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MULTIPLICATIVE BACKGROUND RISK *

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Abstract

Although there has been much attention in recent years on the effects of additive background risks, the same is not true for its multiplicative counterpart. We consider random wealth of the multiplicative form $\tilde{x}\tilde{y}$, where $\tilde{x}$ and $\tilde{y}$ are statistically independent random variables. We assume that $\tilde{x}$ is endogenous to the economic agent, but that $\tilde{y}$ is an exogenous and nontradable background risk, which represents a type of market incompleteness. Our main focus is on how the presence of the multiplicative background risk $\tilde{y}$ affects risk-taking behavior for decisions on the choice of $\tilde{x}$. We characterize conditions on preferences that lead to more cautious behavior.

Keywords: multiplicative risks, background risk, incomplete markets, standard risk aversion, affiliated utility function, multiplicative risk vulnerability

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1. Introduction

Consider a risk-averse economic agent whose preferences can be represented within an expected-utility framework via the continuously differentiable utility function u. The agent must decide upon choice parameters for a random variable representing final wealth, $\tilde{x}$. For example, $\tilde{x}$ might represent wealth from an individual's portfolio of financial assets, or $\tilde{x}$ might represent random corporate profits based on management decisions within the firm.

A fair amount of attention in recent years has examined how decisions on $\tilde{x}$ might be affected by the addition of an additive risk $\tilde{\varepsilon}$, where $\tilde{\varepsilon}$ and $\tilde{x}$ are statistically independent. Thus, final wealth or profits can be written as $\tilde{x} + \tilde{\varepsilon}$. The market is assumed to be incomplete in that $\tilde{\varepsilon}$ is not directly insurable. For example, $\tilde{\varepsilon}$ might represent future wage income subject to human-capital risks; or $\tilde{\varepsilon}$ might represent an exogenous pension portfolio provided by one's employer. Although it is interesting to examine the interdependence between $\tilde{x}$ and $\tilde{\varepsilon}$, the case of independence is of special interest and provides for many interesting observations. In order to focus on the risk effects, rather than wealth effects, it is often assumed that $E\tilde{\varepsilon} = 0$, where E denotes the expectation operator. In such a case, $\tilde{\varepsilon}$ is often called a "background risk." Since any non-zero mean for $\tilde{\varepsilon}$ can be added to the $\tilde{x}$ term, this assumption does not reduce the applicability of the model. Our purpose in the present paper is to examine the effects of introducing a "multiplicative background risk" into the individual's final wealth distribution.

The modern literature on additive background risk stems from the papers of Kihlstrom, et al. (1981), Ross (1981) and Nachman (1982). These papers focus on interpersonal behavior comparisons, mainly addressing the question: "If I am willing to

pay more than you to rid myself of any fair lottery, would I still be willing to do so in the presence of an additive background risk?" Doherty and Schlesinger (1983) incorporated the analysis into intrapersonal models of decision making under uncertainty, focusing on differences in optimal behavior with vs. without a background risk. The literature underwent somewhat of a renaissance in the 1990's thanks to new theoretical tools provided by Pratt and Zeckhauser (1987), Kimball (1990) and Gollier and Pratt (1996).

One canonical hypothesis concerning additive background risk is that the riskiness of $\tilde{\varepsilon}$ leads to a more cautious behavior towards decisions on $\tilde{x}$. For example, Guiso, et al. (1996) use Italian survey data to show that individuals with a riskier perception of their (exogenously managed) pension wealth react by investing relatively more in bonds in their personal accounts. However, this conclusion need not always be the case in theory, unless particular restrictions on preferences are met. Eeckhoudt and Kimball (1992) first examined this direction of research. Rather than review the large body literature for the case of additive background risks, we refer the reader to the excellent comprehensive presentation of this material in Gollier (2001).

Surprisingly, very little attention has been given to the case where the background risk is multiplicative. Indeed, if one were to ask the reader to think of possible types of background risks, we believe that examples with multiplicative types of background risk would be at least as prevalent as additive ones. Our goal in this paper is to provide a theoretical foundation for models with a multiplicative background risk. Under what conditions on preferences will the presence of a multiplicative background risk compel the agent to behave more cautiously in making decisions about the endogenous wealth variable $\tilde{x}$?

To this end, let $\tilde{y}$ be a random variable on a positive support that is statistically independent of $\tilde{x}$. We consider final wealth to be given by the product $\tilde{x}\tilde{y}$. The random variable $\tilde{y}$ is considered to be exogenous to the individual and is not insurable. Numerous examples of such multiplicative risks include the following:

1. Let $\tilde{x}$ be the pre-tax profits of a firm and let $\tilde{y}$ represent the firm's retention rate net of taxes, where tax rates are random due to tax-legislation uncertainty.

2. Let $\tilde{x}$ be the random wealth in an individual's financial portfolio in period one, and let $\tilde{y}$ denote the return on a mandatory (and exogenously managed) annuity account that uses proceeds from $\tilde{x}$ in period two.

3. Let $\tilde{x}$ denote nominal wealth or profit and let $\tilde{y}$ denote an end-of-period price deflator.

4. Let $\tilde{x}$ denote profit in some foreign currency for which forward contracts or options are not available and let $\tilde{y}$ denote the end-of-period exchange rate.

5. Let $\tilde{x}$ denote the random quantity of output for a farm commodity and let $\tilde{y}$ denote an exogenous random per-unit profit.

In order to isolate the risk effects of $\tilde{y}$, we will assume that $E\tilde{y}=1$ throughout this paper. For the case where $\tilde{y}$ has a mean that differs from one, we can incorporate this mean into $\tilde{x}$ via a deterministic scaling effect.[1] Since $\tilde{x}\tilde{y}=\tilde{x}+\tilde{x}(\tilde{y}-1)$, the assumption that $E\tilde{y}=1$, together with the independence of $\tilde{x}$ and $\tilde{y}$, guarantees that $\tilde{x}\tilde{y}$ is riskier than $\tilde{x}$ alone in the sense of Rothschild and Stiglitz (1970). We will refer to $\tilde{y}$, defined in this manner with $E\tilde{y}=1$, as a "multiplicative background risk."

We should point out at the outset that the results for the multiplicative case do not simply mirror those of the additive case. For instance, consider a simple portfolio example with an allocative choice between risky stocks and risk-free bonds. The

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[1] Thus, for instance, in our first example above we can let $\tilde{x}$ represent after-tax profits based on the expected tax rates and let $\tilde{y}$ represent a deviation from the expected after-tax retention rates. Or, in the second example let $\tilde{x}$ denote wealth including expected annuity returns and let $\tilde{y}$ denote a multiplicative excess-return adjustment.

individual has an initial wealth of 100 and the risk-free rate is assumed to be $r_f = 0.05$. The return on the stock portfolio is assumed to be log-binomial with an expected return of $E\tilde{r} = 0.11$ and a standard deviation of $\sigma = 0.20$ (implying that, in a binomial model, stocks either return about 33% or lose about 10%, each with an equally likely chance). Utility is assumed to belong to the HARA class with $u(x) = -\frac{1}{2}(x+a)^{-2}$, where a is a constant chosen such that $x+a$ remains positive over relevant wealth levels. We note that preferences satisfy decreasing absolute risk aversion (DARA) for any choice of a, whereas relative risk aversion will be increasing [decreasing, constant] whenever a is positive [negative, zero]. We examine the addition of two alternative sources of background risk. The first is an additive background risk, for which final wealth is either increased or decreased by 30, each with probability one-half. The second is a multiplicative background risk, for which wealth is either increased or decreased by 30 percent, each with a probability one-half. The optimal portfolio choices are illustrated in the following table.

TABLE 1: **Bond Proportions: Multiplicative vs. Additive Background Risk**
(All utility is DARA within the HARA class, $u(x) = -\frac{1}{2}(x+a)^{-2}$, initial wealth = 100)
(Relative risk aversion is constant for a=0, increasing for a=+25 and decreasing for a=-25)

Utility	Background Risk	Proportion in Bonds
$a = 0$	None	55%
	Additive	66%
	Multiplicative	55%
$a = +25$	None	45%
	Additive	54%
	Multiplicative	41%
$a = -25$	None	66%
	Additive	78%
	Multiplicative	70%

In each case in the above example, the proportion of wealth invested in risk-free bonds increases when an additive background risk is included.[2] Since DARA inside of the HARA class of preferences also implies standard risk aversion (Kimball 1993), we know that bond proportions will always increase with an additive background risk. However, as the example shows, a multiplicative background risk might cause the bond proportion to shrink. In particular, when $a = 25$, so that we have both DARA and increasing relative risk aversion – hardly considered unusual cases – we then have a lower proportion of wealth invested in the risk-free bond. That is, the investor reacts to the multiplicative background risk by taking a more aggressive position in stocks.

Our paper will show how each of the situations in the example above can be determined qualitatively (i.e. whether more or fewer bonds are purchased in the presence of a background risk) before calculating the optimal portfolios. The fact that the qualitative effects might be predetermined by the parameters of the model implies that care must be taken when modeling various economic and/or financial phenomena. For example, seemingly innocuous assumptions made about preferences might actually predispose a model to achieve particular results.

We begin in the next section by introducing the basic framework. We next examine some conditions on preferences that lead to more (or less) cautious behavior towards $\tilde{x}$ in the presence of a multiplicative background risk $\tilde{y}$. In section 4, we derive rather technical necessary and sufficient conditions on preferences such that a multiplicative background risk will always lead to more cautious behavior: a condition that we label "multiplicative risk vulnerability." In section 5, we define the affiliated utility function as the composite of utility with the exponential function. This allows us to translate several results from the case of additive background risk to our model with

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[2] Note that, even for the cases with no background risk, since relative risk aversion is decreasing in a, we have the bond proportion falls as a rises. Our point in the table, however, is to compare the levels of bonds between various types of background risk for a fixed value of a.

multiplicative background risk. In particular, necessary and sufficient conditions on the affiliated utility are presented such that a multiplicative background risk will always lead to more cautious behavior. Section 6 extends the usefulness of the conditions placed on the affiliated utility function by determining equivalent properties of the individual's actual utility function. Section 7 briefly looks at comparative risk aversion, before we offer some concluding remarks.

2. The Basic Model

Consider a risk-averse economic agent with utility function u. We wish to determine how the addition of a multiplicative background risk $\tilde{y}$ affects decision making on $\tilde{x}$. Both $\tilde{x}$ and $\tilde{y}$ are assumed to be strictly positive a.s. Let F and G denote the (cumulative) distribution functions associated with the random variables $\tilde{x}$ and $\tilde{y}$ respectively. Since $\tilde{x}$ and $\tilde{y}$ are independent, we can write expected utility as the iterated integral

$$(1) \qquad Eu(\tilde{x}\tilde{y}) = \int_0^\infty \int_0^\infty u(xy)dG(y)dF(x) \equiv E_F[E_G u(\tilde{x}\tilde{y})].$$

Define the derived utility function, see Nachman (1982)[3], as the interior integral given in equation (1). That is,

$$(2) \qquad v_G(x) \equiv \int_0^\infty u(xy)dG(y) = E_G u(x\tilde{y})$$

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[3] Actually, Nachman considers a more general relationship between $\tilde{x}$ and $\tilde{y}$. We adapt his measure to the case of multiplicative risks. The derived utility function for the additive case is described earlier by Kihlstrom, et al. (1981).

Trivially, $v_G(x)$ is increasing and concave since u is. Thus, equation (1) can be written as $Eu(\tilde{x}\tilde{y}) = E_F v_G(\tilde{x})$. Decisions on $\tilde{x}$ made in the presence of the multiplicative risk $\tilde{y}$ under utility u are isomorphic to decisions made on $\tilde{x}$ in isolation under the risk-averse utility $v_G(x)$. Let $\Gamma(\tilde{x})$ denote the set of positive random variables $\tilde{y}$ such that $\tilde{y}$ is statistically independent from $\tilde{x}$ and $E\tilde{y}=1$. Our focus here is in determining conditions on the utility function u such that the derived utility function, $v_G(x)$, is more risk averse than u for all $\tilde{y} \in \Gamma(\tilde{x})$. In other words, we wish to determine conditions on u that will guarantee that

$$(3) \qquad \frac{-v''_G(x)}{v'_G(x)} \equiv \frac{-E_G[u''(x\tilde{y})\tilde{y}^2]}{E_G[u'(x\tilde{y})\tilde{y}]} \geq \frac{-u''(x)}{u'(x)} \qquad \forall x.^4$$

To avoid excessive notation, we will dispense with the subscripts and simply write $v(x)$ and $Eu(x\tilde{y})$, where we assume $\tilde{y}$ is an arbitrary member of $\Gamma(\tilde{x})$. We will let $r_v(x)$ and $r_u(x)$ denote the measure of absolute risk aversion for v and u respectively, i.e. the left-hand-side and right-hand-side of inequality (3) respectively.

Since we are involved with a multiplicative background risk, it is often convenient to consider the corresponding measures of relative risk aversion, $R_v(x) \equiv x r_v(x)$ and $R_u(x) \equiv x r_u(x)$. Obviously, for any positive wealth level x, $r_v(x) \geq r_u(x)$ if and only if $R_v(x) \geq R_u(x)$.

For arbitrary x, straightforward manipulation of (3) shows that

$$(4) \qquad R_v(x) = E[R_u(x\tilde{y}) \frac{u'(x\tilde{y})\tilde{y}}{E[u'(x\tilde{y})\tilde{y}]}] \equiv \int_0^\infty R_u(xy) d\eta_x(y)$$

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[4] In order to keep the mathematics simple, we will take "more risk averse" to be in the weak sense of Pratt (1964).

where $\eta_x(y) \equiv \int_0^y \dfrac{u'(xt)t\,dG(t)}{E_G[u'(x\tilde{y})\tilde{y}]}$.

Note that $\eta_x(y)$ is itself a well-defined probability distribution. We define $\hat{E}_x$ to denote the expectation operator based on the probability distribution $\eta_x(y)$, which is a type of risk-adjusted probability measure.[5] Thus, we see that relative risk aversion for v is a weighted average of relative risk aversion for u, namely $R_v(x) = \hat{E}[R_u(x\tilde{y})]$.

3. Risk Aversion Properties

From equation (4), it follows trivially that v inherits constant relative aversion (CRRA), whenever u exhibits CRRA. More explicitly, if $R_u(x) = \gamma \quad \forall x$, then $R_v(x) = \gamma \ \forall x$ as well. Since it then also follows that $r_u(x) = r_v(x) \ \forall x$, we see that u and v are equivalent utility representations under CRRA. This is not surprising, since any optimal choice of an endogenous $\tilde{x}$ also will be optimal for $\tilde{x}y$, for every constant positive level of y under CRRA preferences.

From equation (4), we also see that $R_v(x)$ will be everywhere greater than [less than] one if $R_u(x)$ is everywhere greater than [less than] one. This result is more than just a technicality. Since many results in the literature on choice under uncertainty specify a global condition that either $R_u(x) > 1$ or $R_u(x) < 1$, such results also will hold in the presence of a multiplicative background risk, since $R_v(x)$ also will satisfy the appropriate property. More generally, equation (4) provides bounds for $R_v(x)$, such that given any $\tilde{y} \in \Gamma(x)$, with distribution function G,

$$\inf\{R_u(xy)\} \le R_v(x) \le \sup\{R_u(xy)\} \quad \forall y \in Supp(G).$$

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[5] If we have a representative agent model, and if we confine ourselves to a fixed value of x, this measure is simply the "risk-neutral probability measure." The random variable $[u'(x\tilde{y})\tilde{y}]/E[u'(x\tilde{y})\tilde{y}]$ in equation (4) is the Radon-Nikodym derivative of this measure with respect to G, again conditional on a fixed value of x. To simplify notation below, we will write simply $\hat{E}$, since the x subscript should be understood.

We next wish to examine conditions under which (3) holds $\forall \tilde{y} \in \Gamma(\tilde{x})$, i.e., we want to know when v is more risk averse than u. We may consider conditions for which this holds locally, with $r_v(x) \geq r_u(x)$, by examining the equivalent condition $R_v(x) \geq R_u(x)$. Our approach is to consider this last inequality for a particular value of x, by applying η_x as in equation (4). If the value of x chosen is arbitrary, so that $R_v(x) \geq R_u(x) \ \forall x$, then we are done. In the rest of this section, we extend equation (4) to directly obtain sufficient conditions for which v is more (or less) risk averse than u. In the following sections of the paper, we introduce two additional approaches to the problem.

Suppose that $R_u(x)$ is (not necessarily strictly) convex. Since $\eta_x(y)$ is a probability distribution, it follows from Jensen's inequality and equation (4) that

$$(5) \qquad R_v(x) \equiv \hat{E} R_u(x\tilde{y}) \geq R_u(x\hat{E}\tilde{y}),$$

where

$$(6) \qquad \hat{E}\tilde{y} = \int_0^\infty y \, d\eta_x(y) = \int_0^\infty y \frac{u'(xy)y}{E[u'(x\tilde{y})\tilde{y}]} dG(y).$$

Next, note that

$$(7) \qquad \frac{\partial^2 u(xy)}{\partial x \partial y} = \frac{\partial}{\partial y}[u'(xy)y] = u'(xy)[1 - R_u(xy)].$$

The sign of (7) tells us whether increases in the level of y will increase or decrease the marginal utility of x. The derivative in (7) will be everywhere positive [negative] if $R_u(xy) < [>] 1 \ \forall y$ in the support of G. This implies that increases in y reduce the

marginal utility of x whenever $R_u > 1$, and increases in y increase the marginal utility of x whenever $R_u < 1$.

Since $E\left\{\dfrac{u'(x\tilde{y})\tilde{y}}{E[u'(x\tilde{y})\tilde{y}]}\right\} = 1$, we see from (6) and (7), for example, that $R_u > 1$ everywhere implies that the probability measure $\eta_x(y)$ puts relatively more weight on lower values of y than does the true probability measure $G(y)$. The opposite is true if $R_u < 1$. We thus obtain the following result from (6) and (7).

Lemma 1: $\quad \hat{E}\tilde{y} \gtreqless E\tilde{y} = 1 \quad$ if $\quad R_u(xy) \lesseqgtr 1 \quad \forall y \in Supp(G)$.

We are now ready to prove the following result:

Proposition 1: Suppose that $R_u(x)$ is convex and that one of the following conditions holds $\forall (x, y) \in Supp(F) \times Supp(G)$:

      (i) $\;R_u(xy) > 1$ and R_u is decreasing,

or   (ii) $R_u(xy) < 1$ and R_u is increasing.

Then v is more risk averse than u.

Proof: Since $R_u(x)$ is convex, it follows from equation (4) that $R_v(x) \geq R_u(x\hat{E}\tilde{y})$ by Jensen's inequality. If $R_u > 1$, then $\hat{E}\tilde{y} < 1$ from Lemma 1. Hence, $R_u(x\hat{E}\tilde{y}) \geq R_u(x)$ under the assumption of decreasing relative risk aversion (DRRA). If $R_u < 1$, then it follows from Lemma 1 that $\hat{E}\tilde{y} > 1$. Hence, $R_u(x\hat{E}\tilde{y}) \geq R_u(x)$ under the assumption of increasing relative risk aversion (IRRA). Thus we have $R_v(x) \geq R_u(x)$ whenever condition (_i_) or (_ii_) holds. ∎

Interestingly, if we have CRRA preferences, we have already seen that u and v are equivalent regardless of whether or not relative risk aversion exceeds one. If relative risk aversion is increasing in wealth, as originally postulated by Arrow (1971) and empirically

supported by much literature, most recently by Guiso and Paiella (2001), then v will be more risk averse than u whenever R_u is convex and less than 1. If R_u is everywhere greater than 1 and exhibits increasing relative risk aversion, we cannot use Proposition 1 to verify that v is more risk averse than u. Indeed, if we have $R_u>1$ and if R_u is (not necessarily strictly) concave, it is easy to show that v is then less risk averse than u. Indeed, the following two cases are easy to show.

Proposition 2: *Suppose that* $R_u(x)$ *is concave and that one of the following conditions holds* $\forall (x,y) \in Supp(F) \times Supp(G)$ *:*

> *(i)* $R_u(xy) > 1$ *and* R_u *is increasing,*

or   *(ii)* $R_u(xy) < 1$ *and* R_u *is decreasing.*

Then v is less risk averse than u.

Proof: The proof is similar to Proposition 1 and left to the reader. ∎

Of course, whether risk aversion exhibits constant-, increasing-, or decreasing relative risk aversion, or none of these, is an empirical question. Certainly constant relative risk aversion is very common in equilibrium asset-pricing models. But empirical support also exists for both increasing relative risk aversion (e.g. Guiso and Paiella (2001)) and for decreasing relative risk aversion (e.g. Ogaki and Zhang (2001)). Whether relative risk aversion might be concave or convex in wealth has not received much attention at all until fairly recently. For example, Aït-Sahalia and Lo (2000) examine S&P 500 option prices to find some evidence of an oscillating level of relative risk aversion, although they do find R to be decreasing and convex at relatively low levels of wealth.[6] Aït-Sahalia and Lo (2000) also review much of the literature examining whether

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[6] See also Jackwerth (2000).

relative risk aversion is greater- or less-than one, with most support these days finding $R>1$.

To illustrate Proposition 1 and 2, consider the following examples:

Example 1: Let $u(x)=-e^{-kx}$ where $k>0$. This is the case of constant absolute risk aversion (CARA). In this case $R_u'(x)=k$ and $R_u''(x)=0$. Thus, R_u is increasing and is both convex and concave. If we consider $\tilde{x}$ and $\tilde{y}$ such that $xy<1/k$ $\forall (x,y)\in$ Supp$(F)\times$Supp(G), then $R_u(xy)<1$ and v is more risk averse than u by Proposition 1. However, if $xy>1/k$ $\forall (x,y)\in$ Supp$(F)\times$Supp(G), then $R_u(xy)>1$ and v is less risk averse than u by Proposition 2.

Example 2: Let $u(x)=x-kx^2$ where $k>0$. We restrict $x<\frac{1}{2k}$ so that marginal utility is positive. This is the case of quadratic utility. It is straightforward to show that $R_u(x)=2kx(1-2kx)^{-1}$ and that R_u is both strictly increasing and convex. Moreover, $R_u(xy)<1$ if $xy<\frac{1}{4k}$ $\forall (x,y)\in$ Supp(F) $\times$ Supp(G), so that v is more risk averse than u by Proposition 1. In other words, v is more risk averse than u over the first half of the relevant (upward-sloping) range of the quadratic utility function. On the other hand, if $\frac{1}{4k}<xy<\frac{1}{2k}$ $\forall (x,y)\in$ Supp$(F)\times$Supp(G), then $R_u(xy)>1$, but we cannot apply Proposition 1 (since R_u is increasing) or Proposition 2 (since R_u is convex).

Both utility functions above belong to the so-called HARA class of utility, as does CRRA utility.[7] Since we already showed that u and v are equivalent under CRRA, we see that no general results seem to apply to the HARA class of utility. However, we have more tractability in the shape of R_u under HARA. Let $u(x)=\xi(\eta+\frac{x}{\gamma})^{1-\gamma}$, where $\eta+\frac{x}{\gamma}>0$ and $\frac{\xi(1-\gamma)}{\gamma}>0$. Straightforward calculations show that $R_u'(x)=\eta(\eta+\frac{x}{\gamma})^{-2}$ and that

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[7] Utility belongs to the HARA class if $[r(x)]^{-1}$ is linear in x.

$R_u''(x) = -[\frac{2}{\gamma}(\eta + \frac{x}{\gamma})^{-1}]R_u'(x)$. Thus, for the case of constant absolute risk aversion ($\gamma \to \infty$), we obtain $R_u'(x) = k$ and $R_u''(x) = 0$, as in Example 1. If we have increasing absolute risk aversion, then we must have $\gamma < 0$ and $\eta > 0$. It follows that $R_u'(x) > 0$ and $R_u''(x) > 0$, so that we must have R_u increasing and convex, as is the case with our quadratic utility in Example 2. On the other hand, if we have decreasing absolute risk aversion (DARA), then $\gamma > 0$. Hence, $\operatorname{sgn} R_u''(x) = -\operatorname{sgn} R_u'(x)$. Consequently, we must have R_u either (*i*) constant, (*ii*) decreasing and convex, or (*iii*) increasing and concave. Consequently, if preferences are DARA within the class of HARA utility functions, it follows that we might have *v* either more risk averse than *u*, less risk averse than *u* or equally as risk-averse as *u*. In particular, corresponding to cases (*i*) - (*iii*) above:

(i)   If *u* satisfies CRRA, then *v* and *u* are equivalent.

(ii)   If $R_u > 1$, as well as decreasing, then *v* is more risk averse than *u* by Proposition 1.

(iii)   If $R_u > 1$, as well as increasing, then *v* is less risk averse than *u* by Proposition 2.

Note that $R_u > 1$ in our example in the introduction of this paper (see Table 1) and that conditions (*i*), (*ii*) and (*iii*) above apply to the three cases considered in our example, with *a*=0, *a*=-25 and *a*=+25 respectively.

4. Multiplicative Risk Vulnerability

Let the support for $\tilde{y}$ be contained in some positive interval [*a,b*]. As a direct analogue to Gollier and Pratt (1996), who examine the case of additive risks, we define preferences as being *multiplicatively risk vulnerable* if for every positive wealth variable

$\tilde{x}$ and every $\tilde{y} \in \Gamma(\tilde{x})$, that is for every $\tilde{y}$ independent of $\tilde{x}$ with $E\tilde{y} = 1$, the derived utility function v is more risk averse than u. In other words, any (independent) multiplicative background risk with a mean equal to one always causes an individual with multiplicatively risk-vulnerable preferences to behave in a more cautious manner towards risk $\tilde{x}$.[8]

In this section, we present a necessary and sufficient condition for utility to be multiplicatively risk vulnerable. Since this condition is rather complex, we turn in the next section to some sufficient conditions on preferences to guarantee multiplicative risk vulnerability. We also show how our condition relates to the Gollier and Pratt conditions for the case of additive background risks.

Before proceeding, we require the following Theorem, which is due to Gollier and Kimball (1996). A proof of this Theorem also can be found in Gollier (2001).

Diffidence Theorem (Gollier and Kimball): Let Λ denote the set of all random variables with support contained in the interval $[a,b]$ and let f and g be two real-valued functions. The following two conditions are equivalent:

    (i) For any $\tilde{y} \in \Lambda$, $Ef(\tilde{y}) = 0$ $\Rightarrow$ $Eg(\tilde{y}) \geq 0$.

    (ii) $\exists m \in \mathbb{R}$ such that $g(y) \geq mf(y)$ $\forall y \in [a,b]$.

We now are ready to show the following result.

Proposition 3: Utility is multiplicatively risk vulnerable if and only if for every x>0 and every $y \in [a,b]$,

(8) $u'(xy)y[R_u(xy) - R_u(x)] - (y-1)u'(x)xR_u'(x) \geq 0$.

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[8] Although aesthetically unappealing, the limitation to bounded supports is not particularly restrictive. We already limit $\tilde{x}$ and $\tilde{y}$ to be positive, and for any ε>0, we can always find a value for b such that the probability that $\tilde{y} > b$ is less than ε.

Proof: From the definition of multiplicative risk vulnerability, we need to examine properties on preferences such that

(9) $R_v(x) = \dfrac{-E[u"(x\tilde{y})x\tilde{y}^2]}{E[u'(x\tilde{y})\tilde{y}]} \geq R_u(x) \quad \forall x, \forall \tilde{y}$ with $E\tilde{y} = 1$.

This is equivalent to finding conditions on u such that

(10) $E\tilde{y} = 1 \implies -E[u"(x\tilde{y})x\tilde{y}^2] - R_u(x)E[u'(x\tilde{y})\tilde{y}] \geq 0$.

By the Diffidence Theorem, this statement is equivalent to finding a scalar m, such that

(11) $-u"(xy)xy^2 - R_u(x)u'(xy)y \geq m(y-1) \quad \forall y$,

or equivalently,

(12) $[\operatorname{sgn}(y-1)]u'(xy)y\left[\dfrac{R_u(xy) - R_u(x)}{y-1}\right] \geq [\operatorname{sgn}(y-1)]m$.

Considering $y \to 1$, we see that the only candidate for m is

(13) $m = [u'(xy)y\dfrac{dR_u(xy)}{dy}]\Big|_{y=1} = u'(x)xR_u'(x)$.

Replacing m in (11) above completes the proof.   ■

Note that the steps in the proof of Proposition 3 would also follow if we reversed the initial inequality in (9) above.   We thus immediately have the following result, showing a necessary and sufficient condition for v to be less risk averse than u.

Corollary to Proposition 3: *Derived utility v is less risk averse than u, where $\tilde{y} \in \Gamma(\tilde{x})$, if and only if for every x>0 and every $y \in [a,b]$,*

(14) $u'(xy)y[R_u(xy) - R_u(x)] - (y-1)u'(x)xR_u'(x) \leq 0$.

Propositions 1 and 2 can be derived directly from Proposition 3 and its Corollary. While we do not wish to re-prove our earlier results, we nevertheless illustrate one of the proofs here, since it helps to understand the necessary and sufficient conditions above. For the sake of concreteness, let us consider the case where relative risk aversion is convex as in Proposition 1. In addition, assume that $y>1$. The case where $y<1$ is similar. Under the assumptions in Proposition 1(*i*) or (*ii*), it follows that

$$(15) \qquad \left[\frac{R_u(xy) - R_u(x)}{xy - x} \right] \geq R_u{}'(x) \geq \frac{u'(x)}{u'(xy)y} R_u{}'(x).$$

The first inequality above follows from the convexity of relative risk aversion. The second inequality follows from (7) in two particular cases: it follows if relative risk aversion is greater than one and decreasing, or if relative risk aversion is less than one and increasing. But (8) is easily seen to follow from (15), so that Proposition 1 follows. In a similar manner, it is easy to derive Proposition 2 from the Corollary to Proposition 3.

Before continuing further, we should point out that decreasing relative risk aversion is not necessary for Proposition 3 to hold, whereas Gollier and Pratt (1996) provide a condition similar to (8), that together with decreasing absolute risk aversion is necessary and sufficient for (additive) risk vulnerability.[9] The source of this discrepancy is that Gollier and Pratt consider additive background risks with nonpositive means. In particular, their (additive) risk vulnerability is defined as the condition on preferences such that the derived utility function $w(x) \equiv Eu(x + \tilde{\varepsilon})$ is more risk averse than $u(x)$ for any independent additive background risk $\tilde{\varepsilon}$ with $E\tilde{\varepsilon} \leq 0$. The fact that the mean of the background risk can be negative is what requires decreasing absolute risk aversion in their model. In our multiplicative analogue, we could consider multiplicative background

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[9] The additional Gollier and Pratt condition for the additive case, with independent background risk $\tilde{\varepsilon}$, $E\tilde{\varepsilon} = 0$, can be written as $u'(x+\varepsilon)[r_u(x+\varepsilon) - r_u(x)] - \varepsilon u'(x)r_u{}'(x) \geq 0 \ \forall x, \varepsilon$, which is seen to be similar to our condition (8). This condition alone (without assuming decreasing absolute risk aversion) is both necessary and sufficient if we restrict ourselves to zero-mean additive background risks, as the authors point out.

risks $\tilde{y}$ for which $E\tilde{y} \le 1$; that is, all "universally undesirable" multiplicative background risks. In this case, it follows trivially that we would need to add decreasing relative risk aversion to condition (8) to ensure that all of these background risks lead to a more risk-averse behavior.

5. The Affiliated Utility Function

In this section, we obtain additional results by considering $\ln(xy) = \ln x + \ln y$. This allows us to adapt several results from the case of additive background risks. In order to accomplish this, we introduce the affiliated utility function, $\hat{u}$, which we define such that $u(x) = \hat{u}(\ln x)$, for all $x > 0$. Equivalently, we can substitute $\theta = \ln x$ to write $\hat{u}(\theta) \equiv u(e^\theta)$ $\forall \theta \in \mathbb{R}$. In other words, $\hat{u}$ is the composite of u with the exponential function. Although $\hat{u}$ is increasing, it need not be concave. Since $u(xy) = \hat{u}(\ln x + \ln y)$, we will examine the additive risks $\ln \tilde{x} + \ln \tilde{y}$ in this section.

Let $\hat{r}(\theta)$ denote the absolute risk aversion for $\hat{u}(\theta)$, i.e. $\hat{r}(\theta) = -\hat{u}''(\theta)/\hat{u}'(\theta)$. Straightforward calculations show that

(16) $\quad R_u(x) = 1 - \dfrac{\hat{u}''(\ln x)}{\hat{u}'(\ln x)} = 1 + \hat{r}_u(\ln x)$.

Note that $R_u(x) < 1$ implies that $\hat{r}_u(\ln x) < 0$. Thus, if $R_u(x) < 1$ $\forall x < 0$, then $\hat{u}$ exhibits risk-loving behavior and is convex. This is not surprising given the construction of the affiliated utility function.

If u is more concave than the natural logarithm function, $\hat{u}$ will be concave. That is, $\hat{u}$ will be everywhere risk averse if and only if u is everywhere more risk averse

than log utility.[10] If $u(x) = \ln x$, then $\hat{u}$ is risk neutral. Note that $\hat{u}$ does not represent anyone's utility of wealth, however. To refer to $\hat{u}$ as "risk averse, risk loving or risk neutral" is only a technical convenience, since in all cases, we are assuming that true preferences u are risk averse. Still, by examining the nature of $\hat{r}$, we will be able to adapt several existing results on additive background risk to the multiplicative case.

A few examples can help to illustrate the relationship between utility functions and the corresponding affiliated utility functions:

(*i*) If $u(x) = x$, so that preferences are risk neutral, then $\hat{u}(\theta) = e^{\theta}$, which is risk loving with constant absolute risk aversion.

(*ii*) If $u(x) = \frac{1}{1-\gamma} x^{1-\gamma}$, $\gamma > 0$, $\gamma \neq 1$, so that preferences exhibit constant relative risk aversion with $R_u(x) = \gamma$, then $\hat{u}(\theta) = \frac{1}{1-\gamma} e^{(1-\gamma)\theta}$. Note that affiliated utility functions exhibit constant absolute risk aversion of degree γ-1, which is risk averse only if γ>1.

(*iii*)   If $u(x) = x - bx^2$, $b > 0$, $x < \frac{1}{2b}$, so that utility is quadratic, then $\hat{u}(\theta) = e^{\theta} - be^{2\theta}$.

(*iv*)   The above examples are all special cases of HARA utility.   Let $u(x) = \xi(\eta + \frac{x}{\gamma})^{1-\gamma}$, $\eta + \frac{x}{\gamma} > 0$, $\frac{\xi(1-\gamma)}{\gamma} > 0$. Then $\hat{u}(\theta) = \xi\left(\eta + \frac{e^{\theta}}{\gamma}\right)^{1-\gamma}$.

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[10] If we consider background risks for which $E \ln \tilde{y} = 0$, an increase in the riskiness of $\tilde{y}$ will cause the mean of $\tilde{y}$ to increase. Such an increase will represent a mean-utility preserving increase in risk for someone with logarithmic utility. As a result, the change in background risk will be detrimental [beneficial] to someone with $R_u > 1$ [$R_u < 1$]. See Diamond and Stiglitz (1974).

Define $\hat{v}(\ln x) \equiv E\hat{u}(\ln x + \ln \tilde{y})$. From the definition of $v(x)$ in (2) and of $\hat{u}$ above, it follows that $\hat{v}(\ln x) = v(x)$. In a manner analogous to equation (16) we can derive

$$(17) \quad R_v(x) = 1 - \frac{E\hat{u}''(\ln x + \ln \tilde{y})}{E\hat{u}'(\ln x + \ln \tilde{y})} \equiv 1 + \hat{r}_v(\ln x).$$

From (16) and (17), we easily obtain the following result.

Lemma 2: (i) $R_v(x) \geq R_u(x)$ if and only if $\hat{r}_v(\ln x) \geq \hat{r}_u(\ln x)$,

and (ii) $R_t(x)$ is decreasing if and only if $\hat{r}_t(\ln x)$ is decreasing , $t = u,v$.

Equivalent to (i) above, $r_v(x) \geq r_u(x)$ if and only if $\hat{r}_v(\ln x) \geq \hat{r}_u(\ln x)$. For the case where $R_u < 1$, so that $\hat{u}$ is risk loving, we can still interpret $\hat{r}_v > \hat{r}_u$ as meaning "$\hat{v}$ is more risk averse than $\hat{u}$," but in the sense of being less risk loving.

We are now ready to establish an equivalence between the additive risk vulnerability of the affiliated utility $\hat{u}$ and the multiplicative risk vulnerability of u. Consider the set of $\tilde{y} \in \Gamma(\tilde{x})$, so that $E\tilde{y} = 1$. We define $\hat{u}$ as being additively risk vulnerable if $\hat{r}_v(\ln x) \geq \hat{r}_u(\ln x)$ for every x and for every $\tilde{y} \in \Gamma(\tilde{x})$. In other words, $\hat{u}$ is additively risk vulnerable if $\hat{v}(\ln x) \equiv E\hat{u}(\ln x + \ln \tilde{y})$ is more risk averse than $\hat{u}(\ln x)$ for any $\tilde{y}$ with $E\tilde{y} = 1$. Note that, unlike Gollier and Pratt (1996), we do not require that $\hat{u}$ be concave. The fact that risk aversion of $\hat{u}$ is not required becomes important here, since the affiliated utility function $\hat{u}$ is convex whenever $R_u < 1$. In other words, $\hat{u}$ may be additively risk vulnerable even in this case.

From Lemma 2, the following result is immediate:

Proposition 4: *Preferences are multiplicatively risk vulnerable if and only if the affiliated utility function $\hat{u}$ is additively risk vulnerable.*

Using Propositions 3 and 4, we can characterize additive risk vulnerability of $\hat{u}$ by simply translating the inequality in (8) to properties of $\hat{u}$.

Corollary to Proposition 4: *Preferences are multiplicatively risk vulnerable if and only if*
$$(18) \quad \hat{u}'(\ln xy)[\hat{r}_u(\ln xy) - \hat{r}_u(\ln x)] - (y-1)\hat{u}'(\ln x)\hat{r}_u'(\ln x) \geq 0 \quad \forall x, \forall y \in [a,b].$$

If we wish to extend results from the literature on additive background risks to the case of multiplicative ones, we need to relate our setting to that of Gollier and Pratt (1996). In addition to not requiring risk aversion, our definition of additive risk vulnerability differs from Gollier and Pratt in that we restrict ourselves to additive background risks $\ln \tilde{y}$ for which $E\tilde{y} = 1$, whereas Gollier and Pratt consider background risks such that $E \ln \tilde{y} \leq 0$, using our notation.[11] Since $E\tilde{y} = 1 \Rightarrow E \ln \tilde{y} \leq 0$, it follows that $\hat{u}$ will satisfy our condition for additive risk vulnerability, whenever $\hat{u}$ is risk vulnerable in the sense of Gollier and Pratt. In other words, multiplicative risk vulnerability follows whenever $\hat{u}$ is risk vulnerable in the sense of Gollier and Pratt.[12]

Since risk vulnerability, and in particular inequality (18), is not an easy trait to verify, Gollier and Pratt offer us several useful sufficient conditions for risk vulnerability, which they define exclusively for the case where preferences are risk averse. If we restrict utility such that $R_u > 1$, so that the affiliated utility function $\hat{u}$ is risk averse, we

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[11] Perhaps surprisingly, Gollier and Pratt's proof of their necessary and sufficient conditions for their risk vulnerability does not require risk aversion. It requires only that utility be strictly increasing. This might not seem surprising if we note that our proof of Proposition 3 also does not require risk aversion to hold.

[12] This can be seen more formally as follows. Assume, as do Gollier and Pratt, that $\hat{r}_u$ is decreasing. Then their necessary and sufficient condition for $\hat{v}$ to be more risk averse than $\hat{u}$, for all $\tilde{y}$ with $E \ln \tilde{y} \leq 0$ is
$\hat{u}'(\ln xy)[\hat{r}_u(\ln xy) - \hat{r}_u(\ln x)] - (\ln y)\hat{u}'(\ln x)\hat{r}_u'(\ln x) \geq 0 \quad \forall x, \forall y \in [a,b]$. This is equivalent to the condition in footnote 9 above. But since $y - 1 \geq \ln y$ for all y, inequality (18) follows whenever $\hat{r}_u' < 0$.

may apply some of the Gollier and Pratt results to $\hat{u}$. This leads to the following two sufficient conditions on the affiliated utility function $\hat{u}$ to ensure that preferences u are multiplicatively risk vulnerable.

Proposition 5: *Suppose that* $R_u(x) > 1$ $\forall x$. *Then u is multiplicatively risk vulnerable if either*

      (*i*) $\hat{r}_u$ *is decreasing and convex,*

or   (*ii*) $\hat{u}$ *exhibits standard risk aversion (see Kimball, 1993, and below).*

In some instances, we might be able to check the conditions on the affiliated utility function $\hat{u}$ in Proposition 5 directly. However, we typically will find it easier to deal with properties of u directly, rather than properties of $\hat{u}$. We address this issue in the next section.

6. Properties of Utility and Affiliated Utility

In this section, we examine conditions on the utility function u that must hold if its affiliated utility function $\hat{u}$ is additively risk vulnerable. In particular, we first show that $R_u(x)$ is decreasing and convex, whenever $\hat{r}_u(\ln x)$ is decreasing and convex. We then show how there is a close relationship between standard absolute risk aversion of the affiliated utility function $\hat{u}$ and standard relative risk aversion of u.

We have already established in Lemma 2 that $R_u(x)$ is decreasing whenever $\hat{r}_u(\ln x)$ is decreasing. From equation (16), it follows that

(19) $R_u'(x) = \dfrac{1}{x} \ \hat{r}_u'(\ln x)$

and

$$(20) \quad R_u''(x) = \frac{1}{x^2}[\hat{r}_u''(\ln x) - \hat{r}_u'(\ln x)].$$

If $\hat{r}_u(\ln x)$ is decreasing and convex, it follows from equation (20) that $R_u(x)$ is also convex. As a consequence, the conditions holding in Proposition 5(i) imply those of Proposition 1(i), so that Proposition 1 also might be thought of as a corollary to Proposition 5(i).

The property of standard risk aversion, as presented in Kimball (1993), has become an integral part of the literature on behavior under uncertainty as based upon the expected-utility paradigm. It is especially useful since it is easily characterized by decreasing absolute risk aversion and decreasing absolute prudence, where absolute prudence is measured as $p(x) = -\dfrac{u'''(x)}{u''(x)}$. If $u'''(x) > 0$, preferences are said to be prudent. If the affiliated utility function is standard risk averse, which by definition implies that it must be risk averse, we may apply Proposition 5(ii) to conclude that v is more risk averse than u.[13]

We first obtain a preliminary result that will prove useful. Straightforward calculations show that

$$(22) \quad R_u'(x) = \frac{-u''(x)u'(x) - xu'''(x)u'(x) + x[u''(x)]^2}{[u'(x)]^2} = r_u(x)[1 - P_u(x) + R_u(x)],$$

where $P_u(x) \equiv \dfrac{-xu'''(x)}{u''(x)}$ denotes the measure of relative prudence. Consequently, we directly obtain the following result.

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[13] Further properties of standard risk aversion as well as a discussion of much of the literature applying this property can be found in Gollier (2001).

Lemma 3: $R_u'(x) \gtreqqless 0$ *if and only if* $P_u(x) \lesseqqgtr 1 + R_u(x)$.

We already know that the affiliated utility function $\hat{u}$ is risk averse whenever $R_u(x) > 1$. Lemma 4 shows a condition on the underlying preferences that is equivalent to the prudence of $\hat{u}$.

Lemma 4: *The affiliated utility function $\hat{u}$ exhibits prudence, $\hat{u}'''(\theta) > 0$ $\forall \theta$, if and only if $P_u(x) > 3 - \dfrac{1}{R_u(x)}$.*

<u>Proof</u>: Recall that $\hat{u}(\ln x) = u(x)$, so that we obtain the following by differentiating with respect to $\ln x$:

$$\hat{u}'(\ln x) = xu'(x)$$
$$\hat{u}''(\ln x) = xu'(x) + x^2 u''(x)$$
$$\hat{u}'''(\ln x) = xu'(x) + 3x^2 u''(x) + x^3 u'''(x).$$

Thus, dividing $\hat{u}'''(\ln x)$ by $-x^2 u''(x) > 0$ we obtain

$$\hat{u}'''(\ln x) > 0 \quad \Leftrightarrow \quad -\frac{xu'''(x)}{u''(x)} - 3 - \frac{u'(x)}{xu''(x)} > 0 \quad \Leftrightarrow \quad P_u(x) > 3 - \frac{1}{R_u(x)}.$$

$$\blacksquare$$

From Lemmata 3 and 4, we can easily now show the following.

Lemma 5: *If u exhibits decreasing relative risk aversion, the affiliated utility function $\hat{u}$ exhibits prudence.*

<u>Proof</u>: From Lemmata 3 and 4, the conclusion follows if $1 + R_u(x) \geq 3 - [R_u(x)]^{-1}$. Since $R_u(x)$ is positive, this is equivalent to $\{[R_u(x)]^2 - 2R_u(x) + 1\} = [R_u(x) - 1]^2 \geq 0$, which obviously holds. ■

We can use the derivatives in the proof of Lemma 4 to calculate the measure of absolute prudence for the affiliated utility function. In particular, we obtain

$$(23) \qquad \hat{p}(\ln x) \equiv -\frac{\hat{u}'''(\ln x)}{\hat{u}''(\ln x)} = -\frac{x^2 u'''(x) + 2x u''(x)}{x u''(x) + u'(x)} - 1 = \frac{P_u(x) - 2}{1 - (R_u(x))^{-1}} - 1,$$

where the last step follows from dividing both the numerator and denominator in (23) by $xu''(x)$.

We are now ready to prove that standard relative risk aversion of u is a necessary condition for $\hat{u}$ to be standard:

<u>*Proposition 6*</u>: *Suppose that $\hat{u}$ exhibits standard risk aversion. Then $R_u(x) > 1$ and u exhibits standard relative risk aversion; that is, both $P_u(x)$ and $R_u(x)$ are positive and decreasing.*

<u>Proof</u>: From equation (16), we know that $\hat{u}$ risk averse implies that $R_u(x) > 1$. Since $\hat{u}$ exhibits decreasing absolute risk aversion, it follows from Lemma 2 that u exhibits decreasing relative risk aversion. Thus, we must show that u also exhibits positive and decreasing relative prudence. That relative prudence is positive follows easily from Lemma 3.

Differentiating equation (23) with respect to $\ln x$ we obtain

$$\frac{d\hat{p}(\ln x)}{d \ln x} = \frac{x[1 - (R_u(x))^{-1}]P_u'(x) - x[P_u(x) - 2](R_u(x))^{-2} R_u'(x)}{[1 - (R_u(x))^{-1}]^2}.$$

Because $R_u(x) > 1$, it follows that $[R_u(x)]^2 - R_u(x) > 0$ and, from Lemma 3, that $P_u(x) - 2 > 0$. Thus, it follows that $\dfrac{d\hat{p}(\ln x)}{d\ln x}$ is negative if and only if

$$(24) \qquad P_u'(x) < \frac{P_u(x) - 2}{[R_u(x)]^2 - R_u(x)} R_u'(x) < 0. \quad \blacksquare$$

From the proof of Proposition 6, we see that u exhibiting standard relative risk aversion is necessary, but not quite sufficient to imply that the affiliated utility function $\hat{u}$ is standard risk averse. However, we do obtain the following result.

Corollary to Proposition 6: Let $R_u(x) > 1$. _If u exhibits standard relative risk aversion and the inequality in (24) holds, then the affiliated utility function $\hat{u}$ is standard risk averse._

Proof: Since $R_u(x) > 1$, it follows from equation (16) that $\hat{r}(\theta) > 0$. Standard relative risk aversion of u implies, from Lemma 2, that $\hat{u}$ exhibits decreasing absolute risk aversion. It also follows, from Lemma 5, that $\hat{u}''' > 0$. Since (24) holding implies that $\hat{u}$ also exhibits decreasing absolute prudence, the Corollary follows. $\blacksquare$

From Proposition 6 and its Corollary, it follows that whenever condition (24) holds, the following two conditions are equivalent:

(_i_) Utility u is standard relative risk averse with $R_u > 1$, and

(_ii_) Affiliated utility $\hat{u}$ is standard risk averse.

Since (24) might seem a bit opaque, we provide an illustration of a case where it applies in the following example.

<u>Example</u>: Let u belong to the HARA class of utility functions, $u(x) = \xi(\eta + \frac{x}{\gamma})^{1-\gamma}$ and

suppose that $\gamma > 1$. Now $R_u(x) = \dfrac{x}{\eta + \frac{1}{\gamma}x}$. Thus, it follows easily that u exhibits

decreasing relative risk aversion if and only if $\eta < 0$. Hence, $x > \eta + \frac{1}{\gamma}x$, so that

$R_u(x) > 1$. To see that u exhibits standard relative risk aversion, note that

$P_u(x) = \frac{1+\gamma}{\gamma} R_u(x)$. Thus, u exhibits decreasing relative prudence if and only if u exhibits

decreasing relative risk aversion. Thus, u is standard relative risk averse and $R_u(x) > 1$.

We now wish to show that $\hat{u}$ is standard risk averse.

By the Corollary to Proposition 6, we would be done if the inequality in (24)

holds. Since both $P_u'(x) < 0$ and $R_u'(x) < 0$, inequality (24) is equivalent to

$$\frac{1+\gamma}{\gamma} > \frac{P_u(x)-2}{R_u(x)[R_u(x)-1]} = \frac{(\frac{1+\gamma}{\gamma})R_u(x)-2}{R_u(x)[R_u(x)-1]}$$

$$\Leftrightarrow \quad R_u(x)[R_u(x)-1] > R_u(x) - 2(\tfrac{\gamma}{1+\gamma})$$

$$\Leftrightarrow \quad [R_u(x)-1]^2 > \frac{1-\gamma}{1+\gamma}.$$

This last inequality follows, since $\gamma > 1$. Hence, $\hat{u}$ is standard risk averse. It follows from

Proposition 5(*ii*) that the derived utility function v is more risk averse than u.

If under HARA preferences we assume the property that $R_u(x) > 1 \ \forall x \in (0, \infty)$, it

follows that $\gamma > 1$ must hold. Hence, the additional assumption of standard relative risk

aversion of u would be both necessary and sufficient to guarantee standard (absolute) risk

aversion of $\hat{u}$.

7. Comparative Risk Aversion

A key result in the literature on additive background risk is that the properties of constant absolute risk aversion and decreasing absolute risk aversion for utility are carried over to the derived utility function. On the other hand, the property of increasing absolute risk aversion does not always carry over. In this section we show the analogous results for relative risk aversion in the case of a multiplicative background risk. We have already seen that v inherits constant relative risk aversion from u. Indeed, the level of constant risk aversion is identical. To see that the same holds true for decreasing relative risk aversion, we can apply the Diffidence Theorem once again. It is important to note that $E\tilde{y} = 1$ is not required in Proposition 7.

Proposition 7: Let $\tilde{y}$ have a bounded support contained in [a,b]. If u exhibits nonincreasing relative risk aversion, then so does the derived utility function $v(x) \equiv Eu(x\tilde{y})$.

Proof: It follows from Lemma 3, that we need to show that, $\forall x$,

(25) $P_u(x) \geq 1 + R_u(x) \implies P_v(x) \geq 1 + R_v(x)$.

That is, we must show that

(26) $\dfrac{-Eu'''(x\tilde{y})\tilde{y}^3 x}{Eu''(x\tilde{y})\tilde{y}^2} \geq \dfrac{-Eu''(x\tilde{y})\tilde{y}^2 x}{Eu'(x\tilde{y})\tilde{y}} + 1$.

Inequality (26) is equivalent to the following, where λ denotes the value of the right-hand side in (26):

(27) $E[u''(x\tilde{y})\tilde{y}^2 x + (\lambda-1)u'(x\tilde{y})\tilde{y}] = 0 \implies E[u'''(x\tilde{y})\tilde{y}^3 x + \lambda u''(x\tilde{y})\tilde{y}^2] \geq 0$.

By the Diffidence Theorem, (27) will hold if we can find a real number m, such that

(28) $u'''(xy)y^3x + \lambda u''(xy)y^2 \geq m[u''(xy)y^2x + (\lambda - 1)u'(xy)y] \quad \forall y \in [a,b]$.

The left-hand side of (28) can be written as

(29) $\dfrac{xyu''(xy)}{u'(xy)} \dfrac{u'(xy)y}{x}\left[\dfrac{xyu'''(xy)}{u''(xy)} + \lambda\right] = -R_u(xy)\dfrac{u'(xy)y}{x}\left[\lambda - P_u(xy)\right]$.

Since $P_u(x) \geq 1 + R_u(x)$, it follows from (28) and (29) that

(30) $u'''(xy)y^3x + \lambda u''(xy)y^2 \geq -R_u(xy)\dfrac{u'(xy)y}{x}\left[\lambda - 1 - R_u(xy)\right]$.

From (28) and (22), we would be done if we could find an m, such that

(31) $-R_u(xy)\dfrac{u'(xy)y}{x}\left[\lambda - 1 - R_u(xy)\right] \geq m[u''(xy)y^2x + (\lambda - 1)u'(xy)y]$

$= mu'(xy)y[\lambda - 1 - R_u(xy)]$.

This follows by taking $m = (1-\lambda)/x$, since we then obtain (31) is equivalent to

(32) $-R_u(xy)\left[\lambda - 1 - R_u(xy)\right] + (\lambda - 1)[\lambda - 1 - R_u(xy)] = [\lambda - 1 - R_u(xy)]^2 \geq 0$.

Hence, (25) holds and v exhibits decreasing relative risk aversion. ∎

We next turn briefly to examining some interpersonal characteristics of comparative risk aversion. Kihlstrom, et al. (1981) and Ross (1981) examined these for the case of an additive background risk.[14] Their results are special cases of more general

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[14] Actually, Ross considers the background risk to be mean-independent, which is not as restrictive as the assumption of independence.

results found in Nachman (1982). Nachman is one of the few who considers the case of multiplicative background risks as a special case of his general results, albeit briefly. The basic question we address is the following: If agent 1 is more risk averse than agent 2, will this property be preserved in the presence of a multiplicative background risk? That is, if u_1 is more risk averse than u_2, when will it follow that v_1 is also more risk averse than v_2? One result that is quite easy to obtain is the following:

Proposition 8: Let u^a and u^b be risk-averse utility functions such that u^a is more risk averse than u^b, i.e. $R_u^a(x) \geq R_u^b(x) \quad \forall x$. If $\exists \lambda \in \mathbb{R}$ such that $\forall x \quad R_u^a(x) \geq \lambda \geq R_u^b(x)$, then v^a is more risk averse than v^b.

Proof: Follows directly from equation (4). ∎

The proof of Proposition 8 also follows directly from the following more general result, which is due to Nachman (1982). We include the trivial proof above mostly to illustrate how our model can also generate these types of results. We present Nachman's result below for the sake of completeness.

Proposition (Nachman): Let u^a and u^b be risk-averse utility functions such that u^a is more risk averse than u^b, i.e. $R_u^a(x) \geq R_u^b(x) \quad \forall x$. If there exists a function u^c such that $R_u^a(x) \geq R_u^c(x) \geq R_u^b(x) \quad \forall x$ and $R_u^c(x)$ is nonincreasing, then v^a is more risk averse than v^b.

It follows easily from Nachman's result that v^a will be more risk averse than v^b if either of the utility functions, u^a or u^b, exhibits nonincreasing relative risk aversion. This result is a direct counterpart to the result by Kihlstrom, et al. in the case of additive background risk.

8. Concluding Remarks

The notion that markets are complete is a mathematical nicety that does not hold true in practice. Many types of political, human-capital and social risks, as well as some financial risks, are not represented by direct contracts. Obviously, many of these risks might be hedged indirectly via so-called "cross hedging." However, even when such "background risks" are independent of other risks and cannot be "hedged" per se, they still may have an impact upon risk-taking strategies that are within the control of the economic agent. Much has been done over the past twenty years in examining the effects of additive background risks. But surprisingly little has been done to systematically study economic decision making in the presence of a multiplicative background risk.

This paper is a first step towards developing a comprehensive theory of background risk in this direction. As the examples in our introduction illustrate, models with such multiplicative background risks are not hard to find within the literature. Whereas properties of absolute risk aversion play a key role in analyzing the effects of an additive background risk, properties of relative risk aversion are more important in examining behavior in the presence of a multiplicative background risk. However, results for the case of a multiplicative background risk do not simply "mirror" those for the case where the background risk is additive. An understanding of the basic concepts presented here hopefully might help us understand a multitude of results for which standard theories (in the absence of any background risk) yield predictions that seem at odds with everyday observations of reality.

Since risk aversion captures all the essential information about preferences within an expected-utility framework, our focus here has been on comparing risk aversion with and without a multiplicative background risk. As we learn more about these inherent properties, we hopefully will be able to find better models to use in the realm of positive theories.

References

Aït-Sahalia, Y. and A.W. Lo, (2000), "Nonparametric Risk Management and Implied Risk Aversion," *Journal of Econometrics* 94, 9-51.

Arrow, K.J., (1971), *Essays in the Theory of Risk Bearing*, Chicago: Markum Publishing Company.

Diamond, P. and J. Stiglitz, (1974), "Increases in Risk and in Risk Aversion," *Journal of Economic Theory* 8, 337-360.

Doherty, N. A., and H. Schlesinger, (1983), "Optimal Insurance in Incomplete Markets," *Journal of Political Economy*, 91, 1045-1054.

Eeckhoudt, L. and M.S. Kimball, (1992), "Background Risk, Prudence and the Demand for Insurance," in: Dionne, G. (Ed.), *Contributions to Insurance Economics*, Boston: Kluwer Academic Publishers, pp. 239-254.

Gollier, C., (2001), *The Economics of Risk and Time*, Cambridge: MIT Press.

Gollier, C. and M. Kimball (1996), "Towards a Systematic Approach to the Economics of Uncertainty: Characterizing Utility Functions," unpublished working paper, University of Michigan.

Gollier, C., and J. W. Pratt, (1996), "Risk Vulnerability and the Tempering Effect of Background Risk," *Econometrica*, 64, 1109-1124.

Guiso, L., T. Japelli and D. Terlizzese, (1996), "Income Risk, Borrowing Constraints and Portfolio Choice," *American Economic Review* 86, 158-172.

Guiso, L. and M. Paiella, (2001), "Risk Aversion, Wealth and Financial Market Imperfections," CEPR Discussion Paper No. 2728.

Jackwerth, J.C., (2000), "Recovering Risk Aversion from Option Prices and Realized Returns," *Review of Financial Studies* 13, 433-451.

Kihlstrom, R., D. Romer and S. Williams, (1981), "Risk Aversion with Random Initial Wealth," *Econometrica*, 49, 911-920.

Kimball, M.S., (1993), "Standard Risk Aversion," *Econometrica*, 61, 589-611.

Nachman, D., (1982), "Preservation of 'More Risk Averse' under Expectations," *Journal of Economic Theory*, 28, 361-368.

Ogaki, M. and Q. Zhiang, (2001), "Decreasing Relative Risk Aversion and Tests of Risk Sharing," *Econometrica* 69, 515-526.

Pratt, J.W., (1964), "Risk Aversion in the Small and in the Large," *Econometrica* 32, 122-136.

Pratt, J. W. and R. Zeckhauser, (1987), "Proper Risk Aversion," *Econometrica*, 55, 143-154.

Ross, S. A., (1981), "Some Stronger Measures of Risk Aversion in the Small and in the Large with Applications," *Econometrica* 49, 621-638.

Rothschild, M., and J. Stiglitz, (1970), "Increasing Risk: I. A Definition," *Journal of Economic Theory* 2, 225-243.