At some point in our analysis, we will restrict the risk function to be of the negative HARA (hyperbolic absolute risk aversion)-type. This measure still captures major empirical findings on risk perception.

Using this behavioral framework we get two classes of results. First, we determine the shape of the investors' sharing rules in equilibrium. Two definitions of sharing rules will be used. The absolute sharing rule is the function which relates the investor's portfolio payoff to the aggregate payoff, i.e. the exogenously given payoff to all investors. The relative sharing rule is the function which relates an investor's payoff to the payoff of another investor. In the EU context, *Cass* and *Stiglitz* (1970) and *Rubinstein* (1974) showed that for expected utility maximizers using a utility function of the HARA-class with the same exponent, relative sharing rules are linear. In risk-value models where risk is measured by a negative HARA function, relative sharing rules are linear if and only if this risk function is quadratic. Otherwise strictly convex or concave relative sharing rules are obtained. More heterogeneity among investors translates into more concavity or convexity of relative sharing rules.

Nonlinear relative sharing rules relate our model to the literature on portfolio insurance [*Leland* 1980, *Brennan* and *Solanki* 1981, *Benninga* and *Blume* 1985, *Franke*, *Stapleton* and *Subrahmanyam* 1998, *Grossman* and *Zhou* 1996, *Benninga* and *Mayshar* 2000]. *Leland* defines portfolio insurance as a portfolio policy which leads to a convex absolute sharing rule. The intuition behind his concept is that a convex absolute sharing rule gives the investor a higher payoff when the aggregate payoff is low (= low state) as compared to a linear or concave sharing rule. Our results show that an investor with a risk function of the HARA-type who demands a higher expected portfolio return or has a higher initial endowment than another investor buys a sharing rule which is strictly convex relative to that of the other investor. Thus, the first investor buys portfolio insurance from the second investor.

Second, our equilibrium analysis yields a pricing kernel which is declining and convex in aggregate consumption. Hence, as *Dybvig* (1988) points out, there exists a von Neumann-Morgenstern utility function such that a representative investor with this function would imply the same pricing kernel. Hence there would be no need to talk about risk-value models. The important contribution of this paper is, however, to derive the impact of investor heterogeneity on this pricing kernel. It will be shown for HARA-based risk functions that the convexity of the kernel increases monotonically in a simple measure of heterogeneity. More heterogeneity is reflected in more convexity/concavity of relative sharing rules. The more convex/concave relative sharing rules are, the stronger is the need of investors to trade options, the higher option prices will be relative to the price of the underlying asset. Hence the implied volatility derived from the Black-Scholes model increases with heterogeneity. This might explain the observation that stock index options appear to be more expensive than suggested by the Black/Scholes model [*Christensen* and *Prabhala* 1998]. Also, our results could help to explain why the pricing kernels estimated from stock index option prices are leptokurtic and negatively skewed [*Longstaff* 1995, *Brenner* and *Eom* 1996, *Jackwerth* 2000].

The paper is organized as follows. In section 2, we will review some of the theoretical and empirical research in decision theory on how to measure risk. Based on this analysis, we will discuss some general properties a risk measure should have. In section 3, risk-value efficient portfolios are derived. Equilibrium is analysed in sections 4 and 5. In section 4, we first investigate individual sharing rules for a rather general class of risk functions and then for HARA functions. In section 5, results about the pricing kernel are derived. Its convexity will be shown to increase with the heterogeneity of investors. Section 6 summarizes the main results.

2 Risk Measurement

2.1 Background[2]

The separation of value and risk is quite popular in finance. Investors usually talk about the risk of an investment which then is evaluated against its expected return. Thus, decisions are made by evaluating risk and return separately and trading off both components. Taking risk and value separately allows different investors to have different tradeoffs even though their risk measures may be the same, in contrast to EU. The explicit consideration of the investment risk has become even more important in the light of recent regulations which require broker houses to inform their clients about the riskiness of their investments. Risk judgments have also become quite important in bank regulation and management.

Still the most important measure of risk is variance with value being the endogenous benchmark. Variance is easy to use, and risk-value models based on variance as a risk measure are compatible with expected utility. The problem is, however, that variance does not capture peoples risk perception as can be easily demonstrated by the following example:

[2] See *Sarin* and *Weber* (1993) and *E. Weber* (1997) for reviews on risk measurement.

$A = (.5, \$10; .5, \$-10)$ $B = (.2, \$20; .8, \$-5)$ $C = (.8, \$5; .2, \$-20)$.

The vector $(p, \$x; 1-p, \$x')$ denotes an alternative which has a p-chance of getting $\$x$ and a 1-p-chance of getting $\$x'$. All three alternatives have the same expected value and the same variance. However, most subjects judge C to be the most risky alternative: they seem to dislike the relatively large loss potential in alternative C.

To model risk perception, the literature on behavioral decision making has proposed and tested different theories [see *Brachinger* and *Weber* (1997) and *Jia, Dyer* and *Butler* (1999) for an overview]. An exponential model [*Sarin* 1984] and a power function model [*Luce* and *E. Weber* 1986] were found to fit the data quite well [*Keller et al.* 1986 and *E. Weber* and *Bottom* 1990]. These models define risk as the expected value of a function of the outcomes or of the deviations of the outcomes from a possibly endogenous benchmark. Thus, the risk of a random variable e can be written as

$$Risk(e) = E[F(e - \bar{e})] \tag{1}$$

F is a function with $F(0) = 0$, e denotes the random payoff of the alternative and $\bar{e}$ the benchmark. This risk measure is general enough to include risk measures which describe people's risk perception, e.g. the exponential risk measure. *Jia* and *Dyer* (1996) define a standard measure of risk as in equation (1) with $(-F)$ being a von Neumann-Morgenstern utility function. The benchmark $\bar{e}$ equals the expected payoff.

There are two fundamental ways for defining a benchmark: exogenously or endogenously. An exogenous benchmark is set by the investor, e.g., it can be the outcome level the investor wants to surpass. This benchmark can have any sign. This benchmark may be affected by portfolio gains or losses in previous periods as suggested by *Barberis*, *Huang* and *Santos* (2000) and by other factors which have been shown to be important in financial behavior.

An endogenous benchmark depends on the characteristics of the payoff distribution. The most prominent endogenous benchmark is the expected value of the payoffs as in variance and other moments. This benchmark implies that the risk measure is location free, i.e. risk does not change if the return distribution is shifted by adding or subtracting a positive number. Hence risk is independent of the expected payoff. This is a desirable property since the expected payoff (value) is already used as a primitive in the preference function.

2.2 Properties of a Risk Measure

We now describe the risk measure in more detail. Risk will be measured according to equation (1) as the expectation of a function of the deviation of a random variable e from a benchmark $\bar{e}$. In the case of an exogenous benchmark $\bar{e}$ is a given number, in the case of an endogenous benchmark $\bar{e}$ is the expected value $E(e)$.

For simplicity we define the deviation $\hat{e} := e - \bar{e}$.

As a next step we postulate three key properties of the risk function F which are based on the psychological studies cited before:

i) Outcomes above the benchmark reduce risk and outcomes below increase risk, $F(\hat{e}_1) > F(0) > F(\hat{e}_2)$ for $\hat{e}_1 < 0 < \hat{e}_2$. In addition, we require monotonicity: A higher payoff will contribute less to risk than a lower payoff, thus $F' < 0$.

ii) Mean preserving spreads increase risk, thus $F'' > 0$.

iii) The sensitivity to a mean preserving spread is larger in the loss domain (relative to the benchmark) than in the gain domain. Requiring monotonicity then implies that the sensitivity decreases if the payoff increases; thus we require $F''' < 0$.[3]

As proposed by *Jia* and *Dyer* (1996), risk functions with $F' < 0$, $F'' > 0$, and $F''' < 0$ correspond to utility functions with $u' > 0$, $u'' < 0$ and $u''' > 0$, i.e. with positive prudence [*Kimball* 1990].

Standard models of risk perception, e.g. the exponential model, have the properties noted above. Note that variance neither fulfills property i) nor property iii). The three properties imply that alternative C (see example last section) is judged the most risky. Risk judgments found in a number of empirical studies, see, e.g., *Keller, Sarin* and *Weber* (1986), are consistent with the risk ranking implied by these properties. Property iii) is reflected in a variety of empirical results that show that people judge alternatives with potential catastrophic outcomes as being especially risky, see, e.g., *Slovic* (1987).

In the following sections we will derive results using properties i) - iii). The risk function, in general, varies from investor to investor. It is defined on the range $(\underline{\hat{e}}, \overline{\hat{e}})$. In order to guarantee optimal internal solutions to portfolio choice, we add the assumption that $F'(\hat{e}) \rightarrow -\infty$ for $\hat{e} \rightarrow \underline{\hat{e}}$ and $F'(\hat{e}) \rightarrow 0$ for $\hat{e} \rightarrow \overline{\hat{e}}$. At the end, in order to get further results, we need more detailed information about the risk measure. At that point we will assume that the risk function belongs to the set of negative HARA-functions. We consider negative HARA-functions with properties i) - iii). They include

[3]The necessity of $F''' < 0$ can be seen as follows:
Let y, z be two states with the same payoff and the same probability p; in both states the payoff deviates from the expected value by Δ. Their contribution to the total risk of the portfolio is then given by $2pF(\Delta)$. Now replace the payoff deviation Δ in the states y and z by a mean preserving spread around Δ, that is: the deviation from the expected payoff is $\Delta - \alpha$ in state y and $\Delta + \alpha$ in state z ($\alpha > 0$). Notice that the expected payoff is not changed and hence it does not influence the risk contribution of the other states. The new contribution of the states y, z to the total risk is $pF(\Delta - \alpha) + pF(\Delta + \alpha)$. The risk increase, denoted RI_α is then given by: $RI_\alpha(\Delta) = pF(\Delta - \alpha) + pF(\Delta + \alpha) - 2pF(\Delta)$ and the strict convexity of F is equivalent to $RI_\alpha > 0$ for all Δ and all $\alpha > 0$. We require for an increase in Δ that $RI'_\alpha(\Delta) = pF'(\Delta - \alpha) + pF'(\Delta + \alpha) - 2pF'(\Delta) < 0$. This holds iff $F''' < 0$.

the exponential function, an important function to describe people's risk judgments. The negative HARA- class is defined as

$$F(\hat{e}) \ = -\frac{1-\gamma}{\gamma} \left(A + \frac{\hat{e}}{1-\gamma}\right)^{\gamma} \tag{2}$$

where $\gamma \in \mathbb{R} \setminus \{ 0,\ 1 \}$; $A > 0$. In the case $\gamma = 0$ we obtain $F(\hat{e}) = -ln(A + \hat{e})$ and in the case $\gamma = -\infty$ we get $F(\hat{e}) = exp(-B\hat{e})$ with $B > 0$. For $\gamma > -\infty$ the domain of F is constrained by $(A + \hat{e}/(1-\gamma)) > 0$. A has to be sufficiently high. For $\gamma < 1$, we need $inf\ \hat{e} > -A(1-\gamma)$; for $\gamma > 1$, we need $sup\ \hat{e} < -A(1-\gamma)$. Since it makes little sense to constrain $\hat{e}$ from above, we shall mostly assume $\gamma < 1$. We have $F' < 0, F'' > 0$ and

$$F'''\ (\hat{e}) = \frac{\gamma - 2}{1-\gamma} \left(A + \frac{\hat{e}}{1-\gamma}\right)^{\gamma-3} \tag{3}$$

As we require $F'''(\hat{e}) < 0$, $\gamma < 1$ or $\gamma > 2$ is implied. Therefore, we will only consider functions with $\gamma < 1$ or $\gamma > 2$, mostly $\gamma < 1$.

2.3 Risk-value Models and EU Models

So far we have concentrated on modelling risk. We will now briefly discuss the relationship between risk-value models and EU models. Risk-value models combine the primitives risk and value into a preference measure without constraining the tradeoff between value and risk. In the expected utility model the utility function determines both, risk measurement and the tradeoff between risk and value. Thus, the decision maker cannot determine measurement and tradeoff independently.

Jia and *Dyer* (1996) consider risk-value models as defined in equation (1). They show that preference orders which can be derived from a risk-value model can be derived from an expected utility model only if a very strong condition, called risk independence, holds for expected utility. Basically, this condition requires that lotteries with the same expected payoff can be preference ranked by their risk measures such that this ranking does not change when the expected payoff changes.[4] This condition holds only for very few utility functions which, in turn, imply very specific risk measures. Thus, risk-value models are more general. This is in the spirit of behavioral research to allow for a wider range of behavior.

3 Efficient Portfolios

We assume a two date-economy with a perfect and complete capital market. At date 0 investors choose their portfolios which pay off at date 1. A state of nature at date 1 is defined by the exogenously given aggregate payoff ε; i.e. the sum of payoffs to all investors.[5] ε is a positive variable, $\varepsilon \in (\underline{\varepsilon}, \overline{\varepsilon})$ with the probability density being positive for every $\varepsilon \in (\underline{\varepsilon}, \overline{\varepsilon})$. Since we are not interested in time preferences, the whole analysis is done in forward terms. Equivalently, the risk-free rate can be assumed to be zero. Define:

p_ε : = probability density of ε,

e_ε : = number of claims contingent on state ε, purchased by the investor; each claim pays off \$1 if and only if state ε obtains,

π_ε: = forward pricing kernel; for every state ε it denotes the price of a claim contingent on that state divided by the state probability density, $\pi_\varepsilon > 0$; $E(\pi) = 1$.

W_0 : = the investor's initial forward endowment (wealth), $W_0 > 0$. Since the investor is endowed with state-contingent claims, W_0 equals the forward

[4] See *Bell* (1995a, b) for further discussion on this issue.

[5] The market is said to be complete if for every $\varepsilon^0 \in \mathbb{R}^+$ there exists a claim which pays off \$ 1 if $\varepsilon > \varepsilon^0$ and zero otherwise [see *Nachman* 1988]. We assume the existence of $\underline{\varepsilon}, \overline{\varepsilon} \in \mathbb{R}^+ \cup \{\infty\}$such that the state space is identified by $(\underline{\varepsilon}, \overline{\varepsilon})$.

market value of these claims.

R^*: = expected gross portfolio return on the forward endowment required by the investor.

The pricing kernel is assumed to be twice continuously differentiable. To simplify notation, the state index will be dropped unless necessary for clarity of exposition.

In risk-value models the investor, first, derives the set of risk-value efficient portfolios and, second, chooses one of these portfolios according to his tradeoff between risk and value. The advantage of our analysis is that asset pricing in a risk-value equilibrium can be analyzed as in the CAPM-world without making assumptions about this tradeoff.

A risk-value efficient portfolio minimizes the risk subject to the constraint that the expected portfolio payoff $E(e)$ does not fall below some exogenously given value W_0R^* (payoff constraint). Hence, the expected gross portfolio return has to be equal or higher than R^*. In this section, the prices for state-contingent claims are assumed to be exogenously given. Then an efficient portfolio is the solution to the following problem.

Minimize

$$E\ [F\ (\hat{e})] \tag{4}$$

subject to the budget constraint:

$$E\ [e\pi] = W_0 \tag{5}$$

and the payoff constraint:

$$E\ (e) \geq W_0 R^* \tag{6}$$

Varying R^* parametrically allows us to derive all risk-value efficient portfolios. In the following the function

$$f\ (\hat{e}) := -\frac{\partial\ F\ \ (\hat{e})}{\partial\ \hat{e}} \tag{7}$$

will be of special importance. From the properties of the risk function $F(F' < 0,\ F'' > 0$, and $F''' < 0)$, we immediately get:$f > 0,\ f' < 0,\ f'' > 0$.

3.1 Risk-Value Models With an Endogenous Benchmark

Using this notation we can write the first order condition for a solution to the minimization problem (4) - (6) with an endogenous benchmark $\bar{e} = E(e)$, (η is the Lagrange-multiplier of the budget constraint (5) and λ the Lagrange-multiplier of the payoff constraint (6)) as[6] (after dividing by the probability density)

$$-f(\hat{e}_\varepsilon) + E[f\ (\hat{e})] = \eta\pi_\varepsilon + \lambda;\ \forall\ \varepsilon. \tag{8}$$

As $E(\pi) = 1$, taking expectations yields

$$0 = \eta + \lambda \tag{9}$$

[6] The solution of the minimization problem exists and is unique if a) for every e satisfying the constraints (5) and (6) the value of the objective function is finite and b) for every λ, η and e satisfying the first order condition (8) $E[e\pi]$ is finite [*Back* and *Dybvig* 1993].

Since raising R^* raises the risk of the efficient portfolio, $\lambda > 0$ so that, by (9), $\eta < 0$. Substituting λ in equation (8) yields

$$-E[f(\hat{e})] + f(\hat{e}_\varepsilon) = \eta\,[1 - \pi_\varepsilon]\,;\ \forall\,\varepsilon \tag{10}$$

Defining $\theta_\varepsilon = (1 - \pi_\varepsilon)$ this equation can be rewritten as :

$$-E[f(\hat{e})] + f(\hat{e}_\varepsilon) = \ \eta\,\theta_\varepsilon;\,\forall\,\varepsilon \tag{11}$$

The investor's efficient portfolio is characterized by equation (11) and constraints (5) and (6). θ_ε is the difference between the forward price of the risk-free claim, 1, and the probability deflated forward price of a state ε-contingent claim. θ_ε is negative if the probability deflated forward price of a state ε-contingent claim exceeds the forward price of the risk-free claim. θ has zero expectation. The higher θ_ε, the cheaper are the state ε-contingent claims, and the more state -contingent claims the investor buys because a higher level of these claims raises $-f(\hat{e}_\varepsilon)$.

The risk-value efficient frontier can be derived by varying parametrically the required expected return R^*. If R^* is not greater than 1, then the Lagrange multiplier $\lambda = 0$ and risk is zero. All endowment is invested in the risk-free asset so that e_ε is the same for every state. λ grows with R^* ($R^* > 1$), because the objective function is strictly convex. Since $\lambda = dE[F(\hat{e})]/d(W_0R^*)$ for efficient portfolios, it follows that the risk-value efficient frontier is strictly convex as shown in figure 1.

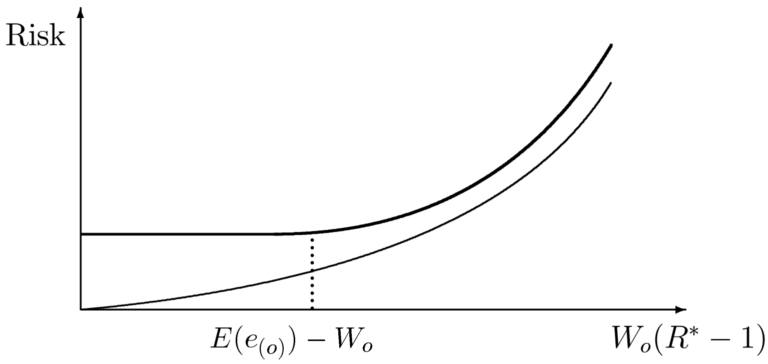


Fig.1 The risk-value efficient frontier depicts the minimal portfolio risk as a function of the required expected portfolio return R^*. The thin curve represents the frontier for an endogenous benchmark, the thick curve for an exogenous benchmark.

3.2 Risk-Value Models With an Exogenous Benchmark

Now consider risk functions with an *exogenous* benchmark $\bar{e}$; $\bar{e} \gtreqless 0$. Then in the first order condition (8) for a risk-value efficient portfolio the second term disappears since portfolio choice has no effect on the benchmark. Hence the first order condition reads:

$$-f(\hat{e}_\varepsilon) = \eta\pi_\varepsilon + \lambda; \forall\varepsilon. \tag{12}$$

Again, $\lambda > 0$ so that $\eta < 0$ follows.

Taking expectations yields

$$-E[f(\hat{e})] = \eta + \lambda \tag{13}$$

so that subtraction of (12) from (13) leads to

$$-E[f(\hat{e})] + f(\hat{e}_\varepsilon) = \eta\theta_\varepsilon;\ \forall\ \varepsilon. \tag{14}$$

Compare the solution of the exogenous benchmark model with that of the endogenous benchmark model. From equation (12), raising the payoff in some state always reduces risk in the case of an exogenous benchmark. From equation (8), for an endogenous benchmark an increase in some payoff lowers [raises] risk if the payoff $\hat{e}_\varepsilon$ is lower [higher] than $f^{-1}(E[f(\hat{e})])$. This follows from the impact of the payoff increase on the benchmark. Yet, in the case of an exogenous benchmark, the risk reduction is higher the lower the payoff. Therefore, the risk reduction induced by raising the payoff in some state minus the expected risk reduction induced by raising the payoff in every state, $-f(\hat{e}_\varepsilon) + E[f(\hat{e})]$, in the exogenous benchmark model has the same properties as the risk reduction in the endogenous benchmark model. This explains why the first order conditions (11) and (14) look precisely the same although the optimal portfolios are different because of the different benchmarks.

In spite of the formal identity of (11) and (14) the optimal solutions to both problems differ substantially. If the payoff constraint (6) is not binding, then the optimal solution to the endogenous benchmark-problem is to buy only the risk free asset. The exogenous benchmark-problem without the payoff constraint is formally the same as the traditional state preference -EU-choice problem Max $E[u(e)]$ subject to the budget constraint (5). This follows since minimizing risk with the risk function being a negative utility function is formally the same as maximizing expected utility. Hence the investor minimizing risk with an exogenous benchmark chooses a risky portfolio $e_{(0)}$ even if the payoff constraint (6) is not binding. This implies $E(e_{(0)}) > W_0$ or, equivalently, an expected gross portfolio return $E(R) > 1$. The payoff constraint becomes binding only if the required expected payoff is raised above this expectation, i.e. if $W_0 R^* > E(e_{(0)})$; (see figure 1).

We summarize the differences between risk-value models and expected utility models as follows.

1. The risk-value model with an exogenous benchmark is formally the same as the expected utility model if the payoff constraint (6) is not binding. If the payoff constraint is binding, then the investor has to take more risk so as to satisfy the payoff constraint. Hence the payoff constraint adds a new element to the optimization. For risk functions with an endogenous benchmark the investor takes risk if and only if the payoff constraint is binding.

2. If the benchmark is endogenous, then risk measurement depends on this endogenous benchmark which conflicts with the axioms of von Neumann/Morgenstern.[7]

[7] A third difference between risk-value models with an endogenous benchmark and EU-models relates to satiation. In the EU-model, marginal utility is always positive, by assumption. In the risk-value model, the investor may be worse off if she receives an additional payoff in some state with a high portfolio return. Consider as an example the preference function $P(\bar{e}, risk)$ with $\alpha > 0$ and $\bar{e} = E(e)$, $P(\bar{e}, risk) = \alpha\bar{e} - risk$, with $risk$ as defined in equation (1). Differentiate the preference function with respect to e_ε . This yields:

$$\frac{\partial P}{\partial e_\varepsilon} = p_\varepsilon \left[\alpha + f(\hat{e}_\varepsilon) - E[f(\hat{e})]\right].$$

If the portfolio payoff is random, there must exist a state ε with $f(\hat{e}_\varepsilon) < E[f(\hat{e})]$ (for example, the state with the lowest π_ε). Then, given a sufficiently small α, we have $\partial P / \partial e_\varepsilon < 0$. Hence an increase in the state ε-portfolio return may reduce the investor's welfare which contradicts the usual assumption of non-satiation. If, however, all prices for state-contingent claims, π_ε, are positive, then the investor always chooses his/her optimal portfolio such that he/she never reaches or crosses satiation. This follows since a risk free-asset exists and the investor can always purchase fewer claims contingent on these critical states, invest the saved money in the risk-free asset and, thereby, increase his/her welfare.

4 Equilibrium: Investors' Sharing Rules

4.1 General Risk Functions

Next equilibrium in the capital market will be investigated. We assume the existence of an equilibrium. If there exist multiple equilibria, we analyse anyone of them. Every investor chooses a risk-value efficient portfolio. All investors are assumed to have homogeneous expectations. First, individual sharing rules will be analysed in equilibrium (Section 4). After considering general risk functions (Section 4.1), we restrict ourselves to HARA-based risk functions (Section 4.2). Second, the equilibrium pricing kernel will be derived and analyzed (Section 5).

Individual investors are indexed by i. Hence all investor-dependent variables have to be indexed by i. Then condition (11) for an endogenous benchmark reads

$$-E[f_i(\hat{e}_i)] + f_i(\hat{e}_{i\varepsilon}) = \eta_i \, \theta_\varepsilon \; ; \quad \forall \, \varepsilon, i. \tag{15}$$

Condition (14) for an exogenous benchmark is the same. Unless stated otherwise, the following results hold for efficient portfolios with an endogenous and with an exogenous benchmark. $\lambda_i > 0$ and $\eta_i < 0$, will be assumed throughout. Then we can derive a proposition which relates investors' portfolio choices to the pricing of state-contingent claims.

Proposition 1 *: For every investor, his/her optimal payoff is decreasing and convex in the probability-deflated price for state-contingent claims.*

Proof. [8]

[8] By the implicit function theorem, $\hat{e}_i\ (\varepsilon)$ is twice continuously differentiable since $f_i\ (\hat{e}_i)$ and $\theta_\varepsilon = \theta(\varepsilon)$ are twice continuously differentiable.

Differentiate equation (15) with respect to θ. Recall, $f_i(\hat{e}_i) > 0$, $f_i'(\hat{e}_i) < 0$, and $f_i''(\hat{e}_i) > 0$. Hence it follows that e_i is an increasing and convex function in θ. As $\theta = 1 - \pi$, e_i is a decreasing, convex function in π. This proves proposition 1 ■

An absolute sharing rule relates investor i's payoff e_i to the aggregate payoff ε. Proposition 1 states $de_i/d\pi < 0$. Aggregation across investors implies $d\varepsilon/d\pi < 0$. Hence it follows that $de_i/d\varepsilon > 0$. The positive slope of the absolute sharing rule does not come as a surprise. This is still in line with EU-theory. The major difference between the risk-value and the EU-model is reflected in the shapes of the sharing rules. Equation (15) implies for two investors i and j in the risk-value model

$$\frac{-E[f_i\ (\hat{e}_i)] + f_i(\hat{e}_{i\varepsilon})}{-\eta_i} = \frac{-E[f_j\ (\hat{e}_j)] + f_j(\hat{e}_{j\varepsilon})}{-\eta_j}\ ; \quad \forall\ \varepsilon. \tag{16}$$

Define $s_i := E[f_i(\hat{e}_i)]/-\eta_i$ to be investor i 's sharing constant. Equation (16) can be written then as

$$-s_i + \frac{f_i(\hat{e}_{i\varepsilon})}{-\eta_i} = -s_j + \frac{f_j(\hat{e}_{j\varepsilon})}{-\eta_j}\ ; \quad \forall\ \varepsilon. \tag{17}$$

The new element in risk-value models as compared to expected utility models are the sharing constants. They are generated in the case of an exogenous benchmark by the payoff constraint and, in the case of an endogenous benchmark, by the impact of a payoff change on the benchmark. In order to grasp the intuition behind the sharing constants, consider the inverse sharing constant - $\eta_i/E[f_i\ (\hat{e}_i)]$. $-\eta_i$ is the efficient increase in risk due to a marginal reduction in the initial endowment available for buying claims, holding the required expected payoff W_0R^* constant. This risk increase depends on the risk function which is determined up to a linear positive transformation. Hence we need to standardize these marginal risk increases to make them comparable across investors. This is done by dividing - η_i through the expected slope of the risk function based on the efficient portfolio, $E[f_i\ (\hat{e}_i)]$. Hence the inverse sharing constant measures the standardized efficient risk increase due to a marginal reduction in the initial endowment.

In order to gain some insight into the mechanics of the risk-value model, we analyse the impact of changes in initial endowment and in the required expected return on an investor's sharing constant. For simplicity of notation, we drop the index i in Lemma 1.

Lemma 1: *Consider risk-value efficient portfolios under the condition* $\lambda > 0$ and $\eta < 0$. *Then, given the prices of state-contingent claims, the sharing constant s declines when*

– *the initial endowment W_0 increases, or*

– *the required expected return R^* increases.*

Proof. See Appendix A.

From Lemma 1 it is apparent that the sharing constants differ across investors. The sharing constants, in fact, prohibit linear sharing rules. This is illustrated by proposition 2 which does not constrain $f''(\hat{e})$ to be positive. It should be noted that each investor has a linear absolute sharing rule if all relative sharing rules are linear, and vice versa.

Proposition 2 *: Let $f''(\hat{e})$ be unconstrained in sign. Then in an equilibrium with risk-value models every investor has a linear (absolute) sharing rule if and only if every investor uses a quadratic function $F(\hat{e})$.*

Proof. See Appendix B.

Proposition 2 provides a strong result about the shape of the sharing rules. In a risk-value world the sharing rules are linear if and only if every investor uses variance or a related quadratic risk measure (which both behaviorally are not appropriate), or, equivalently, if and only if the sharing constant disappears. This is true only for quadratic risk functions. Then $f(\hat{e}) = a + b\hat{e}$ so that $f(\hat{e}) - E(f(\hat{e})) = b(e - E(e)) = b\hat{e}$. In contrast, for EU- models, *Rubinstein* (1974) has shown that linear sharing rules are obtained whenever all investors have a HARA-utility function with the same γ.

In the following, we require again $f''(\hat{e}) > 0$ and analyse the sharing rule of investor i relative to that of investor j, i.e. the relative sharing rule $e_i(e_j)$. In analogy to *Leland* (1980), we say that investor i purchases portfolio insurance from investor j if his sharing rule e_i is strictly convex in e_j. Proposition 3 provides conditions for trading portfolio insurance.

Proposition 3 *: In a risk-value equilibrium the following statements are equivalent:*

– Investor i's sharing rule is strictly convex [linear] [strictly concave] relative to that of investor j.

– The coefficient of absolute prudence of investor i's risk function, $-f_i''(\hat{e}_i)/f_i'(\hat{e}_i)$*, multiplied by* de_i/de_j*, is everywhere greater than [equal to] [smaller than] the coefficient of absolute prudence of investor j's risk function.*

–

$$-\eta_i \frac{f_i''(\hat{e}_i)}{[f_i'(\hat{e}_i)]^2} > [=][<] - \eta_j \frac{f_j''(\hat{e}_j)}{[f_j'(\hat{e}_{jj})]^2}. \tag{18}$$

Proof. See Appendix C.

Proposition 3 provides necessary and sufficient conditions for the shape of an investor's sharing rule relative to that of another one. The shape depends on the investors' coefficients of absolute prudence given efficient portfolios.

These conditions are related to those obtained by *Leland* (1980) in an EU equilibrium.

4.2 HARA-Based Risk Functions

In this section, we derive more specific results assuming that the risk function F belongs to the negative HARA-class. $F(\hat{e})$ is given by equation (2) with $\gamma > 2$ or $\gamma < 1$. γ is assumed to be the same for all investors.

A necessary condition for the existence of an equilibrium in the case of a non-exponential risk function ($\gamma > -\infty$) is

$$\sum_i (A_i + \frac{\hat{e}_{i\varepsilon}}{1-\gamma}) \equiv A + \frac{\hat{e}_\varepsilon}{1-\gamma} > 0, \quad \forall\, \varepsilon \quad \text{with } A \equiv \sum_i A_i \text{ and } \hat{e}_\varepsilon \equiv \sum_i \hat{e}_{i\varepsilon}.$$

This condition must hold because $A_i + \hat{e}_{i\varepsilon} / (1-\gamma) > 0 \quad \forall i, \varepsilon$ is required by the FOC (11) resp. (14). Let $\bar{e} \equiv \sum_i \bar{e}_i$, then $\hat{e}_\varepsilon = \varepsilon - \bar{e}$. Hence an equilibrium requires $A + (\varepsilon - \bar{e}) \ / \ (1-\gamma) > 0$. For the more important case $\gamma < 1$ this implies $\varepsilon \geq \bar{e} - A(1-\gamma), \forall\, \varepsilon$. If $W_0 \equiv \sum_i W_{0i}$, then in equilibrium W_0 is the forward market value of the aggregate payoff ε, i.e. $W_0 = E(\varepsilon\pi(\varepsilon))$.

We first show that investor i's sharing rule relative to that of investor j is either concave, linear or convex.

Proposition 4 *: Consider two investors i and j who measure risk by a negative HARA-function with γ being the same for both. Then the following statements are equivalent:*

– Investor i's sharing rule is strictly convex [linear] [strictly concave] relative to that of investor j.

– Investor i's sharing constant is smaller than [equal to][greater than] that of investor j.

Proof. See Appendix D.

Proposition 4 illustrates our earlier claim that the sharing constants in the optimality condition (11) resp. (14) generate room for a larger variety of sharing rules. While in the EU-model with HARA-utility all investors have linear sharing rules, in the risk-value model one investor's sharing rule relative to another one's is concave, linear or convex. It is the difference between the sharing constants of both investors which determines convexity, linearity and concavity. If all sharing constants are the same, then linearity obtains as in the EU- model, and a representative investor exists. Conversely, heterogeneity of investors may be measured by differences in the sharing constants.

Corollary: *The convexity of investor i's sharing rule relative to that of investor j grows with the difference in the sharing constants* $(s_j - s_i)$.

Proof. Consider a sequence of investors such that $s_j - s_{j+1} = \Delta$ with Δ being a small positive number; $j = 1, ..., J-1$.. Then the convexity of $\hat{e}_{j+1}(\hat{e}_j)$ is positive. Hence the convexity of $\hat{e}_{j+k}(\hat{e}_j)$ must increase in k, and, hence, in $s_j - s_{j+k}$ ■

More insight can be gained by analysing the investor's sharing rule in terms of the gross return R rather than the payoff e. Since $R := e/W_0$, the risk function can also be written as [$\widehat{R} := R - \overline{R}$ with $\overline{R} = E(R)$ for an endogenous benchmark resp. $\overline{R}$ being the exogenous benchmark]

$$E[F(\widehat{R})] = \frac{1-\gamma}{\gamma} E\left(\frac{A}{W_0} + \frac{\widehat{R}}{1-\gamma}\right)^{\gamma}$$

except for the case of the exponential risk function ($\gamma = -\infty$). In this case $E[F(\widehat{R})] = E[exp(-BW_0\widehat{R})]$.

The investor minimizes his risk subject to the budget contraint $E(R\pi) = 1$ and the return constraint $E(R) \geq R^*$. The solution gives the efficient return sharing rule.

The ratio A/W_0 is crucial for risk measurement if $\gamma > -\infty$. Depending on the investor, the ratio A/W_0 might be independent of W_0 or not. Define

$$\frac{A^*}{W_0} = \frac{A}{W_0} - \frac{\overline{R}}{1-\gamma}.$$

Then

$$E[F(\widehat{R})] = -\frac{1-\gamma}{\gamma} E\left(\frac{A^*}{W_0} + \frac{R}{1-\gamma}\right)^{\gamma}. \tag{19}$$

Investors differ in terms of $W_0\ /A^*$ for $\gamma > -\infty$ resp. BW_0 for $\gamma = -\infty$ and the required expected return R^*. Hence the sharing constants of two investors differ because of differences in these parameters. Applying Lemma 1 shows that the sharing constant of an investor declines when W_0/A^* resp. BW_0 or the required expected return R^* increases.

This proves

Proposition 5 *: Given the pricing kernel, the difference between the sharing constants of investors i and j, $(s_i - s_j)$, is monotonically increasing in $(W_{0j}/A_j^* - W_{0i}/A_i^*)$ for $\gamma > -\infty$ resp. $(B_jW_{0j} - B_iW_{0i})$ for $\gamma = -\infty$ and $(R_j^* - R_i^*)$.*

Before we discuss the sharing rules of these investors, it is helpful to understand the impact of W_0/A^* resp. BW_0. The curvature of the risk function, $-f'(\widehat{R})/f(\widehat{R})$, similar to absolute risk aversion, increases monotonically in W_0/A^* resp. BW_0. Therefore we denote W_0/A^* resp. BW_0 as the investor's risk sensitivity. With $1 > \gamma > -\infty$, risk sensitivity is higher for a more aggressive investor, i.e. an investor, who uses a higher exogenous benchmark return or, in the case of an endogenous benchmark, demands a higher expected portfolio return.

Proposition 6 characterizes the sharing rules of investors.

Proposition 6 *: Consider two investors i and j who have HARA-risk functions with the same γ.*

a) Suppose that the risk sensitivity, $W_0\ /A^*$ for $\gamma > -\infty$ resp. BW_0 for $\gamma = -\infty$, *is higher for investor i than for investor j and/or she demands a higher expected portfolio return. Then investor i's sharing rule is strictly convex relative to that of investor j.*

b1) Suppose that for both investors the risk sensitivity is the same, but investor i demands a higher expected portfolio return. Then there exists some portfolio return R^1 *such that*

$$R_i < [=][>] R_j \quad \text{for} \quad R_j < [=][>] R^1.$$

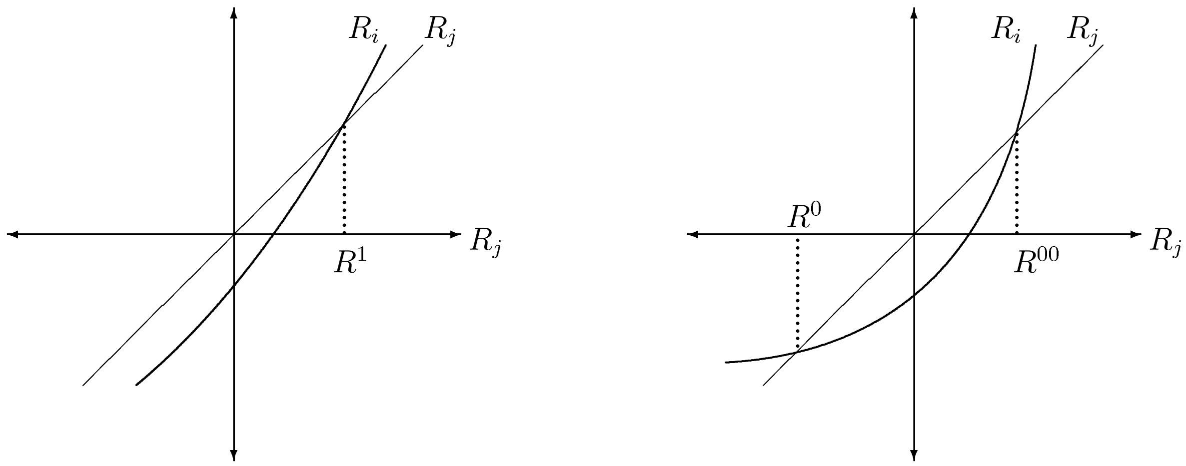
b2) Suppose that the risk sensitivity is higher for investor i but both investors demand the same expected portfolio return. Then there exist R^0 *and* R^{00} *with* $R^0 < R^{00}$ *such that*

$$R_i > R_j \text{ for } \quad R_j < R^0 \text{ and } R_j > R^{00},$$

$$R_i < R_j \quad \text{for } R^0 < R_j < R^{00}.$$

Proof. See Appendix E.

Figure 2



b1) Investor i demands higher expected portfolio return. His sharing rule is convex relative to that of investor j such that R_i and R_j intersect once.

b2) Investor i has a higher risk sensitivity. His sharing rule is convex relative to that of investor j such that both intersect twice.

Proposition 6 is illustrated in Figure 2. If two investors i and j have the same expected portfolio return and the same risk sensitivity, then their portfolio returns are the same in every state. If investor i has a higher risk sensitivity and/or demands a higher expected portfolio return, then her sharing rule is convex relative to that of the other investor. The special cases b1) and b2) will help to understand the intuition behind this result.

Proposition 6 b1) says that an investor i who ceteris paribus demands a higher expected portfolio return than investor j chooses a portfolio such that in the low states ($R_j < R^1$) her portfolio return is lower and in the high states ($R_j > R^1$) it is higher. Hence, a higher expected return forces her to take more risk. Now suppose, she would choose a sharing rule which is linear relative to that of the other investor but has higher slope (such as in EU-portfolio analysis). This would raise her risk dramatically because her return would fall strongly in the low states and the strict concavity of her marginal risk function would reinforce the impact on risk. Therefore, she rebalances her portfolio by buying more claims in the low states yielding a convex sharing rule relative to investor j. This result may be counterintuitive since one may feel that investors who demand a higher expected return are more aggressive and, therefore, should sell portfolio insurance.

Proposition 6 b2) states that, given the same expected return, the more risk sensitive investor chooses a portfolio such that her return is higher in the very low and in the very high states, but lower in between. Thus, the more risk sensitive investor cuts back large negative deviations of her return from the benchmark return in the very low states and, thereby, reduces her risk. Moreover, she raises positive deviations in the very high states which also reduces her risk. In order to obtain the same expected return, she has to accept lower returns in the states in between in which the deviations are small anyway.

Nonlinear sharing rules are also obtained in EU - equilibria based on HARA utility with distortions. *Grossman* and *Zhou* (1996) analyze the equilibrium for two investors who maximize expected utility. Suppose that both optimization problems are identical except that the second investor also makes sure that his random wealth never falls below a given floor. Then this investor buys options from the other one. *Benninga* and *Mayshar* (2000) prove a related result for an equilibrium with two investors who maximize their expected utility without a floor. Both have constant relative risk aversion, but at different levels. Then the investor with higher relative risk aversion buys options from the other investor. *Franke, Stapleton* and *Subrah-*

manyam (1998) obtain a related result when investors have HARA-utility and face background risk.This risk destroys linearity of the sharing rules.A poor investor suffers more from a given level of background risk than a rich investor and, therefore, tends to buy portfolio insurance.

5 The Pricing Kernel and Investor Heterogeneity

5.1 General Risk Functions

In this section we analyze the pricing kernel in a risk-value equilibrium. We first investigate how in equilibrium the price for state-contingent claims is related to the aggregate payoff. Proposition 7 provides the result. Note that proposition 7 is true also if some investors have endogeneous and the others have exogenous benchmarks.

Proposition 7 *: In a risk-value equilibrium the probability-deflated price for state-contingent claims, π_ε, is decreasing and convex in the aggregate payoff ε.*[9]

Proof: The sum of the payoffs of individual investors is the aggregate payoff. Hence proposition 1 implies that the aggregate payoff is decreasing and convex in π. Therefore π must be decreasing and convex in the aggregate payoff ■

The result of proposition 7 does not come as a surprise. It is also obtained in an EU-equilibrium if every investor has a von Neumann-Morgenstern utility function with positive prudence. The differences between the risk-value-

[9]For very high aggregate payoffs the market price for state-contingent claims could be negative. This would violate the assumption of arbitrage-free markets. A negative price for some state-contingent claims requires, however, that every investor is beyond satiation in that state, a possibility that can be safely ignored (see also footnote 7).

and the EU-equilibrium become transparent when we assume HARA-based risk functions.

5.2 HARA-Based Risk Functions

We assume that all investors have HARA-based risk functions with the same γ. We will show that more heterogeneity of investors raises the convexity of the pricing kernel $c(\varepsilon)$:

$$c(\varepsilon) \equiv -\frac{\pi''(\varepsilon)}{\pi'(\varepsilon)}$$

The convexity of the pricing kernel is essential for option pricing. Consider two economies with pricing kernels $\pi_1(\varepsilon)$ and $\pi_2(\varepsilon)$ such that the convexity of the first pricing kernel is higher everywhere and the forward price of the aggregate payoff-distribution is the same, $E[\varepsilon\ \pi_1(\varepsilon)] = E[\varepsilon\pi_2(\varepsilon)]$.Then based upon a result of *Franke, Stapleton* and *Subrahmanyam* (1999) proposition 8 shows that all European options on ε are more expensive under the more convex pricing kernel $\pi_1(\varepsilon)$.

Proposition 8 *: Let $\pi_1(\varepsilon)$ and $\pi_2(\varepsilon)$ be two forward pricing kernels with the underlying asset's forward price $E[\varepsilon\pi_1(\varepsilon)] = E[\varepsilon\pi_2(\varepsilon)]$ being the same. Assume that the convexity of $\pi_1(\varepsilon)$ exceeds that of $\pi_2(\varepsilon)$ for every ε. Then all European options on ε are more expensive under $\pi_1(\varepsilon)$ than under $\pi_2(\varepsilon)$.*

Proof. See Appendix F.

Given the importance of the pricing kernel we investigate it in the risk-value equilibrium. Two investors are said to be heterogeneous if their sharing constants differ. The more they differ, the more convex or concave are the relative sharing rules, the higher is the convexity of the pricing kernel as will be shown in Proposition 9. Heterogenity of investors will be measured by the "variance" of their sharing constants.

Let denote

$$g_{i\varepsilon} = \left(A_i + \frac{\hat{e}_{i\varepsilon}}{1-\gamma}\right) / \left(A + \frac{\hat{e}_{\varepsilon}}{1-\gamma}\right) \quad \text{for} \quad \gamma > -\infty,$$

$$g_{i\varepsilon} = g_i = \frac{1/B_i}{\sum_j 1/B_j} \quad \text{for} \quad \gamma = -\infty,$$

so that $\sum_i g_{i\varepsilon} = 1$. Moreover, define $V(\varepsilon)$, a hyperbolic "variance" measure of the sharing constants, as

$$V(\varepsilon) \equiv \sum_i g_{i\varepsilon} \left(\frac{[f_1(\hat{e}_{1\varepsilon})/-\eta_1 - s_1 + s_i]^{-1}}{\sum_j g_{j\varepsilon}[f_1(\hat{e}_{1\varepsilon})/-\eta_1 - s_1 + s_j]^{-1}} - 1 \right)^2 \tag{20}$$

This variance measure is endogenous since all terms are determined by the equilibrium. It basically measures the differences between the sharing constants which are determined by the differences in risk sensitivities and required portfolio returns (Proposition 5). If all sharing constants are the same, then this "variance" is zero. Thus, this "variance" tends to increase with differences among investors in risk sensitivities and required portfolio returns.

Proposition 9 *: Assume that every investor uses a risk function belonging to the HARA-class with γ being the same for every investor. Then for $\gamma > -\infty$ the convexity of the pricing kernel is*

$$\begin{aligned} c(\varepsilon) &= \left(A + \frac{\hat{e}_{\varepsilon}}{1-\gamma}\right)^{-1} \frac{\gamma-2}{\gamma-1}[1+V(\varepsilon)] \\ &= \left(A + \frac{\hat{e}_{\varepsilon}}{1-\gamma}\right)^{-1} \left[\frac{\gamma-2}{\gamma-1} + \frac{1}{\pi'(\varepsilon)} \sum_i s_i \frac{d^2 e_i}{d\varepsilon^2}\right]; \qquad \forall \varepsilon. \end{aligned} \tag{21}$$

For the exponential risk function ($\gamma = -\infty$) equation (21) holds with $[A + \hat{e}_\varepsilon/(1-\gamma)]^{-1}$ being replaced by $\left[\sum_j 1/B_j\right]^{-1}$.

Proof. See Appendix G.

Proposition 9 reveals the impact of investor heterogeneity on the convexity of the pricing kernel. By equation (21), the heterogeneity is reflected in the variance measure of the sharing constants which varies with ε. If all sharing constants are equal, then $V(\varepsilon) = 0$. Hence the convexity of the pricing kernel is minimal. The higher the variance is, the higher is the convexity of the pricing kernel.

Alternatively, the heterogeneity between investors is reflected in the convexities of their absolute sharing rules. The second part of equation (21) shows that the convexity of the pricing kernel is minimal if all sharing rules are linear. The sum $\sum_i s_i d^2 e_i/d\varepsilon^2$ is a weighted sum of the convexities $d^2 e_i/d\varepsilon^2$. If the weights s_i were the same across investors, then the sum would be zero. Hence it is the difference in weights which makes the sum nonzero. Proposition 5 tells us that the sharing constant s_i is determined by investor i's risk sensitivity and her required portfolio return. The higher these parameters are, the lower is her sharing constant, the more convex is her sharing rule. Consider, for example, an economy with two investors only. Investor i (h) has low (high) risk sensitivity, both demand the same portfolio return R^*. Then investor i (h) buys a concave (convex) sharing rule. Since

$$0 < s_h < s_i \text{ and } -d^2 e_i/d\varepsilon^2 = d^2 e_h/d\varepsilon^2, \quad \sum_j s_j d^2 e_j/d\varepsilon^2 < 0, \ \forall \varepsilon.$$

In general, the inverse relationship between s_i and $d^2 e_i/d\varepsilon^2$ across investors renders the sum $\sum_i s_i d^2 e_i/d\varepsilon^2$ negative. Hence, divided by $\pi'(\varepsilon)$, it is positive.

Interestingly, heterogeneity of investors always raises the convexity of the pricing kernel. This is intuitively appealing since investors being very risk sensitive and/or demanding a high portfolio return have a strong appetite for portfolio insurance, i.e. they have a strong appetite for claims in the very low and in the very high aggregate payoff-states. This makes these claims more expensive relative to claims in the intermediate states.

In order to illustrate this, consider the case $1 > \gamma > -\infty$ and let investor h have the lowest sharing constant. He is very risk sensitive so that his W_0/A^* is high and he demands a high portfolio return R^*. Since his W_0/A^* is high, $-A^* /W_0$ is high. Hence in a state with a very low aggregate payoff, this investor has to buy enough claims so as to assure $R > - A^*(1-\gamma)/W_0$. Therefore he has to buy a substantial fraction of the available claims driving up the price of these claims. Also in the very high payoff-states he buys a very high fraction of the available claims. This can be seen by analyzing $g_{h\varepsilon}$. $g_{h\varepsilon}$ may be interpreted as investor h's fractional purchase of claims on the aggregate payoff, distorted by the constants A_h and A.

Proposition 10 *: Assume $1 > \gamma > -\infty$. Investor h is the unique investor with the lowest sharing constant. Then $g_{h\varepsilon}$ increases monotonically in ε and approaches 1 for $\varepsilon \to \infty$.*

Proof. See Appendix H.

Proposition 10 shows that investor h being very risk sensitive and demanding a high portfolio return buys a high fraction of all claims in the very high-payoff states. In these states claims are relatively cheap enabling the investor to earn a high return.

Since $g_{h\varepsilon} \to 1$ for $\varepsilon \to \infty$, $V(\varepsilon) \to 0$ for $\varepsilon \to \infty$. Investor h dominates pricing in the high payoff-states so that the measure of investor heterogeneity converges to zero. Yet the convexity of the pricing kernel approaches 0 and, thus, becomes minimal since for $\varepsilon \to \infty$,

$$c(\varepsilon) \to \frac{\gamma-2}{\gamma-1}\left(A + \frac{\hat{e}_\varepsilon}{1-\gamma}\right)^{-1} \to \frac{\gamma-2}{\gamma-1}\left(A_h + \frac{\hat{e}_{h\varepsilon}}{1-\gamma}\right)^{-1} \to 0.$$

Similarly, it can be shown in the case of exponential risk functions ($\gamma = -\infty$) for investor h having the lowest sharing constant that $e_{h\varepsilon}/\varepsilon$ approaches 1 for $\varepsilon \to \infty$ and that $V(\varepsilon) \to 0$ for $\varepsilon \to \infty$ implying $c(\varepsilon) \to \left(\sum_i 1/B_i\right)^{-1}$.

It is interesting to compare the convexity in this risk-value equilibrium to that in an EU-equilibrium in which all investors use a HARA-utility function with the same exponent γ. Assume that $\gamma > -\infty$ and the sum $\sum_i A_i$ in the EU-world equals $\left[\sum_i (A_i - \overline{R_i}/[1-\gamma])\right]$ in the risk-value world. If $\gamma = -\infty$, then asssume $\sum_i (1/B_i)$ to be the same in both worlds. Then the convexity of the EU-pricing kernel is the same as that of the risk-value pricing kernel if all sharing rules are linear in the risk-value equilibrium. Since sharing rules are linear in the EU-equilibrium anyway, this finding reinforces the importance of heterogeneity of investors: If in both worlds there exists a representative investor, i.e. if all sharing rules are linear, then the convexity of the pricing kernel is the same in both worlds. Otherwise the convexity is higher in the risk-value world because a representative investor does not exist.

The important result is that all European options are more expensive relative to the underlying asset, the more convex the pricing kernel is. Since convexity in the HARA-based risk-value equilibrium increases with investor heterogeneity, relative option prices increase with investor heterogeneity. Stated differently, the more the sharing rules deviate from linearity, the higher is the investors' need for option trading, the more expensive are European options relative to the underlying asset. Hence investor heterogeneity might provide an explanation for the observation that stock index options appear to be more expensive than suggested by the Black/Scholes model [*Christensen* and *Prabhala* 1998]. Also, our results could help to explain that the pricing kernels estimated from stock index option prices are leptokurtic and negatively skewed [*Longstaff* 1995, *Brenner* and *Eom* 1996, *Jackwerth* 2000].

6 Conclusion

The paper considers portfolio choice and asset pricing in a world where investors' preferences are modelled by a risk-value approach. We consider risk functions with an exogenous and with an endogenous benchmark. Both models yield similar results. In contrast to expected utility, risk-value models do not constrain the tradeoff between value and risk given a risk measure. This approach is consistent with the widely observable separation of value and risk in finance. We have defined properties the risk measure should have. These properties are verified using empirical findings on risk judgments and are related to the properties of utility functions.

Looking at efficient individual sharing rules in risk-value models, the first order conditions display an additional term, the sharing constant. This constant generates a larger variety of sharing rules as compared to expected utility models. In the risk-value world with the risk function being a negative HARA-function, these sharing constants determine whether an investor's sharing rule is convex, linear or concave relative to that of another investor. Highly risk sensitive investors tend to buy portfolio insurance from less risk sensitive investors. Also, the more aggressive investors, i.e. those who demand a higher expected portfolio return, take more risk, but also tend to buy portfolio insurance.

The pricing kernel is declining and convex in the aggregate payoff. This is true also in an expected utility equilibrium if the third derivative of the utility function is positive. Yet asset pricing turns out to be different in a risk-value equilibrium based on HARA-risk functions as compared to an expected utility equilibrium based on HARA-utility functions. In the risk-value world the convexity of the pricing kernel increases with the heterogeneity of investors which is reflected in the non-linearity of their sharing rules. The more heterogeneous investors are, the more demand for options trading exists, the more convex is the pricing kernel, the more expensive are European options relative to the underlying asset.

These results suggest a need for further research on asset pricing and investor heterogeneity in an expected utility framework, but also in others like the risk-value model.

Appendix A: Proof of Lemma 1

First, we consider an increase in W_0 to bW_0 holding the required expected return R^* constant; $b > 1$. Then η changes to $a\eta$; $a > 0$. Define $\hat{e}_1 := (e - \bar{e})$ for the initial endowment W_0 and define $\hat{e}_b := (e - \bar{e})$ for the initial endowment bW_0. Then we need to show that

$$\frac{E[f(\hat{e}_1)]}{-\eta} > \frac{E[f(\hat{e}_b)]}{-a\eta}$$

or

$$aE[f(\hat{e}_1)] > E[f(\hat{e}_b)]. \tag{22}$$

From equation (15) it follows that $\forall\ \varepsilon$,

$$-E[f(\hat{e}_b)] + f(\hat{e}_{b\varepsilon}) = a\eta\ \theta_\varepsilon = a(-E[f(\hat{e}_1)] + f(\hat{e}_{1\varepsilon})). \tag{23}$$

As the mean absolute deviation between payoffs across states has to grow with W_0, the monotonicity of f implies that also the mean absolute deviation $\mid E[f(\hat{e})] - f(\hat{e})] \mid$ has to grow. Hence $a > 1$. Now assume, by contradiction, that inequality (22) is not true. Then equation (23) implies

$$f(\hat{e}_{b\varepsilon}) \geq af(\hat{e}_{1\varepsilon}); \quad \forall\ \varepsilon. \tag{24}$$

As $a > 1$ and $f > 0$, this implies

$$f(\hat{e}_{b\varepsilon}) > f(\hat{e}_{1\varepsilon}); \quad \forall\ \varepsilon.$$

Since $f' < 0$, it follows that $\hat{e}_{b\varepsilon} < \hat{e}_{1\varepsilon}$ and hence

$$\hat{e}_{b\varepsilon} < \hat{e}_{1\varepsilon}; \quad \forall\, \varepsilon,$$

which contradicts the budget constraint (5). Therefore inequality (22) must be true.

Second, we consider an increase in R^* so that $\hat{e}_1$ changes to $\hat{e}_{b^o}$. Then η changes to $a^o\eta$. Hence the sharing constant decreases if inequality (22) holds with a and b being replaced by a^o and b^o. Therefore the same method by which the first part of Lemma 1 has been proven can be applied here. ■

Appendix B: Proof of Proposition 2

a) Sufficiency: Suppose that $F(\hat{e})$ is a quadratic function. Then $f(\hat{e}) = a + b\hat{e}$. Hence (17) implies linear relative sharing rules for two investors i and j. Therefore all absolute rules are also linear.

b) Necessity: Differentiate (15) with respect to ε; this yields

$$f_i'(\hat{e}_{i\varepsilon})\frac{de_i}{d\varepsilon} = \eta_i \,\frac{d\theta}{d\varepsilon} \tag{25}$$

Now suppose a linear absolute sharing rule for every investor: $\hat{e}_{i\varepsilon} = \alpha_i + \beta_i\,\varepsilon$, so that $\frac{de_i}{d\varepsilon} = \beta_i$. Then it follows from (25) for any two investors i and j

$$f_i'(\hat{e}_{i\varepsilon})\frac{\beta_i}{\eta_i} = f_j'(\hat{e}_{j\varepsilon})\frac{\beta_j}{\eta_j}; \quad \forall\, \varepsilon. \tag{26}$$

In the traditional state preference-model, the investor maximizes his expected utility $E[u(e)]$ subject to the budget constraint (5). Let μ denote the Lagrange-multiplier of the budget constraint. Then the first order conditions imply for two investors i and j

$$u_i'(e_{i\varepsilon})\frac{1}{\mu_i} = u_j'(e_{j\varepsilon})\frac{1}{\mu_j}; \quad \forall\, \varepsilon. \tag{27}$$

Cass and *Stiglitz* (1970) have shown that in the EU-model all investors can have linear sharing rules only if $u(e)$ belongs to the HARA class with the exponent γ being the same for all investors. Since (26) is formally the same as (27), it follows for the risk-value model that all investor can have linear sharing rules only if $f(\hat{e})$ belongs to the HARA class with $(\gamma - 1)$ being the same for all investors. Therefore we can rewrite (17) as

$$-\frac{1}{\eta_i}\left(A_i + \frac{\alpha_i + \beta_i\varepsilon}{1-\gamma}\right)^{\gamma-1} + \frac{1}{\eta_j}\left(A_j + \frac{\alpha_j + \beta_j\varepsilon}{1-\gamma}\right)^{\gamma-1} = s_i - s_j; \quad \forall\, \varepsilon.$$

Suppose that $s_i \neq s_j$. Then, the last equation can hold for every ε only if $\gamma = 2$. Hence $F(\hat{e})$ must be quadratic for every investor. ■

Appendix C: Proof of Proposition 3

Differentiating equation (16) with respect to e_j yields

$$\frac{f_i'(\hat{e}_i)}{-\eta_i}\frac{de_i}{de_j} = \frac{f_j'(\hat{e}_j)}{-\eta_j}. \tag{28}$$

Hence de_i/de_j is a constant if and only if $f_j'(\hat{e}_j)/f_i'(\hat{e}_i)$ is a constant. Then investor i's sharing rule is linear relative to that of investor j. Investor i's sharing rule is strictly convex [concave] relative to that of investor j if $f_j'(\hat{e}_j)/f_i'(\hat{e}_i)$ is strictly increasing [decreasing] in e_j and, hence, in the aggregate payoff ε. This result can be restated using the coefficient of the negative third to the second derivative of the F-function, $-f_i''(\hat{e}_i)/f_i'(\hat{e})$. This coefficient is called the coefficient of absolute prudence of the investor's risk function; it is the analogue to Kimball's coefficient of absolute prudence. Multiplying equation (28) by -1, taking logs and differentiating with respect to e_j yields

$$\frac{d\ ln(de_i/de_j)}{de_j} = -\frac{f_j''(\hat{e}_j)}{-f_j'(\hat{e}_j)} + \frac{f_i''(\hat{e}_i)}{-f_i'(\hat{e}_i)}\frac{de_i}{de_j}. \tag{29}$$

This proves the equivalence of the first two statements in Proposition 3. Substituting de_i/de_j in equation (29) from equation (28) shows that $d\ ln(de_i/de_j)/de_j >$ [=] [<] 0 if and only if (18) holds. ■

Appendix D: Proof of Proposition 4

For any HARA function,

$$\frac{f_i''(\hat{e}_i)}{[f_i'(\hat{e}_i)]^2} = \frac{\gamma - 2}{\gamma - 1}/f_i(\hat{e}_i).$$

Hence, by the last statement of Proposition 3, $e_i(e_j)$ is strictly convex [linear] [strictly concave] if and only if

$$\frac{f_i(\hat{e}_i)}{-\eta_i} < \ [=]\ [>] \frac{f_j(\hat{e}_j)}{-\eta_j}; \quad \forall\ e_j.$$

Hence, by equation (17), $e_i(e_j)$ is strictly convex [linear] [strictly concave] if and only if investor i's sharing constant is smaller than [equal to] [greater than] that of investor j. This proves Proposition 4. ■

Appendix E: Proof of Proposition 6

The proof will be shown for $\gamma > -\infty$. It is the same for $\gamma = -\infty$.

a) First, we prove statement a). Consider the minimisation of the objective function (19) s.t. $E[R\pi] = 1$ and $E(R) \geq R^*$.
Hence for two investors i and j with $A^*_{0i}/W_{0i} = A^*_{0j}/W_{0j}$ and $R^*_i = R^*_j$ it follows that the efficient portfolio returns are the same. By Lemma 1, investor i's sharing constant decreases when her initial endowment increases or when she demands a higher expected portfolio return. Then, by Proposition 4, her sharing rule is strictly convex relative to that of investor j.

b1) Now we prove statement b1). As $R_i(R_j)$ is strictly convex, the curves $R_i(R_j)$ and $R_j(R_j)$ can intersect at most twice. They have to intersect at least once, otherwise the budget constraint $E[R\pi] = 1$ cannot hold for both. Hence we have to show that two intersections are impossible. Equation (28) yields in the HARA-case, starting from objective function (19),

$$\frac{dR_i}{dR_j} = \left(\frac{\frac{A^*_i}{W_{0i}} + \frac{R_i}{1-\gamma}}{\frac{A^*_j}{W_{0j}} + \frac{R_j}{1-\gamma}} \right)^{2-\gamma} \frac{\eta_i}{\eta_j}.$$

At an intersection $R_i = R_j$ so that $W_{0i}/A^*_i = W_{0j}/A^*_j$ implies that the bracketed term equals 1. Hence the slope dR_i/dR_j at an intersection is unique. Therefore convexity of $R_i(R_j)$ rules out two intersections. Given one intersection at $R_j = R^1$, a higher expected portfolio return can be obtained only if in the states $R_j < R^1$ relatively expensive claims are sold and in the states $R_j > R^1$ relatively cheap claims are bought. Thus $dR_i/dR_j > 1$ for $R_j = R^1$.

b2) Finally we prove statement b2). The strict convexity of $R_i(R_j)$ implies that $R_i(R_j)$ and $R_j(R_j)$ can intersect at most twice. $R_i^* = R_j^*$ requires at least one intersection. Suppose there exists one intersection only. As $\pi(\varepsilon)$ is monotonically decreasing, purchasing claims in some low states $\varepsilon < \varepsilon^+$ and selling claims in some high states $\varepsilon > \varepsilon^+$ such that this transaction is self-financing implies a lower expected return. Thus, one intersection contradicts $R_i^* = R_j^*$ so that two intersections are implied. ■

Appendix F: Proof of Proposition 8

Franke/Stapleton/Subrahmanyan (1999) have shown the following. Consider two pricing kernels $\pi_1(\varepsilon)$ and $\pi_2(\varepsilon)$ with $E[\pi_1(\varepsilon)] = E[\pi_2(\varepsilon)] = 1$ and $E[\varepsilon\pi_1(\varepsilon)] = E[\varepsilon\pi_2(\varepsilon)]$. Suppose that these kernels intersect twice such that $\pi_1(\varepsilon) > \pi_2(\varepsilon)$ for low and high of levels of ε and $\pi_1(\varepsilon) < \pi_2(\varepsilon)$ in between. Then all European options on ε are more expensive under $\pi_1(\varepsilon)$.
Hence in order to prove Proposition 8 it suffices to show: If $\pi_1(\varepsilon)$ is more convex than $\pi_2(\varepsilon)$ everywhere, then these pricing kernels must intersect twice such that for low and high ε-levels $\pi_1(\varepsilon) > \pi_2(\varepsilon)$ and vice versa in between. *Franke/Stapleton/Subrahmanyan* (1999) have shown that the pricing kernels must intersect at least twice to produce the same forward price of the underlying asset. More than two intersections will be shown to be impossible. $c_1(\varepsilon) > c_2(\varepsilon)\ \forall\varepsilon$ implies

$$-\frac{dln\left[-\pi_1'(\varepsilon)\right]}{d\varepsilon} > -\frac{dln\left[-\pi_2'(\varepsilon)\right]}{d\varepsilon}$$

$$or \qquad -\frac{dln[-\pi_1'(\varepsilon)/-\pi_2'(\varepsilon)]}{d\varepsilon} > 0$$

so that $-\pi_1'(\varepsilon)/-\pi_2'(\varepsilon)$ must decline in ε everywhere. Suppose that $-\pi_1'(\varepsilon)/-\pi_2'(\varepsilon) > 1$ at the first intersection of $\pi_1(\varepsilon)$ and $\pi_2(\varepsilon)$. Hence it must be less than 1 at the second intersection and greater than 1 at the third intersection. But the latter condition contradicts the condition that $-\pi_1'(\varepsilon)/-\pi_2'(\varepsilon)$ declines. Hence a third intersection cannot exist. Also $\pi_1(\varepsilon) > \pi_2(\varepsilon)$ before the first intersection and after the second intersection and vice versa in between follows. ■

Appendix G: Proof of Proposition 9

Differentiate the first order condition (15) with respect to ε. This yields

$$f_i'(\hat{e}_{i\varepsilon})\frac{de_i}{d\varepsilon} = -\eta_i\pi'(\varepsilon); \quad \forall\ i, \varepsilon. \tag{30}$$

Multiply this equation by -1, take logs and differentiate with respect to ε. This yields

$$\frac{f_i''(\hat{e}_{i\varepsilon})}{f_i'(\hat{e}_{i\varepsilon})}\ \frac{de_i}{d\varepsilon} + \frac{d^2e_i/d\varepsilon^2}{de_i/d\varepsilon} = -c(\varepsilon); \quad \forall\ i, \varepsilon. \tag{31}$$

First, we prove the first part of equation (21). Multiply equation (31) by $de_i/d\varepsilon$ and aggregate across investors. This yields

$$\sum_i \frac{f_i''(\hat{e}_{i\varepsilon})}{f_i'(\hat{e}_{i\varepsilon})}\ \left(\frac{de_i}{d\varepsilon}\right)^2 = -c(\varepsilon); \quad \forall\ \varepsilon, \tag{32}$$

since $\sum_i d^2e_i/d\varepsilon^2 = 0$.

In the HARA-case with $\gamma > -\infty$,

$$\frac{f_i''(\hat{e}_{i\varepsilon})}{-f_i'(\hat{e}_{i\varepsilon})} = \frac{\gamma-2}{\gamma-1}\left(A_i + \frac{\hat{e}_{i\varepsilon}}{1-\gamma}\right)^{-1} = \frac{\gamma-2}{\gamma-1}\,\frac{-f_i'(\hat{e}_{i\varepsilon})}{f_i(\hat{e}_{i\varepsilon})}; \quad \forall\, i,\varepsilon \tag{33}$$

Hence

$$c(\varepsilon) = \frac{\gamma-2}{\gamma-1}\sum_i \left(A_i + \frac{\hat{e}_{i\varepsilon}}{1-\gamma}\right)\left(\frac{de_i/d\varepsilon}{A_i + \frac{\hat{e}_{i\varepsilon}}{1-\gamma}}\right)^2; \quad \forall\, \varepsilon \tag{34}$$

Divide (30) by $f_i'(\hat{e}_{i\varepsilon})$ and aggregate. This yields

$$1 = \pi'(\varepsilon)\sum_i -\eta_i/f_i'(\hat{e}_{i\varepsilon}) = \pi'(\varepsilon)\sum_i \left(A_i + \frac{\hat{e}_{i\varepsilon}}{1-\gamma}\right)(\eta_i/f_i(\hat{e}_{i\varepsilon})) \tag{35}$$

Now divide (30) by $f_i(\hat{e}_{i\varepsilon})$. This and (35) yield

$$\frac{f_i'(\hat{e}_{i\varepsilon})}{f_i(\hat{e}_{i\varepsilon})}\,\frac{de_i}{d\varepsilon} = \frac{-\eta_i}{f_i(\hat{e}_{i\varepsilon})}\pi'(\varepsilon) = \frac{-\eta_i/f_i(\hat{e}_{i\varepsilon})}{\sum_j \left(\eta_j/f_j(\hat{e}_{j\varepsilon})\right)\left(A_j + \frac{\hat{e}_{j\varepsilon}}{1-\gamma}\right)} \tag{36}$$

The left hand side of this equation, multiplied by -1, equals the term in the squared bracket of (34). Hence it follows that

$$c(\varepsilon) = \frac{\gamma-2}{\gamma-1}\sum_i \frac{A_i + \frac{\hat{e}_{i\varepsilon}}{1-\gamma}}{\left[\sum_j \left(\eta_j/f_j(\hat{e}_{j\varepsilon})\right)\left(A_j + \frac{\hat{e}_{j\varepsilon}}{1-\gamma}\right)\right]^2}\left(\frac{-\eta_i}{f_i(\hat{e}_{i\varepsilon})}\right)^2; \quad \forall\, \varepsilon. \tag{37}$$

Define $g_{i\varepsilon} \equiv \left(A_i + \frac{\hat{e}_{i\varepsilon}}{1-\gamma}\right) / \left(A + \frac{\hat{e}_{\varepsilon}}{1-\gamma}\right)$ $\forall\, i,\varepsilon$ so that $\sum_i g_{i\varepsilon} = 1\ \forall\, \varepsilon$.

Then (37) yields

$$
\begin{aligned}
c(\varepsilon) &= \frac{\gamma-2}{\gamma-1}\left(A+\frac{\hat{e}_\varepsilon}{1-\gamma}\right)^{-1}\sum_i g_{i\varepsilon}\left(\frac{-\eta_i/f_i(\hat{e}_{i\varepsilon})}{\sum_j g_{j\varepsilon}(-\eta_j/f_j(\hat{e}_{j\varepsilon})}\right)^2 \\
&= \frac{\gamma-2}{\gamma-1}\left(A+\frac{\hat{e}_\varepsilon}{1-\gamma}\right)^{-1}[1+V(\varepsilon)]\,; \quad \forall\,\varepsilon, \qquad (38)
\end{aligned}
$$

with $V(\varepsilon)$ being defined in equation (20). The latter follows from $f_i(\hat{e}_{i\varepsilon})/-\eta_i = f_1(\hat{e}_{1\varepsilon})/-\eta_1 - s_1 + s_i$; $\forall i$. This proves the first part of equation (21).

Now we we prove the second part of equation (21). Substitute $f_i''(\hat{e}_{i\varepsilon})/f_i'(\hat{e}_{i\varepsilon})$ in equation (31) from (33) and multiply the equation by $-(A_i+\hat{e}_{i\varepsilon}/(1-\gamma))$. This yields

$$
\frac{\gamma-2}{\gamma-1}\frac{de_i}{d\varepsilon} - \frac{A_i+\hat{e}_{i\varepsilon}/(1-\gamma)}{de_i/d\varepsilon}\,\frac{d^2e_i}{d\varepsilon^2} = c(\varepsilon)\left(A_i+\frac{\hat{e}_{i\varepsilon}}{1-\gamma}\right); \quad \forall\, i,\varepsilon. \qquad (39)
$$

The factor of $d^2e_i/d\varepsilon^2$ equals the inverse first term in equation (36). Hence it follows from equation (36) and (15) that this factor equals

$$
\frac{f_i(\hat{e}_{i\varepsilon})}{-\eta_i}\,\frac{1}{\pi'(\varepsilon)} = \frac{-1+\pi_\varepsilon+s_i}{\pi'(\varepsilon)}. \qquad (40)
$$

Insert (40) in (39) and obtain after aggregation across investors since $\sum_i d^2e_i/d\varepsilon^2 = 0$,

$$
\frac{\gamma-2}{\gamma-1} + \sum_i \frac{s_i}{\pi'(\varepsilon)}\; d^2e_i/d\varepsilon^2 = c(\varepsilon)\;\left(A+\frac{\hat{e}_\varepsilon}{1-\gamma}\right); \quad \forall\,\varepsilon \qquad (41)
$$

This proves the second part of equation (21). The proof of proposition 9 is the same for $\gamma = -\infty$. ■

Appendix H: Proof of Proposition 10

For a HARA-based risk function with $1 > \gamma > -\infty$ the FOC (11) resp. (14) yields

$$\left(A_i + \frac{\hat{e}_{i\varepsilon}}{1-\gamma}\right)^{\gamma-1} = -\eta_i[\pi_\varepsilon - 1 + s_i]; \quad \forall\ i, \varepsilon, \tag{42}$$

or

$$A_i + \frac{\hat{e}_{i\varepsilon}}{1-\gamma} = (-\eta_i[\pi_\varepsilon - 1 + s_i])^{\frac{1}{\gamma-1}}; \quad \forall\ i, \varepsilon. \tag{43}$$

Aggregating across investors yields $A + \hat{e}_\varepsilon/(1-\gamma)$. Dividing the aggregate equation by $A_i + \hat{e}_{i\varepsilon}/(1-\gamma)$ yields $1/g_{i\varepsilon}$ and hence,

$$1/g_{i\varepsilon} = \sum_j \left(\frac{-\eta_i[\pi_\varepsilon - 1 + s_i]}{-\eta_j[\pi_\varepsilon - 1 + s_j]}\right)^{\frac{1}{1-\gamma}} \tag{44}$$

so that

$$\frac{d(1/g_{i\varepsilon})}{d\varepsilon} = \sum_j \frac{1}{1-\gamma}\left(\frac{-\eta_i}{-\eta_j}\right)^{\frac{1}{1-\gamma}} \left(\frac{\pi_\varepsilon - 1 + s_i}{\pi_\varepsilon - 1 + s_j}\right)^{\frac{1}{1-\gamma}-1} \frac{s_j - s_i}{(\pi_\varepsilon - 1 + s_j)^2}\ \pi'(\varepsilon). \tag{45}$$

Assume $\gamma < 1$ and $s_h \leq s_j\ \forall i$. Then $\pi'(\varepsilon) < 0$ implies $d(1/g_{h\varepsilon})/d\varepsilon < 0$ so that $dg_{h\varepsilon}/d\varepsilon > 0$.

It remains to be shown for $\gamma < 1$ that $g_{h\varepsilon} \to 1$ for $\varepsilon \to \infty$. For $\varepsilon \to \infty$, $\hat{e}_{i\varepsilon} \to \infty$ for at least one investor. By equation (43), $\hat{e}_{i\varepsilon} \to \infty$ if $\pi_\varepsilon - 1 + s_i \to 0$. Since $\pi_\varepsilon - 1 + s_i > 0 \ \forall i, \varepsilon$ and $s_h < s_j \ \forall j, \ j \neq h, \pi_\varepsilon - 1 + s_i \to 0$ for $\varepsilon \to \infty$ must hold for $i = h$ only. Hence $\hat{e}_{h\varepsilon} \to \infty$ and, therefore, by equation (44), $g_{h\varepsilon} \to 1$ for $\varepsilon \to \infty$. It follows from $\pi_\varepsilon - 1 + s_h \to 0$ for $\varepsilon \to \infty$ and $\pi_\varepsilon \geq 0$ that s_h must not exceed 1. ■

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