A Quasilinear Parabolic Equation with Quadratic Growth of the Gradient modeling Incomplete Financial Markets

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Abstract. We consider a quasilinear parabolic equation with quadratic gradient terms. It arises in the modelling of an optimal portfolio which maximizes the expected utility from terminal wealth in incomplete markets consisting of risky assets and non-tradable state variables. The existence of solutions is shown by extending the monotonicity method of Frehse. Furthermore, we prove the uniqueness of weak solutions under a smallness condition on the derivatives of the covariance matrices with respect to the solution. The influence of the non-tradable state variables on the optimal value function is illustrated by a numerical example.

1 Introduction

One fundamental problem in mathematical finance is the problem of portfolio selection, i.e., an agent invests in a market trying to maximize the expected utility of his or her terminal wealth [21]. For a *complete* market this problem was solved in [27, 28], deriving a nonlinear PDE (Bellman equation) for the value function of the optimization problem, i.e. the utility of the optimal portfolio.

The maximization of expected utility from terminal wealth in *incomplete* markets has been studied in [23, 25]. The author in [25] considers an arbitrage-free continuous time

market model with unrestricted trading and a fixed time horizon, i.e. $t \in [0, T]$. The market consists of a riskless bond, d risky assets and d' non-tradable state variables and hence is incomplete. Examples for such state variables are credit risks of a bank or an employee's personal income, which usually cannot be traded. The optimization problem is to find a portfolio strategy which maximizes the expected utility from terminal wealth over the set of self-financing portfolios with initial capital $x > 0$ and non-negative wealth, denoted by $\mathcal{X}(x) = \{X(t) \geq 0 : X(0) = x\}$, using isoelastic utility functions with constant relative risk aversion,

$$U^{(p)}(x) = \mathrm{sgn}(1-p)\frac{x^p}{p}, \quad U^0(x) = \ln x,$$

with $x > 0$ and exponent $p \notin \{0, 1\}$. The optimal value function of this problem is defined by

$$v(x) = \sup_{X \in \mathcal{X}(x)} E[U^{(p)}(X(T))].$$

Solving this optimization problem with $p < 1$ is an approach for finding portfolios of optimal expected growth [20, 21, 23]. For $p = 2$ the problem is related to the mean variance hedging problem [17, 24, 30].

Following a stochastic duality approach, the existence of an optimal (locally efficient) portfolio is proved in [25]. The relationship between the optimal portfolio and the optimal martingal measure for the dual problem is characterized by a backward stochastic differential equation. For a Markovian market with d price processes $S_t^{(i)}$ and d' state variable processes $S_t'^{(j)}$ satisfying the stochastic differential equations

$$dS_t^{(i)} = \mu^{(i)}(S_t^{(i)})\,dt + \sigma^{(i)}(S_t^{(i)})\,dW_t^{(i)}, \quad i = 1, \ldots, d,$$
$$dS_t'^{(j)} = \mu'^{(j)}(S_t'^{(j)})\,dt + \sigma'^{(j)}(S_t'^{(j)})\,dW_t'^{(j)}, \quad j = 1, \ldots, d',$$

where $W_t^{(i)}$ and $W_t'^{(j)}$ are correlated Wiener processes, the following quasilinear parabolic PDE for the logarithm of the optimal value function has been derived in [25]:

$$\partial_t u - \frac{1}{2}\sum_{i,j=1}^{d} c_{ij}(u)\partial_i\partial_j u - \frac{1}{2}\sum_{i,j=1}^{d'} c'_{ij}(u)\partial'_i\partial'_j u$$

$$= \mu \cdot \nabla u + \mu' \cdot \nabla' u + q(\mu - rS) \cdot \nabla u - \frac{q}{2}\beta(u)^2 + pr \qquad \text{in } \hat{\Omega} \times (0, T), \qquad \text{(1a)}$$

$$+ \frac{1}{2(p-1)}(\nabla u)^\top C(u)\nabla u - \frac{1}{2}(\nabla' u)^\top C'(u)\nabla' u,$$

$$u(S, S', t) = u_D(S, S', t) \qquad\qquad\qquad\qquad\qquad\qquad \text{on } \partial\hat{\Omega} \times (0, T), \quad \text{(1b)}$$

$$u(S, S', 0) = u_0(S, S') \qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{in } \hat{\Omega}, \qquad\qquad \text{(1c)}$$

where $u = u(S, S', t)$ is the logarithm of the optimal value function, either $\hat{\Omega} = \Omega \times \Omega' \subset \mathbb{R}^d \times \mathbb{R}^{d'}$ is a bounded domain or $\hat{\Omega} = \mathbb{R}^d \times \mathbb{R}^{d'}$, and $T > 0$. We use the notations

$\partial_t = \partial/\partial t$ and $\nabla = (\partial_1, \ldots, \partial_d)$, $\nabla' = (\partial'_1, \ldots, \partial'_{d'})$ with the partial derivatives $\partial_i = \partial/\partial S_i$, $\partial'_i = \partial/\partial S'_i$. Furthermore,

- $C = (c_{ij}(S, t, u))_{i,j} : \Omega \times (0, T) \times \mathbb{R} \to \mathbb{R}^{d \times d}$ and $C' = (c'_{ij}(S', t, u))_{i,j} : \Omega' \times (0, T) \times \mathbb{R} \to \mathbb{R}^{d' \times d'}$ are the symmetric and positive definite covariance matrices of the risky assets and the non-tradable state variables, respectively;

- $\mu(S, t) : \Omega \times (0, T) \to \mathbb{R}^d$ and $\mu'(S', t) : \Omega' \times (0, T) \to \mathbb{R}^{d'}$ are the expected returns;

- $r(S, S', t) : \Omega \times \Omega' \times (0, T) \to \mathbb{R}$ is the riskless interest rate;

- $\beta(S, S', t, u)^2 = (\mu - rS)^\top C^{-1} (\mu - rS)$ is the square of the risk premium;

- $p \notin \{0, 1\}$ is the exponent of the utility function and $q \in \mathbb{R}$ is given by $1/p + 1/q = 1$.

In the case $p = 0$, which relates to the logarithmic utility function $U^0(x) = \ln x$, the optimization problem is also known as maximizing the Kelly criterion [16, 19, 21]. Note that if $p = 0$, the quadratic terms in (1a) can be removed by an exponential transformation.

The solution u of (1a) allows to construct the optimal portfolio π^*. Indeed, the optimal portfolio strategy is given by $H(S, S', t) = (1-p)^{-1}(\lambda - \nabla u)$ [25] (where $\lambda = C^{-1}(\mu - rS)$), and the optimal portfolio equals $\pi^* = H \cdot S$. The components of the vector $H(S, S', t)$ are the shares of the underlyings in the portfolio. Recall that for Merton's model it holds $H(S, S', t) = (1 - p)^{-1}\lambda$ [29], and the portfolios coincide if u is constant with respect to the asset prices. This is the case if, for instance, the expression $pr - q\beta^2/2$ and the initial data u_0 is constant in $\hat{\Omega} \times (0, T)$ since then, equation (1a) has the solution $u(S, S', t) = (pr - q\beta^2/2)t + u_0$.

Up to now, the question of well-posedness of problem (1) has not been studied in the literature. The main aim of this paper is to prove the existence and uniqueness of generalized Sobolev solutions to the initial-boundary-value problem (1) and to the Cauchy problem (1a), (1c) in $\hat{\Omega} = \mathbb{R}^d \times \mathbb{R}^{d'}$.

The main mathematical difficulty is the treatment of the terms with the quadratic gradients. In order to show the existence of solutions usually an approximate problem is solved (for instance, with linearly growing gradient terms) and appropriate a priori estimates independent of the approximation parameter are derived. In the mathematical literature there are two approaches to obtain uniform a priori estimates. The first idea is to establish L^∞ bounds (for instance, from a maximum principle) which lead to H^1 bounds [7, 8, 9, 10, 13, 14, 18, 26]. The second idea is to derive H^1 bounds directly without L^∞ bounds if a sign condition of the form $f(u, \nabla u)u \geq 0$ (where f is a function with quadratic growth) is fulfilled [2, 5, 6, 31]. Another interesting work [22] studies the connections of backward stochastic differential equations and partial differential equations with quadratic growth of the gradient similar to (1) (and their viscosity and Sobolev solutions). However, the results presented here are not covered by those in [22], as we consider nonlinear covariance matrices.

We adopt some of the methods of the literature mentioned above and generalize them slightly to deal with our problem. Clearly, our results can be extended to more general

equations fulfilling similar regularity and growth conditions, but the emphasis of this work is placed on studying the particular problem (1).

We prove the existence of generalized solutions by first proving uniform L^∞ bounds for an approximate problem. In fact, it is easy to see that *smooth* solutions of (1a) attain their extremal values on the parabolic boundary of the domain if $-q\beta^2/2 + pr = 0$. Using Stampacchia's truncation technique, we show L^∞ bounds for *generalized* solutions of (1a). Then uniform H^1 bounds are derived using nonlinear test functions of the type $\sinh(\lambda u)$ for sufficiently large $\lambda > 0$. The uniform H^1 bounds only imply *weak* convergence in H^1 of the sequence of approximating solutions. However, the quasilinear structure of the problem requires that the sequence converges *strongly* in H^1. This is achieved by employing the monotonicity method of Frehse [15], originally used for *elliptic* problems, which we extend to *parabolic* equations (section 2). Moreover, we show the existence of solutions to the whole-space problem (1a), (1c) which is the original formulation in [25] (section 3). Note that, although the sign of one of the quadratic terms depends on whether $p < 1$ or $p > 1$, the proofs of these results hold for arbitrary values of p and, in fact, do not rely on the sign of $(1 - p)$ at all.

Our second main result is a proof of the uniqueness of generalized solutions to (1). The uniqueness proof has to overcome the difficulties arising from both the quadratic gradient terms and the quasilinearity. In order to deal with the quadratic gradients, the uniqueness of solutions of often shown in the space of functions whose gradient lies in a smaller space than L^2 (for instance in L^∞) [11, 33]. Quasilinear terms can be handled using duality methods [1]. However, there are much less uniqueness results (and techniques) for problems with *both* difficulties. We are only aware of the paper of Barles and Murat [3], where the uniqueness of weak solutions to general elliptic problems is proved under a structure condition on the nonlinearities. We adapt their method in order to show the uniqueness of generalized solutions to (1) either if the covariance matrices C and C' do *not* depend on S and S', respectively, or if $p < 1$ and some (smallness) conditions on the derivatives of C and C' with respect to u are satisfied (section 4). Notice that we do *not* need regularity assumptions on the solution.

Finally, we present some numerical results by solving problem (1) with a finite element method for two risky assets and one state variable (section 5). The experiments are showing that the optimal value function varies only slowly with respect to the state variable.

2   Existence of solutions

In this section we prove the existence of (generalized) solutions to (1). Let $Q_T = \hat\Omega \times (0, T)$. We call u a (generalized) solution of (1) if $u - u_D \in L^2(0, T; H^1_0(\hat\Omega))$, $u \in H^1(0, T; H^{-1}(\hat\Omega))$, u fulfills the initial condition (1c) in the sense of $L^2(\hat\Omega)$ and

$$\int_0^T \langle u_t, \phi \rangle \, dt + \frac{1}{2} \int_{Q_T} (\nabla \phi)^\top C(u) \nabla u \, dx \, dt + \frac{1}{2} \int_{Q_T} (\nabla' \phi)^\top C'(u) \nabla' u \, dx \, dt$$

$$= \int_{Q_T} (\mu \cdot \nabla u + \mu' \cdot \nabla' u + q(\mu - rS) \cdot \nabla u - \frac{q}{2}\beta(u)^2 + pr)\phi \, dx \, dt \qquad (2)$$

$$+ \frac{1}{2(p-1)} \int_{Q_T} (\nabla u)^{\top} C(u) \nabla u \phi \, dx \, dt - \frac{1}{2} \int_{Q_T} (\nabla' u)^{\top} C'(u) \nabla' u \phi \, dx \, dt$$

$$- \frac{1}{2} \int_{Q_T} ((\operatorname{div} C)(u) \cdot \nabla u + (\operatorname{div}' C')(u) \cdot \nabla' u)\phi \, dx \, dt$$

holds for any $\phi \in L^{\infty}(Q_T) \cap L^2(0,T;H_0^1(\hat{\Omega}))$. Here, $u_t = \partial_t u$, $(\operatorname{div} C)(u)$ denotes the vector with components $((\operatorname{div} C)(u))_j = \sum_{i=1}^{d} \partial c_{ij}(u)/\partial S_i$ (analogously for $\operatorname{div}' C'(u)$) and $\langle \cdot, \cdot \rangle$ is the dual product between $H^{-1}(\hat{\Omega})$ and $H_0^1(\hat{\Omega})$. The notion of solution for the whole-space problem is analogous.

The basic hypotheses for the initial-boundary-value problem are as follows:

(H1) Domain: $\hat{\Omega} = \Omega \times \Omega' \subset \mathbb{R}^d \times \mathbb{R}^{d'}$ is a bounded domain with boundary $\partial\hat{\Omega} \in C^1$, $d \geq 1$, $d' \geq 0$.

(H2) Coercivity: $\exists \alpha, \alpha' > 0 : \forall \xi \in \mathbb{R}^n \backslash \{0\} : \forall S, S', t, u :$

$$\xi^{\top} C(S,t,u)\xi \geq \alpha \quad \text{and} \quad \xi^{\top} C'(S',t,u)\xi \geq \alpha'.$$

(H3) Symmetry: $c_{ij} = c_{ji}$ for all $i,j \in \{1, \ldots, d\}$ and $c'_{ij} = c'_{ji}$ for all $i,j \in \{1, \ldots, d'\}$.

(H4) Data: $C(\cdot, \cdot, u), C(\cdot, \cdot, u) \in L^{\infty}(0,T;W^{1,\infty}(\Omega))$ for all $u \in \mathbb{R}$ and $C(S,t,\cdot), C'(S',t,\cdot) \in C^1(\mathbb{R}) \cap W^{1,\infty}(\mathbb{R})$ for all S, S', t,
$p \in \mathbb{R} \backslash \{0,1\}$, $\mu \in L^{\infty}(0,T;L^{\infty}(\Omega))$, $\mu' \in L^{\infty}(0,T;L^{\infty}(\Omega'))$, $r \in L^{\infty}(0,T;L^{\infty}(\hat{\Omega}))$, $u_D \in L^2(0,T;H^2(\hat{\Omega})) \cap L^{\infty}(0,T;L^{\infty}(\hat{\Omega})) \cap H^1(0,T;L^1(\hat{\Omega}))$, $u_0 \in L^{\infty}(\hat{\Omega}) \cap H^1(\hat{\Omega})$.

First we prove that there exists a solution of a truncated approximate problem. Define $s_K = \max(-K_2, \min(s, K_1))$ for $s \in \mathbb{R}$, where

$$K_1 = K_1(t) = (t+1)\overline{M}, \quad K_2 = K_2(t) = (t+1)\underline{M}$$

and

$$\overline{M} = \max\{\sup_{\hat{\Omega}} u_0, \sup_{\partial\hat{\Omega}\times(0,T)} u_D, M_2(r,\beta,p)\}, \quad \underline{M} = \min\{\inf_{\hat{\Omega}} u_0, \inf_{\partial\hat{\Omega}\times(0,T)} u_D, M_1(r,\beta,p)\},$$

with

$$M_1(r,\beta,p) = -\sup_{S,S',t,u}\left(\frac{q}{2}\beta(S,S',t,u)^2 - pr(S,S',t)\right),$$

$$M_2(r,\beta,p) = -\inf_{S,S',t,u}\left(\frac{q}{2}\beta(S,S',t,u)^2 - pr(S,S',t)\right).$$

Consider the approximate problem

$$
\int_0^T \langle u_t^\varepsilon, \phi \rangle \, dt + \frac{1}{2} \int_{Q_T} (\nabla\phi)^\top C(u^\varepsilon) \nabla u^\varepsilon \, dx \, dt + \frac{1}{2} \int_{Q_T} (\nabla'\phi)^\top C'(u^\varepsilon) \nabla' u^\varepsilon \, dx \, dt
$$

$$
= \int_{Q_T} (\mu \cdot \nabla u^\varepsilon + \mu' \cdot \nabla' u^\varepsilon + q(\mu - rS) \cdot \nabla u^\varepsilon - \frac{q}{2}\beta(u^\varepsilon)^2 + pr)\phi \, dx \, dt
$$

$$
+ \frac{1}{2(p-1)} \int_{Q_T} \frac{(\nabla u^\varepsilon)^\top C(u^\varepsilon) \nabla u_K^\varepsilon}{1 + \varepsilon(\nabla u^\varepsilon)^\top C(u^\varepsilon)\nabla u^\varepsilon} \phi \, dx \, dt \tag{3}
$$

$$
- \frac{1}{2} \int_{Q_T} \frac{(\nabla' u^\varepsilon)^\top C'(u^\varepsilon) \nabla' u_K^\varepsilon}{1 + \varepsilon(\nabla' u^\varepsilon)^\top C'(u^\varepsilon)\nabla' u^\varepsilon} \phi \, dx \, dt
$$

$$
- \frac{1}{2} \int_{Q_T} ((\operatorname{div} C)(u^\varepsilon) \cdot \nabla u^\varepsilon + (\operatorname{div}'C')(u^\varepsilon) \cdot \nabla' u^\varepsilon)\phi \, dx \, dt
$$

for any $\phi \in L^2(0,T; H_0^1(\hat{\Omega}))$ and $\varepsilon > 0$ subject to boundary and initial conditions (1b), (1c).

Lemma 1 *There exists a solution u^ε of (3), (1b), (1c) such that $u^\varepsilon - u_D \in L^2(0,T; H_0^1(\hat{\Omega}))$ and $u^\varepsilon \in L^2(0,T; H^2(\hat{\Omega})) \cap H^1(0,T; L^2(\hat{\Omega}))$.*

Proof. We use a fixed point argument. For given $w \in L^2(0,T; H^1(\hat{\Omega}))$ we consider the linear equation

$$
\int_0^T \langle u_t^\varepsilon, \phi \rangle \, dt + \frac{1}{2} \int_{Q_T} (\nabla\phi)^\top C(w) \nabla u^\varepsilon \, dx \, dt + \frac{1}{2} \int_{Q_T} (\nabla'\phi)^\top C'(w) \nabla' u^\varepsilon \, dx \, dt
$$

$$
= \int_{Q_T} (\mu \cdot \nabla u^\varepsilon + \mu' \cdot \nabla' u^\varepsilon + q(\mu - rS) \cdot \nabla u^\varepsilon - \frac{q}{2}\beta(w)^2 + pr)\phi \, dx \, dt
$$

$$
+ \frac{1}{2(p-1)} \int_{Q_T} \frac{(\nabla w)^\top C(w) \nabla w_K}{1 + \varepsilon(\nabla w)^\top C(w)\nabla w} \phi \, dx \, dt
$$

$$
- \frac{1}{2} \int_{Q_T} \frac{(\nabla' w)^\top C'(w) \nabla' w_K}{1 + \varepsilon(\nabla' w)^\top C'(w)\nabla' w} \phi \, dx \, dt
$$

$$
- \frac{1}{2} \int_{Q_T} ((\operatorname{div} C)(w) \cdot \nabla u^\varepsilon + (\operatorname{div}'C')(w) \cdot \nabla' u^\varepsilon)\phi \, dx \, dt \tag{4}
$$

for any $\phi \in L^2(0,T; H_0^1(\hat{\Omega}))$ subject to the boundary and initial conditions (1b), (1c).

Since

$$0 \leq \frac{(\nabla w)^\top C(w) \nabla w_K}{1 + \varepsilon (\nabla w)^\top C(w) \nabla w} \leq \frac{1}{\varepsilon}, \quad 0 \leq \frac{(\nabla' w)^\top C'(w) \nabla' w_K}{1 + \varepsilon (\nabla' w)^\top C'(w) \nabla' w} \leq \frac{1}{\varepsilon}, \tag{5}$$

(4) is a linear parabolic equation with bounded coefficients and bounded inhomogeneity. By standard results [12], (4) admits a unique solution u^ε such that $u^\varepsilon - u_D \in L^2(0, T; H_0^1(\hat{\Omega}))$, $u^\varepsilon \in L^2(0, T; H^2(\hat{\Omega})) \cap H^1(0, T; L^2(\hat{\Omega}))$. Thus the fixed point operator

$$S : L^2(0, T; H^1(\hat{\Omega})) \to L^2(0, T; H^1(\hat{\Omega})), \quad w \mapsto u^\varepsilon,$$

is well defined and $S(L^2(0, T; H^1(\hat{\Omega}))) \subset L^2(0, T; H^2(\hat{\Omega})) \cap H^1(0, T; L^2(\hat{\Omega}))$. The following estimate holds [12]

$$\|u^\varepsilon\|_{L^2(0,T;H^2(\hat{\Omega}))} + \|u^\varepsilon\|_{L^\infty(0,T;H^1(\hat{\Omega}))} + \|u_t^\varepsilon\|_{L^2(0,T;L^2(\hat{\Omega}))} \leq c,$$

where in general $c > 0$ is a generic constant depending on ε, the data and on the inhomogeneity. Here, in fact, the inhomogeneity is bounded independently of w. Thus c only depends on ε and the data, but not on w. In view of the compact embedding $L^2(0, T; H^2(\hat{\Omega})) \cap H^1(0, T; L^2(\hat{\Omega}))) \subset L^2(0, T; H^1(\hat{\Omega}))$ [32], S is compact in $L^2(0, T; H^1(\hat{\Omega}))$. Standard arguments show that S is continuous. The hypotheses for Schauder's fixed point theorem are fulfilled and (3), (1b), (1c) admits at least one solution u^ε. ∎

The existence proof for the original problem is based on the following uniform a priori estimates.

Lemma 2 *Let u^ε be a generalized solution to (3), (1b), (1c) in $(0, T)$. Then there exist constants $\overline{K}, \underline{K} > 0$ (independent of ε) such that*

$$\underline{K} \leq u^\varepsilon \leq \overline{K},$$

where $\underline{K} = \min_{0 \leq t \leq T} K_2(t)$, $\overline{K} = \max_{0 \leq t \leq T} K_1(t)$.

Remark 3 *The sign of one of the quadratic terms depends on whether $p < 1$ or $p > 1$. Without truncation in the quadratic terms it is easy to obtain upper or lower L^∞ estimates for $p < 1$ and $p > 1$, respectively, using standard test functions, but it is not possible to obtain the missing lower (upper) estimate in this way. Our proof does not rely on the sign of $(1 - p)$, since by truncating the solution and choosing appropriate test functions, these terms vanish completely.*

Proof. Let $\varphi(u^\varepsilon) := u^\varepsilon - K_1(t)$. Using $\varphi(u^\varepsilon)^+ := \max(0, \varphi(u^\varepsilon)) \in L^2(0, T; H_0^1(\hat{\Omega}))$ as a test

function in (3) yields, in view of $\nabla u_K^\varepsilon \varphi(u^\varepsilon)^+ \equiv 0$,

$$
\frac{1}{2} \int_{\hat{\Omega}} (\varphi(u^\varepsilon)^+(t)^2 - \underbrace{\varphi(u_0^\varepsilon)^{+2}}_{=0}) \, dx + \frac{1}{2} \int_{Q_T} (\nabla \varphi(u^\varepsilon)^+)^\top C(u^\varepsilon) \nabla u^\varepsilon \, dx \, dt
$$

$$
+ \frac{1}{2} \int_{Q_T} (\nabla' \varphi(u^\varepsilon)^+)^\top C'(u^\varepsilon) \nabla' u^\varepsilon \, dx \, dt
$$

$$
= \int_{Q_T} (\mu \cdot \nabla u^\varepsilon + \mu' \cdot \nabla' u^\varepsilon + q(\mu - rS) \cdot \nabla u^\varepsilon - \underbrace{(\frac{q}{2} \beta(u^\varepsilon)^2 - pr + \overline{M})}_{\geq 0})) \, \varphi(u^\varepsilon)^+ \, dx \, dt
$$

$$
- \frac{1}{2} \int_{Q_T} ((\operatorname{div} C)(u^\varepsilon) \cdot \nabla u^\varepsilon + (\operatorname{div}' C')(u^\varepsilon) \cdot \nabla' u^\varepsilon) \, \varphi(u^\varepsilon)^+ \, dx \, dt
$$

$$
\leq \int_{Q_t} (\mu \cdot \nabla \varphi(u^\varepsilon)^+ + \mu' \cdot \nabla' \varphi(u^\varepsilon)^+ + q(\mu - rS) \cdot \nabla \varphi(u^\varepsilon)^+) \, \varphi(u^\varepsilon)^+ \, dx \, dt
$$

$$
- \frac{1}{2} \int_{Q_T} ((\operatorname{div} C)(u^\varepsilon) \cdot \nabla \varphi(u^\varepsilon)^+ + (\operatorname{div}' C')(u^\varepsilon) \cdot \nabla' \varphi(u^\varepsilon)^+) \, \varphi(u^\varepsilon)^+ \, dx \, dt
$$

$$
=: I. \tag{6}
$$

We use Young's inequality and (H4) to estimate the right hand side:

$$
I \leq \int_{Q_T} (\delta |\nabla \varphi(u^\varepsilon)^+|^2 + \delta |\nabla' \varphi(u^\varepsilon)^+|^2 + \frac{c}{\delta} (\varphi(u^\varepsilon)^+)^2) \, dx \, dt,
$$

where $\delta > 0$, and $c > 0$ is a constant independent of ε and varying in the following from occurrence to occurrence.

We use the coercivity (H2) of C and C' to estimate the left hand side of (6) from below. Then the gradient terms on the right hand side can be controlled, for sufficiently small $\delta > 0$, by the left hand side. More precisely, we obtain

$$
\frac{1}{2} \int_{\hat{\Omega}} (\varphi(u^\varepsilon)^+(t))^2 \, dx + \frac{1}{2} \int_{Q_T} \underbrace{(\alpha - 2\delta)}_{\geq 0} |\nabla \varphi(u^\varepsilon)^+|^2 \, dx \, dt
$$

$$
+ \frac{1}{2} \int_{Q_T} \underbrace{(\alpha' - 2\delta))}_{\geq 0} |\nabla' \varphi(u^\varepsilon)^+|^2 \, dx \, dt
$$

$$
\leq \frac{2c}{\delta} \int_{Q_T} (\varphi(u^\varepsilon)^+)^2 \, dx,
$$

which implies

$$
\frac{1}{2} \int_{\hat{\Omega}} (\varphi(u^\varepsilon)^+(t))^2 \, dx \leq \frac{2c}{\delta} \int_{Q_T} (\varphi(u^\varepsilon)^+)^2 \, dx,
$$

and applying Gronwall's lemma yields $u^\varepsilon \leq K_1 \leq \overline{K}$ a.e. in $\hat{\Omega} \times (0,T)$.

In order to derive the lower bound set $\varphi(u^\varepsilon) := u^\varepsilon - K_2$. Using $\varphi(u^\varepsilon)^- := \min(0, \varphi(u^\varepsilon))$ $\in L^2(0,T; H_0^1(\hat{\Omega}))$ as a test function in (3) yields

$$\frac{1}{2}\int\limits_{\hat{\Omega}} ((\varphi(u^\varepsilon)^-(t))^2 - \underbrace{\varphi(u_0^\varepsilon)^{-2}}_{=0}) \, dx + \frac{1}{2}\int\limits_{Q_T} (\nabla\varphi(u^\varepsilon)^-)^\top C(u^\varepsilon)\nabla u^\varepsilon \, dx\,dt$$

$$+ \frac{1}{2}\int\limits_{Q_T} (\nabla'\varphi(u^\varepsilon)^-)^\top C'(u^\varepsilon)\nabla' u^\varepsilon \, dx\,dt$$

$$= \int\limits_{Q_T} (\mu \cdot \nabla u^\varepsilon + \mu' \cdot \nabla' u^\varepsilon + q(\mu - rS) \cdot \nabla u^\varepsilon - \underbrace{(\frac{q}{2}\beta(u^\varepsilon)^2 - pr + \underline{M})}_{\leq 0}) \, \varphi(u^\varepsilon)^- \, dx\,dt$$

$$- \frac{1}{2}\int\limits_{Q_T} ((\operatorname{div} C)(u^\varepsilon) \cdot \nabla u^\varepsilon + (\operatorname{div}'C')(u^\varepsilon) \cdot \nabla' u^\varepsilon) \, \varphi(u^\varepsilon)^- \, dx\,dt$$

$$\leq \int\limits_{Q_T} (\mu \cdot \nabla\varphi(u^\varepsilon)^- + \mu' \cdot \nabla'\varphi(u^\varepsilon)^- + q(\mu - rS) \cdot \nabla\varphi(u^\varepsilon)^-) \, \varphi(u^\varepsilon)^- \, dx\,dt$$

$$- \frac{1}{2}\int\limits_{Q_T} ((\operatorname{div} C)(u^\varepsilon) \cdot \nabla\varphi(u^\varepsilon)^- + (\operatorname{div}'C')(u^\varepsilon) \cdot \nabla'\varphi(u^\varepsilon)^-) \, \varphi(u^\varepsilon)^- \, dx\,dt.$$

We can estimate similarly as above and applying Gronwall's lemma yields $u^\varepsilon \geq K_2 \geq \underline{K}$ a.e. in $\hat{\Omega} \times (0,T)$. ∎

Lemma 4 *Let u^ε be a weak solution to* (3), (1b), (1c)*. Then there exists a constant $k > 0$ (independent of ε) such that*

$$\|u^\varepsilon\|_{L^2(0,T;H^1(\hat{\Omega}))} \leq k.$$

Proof. Inspired by [15], we use $\sinh(\lambda u^\varepsilon) - \sinh(\lambda u_D)$, $\lambda > 0$, as a test function in (3) to obtain

$$\int\limits_0^T \langle u_t^\varepsilon, \sinh(\lambda u^\varepsilon) - \sinh(\lambda u_D)\rangle \, dt + \frac{1}{2}\int\limits_{Q_T} \lambda \cosh(\lambda u^\varepsilon)(\nabla u^\varepsilon)^\top C(u^\varepsilon)\nabla u^\varepsilon \, dx\,dt$$

$$+ \frac{1}{2}\int\limits_{Q_T} \lambda \cosh(\lambda u^\varepsilon)(\nabla' u^\varepsilon)^\top C'(u^\varepsilon)\nabla' u^\varepsilon \, dx\,dt$$

$$= \int\limits_{Q_T} (\mu \cdot \nabla u^\varepsilon + \mu' \cdot \nabla' u^\varepsilon + q(\mu - rS) \cdot \nabla u^\varepsilon - \frac{q}{2}\beta(u^\varepsilon)^2 + pr)(\sinh(\lambda u^\varepsilon) - \sinh(\lambda u_D)) \, dx\,dt$$

$$+ \frac{1}{2(p-1)}\int\limits_{Q_T} \frac{(\nabla u^\varepsilon)^\top C(u^\varepsilon)\nabla u^\varepsilon}{1 + \varepsilon(\nabla u^\varepsilon)^\top C(u^\varepsilon)\nabla u^\varepsilon} \, (\sinh(\lambda u^\varepsilon) - \sinh(\lambda u_D)) \, dx\,dt$$

$$-\frac{1}{2}\int_{Q_T}\frac{(\nabla'u^\varepsilon)^\top C'(u^\varepsilon)\nabla'u^\varepsilon}{1+\varepsilon(\nabla'u^\varepsilon)^\top C'(u^\varepsilon)\nabla'u^\varepsilon}\left(\sinh(\lambda u^\varepsilon)-\sinh(\lambda u_D)\right)dx\,dt$$

$$-\frac{1}{2}\int_{Q_T}((\operatorname{div}C)(u^\varepsilon)\cdot\nabla u^\varepsilon+(\operatorname{div}'C')(u^\varepsilon)\cdot\nabla'u^\varepsilon)(\sinh(\lambda u^\varepsilon)-\sinh(\lambda u_D))\,dx\,dt$$

$$+\frac{1}{2}\int_{Q_T}\lambda\cosh(\lambda u_D)\left[(\nabla u_D)^\top C(u^\varepsilon)\nabla u^\varepsilon+(\nabla'u_D)^\top C'(u^\varepsilon)\nabla'u^\varepsilon\right]dx\,dt.$$

Since u^ε is uniformly bounded in $L^\infty(Q_T)$ and $|\sinh(x)|\leq\cosh(x)$, $x\in\mathbb{R}$, we obtain

$$\frac{1}{2}\int_{Q_T}\lambda\cosh(\lambda u^\varepsilon)\left[(\nabla u^\varepsilon)^\top C(u^\varepsilon)\nabla u^\varepsilon+(\nabla'u^\varepsilon)^\top C'(u^\varepsilon)\nabla'u^\varepsilon\right]dx\,dt$$

$$\leq\int_{Q_T}|(\mu\cdot\nabla u^\varepsilon+\mu'\cdot\nabla'u^\varepsilon+q(\mu-rS)\cdot\nabla u^\varepsilon)(\sinh(\lambda u^\varepsilon)-\sinh(\lambda u_D))|\,dx\,dt$$

$$+\underbrace{\int_{Q_T}|\frac{q}{2}\beta(u^\varepsilon)^2-pr|(\cosh(\lambda u^\varepsilon)+\cosh(\lambda u_D))\,dx\,dt}_{\leq L_1}$$

$$+\frac{1}{2|p-1|}\int_{Q_T}(\nabla u^\varepsilon)^\top C(u^\varepsilon)\nabla u^\varepsilon\,(\cosh(\lambda u^\varepsilon)+\cosh(\lambda u_D))\,dx\,dt$$

$$+\frac{1}{2}\int_{Q_T}(\nabla'u^\varepsilon)^\top C'(u^\varepsilon)\nabla'u^\varepsilon\,(\cosh(\lambda u^\varepsilon)+\cosh(\lambda u_D))\,dx\,dt$$

$$+\frac{1}{2}\int_{Q_T}|(\operatorname{div}C)(u^\varepsilon)\cdot\nabla u^\varepsilon+(\operatorname{div}'C')(u^\varepsilon)\cdot\nabla'u^\varepsilon||\sinh(\lambda u^\varepsilon)-\sinh(\lambda u_D)|\,dx\,dt$$

$$+\frac{1}{2}\int_{Q_T}\lambda\cosh(\lambda u_D)\left[|(\nabla u_D)^\top C(u^\varepsilon)\nabla u^\varepsilon|+|(\nabla'u_D)^\top C'(u^\varepsilon)\nabla'u^\varepsilon|\right]dx\,dt$$

$$+\underbrace{\frac{1}{\lambda}\int_{\hat\Omega}|\cosh(\lambda u^\varepsilon)(t)-\cosh(\lambda u_0)|\,dx}_{\leq L_2}+\underbrace{\int_{Q_T}|u^\varepsilon\cosh(\lambda u_D)u_{D,t}|\,dx\,dt}_{\leq L_3}.$$

Here we use the assumption that $u_D\in H^1(0,T;L^1(\hat\Omega))$. Choosing λ sufficiently large and using Young's inequality for some $\delta>0$, we can further estimate

$$\frac{1}{2}\int_{Q_T}\underbrace{\left(\lambda\cosh(\lambda u^\varepsilon)-\frac{1}{|p-1|}(\cosh(\lambda u^\varepsilon)+\cosh(\lambda u_D))\right)}_{=:\kappa>0}(\nabla u^\varepsilon)^\top C(u^\varepsilon)\nabla u^\varepsilon\,dx\,dt$$

$$+ \frac{1}{2} \int_{Q_T} \underbrace{(\lambda \cosh(\lambda u^\varepsilon) - (\cosh(\lambda u^\varepsilon) + \cosh(\lambda u_D)))}_{=: \kappa' > 0} (\nabla' u^\varepsilon)^\top C'(u^\varepsilon) \nabla' u^\varepsilon \, dx \, dt$$

$$\leq L_1 + L_2 + L_3 + \int_{Q_T} |\mu \cdot \nabla u^\varepsilon + \mu' \cdot \nabla' u^\varepsilon + q(\mu - rS) \cdot \nabla u^\varepsilon|| \sinh(\lambda u^\varepsilon) - \sinh(\lambda u_D)| \, dx \, dt$$

$$+ \frac{1}{2} \int_{Q_T} |(\operatorname{div} C)(u^\varepsilon) \cdot \nabla u^\varepsilon + (\operatorname{div}'C')(u^\varepsilon) \cdot \nabla' u^\varepsilon|(\cosh(\lambda u^\varepsilon) + \cosh(u_D)) \, dx \, dt$$

$$+ \frac{1}{2} \int_{Q_T} \lambda \cosh(\lambda u_D) \Big[\underbrace{|(\nabla u_D)^\top C(u^\varepsilon) \nabla u^\varepsilon|}_{\leq ||C||_2 |\nabla u_D||\nabla u^\varepsilon|} + \underbrace{|(\nabla' u_D)^\top C'(u^\varepsilon) \nabla' u^\varepsilon|}_{\leq ||C'||_2 |\nabla' u_D||\nabla' u^\varepsilon|} \Big] \, dx \, dt$$

$$\leq L_1 + L_2 + L_3 + \int_{Q_T} \Big(\delta |\nabla u^\varepsilon|^2 + \frac{c}{\delta} (\cosh(\lambda u^\varepsilon) + \cosh(\lambda u_D))^2 \Big) \, dx \, dt$$

$$+ \int_{Q_T} \Big(\delta |\nabla' u^\varepsilon|^2 + \frac{c}{\delta} (\cosh(\lambda u^\varepsilon) + \cosh(\lambda u_D))^2 \Big) \, dx \, dt$$

$$+ \int_{Q_T} \Big(\delta(|\nabla u^\varepsilon|^2 + |\nabla' u^\varepsilon|^2) + \frac{1}{\delta} (|\operatorname{div} C(u^\varepsilon)|^2 + |\operatorname{div}'C'(u^\varepsilon)|^2) \Big) (\cosh(\lambda u^\varepsilon) + \cosh(\lambda u_D)) \, dx \, dt$$

$$+ \frac{1}{2} \int_{Q_T} \lambda \cosh(\lambda u_D) \Big[||C||_2 (\frac{1}{\delta} |\nabla u_D|^2 + \delta |\nabla u^\varepsilon|^2) + ||C'||_2 (\frac{1}{\delta} |\nabla' u_D|^2 + \delta |\nabla' u^\varepsilon|^2) \Big] \, dx \, dt,$$

where $||\cdot||_2$ denotes the matrix norm defined by $||C||_2 = \sup_{|x|=1} |Cx|$ and $|\cdot|$ is the euclidian norm. For sufficiently small $\delta > 0$ the gradient terms on the right hand side can now be estimated by the left hand side using the coercivity (H2) of C and C':

$$\frac{1}{2} \int_{Q_T} \Big\{ \alpha \kappa - \delta \big[2c + 2\cosh(\lambda u^\varepsilon) + 2\cosh(\lambda u_D) + \lambda ||C||_2 \cosh(\lambda u_D) \big] \Big\} |\nabla u^\varepsilon|^2 \, dx \, dt$$

$$+ \frac{1}{2} \int_{Q_T} \Big\{ \alpha' \kappa' - \delta \big[2c + 2\cosh(\lambda u^\varepsilon) + 2\cosh(\lambda u_D) + \lambda ||C'||_2 \cosh(\lambda u_D) \big] \Big\} |\nabla' u^\varepsilon|^2 \, dx \, dt$$

$$\leq L_1 + L_2 + L_3 + \int_{Q_T} \frac{2}{\delta} (\cosh(\lambda u^\varepsilon) + \cosh(\lambda u_D))^2 \, dx \, dt$$

$$+ \int_{Q_T} \frac{1}{\delta} (|(\operatorname{div} C)(u^\varepsilon)|^2 + |(\operatorname{div}'C')(u^\varepsilon)|^2)(\cosh(\lambda u^\varepsilon) + \cosh(\lambda u_D)) \, dx \, dt,$$

$$+ \frac{1}{2} \int_{Q_T} \lambda \cosh(\lambda u_D) \Big[||C||_2 \frac{1}{\delta} |\nabla u_D|^2 + ||C'||_2 \frac{1}{\delta} |\nabla' u_D|^2 \Big] \, dx \, dt$$

By Lemma 2, the right hand side is bounded and we conclude

$$\int\limits_{Q_T} (|\nabla u^\varepsilon|^2 + |\nabla' u^\varepsilon|^2)\, dx\, dt \leq k.$$

Due to Poincaré's inequality we obtain the desired H^1-bound. ∎

The main result of this section is the following theorem.

Theorem 5 *Let (H1)–(H4) hold. Then there exists a solution u of (1) such that $u - u_D \in L^\infty(0, T; L^\infty(\hat{\Omega})) \cap L^2(0, T; H_0^1(\hat{\Omega}))$ and $u \in H^1(0, T; H^{-1}(\hat{\Omega}))$.*

Proof. Let u^ε be a solution of (3), (1b), (1c). In view of Lemma 4, $\|u^\varepsilon\|_{L^2(0,T;H^1(\hat{\Omega}))}$ is uniformly bounded and we can extract a subsequence u^ε (not relabeled) such that, as $\varepsilon \to 0$,

$$u^\varepsilon \rightharpoonup u \quad \text{in } L^2(0, T; H^1(\hat{\Omega})), \tag{7}$$

using, e.g., [34, Theorem 21.D]. Since also $\|u_t^\varepsilon\|_{L^2(0,T;H^{-1}(\hat{\Omega}))}$ is uniformly bounded, again for a subsequence which is not relabeled,

$$u_t^\varepsilon \rightharpoonup u_t \quad \text{in } L^2(0, T; H^{-1}(\hat{\Omega})). \tag{8}$$

By Aubin's lemma [32] we obtain

$$u^\varepsilon \to u \quad \text{in } L^2(0, T; L^2(\hat{\Omega})), \tag{9}$$

In order to pass to the limit as $\varepsilon \to 0$ in the quadratic gradient terms of the truncated approximate equation (3) we need the strong convergence of $u^\varepsilon \to u$ in $L^2(0, T; H^1(\hat{\Omega}))$. The proof of this result is the main step of the proof.

To establish the strong convergence of $u^\varepsilon \to u$ we use the so-called monotonicity method of Frehse [15], extended here to parabolic problems. Let $\bar{u}^\varepsilon = u^\varepsilon - u$ and choose $\sinh(\lambda \bar{u}^\varepsilon)$, $\lambda > 0$, as a test function in the approximate problem (3):

$$\int\limits_0^T \langle u_t^\varepsilon, \sinh(\lambda \bar{u}^\varepsilon) \rangle\, dt + \frac{1}{2} \int\limits_{Q_T} \lambda \cosh(\lambda \bar{u}^\varepsilon)(\nabla \bar{u}^\varepsilon)^\top C(u^\varepsilon) \nabla u^\varepsilon\, dx\, dt$$

$$+ \frac{1}{2} \int\limits_{Q_T} \lambda \cosh(\lambda \bar{u}^\varepsilon)(\nabla' \bar{u}^\varepsilon)^\top C'(u^\varepsilon) \nabla' u^\varepsilon\, dx\, dt$$

$$= \int\limits_{Q_T} \left(\mu \cdot \nabla u^\varepsilon + \mu' \cdot \nabla' u^\varepsilon + q(\mu - rS) \cdot \nabla u^\varepsilon - \frac{q}{2} \beta(u^\varepsilon)^2 + pr \right) \sinh(\lambda \bar{u}^\varepsilon)\, dx\, dt$$

$$+ \frac{1}{2(p-1)} \int\limits_{Q_T} \frac{(\nabla u^\varepsilon)^\top C(u^\varepsilon) \nabla u^\varepsilon}{1 + \varepsilon (\nabla u^\varepsilon)^\top C(u^\varepsilon) \nabla u^\varepsilon} \sinh(\lambda \bar{u}^\varepsilon)\, dx\, dt \tag{10}$$

$$-\frac{1}{2}\int\limits_{Q_T}\frac{(\nabla'u^\varepsilon)^\top C'(u^\varepsilon)\nabla'u^\varepsilon}{1+\varepsilon(\nabla'u^\varepsilon)^\top C'(u^\varepsilon)\nabla'u^\varepsilon}\ \sinh(\lambda\bar{u}^\varepsilon)\ dx\,dt$$

$$-\frac{1}{2}\int\limits_{Q_T}((\operatorname{div}C)(u^\varepsilon)\cdot\nabla u^\varepsilon+(\operatorname{div}'C')(u^\varepsilon)\cdot\nabla'u^\varepsilon)\ \sinh(\lambda\bar{u}^\varepsilon)\ dx\,dt.$$

The left hand side of this equation can be written as follows:

$$\int\limits_0^T\langle\bar{u}_t^\varepsilon,\sinh(\lambda\bar{u}^\varepsilon)\rangle\ dt+\int\limits_0^T\langle u_t,\sinh(\lambda\bar{u}^\varepsilon)\rangle\ dt+\frac{1}{2}\int\limits_{Q_T}\lambda\cosh(\lambda\bar{u}^\varepsilon)(\nabla\bar{u}^\varepsilon)^\top C(u^\varepsilon)\nabla\bar{u}^\varepsilon\ dx\,dt$$

$$+\frac{1}{2}\int\limits_{Q_T}\lambda\cosh(\lambda\bar{u}^\varepsilon)(\nabla'\bar{u}^\varepsilon)^\top C'(u^\varepsilon)\nabla'\bar{u}^\varepsilon\ dx\,dt \tag{11}$$

$$+\frac{1}{2}\int\limits_{Q_T}\lambda\cosh(\lambda\bar{u}^\varepsilon)\Big[(\nabla\bar{u}^\varepsilon)^\top C(u^\varepsilon)\nabla u+(\nabla'\bar{u}^\varepsilon)^\top C'(u^\varepsilon)\nabla'u\Big]\ dx\,dt.$$

We claim that the first term is non-negative. Indeed, let $u^\delta\in C^1([0,T];H^1(\hat{\Omega}))$ be a sequence such that $u^\delta\to u$ in $L^2(0,T;H^1(\hat{\Omega}))\cap H^1(0,T;H^{-1}(\hat{\Omega}))$ as $\delta\to 0$ and $u^\delta(0)=u_0$. Then

$$\int\limits_0^T\int\limits_{\hat{\Omega}}(u^\varepsilon-u^\delta)_t\sinh(\lambda(u^\varepsilon-u^\delta))\ dt$$

$$=\frac{1}{\lambda}\int\limits_{\hat{\Omega}}\cosh(\lambda(u^\varepsilon-u^\delta)(T))\ dx-\frac{1}{\lambda}\int\limits_{\hat{\Omega}}\cosh(\lambda(u^\varepsilon-u^\delta)(0))\ dx$$

$$=\frac{1}{\lambda}\int\limits_{\hat{\Omega}}(\cosh(\lambda(u^\varepsilon-u^\delta)(T))-1)\ dx\geq 0,$$

and letting $\delta\to 0$ shows that

$$\int\limits_0^T\langle\bar{u}_t^\varepsilon,\sinh(\lambda\bar{u}^\varepsilon)\rangle\geq 0.$$

The quadratic gradient terms on the right hand side of (10) can be estimated as

$$\frac{(\nabla u^\varepsilon)^\top C(u^\varepsilon)\nabla u^\varepsilon}{1+\varepsilon(\nabla u^\varepsilon)^\top C(u^\varepsilon)\nabla u^\varepsilon}$$

$$\leq(\nabla\bar{u}^\varepsilon)^\top C(u^\varepsilon)\nabla\bar{u}^\varepsilon+(\nabla u)^\top C(u^\varepsilon)\nabla u+(\nabla\bar{u}^\varepsilon)^\top C(u^\varepsilon)\nabla u+(\nabla u)^\top C(u^\varepsilon)\nabla\bar{u}^\varepsilon$$

and likewise for the ∇' terms. Taking the modulus and choosing λ sufficiently large, (10) and (11) become

$$\frac{1}{2}\int_{Q_T}(\lambda - \frac{1}{|p-1|})\cosh(\lambda\bar{u}^\varepsilon)(\nabla\bar{u}^\varepsilon)^\top C(u^\varepsilon)\nabla\bar{u}^\varepsilon \, dx \, dt$$

$$+\frac{1}{2}\int_{Q_T}(\lambda - 1)\cosh(\lambda\bar{u}^\varepsilon)(\nabla'\bar{u}^\varepsilon)^\top C'(u^\varepsilon)\nabla'\bar{u}^\varepsilon \, dx \, dt$$

$$\leq \int_{Q_T}|(\mu\cdot\nabla\bar{u}^\varepsilon + \mu'\cdot\nabla'\bar{u}^\varepsilon + q(\mu - rS)\cdot\nabla\bar{u}^\varepsilon)\sinh(\lambda\bar{u}^\varepsilon)| \, dx \, dt$$

$$+\int_{Q_T}|(\mu\cdot\nabla u + \mu'\cdot\nabla'u + q(\mu - rS)\cdot\nabla u - \frac{q}{2}\beta(u^\varepsilon)^2 + pr)\sinh(\lambda\bar{u}^\varepsilon)| \, dx \, dt$$

$$+\frac{1}{2|p-1|}\int_{Q_T}|[(\nabla u)^\top C(u^\varepsilon)\nabla u^\varepsilon + (\nabla u^\varepsilon)^\top C(u^\varepsilon)\nabla u]\sinh(\lambda\bar{u}^\varepsilon)| \, dx \, dt$$

$$+\frac{1}{2}\int_{Q_T}|[(\nabla'u)^\top C'(u^\varepsilon)\nabla'u^\varepsilon + (\nabla'u^\varepsilon)^\top C'(u^\varepsilon)\nabla'u]\sinh(\lambda\bar{u}^\varepsilon)| \, dx \, dt$$

$$+\frac{1}{2|p-1|}\int_{Q_T}|(\nabla u)^\top C(u^\varepsilon)\nabla u \sinh(\lambda\bar{u}^\varepsilon)| \, dx \, dt$$

$$+\frac{1}{2}\int_{Q_T}|(\nabla'u)^\top C'(u^\varepsilon)\nabla'u \sinh(\lambda\bar{u}^\varepsilon)| \, dx \, dt$$

$$+\frac{1}{2}\int_{Q_T}|[(\operatorname{div}C)(u^\varepsilon)\cdot\nabla\bar{u}^\varepsilon + (\operatorname{div}'C')(u^\varepsilon)\cdot\nabla'\bar{u}^\varepsilon]\sinh(\lambda\bar{u}^\varepsilon)| \, dx \, dt$$

$$+\frac{1}{2}\int_{Q_T}|[(\operatorname{div}C)(u^\varepsilon)\cdot\nabla u + (\operatorname{div}'C')(u^\varepsilon)\cdot\nabla'u]\sinh(\lambda\bar{u}^\varepsilon)| \, dx \, dt$$

$$+\frac{1}{2}\int_{Q_T}\lambda\cosh(\lambda\bar{u}^\varepsilon)\Big[|(\nabla\bar{u}^\varepsilon)^\top C(u^\varepsilon)\nabla u| + |(\nabla'\bar{u}^\varepsilon)^\top C'(u^\varepsilon)\nabla'u|\Big] \, dx \, dt$$

$$+\int_0^T|\langle u_t, \sinh(\lambda\bar{u}^\varepsilon)\rangle| \, dt$$

$$=:I_1 + \cdots + I_{10}, \tag{12}$$

where we have used again $|\sinh(x)| \leq \cosh(x)$, $x \in \mathbb{R}$.

We need to show that the right hand side of (12) converges to zero. In view of (9) and since $\bar{u}^\varepsilon$ is uniformly bounded in $L^\infty(Q_T)$, it holds

$$\sinh(\lambda\bar{u}^\varepsilon) \to 0 \quad \text{in } L^2(0, T; L^2(\hat{\Omega})), \tag{13}$$

$$\sinh(\lambda\bar{u}^\varepsilon) \rightharpoonup 0 \quad \text{in } L^2(0,T;H^1(\hat{\Omega})),$$

which implies that $I_2, I_5, I_6, I_8, I_{10} \to 0$ as $\varepsilon \to 0$. In view of (7) and (13), we obtain $I_1 \to 0$.

The treatment of the integrals I_3, I_4, I_7 and I_9 is more delicate. In view of (9) and since $\bar{u}^\varepsilon \in L^\infty(Q_T)$ uniformly, $\cosh(\lambda\bar{u}^\varepsilon) \to 1$ in $L^2(0,T;L^2(\hat{\Omega}))$ and a.e. in Q_T. Since $\nabla\bar{u}^\varepsilon$ is uniformly bounded in $L^2(0,T;L^2(\hat{\Omega}))$, it holds for a subsequence (not relabeled),

$$\nabla\cosh(\lambda\bar{u}^\varepsilon) \rightharpoonup \nabla z \quad \text{in } L^2(0,T;L^2(\hat{\Omega}))$$

for some z. From identifying $z = 1$ it follows

$$\nabla\cosh(\lambda\bar{u}^\varepsilon) \rightharpoonup 0 \quad \text{in } L^2(0,T;L^2(\hat{\Omega})).$$

Thus

$$\int\limits_{Q_T} (\nabla u)^\top C(u^\varepsilon)\nabla u^\varepsilon \sinh(\lambda\bar{u}^\varepsilon)\,dx\,dt$$

$$= \frac{1}{\lambda}\int\limits_{Q_T} (\nabla u)^\top C(u^\varepsilon)\nabla\cosh(\lambda\bar{u}^\varepsilon)\,dx\,dt + \int\limits_{Q_T} (\nabla u)^\top C(u^\varepsilon)\nabla u \sinh(\lambda\bar{u}^\varepsilon)\,dx\,dt$$

$$\to 0 \quad \text{as } \varepsilon \to 0.$$

All terms in I_3, I_4, I_7 and I_9 can be treated similarly showing that the right hand side of (12) converges to zero as $\varepsilon \to 0$.

Employing the coercivity (H2) of C, C' and choosing $\lambda > 0$ sufficiently large, we obtain

$$\lim_{\varepsilon\to 0}\int\limits_{Q_T}\left(|\nabla\bar{u}^\varepsilon|^2 + |\nabla'\bar{u}^\varepsilon|^2\right)dx\,dt \le 0.$$

Thus we obtain

$$\nabla\bar{u}^\varepsilon \to 0, \ \nabla'\bar{u}^\varepsilon \to 0 \quad \text{in } L^2(0,T;L^2(\hat{\Omega})) \text{ as } \varepsilon \to 0,$$

which implies

$$u^\varepsilon \to u \quad \text{in } L^2(0,T;H^1(\hat{\Omega})) \text{ as } \varepsilon \to 0.$$

We can pass to the limit as $\varepsilon \to 0$ in (3) and obtain the existence of a solution u of problem (2). ∎

Remark 6 As the solution of (1) lies a posteriori in the space $L^\infty(Q_T)$, the regularity assumptions on the covariance matrices with respect to u can be relaxed. Indeed, by using a truncation argument by Stampacchia, it is not difficult to see that the hypothesis $C(S,t,\cdot), C'(S',t,\cdot) \in C^1(\mathbb{R})$ for all S, S', t is sufficient.

3   The Cauchy problem

We consider the Cauchy problem (1a), (1c) in $R_T = \mathbb{R}^{d+d'} \times (0,T)$. The L^∞ bound for the solutions of problem (1) of section 2 depends on $\mu - rS$ which is not bounded if $S \in \mathbb{R}^d$. Therefore, we need the following assumption.

(H5) $\exists\, M > 0 :\ \sup_{(S,S',t) \in R_T} |\mu(S,t) - r(S,S',t)S| \leq M.$

This assumption can be interpreted as follows: the relative return μ/S tends to the riskless interest rate r for large asset prices. This is known to be the case if the economic model consists of a representative investor with decreasing relative risk aversion or of multiple heterogeneous investors all of whom have constant relative risk aversion [4].

In the proof of Lemma 4 we made use of Poincaré's inequality to obtain the H^1 estimates. Since Poincaré's inequality is of no use now, we still lack an L^2 estimate for an H^1 estimate independent of $\hat{\Omega}$. It is provided by the following lemma.

Lemma 7 *Let (H1)–(H5) hold and let u be a weak solution to (1) such that $u_D = 0$. Then there exists a constant $L > 0$ (not depending on u) such that*

$$\|u\|_{L^\infty(0,T;L^p(\hat{\Omega}))} \leq L \quad \forall\, p < \infty.$$

Proof. As $u \in L^\infty(Q_T)$ and the L^∞ bound is independent of $\hat{\Omega}$ (because of (H5)) it suffices to prove that

$$\|u\|_{L^\infty(0,T;L^1(\hat{\Omega}))} \leq c, \tag{14}$$

for some $c > 0$, since then the result follows from interpolation. The idea of the proof of (14) is to use a smooth and monotone approximation of the sign function $\mathrm{sign}(u)$ as a test function in the weak formulation of (1).

Let η be convex and smooth such that

$$\eta(0) = 0, \quad \eta'(0) = 0, \quad \eta(x) = |x| - 0.5 \quad \text{for } |x| \geq 1$$

and define for $\delta > 0$

$$\eta_\delta(x) = \delta\eta\left(\frac{x}{\delta}\right), \quad x \in \mathbb{R}.$$

By construction of η_δ,

$$\eta_\delta(u) \leq |u| \quad \text{and} \quad \eta_\delta(u) \to |u| \quad \text{a.e. in } Q_T.$$

Using dominated convergence this implies

$$\eta_\delta(u) \to |u| \quad \text{in } L^2(0,T;L^1(\hat{\Omega})) \text{ as } \delta \to 0.$$

Use $\eta'_\delta(u)$ as a test function in (2) to obtain

$$
\int\limits_0^T \langle u_t, \eta'_\delta(u) \rangle \, d\tau + \frac{1}{2} \int\limits_{Q_T} \underbrace{\eta''_\delta(u)(\nabla u)^\top C(u) \nabla u}_{\geq 0} \, dx \, dt + \frac{1}{2} \int\limits_{Q_T} \underbrace{\eta''_\delta(u)(\nabla' u)^\top C'(u) \nabla' u}_{\geq 0} \, dx \, dt
$$

$$
= \int\limits_{Q_T} (\mu \cdot \nabla u + \mu' \cdot \nabla' u + q(\mu - rS) \cdot \nabla u) \, \eta'_\delta(u) \, dx \, dt - \int\limits_{Q_T} (\frac{q}{2}\beta(u)^2 - pr) \, \eta'_\delta(u) \, dx \, dt \quad (15)
$$

$$
+ \frac{1}{2(p-1)} \int\limits_{Q_T} \underbrace{(\nabla u)^\top C(u) \nabla u}_{\leq \|C(u)\|_2 |\nabla u|^2} \, \eta'_\delta(u) \, dx \, dt - \frac{1}{2} \int\limits_{Q_T} \underbrace{(\nabla' u)^\top C'(u) \nabla' u}_{\leq \|C'(u)\|_2 |\nabla' u|^2} \, \eta'_\delta(u) \, dx \, dt
$$

$$
- \frac{1}{2} \int\limits_{Q_T} ((\operatorname{div} C(u)) \cdot \nabla u + (\operatorname{div}' C'(u)) \cdot \nabla' u) \, \eta'_\delta(u) \, dx \, dt.
$$

Since $u \in L^2(0,T; H^1(\hat{\Omega})) \cap L^\infty(Q_T)$, $u_t \in L^2(0,T; H^{-1}(\hat{\Omega}))$ and η'_δ is smooth it holds [34, Prop. 23.20]

$$
\int\limits_0^T \langle u_t, \eta'_\delta(u) \rangle \, d\tau = \int\limits_{\hat{\Omega}} \eta_\delta(u(T)) \, dx - \int\limits_{\hat{\Omega}} \eta_\delta(u_0) \, dx.
$$

Since $|\eta'_\delta(u)| \leq 1$, the right hand side of (15) is bounded independently of δ (and $\hat{\Omega}$) and we obtain, after letting $\delta \to 0$,

$$
\int\limits_{\hat{\Omega}} |u(T)| \, dx - \int\limits_{\hat{\Omega}} |u_0| \, dx \leq c.
$$

This yields (14) for some constant $c = c(T)$. ∎

We are now able to prove the following theorem.

Theorem 8 *Let (H1)–(H5) hold. Then there exists a solution u of the Cauchy problem (1a), (1c) such that $u \in L^2(0,T; H^1(\mathbb{R}^{d+d'})) \cap L^\infty(R_T)$ and $u \in H^1(0,T; H^{-1}(\mathbb{R}^{d+d'}))$.*

Proof. Let $(\hat{\Omega}^n)_n$ be a sequence of domains with smooth boundaries $\partial\hat{\Omega}^n$ satisfying $\hat{\Omega}^n \subset \hat{\Omega}^{n+1}$ and tending to $\mathbb{R}^{d+d'}$ in the set-theoretical sense as $n \to \infty$. By theorem 5, in each of the cylinders $Q_T^n := \hat{\Omega}^n \times (0,T)$ there exists a solution $u^n \in L^2(0,T; H_0^1(\hat{\Omega}^n)) \cap L^\infty(Q_T^n)$ satisfying $u^n(0) = u_0|_{\hat{\Omega}^n}$. Under the additional assumption (H5) the constants c in the proof of Lemma 2 are independent of $\hat{\Omega}^n$, implying that these solutions are uniformly bounded in L^∞, i.e., it holds

$$
\|u^n\|_{L^\infty(Q_T^n)} \leq K,
$$

where $K > 0$ is independent of $n \in \mathbb{N}$. Furthermore, the estimates in the proof of Lemma 4 are independent of $\hat{\Omega}^n$ if (H5) holds. In view of Lemma 7 we have for $n \geq m$

$$
\|u^n\|_{L^2(0,T; H_0^1(\hat{\Omega}^m))} \leq c \quad (16)
$$

with c independent of n, m.

We can extract a subsequence $(u^{n,m})$ of (u^n) that converges weakly to some $u^{(m)} \in L^2(0, T; H_0^1(\hat{\Omega}^m)) \cap L^\infty(Q_T^m)$ as $n \to \infty$. Following the lines of the proof of Theorem 5 we can see that in fact $u^{n,m} \to u^{(m)}$ strongly in $L^2(0, T; H_0^1(\hat{\Omega}^m))$ and therefore also a.e. in Q_T^m. We have the following diagonal scheme

$$
\begin{array}{cccccl}
u^{1,1}, & u^{2,1}, & u^{3,1}, & \ldots & \to & u^{(1)} = u|_{Q_T^1} \\
 & u^{2,2}, & u^{3,2}, & \ldots & \to & u^{(2)} = u|_{Q_T^2} \\
 & & u^{3,3}, & \ldots & \to & u^{(3)} = u|_{Q_T^3} \\
 & & & \ddots & & \vdots
\end{array}
$$

More precisely, there exists a subsequence $u^{n,1}$ of u^n that converges strongly to some $u^{(1)}$ in $L^2(0, T; H_0^1(\hat{\Omega}^1))$ (and a.e. in Q_T^1). Furthermore, from this subsequence, we can select a subsequence $u^{n,2}$ that converges strongly to some $u^{(2)}$ in $L^2(0, T; H_0^1(\hat{\Omega}^2))$ with $u^{(2)}|_{Q_T^1} = u^{(1)}$, etc. The diagonal sequence $u^{n,n}$ tends to some $u \in L^2(0, T; H_0^1(\mathbb{R}^{d+d'})) \cap L^\infty(R_T)$ which is a solution to the Cauchy problem. ∎

4 Uniqueness of solutions

In this section let either $\hat{\Omega} \subset \mathbb{R}^{d+d'}$ be a bounded domain or $\hat{\Omega} = \mathbb{R}^{d+d'}$.

Lemma 9 *Assume (H1)–(H4) and one of the following additional assumptions:*

(H6) *the matrices $C = C(S, t)$, $C' = C'(S', t)$ do not depend on u,*

or

(H7) *$p < 1$, the matrices $\partial C/\partial u$, $\partial C'/\partial u$ are positive semi-definite, the derivatives $\partial(\text{div } C)/\partial u$, $\partial(\text{div }'C')/\partial u$ are uniformly bounded with respect to S, t and S', t, respectively, and $\|\partial C/\partial u\|_2$, $\|\partial C'/\partial u\|_2$ are sufficiently small (more precisely, we assume that (23) holds; see below).*

If $\hat{\Omega} = \mathbb{R}^{d+d'}$ we also assume (H5). Then the problem (1) has a unique solution in the space of generalized solutions.

Proof. Let u be a solution of (1). We introduce the transformation $u = \varphi(v) = -\ln(e^{-KAv} + 1/K)/A$ for some constants $A, K > 0$, which are chosen later. Using the test function

$\phi = \psi/\varphi'(v)$ for arbitrary $\psi \in L^2(0,T; H_0^1(\hat{\Omega})) \cap L^\infty(0,T; L^\infty(\hat{\Omega}))$ in (2) yields

$$0 = \int_0^t \langle v_t, \psi \rangle \, dt + \frac{1}{2} \int_{Q_T} \left[(\nabla \psi)^\top C(\varphi(v)) \nabla v + (\nabla' \psi)^\top C'(\varphi(v)) \nabla' v \right] dx \, dt$$

$$- \frac{1}{2} \int_{Q_T} \left[\frac{\varphi''(v)}{\varphi'(v)} (\nabla v)^\top C(\varphi(v)) \nabla v + \frac{\varphi''(v)}{\varphi'(v)} (\nabla' v)^\top C'(\varphi(v)) \nabla' v \right] \psi \, dx \, dt$$

$$- \int_{Q_T} \frac{1}{\varphi'(v)} \left[\mu \cdot \nabla \varphi(v) + \mu' \cdot \nabla' \varphi(v) + q(\mu - rS) \cdot \nabla \varphi(v) - \frac{q}{2} \beta(\varphi(v))^2 + pr \right] \psi \, dx \, dt$$

$$- \frac{1}{2(p-1)} \int_{Q_T} \frac{1}{\varphi'(v)} (\nabla \varphi(v))^\top C(\varphi(v)) \nabla \varphi(v) \, \psi \, dx \, dt$$

$$+ \frac{1}{2} \int_{Q_T} \frac{1}{\varphi'(v)} (\nabla' \varphi(v))^\top C'(\varphi(v)) \nabla' \varphi(v) \, \psi \, dx \, dt$$

$$+ \frac{1}{2} \int_{Q_T} \frac{1}{\varphi'(v)} ((\operatorname{div} C)(\varphi(v)) \cdot \nabla \varphi(v) + (\operatorname{div}' C')(\varphi(v)) \cdot \nabla' \varphi(v)) \, \psi \, dx \, dt.$$

The transformed problem is of the form

$$v_t - \operatorname{div}_{\hat{\xi}} (a((S,S'), t, v, (\nabla v, \nabla' v))) + b((S,S'), t, v, (\nabla v, \nabla' v)) = 0, \qquad (17)$$

with

$$a(\hat{S}, t, v, \hat{\xi}) = \begin{pmatrix} C(\varphi(v))\xi \\ C'(\varphi(v))\xi' \end{pmatrix},$$

$$b(\hat{S}, t, v, \hat{\xi}) = \left[-\frac{\varphi''(v)}{2\varphi'(v)} \xi^\top C(\varphi(v)) - \mu - q(\mu - rS) - \frac{\varphi'(v)}{2(p-1)} \xi^\top C(\varphi(v)) + \frac{1}{2} \operatorname{div} C(\varphi(v)) \right] \xi$$

$$+ \left[-\frac{\varphi''(v)}{2\varphi'(v)} \xi'^\top C'(\varphi(v)) - \mu' + \frac{\varphi'(v)}{2} \xi'^\top C'(\varphi(v)) + \frac{1}{2} \operatorname{div}' C'(\varphi(v)) \right] \xi'$$

$$+ \frac{1}{\varphi'(v)} \left(\frac{q}{2} \beta(\varphi(v))^2 - pr \right),$$

where $\hat{S} = (S, S')^\top$, $\hat{\xi} = (\xi, \xi')^\top$ and $\operatorname{div}_{\hat{\xi}} = (\operatorname{div}_\xi, \operatorname{div}_{\xi'})$ is the vectorized divergence operator.

Let u_1, u_2 be two solutions of (2) satisfying the same initial condition (1c) and set $u := u_1 - u_2$ and $u = \varphi(v)$. Using $(v^+)^n = (\max(0,v))^n$, $n \in \mathbb{N}$, as a test function in the equations satisfied by u_1 and u_2, respectively, and subtracting these two equations we get

$$0 = \int_0^t \langle v_t, (v^+)^n \rangle \, dt$$

$$+ \int\limits_{Q_T} n(v^+)^{n-1}(\nabla v^+, \nabla' v^+) \cdot [a(\hat{S}, t, u_1, (\nabla u_1, \nabla' u_1)) - a(\hat{S}, t, u_2, (\nabla u_2, \nabla' u_2))] \, dx \, dt$$

$$+ \int\limits_{Q_T} [b(\hat{S}, t, u_1, (\nabla u_1, \nabla' u_1)) - b(\hat{S}, t, u_2, (\nabla u_2, \nabla' u_2))](v^+)^n \, dx \, dt. \tag{18}$$

The difference in a can be expressed as

$$a(\hat{S}, t, u_1, (\nabla u_1, \nabla' u_1)) - a(\hat{S}, t, u_2, (\nabla u_2, \nabla' u_2))$$

$$= \int\limits_0^1 \frac{\partial}{\partial \tau} a(\hat{S}, t, \tau u_1 + (1 - \tau)u_2, (\nabla(\tau u_1 + (1 - \tau)u_2), \nabla'(\tau u_1 + (1 - \tau)u_2))) \, d\tau$$

$$= \int\limits_0^1 \left[\frac{\partial a}{\partial v}(\hat{S}, t, u_\tau, (\nabla u_\tau, \nabla' u_\tau))v + \frac{\partial a}{\partial \hat{\xi}}(\hat{S}, t, u_\tau, (\nabla u_\tau, \nabla' u_\tau))(\nabla v, \nabla' v)^\top \right] d\tau,$$

where $u_\tau = \tau u_1 + (1 - \tau)u_2$ and similarly for the difference in b. Using these expressions in (18) we obtain

$$\int\limits_0^t \langle v_t, (v^+)^n \rangle \, dt + \int\limits_{Q_T} \int\limits_0^1 n(v^+)^{n-1}(\nabla v^+, \nabla' v^+) \cdot \left[\frac{\partial a}{\partial v}v + \frac{\partial a}{\partial \hat{\xi}} \cdot (\nabla v, \nabla' v)^\top \right] d\tau \, dx \, dt$$

$$+ \int\limits_{Q_T} \int\limits_0^1 \left[\frac{\partial b}{\partial v}v + \frac{\partial b}{\partial \hat{\xi}} \cdot (\nabla v, \nabla' v)^\top \right](v^+)^n \, d\tau \, dx \, dt = 0,$$

omitting the arguments, where $\partial b/\partial \hat{\xi}$ is the vector containing the partial derivatives of b with respect to ξ and ξ'. Using (H2) this leads to

$$\frac{1}{n+1} \int\limits_{\hat{\Omega}} (v^+)^{n+1}(t) \, dx$$

$$+ \int\limits_{Q_T} \int\limits_0^1 n(v^+)^{n-1}\left(\alpha|\nabla v^+|^2 + \alpha'|\nabla' v^+|^2 + (\nabla v^+, \nabla' v^+) \cdot \frac{\partial a}{\partial v}v \right) d\tau \, dx \, dt$$

$$+ \int\limits_{Q_T} \int\limits_0^1 \left[\frac{\partial b}{\partial v}v + \frac{\partial b}{\partial \hat{\xi}} \cdot (\nabla v^+, \nabla' v^+) \right](v^+)^n \, d\tau \, dx \, dt \leq 0.$$

Employing Young's inequality with $\varepsilon = \min(\alpha, \alpha')/2$ we get

$$
\frac{1}{n+1}\int_{\hat\Omega} (v^+)^{n+1}(t)\,dx + \int_{Q_T}\int_0^1 \underbrace{n(v^+)^{n-1}\Big[(\alpha-\varepsilon)|\nabla v^+|^2 + (\alpha'-\varepsilon)|\nabla' v^+|^2\Big]}_{\geq 0}
$$

$$
\leq \int_{Q_T} (v^+)^{n+1}\int_0^1 \underbrace{\left[-\frac{\partial b}{\partial v} + \frac{n}{2\varepsilon}\Big|\frac{\partial a}{\partial v}\Big|^2 + \frac{1}{2\varepsilon n}\Big|\frac{\partial b}{\partial \xi}\Big|^2 + \frac{1}{2\varepsilon n}\Big|\frac{\partial b}{\partial \xi'}\Big|^2\right]}_{=:F(S,S',t,\xi,\xi',\tau)}\,d\tau\,dx\,dt \tag{19}
$$

The idea now is to show that $F(S, S', t, \xi, \xi', \tau)$ is bounded. This idea has been first used by Barles and Murat [3]. In the case (H7) with covariance matrices depending on u, we will make explicit use of the sign of $(1 - p)$ to obtain the necessary estimates. A computation leads to

$$
\begin{aligned}
\frac{\partial b}{\partial v}&(\hat S, t, v, (\nabla v, \nabla' v))\\
&= -\frac{1}{2}\Big(\frac{\varphi''}{\varphi'}\Big)'(v)\Big(\frac{(\nabla\varphi(v))^\top C(\varphi(v))\nabla\varphi(v)}{\varphi'(v)^2} + \frac{(\nabla'\varphi(v))^\top C'(\varphi(v))\nabla'\varphi(v)}{\varphi'(v)^2}\Big)\\
&\quad - \frac{1}{2}\frac{\varphi''}{\varphi'}(v)\Big(\frac{(\nabla\varphi(v))^\top\frac{\partial C}{\partial v}(\varphi(v))\nabla\varphi(v)}{\varphi'(v)^2} + \frac{(\nabla'\varphi(v))^\top\frac{\partial C'}{\partial v}(\varphi(v))\nabla'\varphi(v)}{\varphi'(v)^2}\Big)\\
&\quad + \frac{\varphi''}{2\varphi'(v)^2}\Big(\frac{1}{p-1}(\nabla\varphi(v))^\top C(\varphi(v))\nabla\varphi(v) + (\nabla'\varphi(v))^\top C'(\varphi(v))\nabla'\varphi(v)\Big) \quad (20)\\
&\quad - \frac{1}{2}\Big(\frac{1}{p-1}(\nabla\varphi(v))^\top\frac{\partial C}{\partial u}(\varphi(v))\nabla\varphi(v) - (\nabla'\varphi(v))^\top\frac{\partial C'}{\partial u}(\varphi(v))\nabla'\varphi(v)\Big)\\
&\quad + \frac{1}{2}\Big(\frac{\partial(\mathrm{div}\,C)}{\partial u}(\varphi(v))\nabla\varphi(v) + \frac{\partial(\mathrm{div}'C')}{\partial u}(\varphi(v))\nabla'\varphi(v)\Big)\\
&\quad + \frac{q}{2}(\mu - rS)^\top\frac{\partial C^{-1}}{\partial u}(\varphi(v))(\mu - rS) - \frac{\varphi''}{\varphi'(v)^2}\Big(\frac{q}{2}\beta(\varphi(v))^2 - pr\Big),
\end{aligned}
$$

recalling the definition $\beta^2(\varphi(v)) = (\mu - rS)^\top C^{-1}(\varphi(v))(\mu - rS)$, and

$$
\begin{aligned}
\frac{\partial b}{\partial \xi}(\hat S, t, v, \nabla v) &= -\frac{\varphi''}{\varphi'}(v)C(\varphi(v))\nabla v - \mu - q(\mu - rS) - \frac{1}{p-1}C(\varphi(v))\nabla\varphi(v)\\
&\quad + \frac{1}{2}\mathrm{div}\,C(\varphi(v)), \tag{21}\\
\frac{\partial b}{\partial \xi'}(\hat S, t, v, \nabla' v) &= -\frac{\varphi''}{\varphi'}(v)C'(\varphi(v))\nabla' v - \mu' + C'(\varphi(v))\nabla'\varphi(v) + \frac{1}{2}\mathrm{div}'C'(\varphi(v))
\end{aligned}
$$

We want to obtain expressions in terms of the original variable u. Using

$$
\varphi'(v) = K - e^{Au}, \quad \frac{\varphi''}{\varphi'}(v) = -Ae^{Au}, \quad \Big(\frac{\varphi''}{\varphi'}\Big)'(v)\frac{1}{\varphi'(v)} = -A^2 e^{Au},
$$

we obtain from (20)

$$
\frac{\partial b}{\partial v}(\hat{S}, t, v, (\nabla v, \nabla' v))
$$

$$
= \frac{e^{Au}}{2(K - e^{Au})} \Big[A^2 \big((\nabla u)^\top C(u) \nabla u + (\nabla' u)^\top C'(u) \nabla' u \big) + A \big((\nabla u)^\top \frac{\partial C}{\partial u}(u) \nabla u
$$

$$
+ (\nabla' u)^\top \frac{\partial C'}{\partial u}(u) \nabla' u - \frac{1}{p-1}(\nabla u)^\top C(u) \nabla u - (\nabla' u)^\top C'(u) \nabla' u \big) \Big]
$$

$$
+ \frac{1}{2} \Big(\frac{1}{1-p}(\nabla u)^\top \frac{\partial C}{\partial u}(u) \nabla u + (\nabla' u)^\top \frac{\partial C'}{\partial u}(u) \nabla' u + \frac{\partial(\operatorname{div} C)}{\partial u}(u) \nabla u + \frac{\partial(\operatorname{div}' C')}{\partial u}(u) \nabla' u \Big)
$$

$$
+ \frac{q}{2}(\mu - rS)^\top \frac{\partial C^{-1}}{\partial u}(u)(\mu - rS) + \frac{A e^{Au}}{K - e^{Au}} \Big(\frac{q\beta^2}{2}(\varphi(v)) - pr \Big)
$$

$$
\geq \frac{e^{Au}}{2(K - e^{Au})} \Big[(A^2 - \frac{1}{p-1})(\nabla u)^\top C(u) \nabla u + (A^2 - 1)(\nabla' u)^\top C'(u) \nabla' u
$$

$$
+ A(\nabla u)^\top \frac{\partial C}{\partial u}(u) \nabla u + A(\nabla' u)^\top \frac{\partial C'}{\partial u}(u) \nabla' u - c \Big]
$$

$$
+ \frac{1}{2} \frac{\partial(\operatorname{div} C)}{\partial u}(u) \nabla u + \frac{1}{2} \frac{\partial(\operatorname{div}' C')}{\partial u}(u) \nabla' u,
$$

for some $c > 0$ and using (H7) (in particular, we use here $p < 1$ since then $1/(1-p) > 0$). For the last two terms we use Young's inequality, for some $\delta > 0$:

$$
\frac{1}{2} \frac{\partial(\operatorname{div} C)}{\partial u}(u) \nabla u + \frac{1}{2} \frac{\partial(\operatorname{div}' C')}{\partial u}(u) \nabla' u \geq -\frac{\delta}{4}(|\nabla u|^2 + |\nabla' u|^2) - c(\delta), \qquad (22)
$$

where $c(\delta) > 0$ is a constant which depends on δ and the L^∞ norm of $\partial(\operatorname{div} C)/\partial u$ and $\partial(\operatorname{div}' C')/\partial u$. Now choose $A^2 > \max\{1, 1/(p-1)\}$. In view of (H2) and (H6) or (H7), respectively, we can estimate for sufficiently large $K > 0$ and sufficiently small $\delta > 0$,

$$
\frac{\partial b}{\partial v}(\hat{S}, t, v, (\nabla v, \nabla' v)) \geq \eta |\nabla u|^2 + \eta' |\nabla' u|^2 - c
$$

for some $\eta = \eta(\alpha, K, A, \delta)$, $\eta' = \eta'(\alpha', K, A, \delta) > 0$ and $c > 0$. Notice that $u \in L^\infty(Q_T)$.

The derivatives (21) in the original variable

$$
\frac{\partial b}{\partial \xi}(\hat{S}, t, v, \nabla v) = \frac{A e^{Au}}{K - e^{Au}} C(u) \nabla u - \mu - q(\mu - rS) - \frac{1}{p-1} C(u) \nabla u + \frac{1}{2}(\operatorname{div} C)(u),
$$

$$
\frac{\partial b}{\partial \xi'}(\hat{S}, t, v, \nabla' v) = \frac{A e^{Au}}{K - e^{Au}} C'(u) \nabla' u - \mu' + C'(u) \nabla' u + \frac{1}{2}(\operatorname{div}' C')(u)
$$

can be estimated as follows:

$$\left|\frac{\partial b}{\partial \xi}(\hat{S}, t, v, \hat{\xi})\right|^2 + \left|\frac{\partial b}{\partial \xi'}(\hat{S}, t, v, \hat{\xi})\right|^2$$

$$\leq \left(\frac{e^{2Au}}{(K - e^{Au})^2} + \frac{1}{(p-1)^2}\right)||C(u)||_2^2|\nabla u|^2 + \left(\frac{e^{2Au}}{(K - e^{Au})^2} + 1\right)||C'(u)||_2^2|\nabla' u|^2$$

$$+ (|\mu|^2 + q^2|\mu - rS|^2 + \frac{1}{4}|\operatorname{div} C(u)|^2) + (|\mu'|^2 + \frac{1}{4}|\operatorname{div}' C'(u)|^2)$$

$$\leq L_1(|\nabla u|^2 + |\nabla' u|^2) + L_2,$$

for some positive constants $L_1 = L_1(A, K, p, ||C||_2, ||C'||_2)$ and $L_2 = L_2(A, K, q)$. Further we can estimate

$$\left|\frac{\partial a}{\partial v}\right|^2 = \left|\begin{pmatrix}\frac{\partial C}{\partial u}\nabla u \\ \frac{\partial C'}{\partial u}\nabla' u\end{pmatrix}\right|^2 \leq \left\|\frac{\partial C}{\partial u}\right\|_2^2|\nabla u|^2 + \left\|\frac{\partial C'}{\partial u}\right\|_2^2|\nabla' u|^2.$$

This implies

$$F(S, S', t, \xi, \xi', \tau) \leq -\eta|\nabla u|^2 - \eta'|\nabla' u|^2 + c + \frac{n}{2\varepsilon}\left(\left\|\frac{\partial C}{\partial u}\right\|_2^2|\nabla u|^2 + \left\|\frac{\partial C'}{\partial u}\right\|_2^2|\nabla' u|^2\right)$$

$$+ \frac{1}{2\varepsilon n}\left(L_1(|\nabla u|^2 + |\nabla' u|^2) + L_2\right).$$

Choosing $n > L_1/(2\varepsilon \min(\eta, \eta'))$ and assuming that

$$\left\|\frac{\partial C}{\partial u}\right\|_2^2 \leq \frac{2\varepsilon\eta}{n} - \frac{L_1}{n^2} \quad \text{and} \quad \left\|\frac{\partial C'}{\partial u}\right\|_2^2 \leq \frac{2\varepsilon\eta'}{n} - \frac{L_1}{n^2}, \tag{23}$$

we conclude that $F(S, S', t, \xi, \xi', \tau) \leq L_2/(2\varepsilon n) + c$ and applying Gronwall's lemma in (19) yields $v \leq 0$ in Q_T. This implies that $u_1 - u_2 = u \leq \varphi^{-1}(0) = -\ln(1 - 1/K)/KA$ for all sufficiently large $K > 0$. Thus, after letting $K \to \infty$, $u_1 - u_2 \leq 0$ in Q_T. In a similar way, we can use the test function $(\min(0, v))^n$ for odd $n \in \mathbb{N}$ to prove that $u_1 - u_2 \geq 0$ in Q_T. Hence $u_1 = u_2$ in Q_T which completes the proof. ∎

Combining Lemma 9 and Theorem 5 yields the following theorem.

Theorem 10 *Let (H1)–(H4) and either (H6) or (H7) hold. If $\hat{\Omega} = \mathbb{R}^{d+d'}$ we assume additionally (H5). Then there exists a unique solution to (1) such that $u - u_D \in L^2(0, T; H_0^1(\hat{\Omega}))$ $\cap L^\infty(0, T; L^\infty(\hat{\Omega}))$, $u \in H^1(0, T; H^{-1}(\hat{\Omega}))$.*

5 Numerical illustration

In this section we present a numerical example showing the influence of the non-tradable state variables on the value function. We consider the case $d = 2$ and $d' = 1$, i.e. two

risky assets S_1, S_2 and one non-tradable state variable $S' = S_3$. Thus we have to solve a three-dimensional parabolic problem. We consider the following covariance matrices

$$C(S_1, S_2) = \begin{pmatrix} 0.04S_1^2 & -0.01S_1S_2 \\ -0.01S_1S_2 & 0.005S_2^2 \end{pmatrix}, \quad C'(S_3) = \left(0.05S_3^2 \right).$$

The returns are defined as the Ornstein-Uhlenbeck-type drifts

$$\begin{pmatrix} \mu_1(S_1) \\ \mu_2(S_2) \\ \mu_3(S_3) \end{pmatrix} = \begin{pmatrix} ((6 - S_1) + 0.2)S_1 \\ ((4 - S_2) + 0.1)S_2 \\ ((4 - S_3) + 0.3)S_3 \end{pmatrix},$$

and the interest rate is set to zero. As an initial condition we choose $u_0(S_1, S_2, S_3) = 0$ which corresponds to the initial capital $x = 1$. The risk aversion parameter is taken to be $p = 0.5$. We use quadratic finite elements and a standard Runge-Kutta time discretization as provided by the FEMLAB package for MATLAB to compute the numerical solution. We choose our computational domain as $[2, 10] \times [2, 6] \times [2, 12]$ and the time horizon as $[0, 0.8]$. We used approximately 23,000 3D elements to solve the problem (1).

Figure 1 shows the contour plots of the solution at times $t = 0.1, 0.4, 0.8$ for various values of the state variable S_3. The solution $(S_1, S_2) \mapsto u(S_1, S_2, S_3)$ has a local minimum at $S^* = (S_1^*, S_2^*) = (6.2, 4.1)$, since the expected return of investments in the two assets is zero at this point and the interest rate vanishes. The qualitative behavior of the solution in the variable S_1, S_2 is similar for different values of S_3. The variation with respect to S_3 is of the order of several percent. More precisely, for the values shown in Figure 1, the maximal relative difference to the minimum $S_3^* = 4.3$ at time $t = 0.8$ equals

$$\sup_{(S_1, S_2) \in (2,10) \times (2,6)} \frac{|u(S_1, S_2, S_3, 0.8) - u(S_1, S_2, S_3^*, 0.8)|}{|u(S_1, S_2, S_3^*, 0.8)|} \approx \begin{cases} 9.5\% & : S_3 = 2 \\ 18\% & : S_3 = 12. \end{cases}$$

For asset prices larger than S^* the returns are increasing and hence the solution u, which relates to the utility of the optimal portfolio, too. In that region the partial derivatives of u with respect to S_1 and S_2 are positive. Thus the optimal portfolio strategy $H(S_1, S_2, S_3) = (1 - p)^{-1}(\lambda - \nabla u)$ (the shares of the underlyings S_1 and S_2) has negative components for sufficiently large asset prices. This indicates short selling for the optimal portfolio, which is permitted in the model.

For asset prices smaller than S^* the partial derivatives with respect to S_1 and S_2 are negative and thus, the optimal portfolio strategy H has increasing components. This gives informations on how to change the shares of the portfolio consisting of the two assets and the bond.

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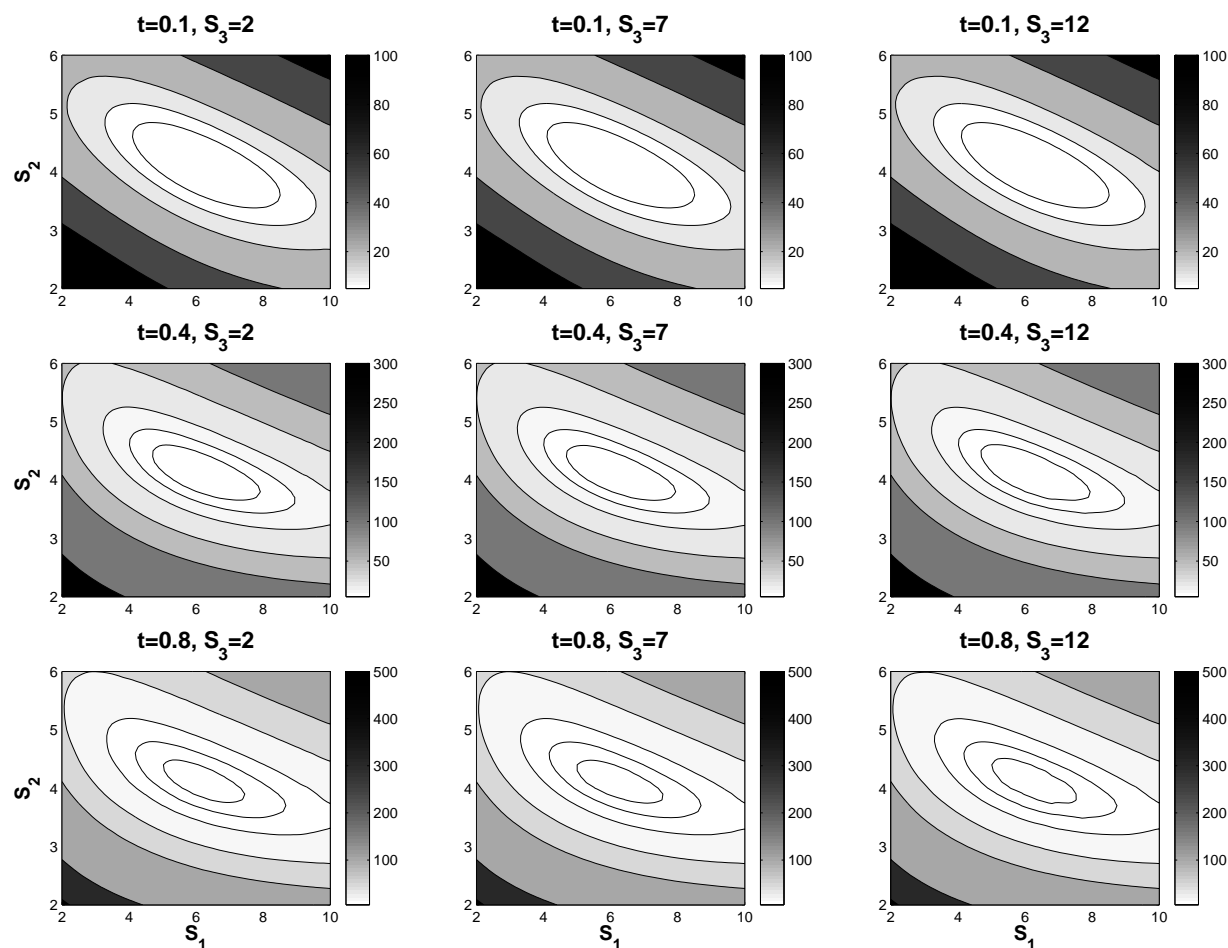
Figure 1: Contour plots of the numerical solution at times $t = 0.1, 0.4, 0.8$ and for fixed $S_3 = 2, 7, 12$.

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