

# Hydrodynamics from kinetic models of conservative economies

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**Abstract.** In this paper, we introduce and discuss the passage to hydrodynamic equations for kinetic models of conservative economies, in which the density of wealth depends on additional parameters, like the propensity to invest. As in kinetic theory of rarefied gases, the closure depends on the knowledge of the homogeneous steady wealth distribution (the Maxwellian) of the underlying kinetic model. The collision operator used here is the Fokker-Planck operator introduced by J.P. Bouchaud and M. Mezard in [4], which has been recently obtained in a suitable asymptotic of a Boltzmann-like model involving both exchanges between agents and speculative trading by S. Cordier, L. Pareschi and one of the authors [11]. Numerical simulations on the fluid equations are then proposed and analyzed for various laws of variation of the propensity.

**Keywords.** Wealth and income distributions, Boltzmann equation, hydrodynamics, Euler equations

## 1 Introduction

In recent years, the study of the evolution of the distribution of wealth in a simple market economy has often been faced by means of methods borrowed from the kinetic theory of rarefied gases [15, 12, 7, 6, 14, 24, 10, 11]. In most of the underlying kinetic models of Boltzmann type the market is represented like an ideal gas, where each molecule is identified with an agent, and each trading event between two agents is considered to be an elastic or money conserving collision between two molecules. The founding idea is that a trading market composed by a sufficiently large number of agents can be described using the laws of statistical mechanics as it happens in a physical system composed of many interacting particles. If one agrees with the claim that there are deep analogies between economics and physics, then various well established physical methods could be applied in analyzing wealth distributions in economies. In particular, by identifying money in a closed economy with energy, the application of statistical physics methods makes it possible to understand better the

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development of tails in wealth distributions of real economies. In kinetic models of simple market economies, in fact, the knowledge of the large-wealth behavior of the steady state density is of primary importance, since it determines *a posteriori* if the model fits data of real economies. By identifying wealth with energy makes it clear that the problem of the description of the large-time behavior of the wealth in a kinetic model of the type considered in [15, 12, 7, 6, 14, 24, 10, 11] is the analogue of the problem of the description of the large-time behavior of the density in the spatially homogeneous Boltzmann equation, both in the elastic [5] and inelastic cases [2, 3]. In particular, for nonconservative kinetic models, this analogy has been recently outlined in [20]. As a matter of fact, in statistical physics, the knowledge of the large time behavior of the density of energy, and its approach to a universal profile are subsequently used to construct hydrodynamic equations, which describe the space-time evolution of macroscopic observables. A similar procedure could be in principle applied to the evolution of the density of wealth, in case this density depends on other important parameters (typically the space variable in a physical system).

In what follows, we will try to answer this question, by assuming that each agent is identified by two main variables, the first given by his wealth, the second by his propensity to invest (in the hope of making a profit). As in a physical system, where each particle is identified by its position and velocity, and the position depends on the velocity through the classical equations of motion, we will assume that the propensity still depends on the wealth through a suitable *equation of motion*. This relationship between propensity and wealth is largely formal, and could be modified in many ways. Nevertheless, it is quite interesting to observe how the behavior of the hydrodynamic equations we obtained depends on the assumption of this *equation of motion*.

We have been interested in this problem after the reading of a recent paper of Y. Wang, N. Ding, L. Zhang [28], who discussed the concept of the statistical description of the velocity of money circulation. This concept is based on holding time of money which is defined as time interval between two transactions. Although this concept is kept in mind when economists think of the velocity, even the term referring to this kind of time interval has been mentioned in several cases, it is somewhat new to them since there has been no explicit specification of it in economics. While there exists a similar term in physics which is measured by the Knudsen number, namely the mean free time between two subsequent collisions of a molecule [5]. Recently, several efforts have been devoted to measure the waiting time distributions in financial markets, see e.g. [17, 22]. In the process of money circulation, not only the amount of money each agent holds can be considered as random variable, but also the holding time between two transactions varies randomly. The theoretical investigation and the numerical simulations in [28] led to the conclusion that the velocity of money is proportional to the share for exchange, and, most important, reversely proportional to number of agents, and independent of the average amount of money.

Using this result in a kinetic model of Boltzmann type, shows that the velocity of relaxation to the steady distribution of wealth is inversely proportional to

the velocity of money circulation, which justifies an hydrodynamical description when the same velocity is sufficiently high.

The paper is organized as follows. In the next section, we will briefly introduce a non-homogeneous Boltzmann type equation, in which the density of agents depends both on wealth and propensity. The equation is described by a coupling of *transport* and collisions, where the collision operator is described in terms of a Fokker-Planck type collision operator first obtained by J.P. Bouchaud and M. Mézard [4] from probabilistic arguments and subsequently deduced by S. Cordier, L. Pareschi and one of the authors [11] from a Boltzmann model involving both exchanges between agents and speculative trading. The stability properties of the homogeneous steady state are subsequently dealt with in Section 3, together with the relevant closure relations. The universal validity of these closure relations is subsequently discussed in Section 4. The equation of motion linking wealth and saving propensity, and the consequent fluid dynamical equations are introduced in Section 5. Finally, Section 6 deals with numerical simulation and comments.

## 2 Inhomogeneous kinetic models for the evolution of wealth

As briefly discussed in the introduction, the study of the time-evolution of the wealth distribution among individuals in a simple economy and the explanation of the formation of tails in this distribution has been achieved by means of kinetic collision-like models. Although the approaches are different they seem to share some common features. Almost all of these models identify a very important variable for the shape of the wealth distribution, which is usually called the saving propensity to trade or the saving rate, respectively. This parameter can both enter into the collision rule as a constant factor [7], or it can be chosen as a random quantity [8]. Other studies include the saving propensity as an independent variable [9], without questioning on the relationship between wealth and saving. The previous approaches to study both wealth and saving distributions show that in any case it could be reasonable to introduce other types of propensities into the game, which are not directly connected to the microscopic binary trade, while they could be important in the evolution of wealth in a market of agents. Among others, one can assume that the evolution of the density of wealth is heavily dependent on the propensity to invest, and at the same time that this propensity is closely related to the amount of money one agents has to deal with. In this case, we are led to study the evolution of the distribution function as a function depending on the propensity  $x \in [0, 1]$ , wealth  $w \in \mathbb{R}_+$  and time  $t \in \mathbb{R}_+$ ,  $f = f(x, w, t)$ . In analogy with the classical kinetic theory of rarefied gases, we emphasize the role of the different parameters by identifying the velocity with the wealth, and the position with the saving propensity. By doing this, we assume at once that the variation of the distribution  $f(x, w, t)$  with respect to the wealth parameter  $w$  will depend

on *collisions* between agents, while the change of distributions in terms of the propensity  $x$  depends on the *transport* term, which contains the equation of motion, namely the law of variation of  $x$  with respect to time,

$$\frac{dx}{dt} = \Phi(x, w). \quad (2.1)$$

The time-evolution of the distribution will obey a non-homogenous Boltzmann-like equation, given by

$$\frac{\partial}{\partial t} f(x, w, t) + \Phi(x, w) \frac{\partial}{\partial x} f(x, w, t) = \frac{1}{\tau} \mathcal{Q}(f)(x, w, t). \quad (2.2)$$

In (2.2)  $\Phi$  is the law of variation of the propensity to invest given in (2.1), while  $\mathcal{Q}$  represents the collision operator which describes the change of density due to binary trades. Finally  $\tau$  is a suitable relaxation time, depending on the velocity of money circulation [28]. Note that in physical applications where no forces are present, the transport term is simply  $\Phi(x, w) = w$ .

The goal of a kinetic model of a simple market economy is to describe the evolution of the distribution of wealth by means of *microscopic* interactions among agents or individuals which perform exchange of money. Each trade can indeed be interpreted as an interaction where a fraction of the money changes hands. One generally assumes that this wealth after the interaction is non negative, which corresponds to impose that no debts are allowed. This rule emphasizes the difference between economic interactions, where not all outcomes are permitted, and the classical interactions between molecules. In any trading, savings come naturally [23]. In a real society or economy, the saving propensity is a very inhomogeneous parameter, and the interest of saving varies from person to person, according to their wealths. To move a step closer to the real situation, one has to introduce a saving factor widely distributed within the population [8, 9], and responsible of different outcomes into binary trades. The evolution of money in such a trading can be written as [9]

$$\begin{aligned} v^* &= \gamma v + \epsilon(\gamma, \mu) [(1 - \gamma)v + (1 - \mu)w], \\ w^* &= \mu w + (1 - \epsilon(\gamma, \mu)) [(1 - \gamma)v + (1 - \mu)w]. \end{aligned} \quad (2.3)$$

Here  $(\gamma, w)$  and  $(\mu, v)$  denote the saving propensities and wealths of agents before collisions. In a single collision it is assumed that the agents maintain their saving propensities fixed, so that the post-collision parameters are  $(\gamma, w^*)$  and  $(\mu, v^*)$ . Moreover  $\epsilon(\gamma, \mu)$  denotes a random fraction, coming from the stochastic nature of the trading. A slightly different mechanism was considered in [11]. Here the trade between two agents has been described by

$$\begin{aligned} v^* &= (1 - \gamma)v + \gamma w + \eta v, \\ w^* &= \gamma v + (1 - \gamma)w + \tilde{\eta} w. \end{aligned} \quad (2.4)$$

In (2.4) the trade depends on a single saving rate  $\gamma \in (0, 1)$ , while the risks of the market are described by  $\eta$  and  $\tilde{\eta}$ , equally distributed random variables with zero mean and variance  $\sigma^2$ .

A Boltzmann-like collision operator can be easily derived by standard methods of kinetic theory, considering that the change in time of  $f(x, w, t)$  due to binary trades depends on a balance between the gain and loss of agents with wealth  $w$ . This operator reads

$$\mathcal{Q}(w) = \left\langle \int_0^1 dy \int_{\mathbb{R}_+} dv \left( \frac{1}{J} f(v_*) f(w_*) - f(v) f(w) \right) \right\rangle. \quad (2.5)$$

In (2.5)  $(v_*, w_*)$  denote the pre-trade pair that produces the post-trade pair  $(v, w)$ , following rules like (2.3) or (2.4), while  $J$  denotes the Jacobian of the transformation of  $(v, w)$  into  $(v^*, w^*)$ . Finally,  $\langle \cdot \rangle$  denotes the operation of mean with respect to possible random quantities (like  $\epsilon(\gamma, \mu)$  or  $\eta, \tilde{\eta}$ ). A useful way of writing the collision operator (2.5), that allows to avoid the Jacobian, is the so-called weak form. It corresponds to consider, for all smooth functions  $\phi(w)$ ,

$$\begin{aligned} & \int_{\mathbb{R}_+} \mathcal{Q}(f, f)(w) \phi(w) dw = \\ & \frac{1}{2} \left\langle \int_0^1 dy \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} dv dw (\phi(v^*) + \phi(w^*) - \phi(v) - \phi(w)) f(y, v) f(x, w) \right\rangle. \end{aligned} \quad (2.6)$$

Setting  $\phi(w) = 1$  into (2.6) and denoting the mass by

$$\rho(f) = \int_{\mathbb{R}_+} f(w) dw$$

implies

$$\frac{d\rho(f)}{dt} = \int_{\mathbb{R}_+} \mathcal{Q}(f, f)(w) dw = 0.$$

Likewise, if there is pointwise conservation of wealth in each binary trade (like in (2.3))

$$v^* + w^* = v + w,$$

or, more generally, conservation in mean (like in (2.4))

$$\langle v^* + w^* \rangle = v + w,$$

the same conservation reflects to (2.6). Choosing  $\phi(w) = w$ , and denoting the mean wealth by

$$m(f) = \frac{1}{\rho(f)} \int_{\mathbb{R}_+} w f(w) dw$$

we obtain

$$\frac{dm(f)}{dt} = \int_{\mathbb{R}_+} \mathcal{Q}(f, f)(x, w) w dw = 0.$$

In the continuous trading limit ( $\gamma \rightarrow 0, \sigma^2/\gamma \rightarrow \lambda$ ), it has been shown in [11] that the collision operator (2.5) is well described by the Fokker-Planck collision operator

$$\mathcal{P}(w) = \frac{\lambda}{2} \frac{\partial^2}{\partial w^2} (w^2 f(w)) + \frac{\partial}{\partial w} (w - m(f)) f(w), \quad (2.7)$$

where, as before,  $m(f)$  is the mean wealth of  $f(w)$ . The key parameter  $\lambda$  is obtained as the limit of the quotient of the variance and the saving rate. The same equation has been obtained before by considering the mean-field limit in a trading model described by stochastic differential equations [4]. The homogenous kinetic equation

$$\frac{\partial}{\partial t} f(w, t) = \mathcal{P}(w, t), \quad (2.8)$$

is such that both the mass and the mean wealth  $m(f)$  are conserved in time. Moreover, for any initial density  $f(w, t = 0) = f_0(w)$  with mass  $\rho$  and mean  $m$ , equation (2.8) has a unique stationary state, from now on called Maxwellian state  $M_{\rho, m}(w)$  given by

$$M_{\rho, m}(w) = \rho \frac{((\mu - 1)m)^\mu}{\Gamma(\mu - 1)} \frac{1}{w^{1+\mu}} \exp\left(-\frac{(\mu - 1)m}{w}\right), \quad (2.9)$$

where

$$\mu = 1 + \frac{2}{\lambda} > 1.$$

Therefore the Maxwellian distribution exhibits a Pareto power law tail for large  $w$ 's. In particular, higher moments of the equilibrium Maxwellian are given in terms of mass  $\rho$  and mean  $m$ . The second moment can be easily evaluated considering that in equilibrium, i.e. as  $t \rightarrow \infty$ , one has

$$\begin{aligned} 0 &= \frac{\lambda}{2} \int_{\mathbb{R}_+} w^2 \frac{\partial^2}{\partial w^2} (w^2 M_{\rho, m}(w)) dw + \int_{\mathbb{R}_+} w^2 \frac{\partial}{\partial w} [M_{\rho, m}(w)(w - m)] dw \\ &= \lambda \int_{\mathbb{R}_+} w^2 M_{\rho, m}(w) dw - 2 \int_{\mathbb{R}_+} w(w - m) M_{\rho, m}(w) dw \\ &= (\lambda - 2) \int_{\mathbb{R}_+} w^2 M_{\rho, m}(w) dw + 2m \int_{\mathbb{R}_+} w M_{\rho, m}(w) dw \\ &= (\lambda - 2) \int_{\mathbb{R}_+} w^2 M_{\rho, m}(w) dw + 2\rho m^2. \end{aligned}$$

Thus, if  $\lambda < 2$ , the second moment of the Maxwellian is bounded, and

$$\int_{\mathbb{R}_+} w^2 M_{\rho, m}(w) dw = \frac{2}{2 - \lambda} \rho m^2. \quad (2.10)$$

In what follows, we will assume that in a closed economy the Maxwellian distribution  $M_{\rho, m}$ , equilibrium solution of the Fokker-Planck equation (2.8), plays the same role as played by the Maxwell distribution in kinetic theory of rarefied gases. However, on the contrary to what happens in classical kinetic theory,

where the equilibrium Maxwellian has all moments bounded, in this case the number of moments bounded in the equilibrium depends on the parameter  $\lambda$  in front of the second-order term in (2.7). Formation of tails and the stability properties of tailed equilibria have been recently studied in [20] in terms of a Fourier-based metric. We refer to [20] for details and references to this topic.

### 3 The passage to hydrodynamic equations

#### 3.1 The Euler equations

The discussion of the previous section enlightened the main properties of the collision operator (2.7), like the existence of a unique Maxwellian equilibrium with tails, and the consequent possibility to obtain higher order moments from the first two (mass and mean wealth). Like in classical kinetic theory of rarefied gases, these properties are the basis of the construction of a reasonable hydrodynamics for the evolution of the propensity. The underlying kinetic model is obtained by substituting the Fokker-Planck operator into the Boltzmann equation (2.2)

$$\frac{\partial}{\partial t} f(x, w, t) + \Phi(x, w) \frac{\partial}{\partial x} f(x, w, t) = \frac{1}{\tau} \mathcal{P}(f)(x, w, t). \quad (3.1)$$

The  $\tau$ -parameter (the analogous of the Knudsen number) represents a suitable relaxation time, and has to be assumed small in fluid dynamical regimes. A clear understanding of the derivation of macroscopic equations in kinetic theory can be obtained through the use of the splitting method, very popular in the numerical approach to the Boltzmann equation [13, 19]. If at each time step we consider sequentially the transport and relaxation operators in the Boltzmann equation (3.1), during this short time interval we recover the evolution of the density from the joint action of the relaxation

$$\frac{\partial f}{\partial t} = \frac{1}{\tau} \mathcal{P}(x, w, t), \quad (3.2)$$

and transport

$$\frac{\partial f}{\partial t} + \Phi(x, w) \frac{\partial}{\partial x} f(x, w, t) = 0. \quad (3.3)$$

As in classical kinetic theory, where the energy is conserved in collisions, the conservation of the mean wealth in the relaxation step is enough to guarantee that (3.2) pushes the solution towards the Maxwellian equilibrium with the same mass and mean of the initial datum. Then, if  $\tau$  is sufficiently small, one can easily argue that the solution to (3.2) is *sufficiently close* to the Maxwellian, and this Maxwellian can be used into the transport step (3.3) to close the equations. In more details, since the Fokker-Planck operator (2.7) is both mass and momentum preserving, integrating equation (3.1) with respect to the wealth velocity  $w$ , using as test functions  $\phi(w) = 1, w$  respectively we obtain

$$\int_{\mathbb{R}_+} \left( \frac{\partial f}{\partial t} + \Phi(x, w) \frac{\partial}{\partial x} f(x, w, t) \right) dw = 0, \quad (3.4)$$

and

$$\int_{\mathbb{R}_+} w \left( \frac{\partial f}{\partial t} + \Phi(x, w) \frac{\partial}{\partial x} f(x, w, t) \right) dw = 0, \quad (3.5)$$

Let us fix the law  $\Phi$  to be linearly dependent on  $w$ ,

$$\Phi(x, w) = (w - \chi \bar{w}) \mu(x), \quad (3.6)$$

where  $\chi$  is a positive constant and  $\bar{w}$  represent a suitable fixed value of the wealth. Then, we obtain from (3.4), (3.5) the equations

$$\frac{\partial \rho}{\partial t} + \mu(x) \frac{\partial}{\partial x} [\rho(m - \chi \bar{w})] = 0, \quad (3.7)$$

$$\frac{\partial(\rho m)}{\partial t} + \mu(x) \frac{\partial}{\partial x} \left[ \int_{\mathbb{R}_+} w^2 f(x, w, t) dw - \chi \bar{w} \rho m \right] = 0. \quad (3.8)$$

In (3.7), (3.8) we defined as macroscopic variables, the local density of agents with propensity  $x$  at time  $t$ , given by

$$\rho(x, t) = \int_{\mathbb{R}_+} f(x, w, t) dw, \quad (3.9)$$

and the local mean

$$m(x, t) = \frac{1}{\rho(x, t)} \int_{\mathbb{R}_+} w f(x, w, t) dw. \quad (3.10)$$

Equation (3.8) depends on the second moment of the density. Using the equilibrium Maxwellian (2.10), however, we can express this second moment in terms of the first two. By this relationship we finally obtain the following system of equations

$$\frac{\partial \rho}{\partial t} + \mu(x) \frac{\partial}{\partial x} [\rho(m - \chi \bar{w})] = 0, \quad (3.11)$$

$$\frac{\partial(\rho m)}{\partial t} + \mu(x) \frac{\partial}{\partial x} \left[ \rho m \left( \frac{2}{2 - \lambda} m - \chi \bar{w} \right) \right] = 0, \quad (3.12)$$

which have to be solved on  $(0, 1) \times (0, T)$  with appropriate boundary and initial conditions. Using (3.11) we can rewrite the second equation as

$$\frac{\partial m}{\partial t} + \mu(x)(m - \chi \bar{w}) \frac{\partial m}{\partial x} + \frac{\lambda}{2 - \lambda} \frac{1}{\rho} \frac{\partial}{\partial x} [\rho m^2] = 0. \quad (3.13)$$

### 3.2 The mathematical structure of Euler equations

In this short section, we will show that the hydrodynamic equations (3.11), (3.12) can be written in symmetric hyperbolic form. Multiplying (3.11) by  $m$  and subtracting it from (3.12) we obtain

$$\begin{aligned} & \rho m_t + \mu(x) \left[ -(\rho_x m + \rho m_x) m \right. \\ & \left. + \chi \bar{w} m \rho_x + (\rho_x m + 2 \rho m_x) \frac{2}{2 - \lambda} m - (\rho_x m + \rho m_x) \chi \bar{w} \right] = 0. \end{aligned} \quad (3.14)$$



Define  $u = (\rho, m)$ . Then a direct computation shows that the system (3.11), (3.14) can be written in compact form as

$$A^0(t, x, u) \frac{\partial u}{\partial t} + A^1(t, x, u) \frac{\partial u}{\partial x} = 0. \quad (3.15)$$

In (3.15)  $A^0$  and  $A^1$  denote the with symmetric matrices

$$A^0(t, x, u) = \begin{pmatrix} \frac{1}{\rho} \frac{\lambda}{2-\lambda} m^2 & 0 \\ 0 & \rho \end{pmatrix}, \quad (3.16)$$

$$A^1(t, x, u) = \begin{pmatrix} \rho(m - \chi \bar{w}) \frac{\lambda}{2-\lambda} m^2 & \frac{\lambda}{2-\lambda} m^2 \\ \frac{\lambda}{2-\lambda} m^2 & (1 + \frac{4}{2-\lambda}) \rho m - \rho \chi \bar{w} \end{pmatrix}. \quad (3.17)$$

Note that, for all  $\rho, m \in G$  belonging to a suitable set  $G$ ,  $A^0(t, x, u)$  is uniformly positive definite provided  $\lambda < 2$ . Due to their structure, suitable numerical methods are available [16].

## 4 Universality of the closure

One of the main advantages linked to the use of the Fokker-Planck collision operator (2.7) is that it is immediate to recover its steady state, namely the Maxwellian (2.9). Unlikely, if we use a different collision operator like the Boltzmann operator (2.5), while it is possible to prove that in case of conservative trades there exists a unique steady state [18], the explicit form of this Maxwellian is unknown. This problem is not present in classical elastic kinetic theory of rarefied gases, where the Maxwellian is uniquely defined independently of the choice of the binary collision operator [5]. This fact causes a first serious problem in the justification of the validity of the closure, which in principle has to be independent of the choice of the underlying microscopic model of collisions, except, eventually, for constant parameters. Using the results of [18], however, we can easily conclude that the closure law

$$\int_{\mathbb{R}_+} w^2 M_{\rho, m}(w) dw = C \rho m^2. \quad (4.1)$$

where  $C = 2/(2 - \lambda)$  in (2.10), has a universal validity, and the type of binary trade used into the collision operator (2.5) is reflected only by the precise value of the constant  $C$ . Following [18], let us suppose that the (conservative) binary interactions are described by the rules

$$v^* = p_1 v + q_1 w, \quad w^* = p_2 v + q_2 w, \quad (4.2)$$

where

$$\langle p_1 + p_2 \rangle = 1, \quad \langle q_1 + q_2 \rangle = 1. \quad (4.3)$$

We remark that both trades (2.3) and (2.4) satisfy assumption (4.3). In this case, application of formula (2.6) with  $\phi(w) = w^n$  allows to compute recursively

the evolution of the principal moments

$$M_n(t) = \int_{\mathbb{R}_+} w^n f(w, t) dw$$

with  $n \geq 2$  (see [18] for details). One obtains

$$\begin{aligned} \frac{d}{dt} M_n(t) &= \frac{1}{2} \langle (p_1^n + p_2^n - 1) + (q_1^n + q_2^n - 1) \rangle M_n(t) + \\ &\frac{1}{2} \sum_{k=1}^{n-1} \binom{n}{k} \langle p_1^k q_1^{n-k} + p_2^k q_2^{n-k} \rangle M_k(t) M_{n-k}(t). \end{aligned} \quad (4.4)$$

Considering that the first moment is conserved,  $M_1(t) = m$ , equation (4.4) also furnishes a recursive computation of the principal moments of the stationary solution

$$M_n = \rho \frac{\sum_{k=1}^{n-1} \binom{n}{k} \langle p_1^k q_1^{n-k} + p_2^k q_2^{n-k} \rangle M_k M_{n-k}}{2 - \langle p_1^n + p_2^n + q_1^n + q_2^n \rangle}. \quad (4.5)$$

In particular,

$$M_2 = \int_{\mathbb{R}_+} w^2 M_{\rho, m}(w) dw = \rho m^2 \frac{2 \langle p_1 q_1 + p_2 q_2 \rangle}{2 - \langle p_1^2 + p_2^2 + q_1^2 + q_2^2 \rangle}, \quad (4.6)$$

that coincides with the law (4.1), in which

$$C = \frac{2 \langle p_1 q_1 + p_2 q_2 \rangle}{2 - \langle p_1^2 + p_2^2 + q_1^2 + q_2^2 \rangle}. \quad (4.7)$$

If we consider the trade (2.4), where

$$\begin{aligned} p_1 &= 1 - \gamma + \eta, & q_1 &= \gamma, \\ p_2 &= \gamma, & q_2 &= 1 - \gamma + \tilde{\eta}. \end{aligned} \quad (4.8)$$

we obtain for  $C$  the value

$$C = \frac{2\gamma(1-\gamma)}{2\gamma(1-\gamma) - \sigma^2}. \quad (4.9)$$

Note that this value corresponds to the choice

$$\lambda = \frac{\sigma^2}{\gamma(1-\gamma)}$$

in (2.10). This result enlightens the meaning of the constant  $\lambda$  appearing in the Fokker-Planck equation (2.7) in terms of the underlying binary trade (2.4). We remark that also in this case the stationary state of the Boltzmann equation

possesses tails. Considering now the trade (2.3), where for simplicity  $\epsilon(\gamma, \mu) = 1/2$ ,

$$\begin{aligned} p_1 &= (1 + \gamma)/2, & q_1 &= (1 - \mu)/2, \\ p_2 &= (1 - \gamma)/2, & q_2 &= (1 + \mu)/2, \end{aligned} \tag{4.10}$$

which implies

$$C = \frac{2(1 - \gamma\mu)}{2 - (\gamma^2 + \mu^2)}. \tag{4.11}$$

This corresponds to the choice

$$\lambda = \frac{(\gamma - \mu)^2}{1 - \gamma\mu}$$

in (2.10). In this second case, however, the constant  $\lambda$  is always strictly less than 2. This is related to the fact that, for pointwise collisions like the one defined by (4.10), the stationary solution has all moments bounded [18]. In conclusion, the previous analysis shows the universality of the closure of hydrodynamic equations, at least in the well-defined case of conservative economies.

## 5 Which law for the propensity to trade?

To proceed and to obtain (at least numerical) results on the time-evolution of the macroscopic quantities, it is necessary to set the two variables  $x$  and  $w$  into relation. In classical hydrodynamics, where the variables are position and velocity this relation is obvious, since velocity is the time derivative of position. In absence of forces, it corresponds to choose  $\Phi(x, w) = w$ . To find an analogue for our economic setting, namely a law for the propensity to trade, we resort to some arguments within the concepts of opinion formation.

In [1], attention has been focused on two aspects, which in principle could be responsible of the formation of coherent structures. The first one is the remarkably simple compromise process, in which pairs of agents reach a fair compromise after exchanging opinions. The second is the diffusion process, which allows individual agents to change their opinions in a random diffusive fashion. While the compromise process has its basis on the human tendency to settle conflicts, diffusion accounts for the possibility that people may change opinion through a global access to information. In the present time, this aspect is gaining in importance due to the emerging of new possibilities (among them electronic mail and web navigation [21]).

Following this line of thought, in [27] a class of kinetic models of opinion formation, based on two-body interactions involving both compromise and diffusion properties in exchanges between individuals have been introduced. These models are described by partial differential equation of Fokker-Planck type for the distribution of opinion among individuals. Similar diffusion equations were

obtained recently in [25] as the mean field limit of the Ochrombel simplification of the Sznajd model [26].

The equilibrium state of the Fokker-Planck equation can be computed explicitly and, in absence of internal points in which diffusion is missing, is in most cases well represented by a Beta distribution

$$B(x; \alpha, \beta) = \frac{x^\alpha(1-x)^\beta}{\int_0^1 u^\alpha(1-u)^\beta du} = \frac{\Gamma(\alpha + \beta + 2)}{\Gamma(\alpha + 1)\Gamma(\beta + 1)} x^\alpha(1-x)^\beta \quad (5.1)$$

where  $\alpha, \beta > -1$ , and  $\Gamma$  is the gamma function. In what follows, we assume that the stationary profile for the distribution of our propensity to trade follows a law of type (5.1). Taking into account that this stationary profile is stable, a highly reasonable hypothesis is to assume that the rate of variation of the propensity is proportional to the density of people having that propensity. In this case, in order to maintain the lower and upper bounds of  $x(t)$  we assume the law

$$\Phi(x, w) = x^\alpha(1-x)^\beta H(w), \quad (5.2)$$

where  $\alpha, \beta > 0$ , and the coefficient  $H(w)$  takes into account the dependence of the law of variation on the (relative) wealth. We remark that, under the new bounds on  $\alpha, \beta$ , people with propensity to trade close to zero or one are more stable in their propensity, while people with intermediate propensity have more inclination to change their idea.

Last, the form of  $H(w)$  can be deduced owing to the following arguments from microeconomic theory.

Consider the case of a rational investor seeking to maximize his utility from wealth after each trade. He can choose a combination of saving his current wealth and the return from a trade, thus, since  $\eta$  has zero mean, his expected post-trade wealth is

$$\langle w^* \rangle = w + x(v - w) = (1 - x)w + xv.$$

The investor's choice of his propensity to invest can be interpreted as the choice of combinations of two options,  $w$  and  $v$ . The investor tries to maximize his expected utility from post-trade wealth  $\langle u(w^*) \rangle$ , where  $u$  is a utility function characterizing the investor's satisfaction from wealth, e.g. the Cobb-Douglas function

$$u(w, v) = w^\delta v^{1-\delta}, \quad 0 \leq \delta \leq 1.$$

The projection of this function onto the  $w$ - $v$ -plane gives level curves of constant utility, the so-called *indifference curves*. The investor is indifferent to the different combinations of  $v$  and  $w$  on such a curve, or in other words, at each point on an indifference curve he has no preference for one combination over another. A rational investor will choose a value for  $x$  that maximizes his utility from post-trade wealth. Therefore it is clear, that  $x$  should be a function of the investor's wealth  $w$  or his relative wealth  $w - \chi \bar{w}$ , if utility from wealth is measured with respect to the mean wealth  $\chi \bar{w}$  as a reference value ( $\chi$  denotes here a suitable

constant). In other words, the fundamental law of physics about position and velocity and their relation is replaced here by an economic law that relates a rational investor's propensity to invest and his wealth based on the principle of utility maximization.

In general, the optimal choice of  $x$  depends on the underlying utility function and this can lead to quite complex and non-linear relationships between propensity to invest  $x$  and wealth  $w$ . However, recall that we are considering a regime where the time between trades  $\tau$  is very small. If we assume that  $\Phi$  is smooth enough, we can approximate it by a linear relation and ignore terms of higher order. Furthermore, empirical results [23] from economic literature suggest, that typically individuals have decreasing absolute risk aversion.

A simple way to take into account these facts is to relate the time variation of the propensity to the relative (with respect to the mean) wealth. A way to cope with these demands is to introduce the following law

$$\Phi(x, w) = \pm \vartheta x^\alpha (1-x)^\beta (w - \chi \bar{m}(t)), \quad (5.3)$$

where  $\bar{m}(t) = \int_0^1 m(x, t) dx$  denotes the mean wealth at time  $t$ , and  $\vartheta$  is a positive constant. Let us remark that the choice  $\alpha = \beta = 1$  implies an exponential decay of  $x(t)$  towards one of the two extremal points. Since  $x(t) < 1$ , its variation with respect to time can be controlled by a suitable choice of these parameters. Let us note further that the choice of a positive (negative) sign into (5.3) implies that individuals with a higher (lower) wealth will be more (less) willing to trade than individuals with lower (higher) wealth. Clearly, this is only one choice among many possibilities. However, it seems a promising, quite natural approach, which is at the same time flexible enough, and sufficiently easy to be tractable from a numerical point of view. Using this choice in (3.11) and (3.12), and absorbing  $\vartheta$  into time, we arrive to the following system

$$\frac{\partial \rho}{\partial t} + x^\alpha (1-x)^\beta \frac{\partial}{\partial x} [\rho(m - \chi \bar{m})] = 0, \quad (5.4)$$

$$\frac{\partial \rho m}{\partial t} + x^\alpha (1-x)^\beta \frac{\partial}{\partial x} \left[ \rho m \left( \frac{2}{2-\lambda} m - \chi \bar{m} \right) \right] = 0. \quad (5.5)$$

As before, using (5.4) we can rewrite the second equation as

$$\frac{\partial m}{\partial t} + x^\alpha (1-x)^\beta (m - \chi \bar{m}) \frac{\partial m}{\partial x} + \frac{\lambda}{2-\lambda} \frac{1}{\rho} \frac{\partial}{\partial x} [\rho m^2] = 0. \quad (5.6)$$

## 6 Numerical results

To solve (5.4), (5.5) numerically, we use a standard finite element method. We choose quadratic Lagrangian elements on a uniform grid with 480 nodes. We use the initial conditions

$$\rho_0(x) = 0.1, \quad m_0(x) = x(1-x), \quad x \in (0, 1).$$

At the boundaries we use homogenous Neumann conditions for  $\rho$  and homogenous Dirichlet conditions for  $m$ . If not mentioned otherwise, we choose  $\alpha = \beta = 2$ ,  $\chi = 1$ ,  $\lambda = 1$  and the final time is  $T = 15$ .

Figure 6.1 displays the numerical solution for different values of  $\lambda$ . Recall that a higher  $\lambda$  corresponds to tails with a lower Pareto index, and this corresponds to a society with a strong economy. We can observe the same effect here at the macroscopic level. As  $\lambda$  increases, the density of agents with a high propensity to trade increases while the wealth density is increasing for larger propensities and decreasing for smaller propensities, i.e. a large fraction of the total wealth is owned by a small group of agents.

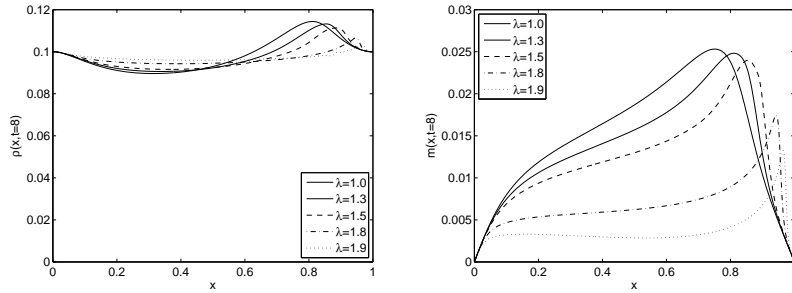


Figure 6.1: Influence of different values for  $\lambda$  ( $\lambda = 1, 1.3, 1.5, 1.8, 1.9$ )

Figure 6.2 shows the influence of different variants of law (5.3). For a value of  $\beta = 1.5$  agents with wealth above the mean wealth increase their propensity to trade which leads to a peak formation in the density  $\rho$  close to  $x = 1$ . For higher values of  $\beta$  the propensity to invest grows slower when above the average wealth and the peak is less pronounced. We present no results concerned with the variation of  $\alpha$ , but clearly they are similar since the law (5.3) is symmetric with respect to these parameters. Therefore, increasing  $\beta$  or  $\alpha$  means to slow down the movement of agents' propensity towards the extremal points, where people have zero or maximal propensity to invest.

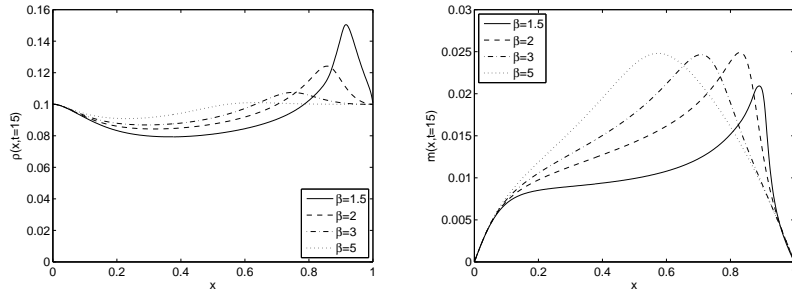


Figure 6.2: Influence of different values for  $\beta$  ( $\beta = 1.5, 2, 3, 5$ )

The influence of the parameter  $\chi$  can be observed in Figure 6.3. For high values of  $\chi$  only very wealthy agents increase their propensity to trade, all other decrease it. This results in a peak formation at lower levels of  $x$  with agents saving most of their wealth. The influence of the different values of  $\chi$  clearly indicate that if the propensity increases in a larger interval, also the density and the mean wealth move left, and there is the possibility to increase the wealth.

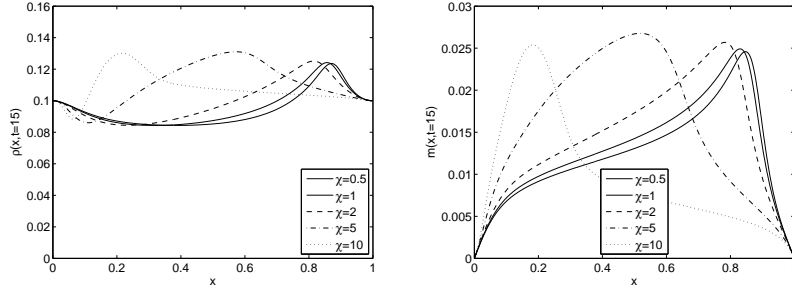


Figure 6.3: Influence of different values for  $\chi$  ( $\chi = 0.5, 1, 2, 5, 10$ )

Obviously, the reference for measuring the wealth, controlled by  $\chi$  and  $\bar{m}(t)$  plays an important role. To model a real economy, one could generalize the law and replace the constant  $\chi$  by a function of time  $\chi(t)$ . For example, one can have agents whose perception of wealth is increasing or decreasing over time by choosing an increasing or decreasing  $\chi(t)$ . Seasonal effects can be modeled by introducing a factor  $\chi(t) = \bar{\chi}(2 - \cos(ct))$ , where  $\bar{\chi}$  is a long-run mean and  $c$  is an annual constant. Figure 6.4 displays the difference in the solutions in two simulations, one carried out using a time-varying factor  $\chi(t) = \bar{\chi}(2 - \cos(ct))$  with  $\bar{\chi} = 1$  and  $c = 1$  and the other with with constant  $\chi = 1$ .

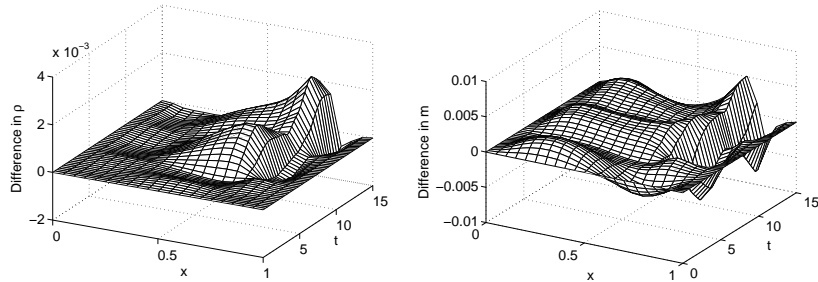


Figure 6.4: Influence of a seasonal effect modelled by a time-varying  $\chi(t)$ . Displayed are the differences in  $\rho$  (left) and  $m$  (right) from one simulation with  $\chi(t) = \bar{\chi}(2 - \cos(ct))$  with  $\bar{\chi} = 1$ ,  $c = 1$  and another one with constant  $\chi = 1$ .

The previous examples enlighten the influence of the law (2.1) in the evolution of the macroscopic quantities. Clearly, the proposed law (5.3) constitutes

only a prototype towards a better understanding of the whole matter. Changing the law (5.3) allows to clarify the role of the various parameters involved. By maintaining the linearity with respect to the wealth parameter  $w$ , we consider in what follows the law

$$\tilde{\Phi}(x, w) = \nu(x)(w - \bar{m}(t)), \quad (6.1)$$

with  $\nu(x) = cx^\alpha(1/2 - x)^\gamma(1 - x)^\beta$ . Law (6.1) assumes as hypothesis the (reasonable?) fact that in correspondence to some point (in this case  $x = 1/2$ ) the propensity tends to stabilize. Figure 6.5 shows a comparison of  $\mu(x)$ ,  $\nu(x)$  corresponding to the same values of  $\alpha$  and  $\beta$ . For numerical simulation we choose

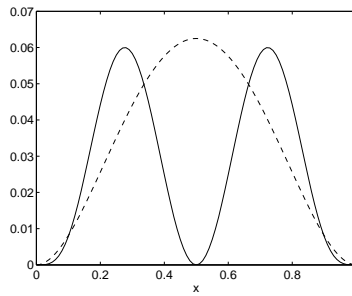


Figure 6.5: Functions  $\mu(x)$  and  $\nu(x)$  used in the numerical illustration.

$\alpha = \beta = 4$ ,  $\gamma = 2$ ,  $c = 750$ ,  $T = 5$  and  $\lambda = 1$  and the same initial and boundary conditions as above. Figure 6.6 shows the plot of the densities  $\rho$  and  $m$  at time  $T = 5$  resulting from the computations with the different laws.

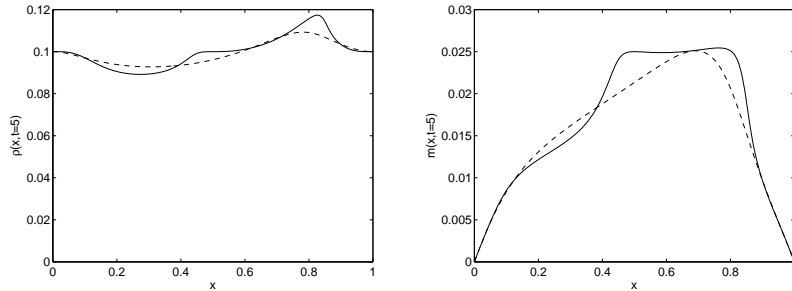


Figure 6.6: Influence of different laws for propensity: solid lines correspond to law (6.1), broken lines to (5.3).



## 7 Conclusions

We formulated a model for the temporal evolution of the density of agents in a market, where the density itself depends both on the propensity to trade and wealth, which are here the analogues of position and velocity in classical statistical mechanics. The underlying inhomogeneous Boltzmann equation is then used as the starting point for the derivation of suitable hydrodynamic equations, which are the Euler equations for the economic system. The equilibrium state which is at the basis of the closure is the stationary solution of a Fokker-Planck equation derived recently both as the mean field limit of a stochastic differential equation [4] and the quasi-invariant limit of a model Boltzmann equation based on binary interactions among agents [11]. The interesting property of this stationary solution (the analogous of the Maxwell distribution in classical kinetic theory of rarefied gases) is that it is known analytically, and possesses Pareto tails of a given order. A detailed discussion, however, shows that the closure relations have a universal character, which implies that the macroscopic equations are invariant with respect to the choice of the conservative kinetic collision operator. The link between wealth and propensity to trade is postulated on the basis of some recent results on opinion formation [25, 1, 27]. The variation in time of the propensity is represented by a logistic-type first-order differential equation which depends linearly on the wealth variable. Various numerical examples are then presented, to show the dependence of the evolution of the macroscopic variables on the empirical law of propensity.

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