A NOTE ON MEAN-VARIANCE HEDGING OF NON-ATTAINABLE CLAIMS

MICHAEL KOHLMANN AND BERNHARD PEISL

ABSTRACT. A market is described by two correlated asset prices. But only one of them is traded while the contingent claim is a function of both assets. We solve the mean-variance hedging problem completely and prove that the optimal strategy consists of a modified pure hedge expressible in terms of the obervation process and a Merton-type investment.

1. Introduction

On the basis of [7] we solve the continuous time hedging problem with a mean-variance objective for general claims. The market is given by one bond and two asset processes, but only one of them is tradable. The target at time T is a function of both assets.

This incomplete market model was treated in [2], [11]. These authors use techniques from martingale theory and orthogonal projection to derive the optimal strategy. We make use of the concept of Backward Stochastic Differential Equations (BSDE for short) which is relatively new and extremely powerful in financial applications [3] to achieve similar results in a more general reading.

2. Preliminaries

The market under consideration is described by the following tools: the random input comes from two independent Brownian motions (w^1, w^2) on a given probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ where $(\mathcal{F}_t)_{t\in[0,T]}$ is the augmentation of the filtration generated by (w^1, w^2) with $T > 0$ the fixed time horizon, and $\mathcal{F} = \mathcal{F}_T$. Then $(\mathcal{F}_t)$ is right-continuous and satisfies the usual conditions.

Received by the editors dec 09, 1999.

1991 *Mathematics Subject Classification.* 60H10, 90A09.

Key words and phrases. backward stochastic differential equations, mean variance hedging, nontradable claims.

This work is partially supported by the Center of Finance and Econometrics, project: mathematical finance, december 19,1999.

With (w^1, w^2) we construct the following standard Brownian motion (w_t) given by

$$w_t = \int_0^t \rho_s \, dw_s^1 + \int_0^t \sqrt{1-\rho_s^2} \, dw_s^2 \quad , \qquad t \in [0,T]$$

where ρ is a bounded adapted process with $|\rho(s)| \leq 1$ for $s \in [0,T]$.

For $t \in [0,T]$ the bond is given by

$$dP_0(t) = r(t) \, P_0(t) \, dt \quad , \quad P_0(0) > 0,$$

and the stock by

$$dP_1(t) = P_1(t) \, (b(t) \, dt + \sigma(t) \, dw_t) \quad , \quad P_1(0) > 0,$$

with r, b, σ real-valued bounded deterministic functions. Moreover we assume $\sigma^2 \geq \delta > 0$. Hence for $t \in [0,T]$ the wealth process is given by

$$\begin{aligned} dx(t) &= (r(t) \, x(t) + (b(t) - r(t)) \, \pi(t)) \, dt + \sigma(t) \, \pi(t) \, dw_t \\ x(0) &= x > 0 \end{aligned}$$

where $\pi \in L^2_{\mathcal{F}}(0,T;\mathbb{R})$ (we denote by $L^2_{\mathcal{F}}(0,T;\mathbb{R})$ the set of all $\mathbb{R}$-valued, measurable stochastic processes $\varphi(t)$ adapted to $\mathcal{F}_t$ such that $E\left[\int_0^T |\varphi(t)|^2 \, dt\right] < \infty$) is the quantity invested in the stock. With the risk premium process

$$\theta := \frac{b-r}{\sigma}$$

we can write this as

$$\begin{aligned} dx(t) &= (r(t) \, x(t) + \theta(t)\sigma(t) \, \pi(t)) \, dt + \sigma(t) \, \pi(t) \, dw_t \\ x(0) &= x > 0. \end{aligned}$$

We call $\pi(t)$, $t \in [0,T]$, the portfolio or the hedging strategy of an agent and a twice integrable predictable strategy will be considered as admissible.

The contingent claim ξ is given as a twice integrable $\mathcal{F}_T$-measurable square integrable random variable. Our objective is, for each initial value $x > 0$ and the contingent claim ξ, to choose a hedging strategy $\pi \in L^2_{\mathcal{F}}(0,T;\mathbb{R})$ so as to minimize

$$J(x,\pi) := \frac{1}{2} E\left[x^{x,\pi}(T) - \xi\right]^2$$

over all hedging strategies where $x^{x,\pi}(T)$ denotes the terminal value of the wealth process associated with the initial value x and the hedging strategy π. This problem was treated in detail in [7] where ξ

was assumed to be $\mathcal{F}_T^w$-measurable. Here we shall assume that $\xi = g(P_1(T), S_T)$ with g a bounded Borel function and

$$\begin{aligned} dS_t &= S_t\left(\mu_t\,dt + \alpha_t\,dw_t^1\right) \\ S_0 &> 0 \end{aligned}$$

where μ and α are real-valued bounded deterministic functions.

Remark 1. (i) In economical terms we may interpret the problem in the following way: An agent is only allowed to trade the stock P_1 in order to hedge a function of S_T. This is a typical situation in markets where the option is written on a non-tradable market instrument, for example a market index. This problem has been treated by martingale methods earlier in [2], [11].

(ii) In the above problem we have the technical difficulty that we have to deal with backward stochastic differential equations where the terminal condition is not measurable with respect to the Brownian motion of the underlying risky asset. Economically this means that the contingent claim ξ is not attainable by a wealth process (x_t). A similar problem is described in [8].

3. Some results on Backward Stochastic Differential Equations

When we try to model the price $(p(t))_{0\le t\le T}$ for ξ we would like to write this price in the usual way as

$$\begin{aligned} dp(t) &= \left(r(t)\ p(t) + \theta(t)\ z_t^1\right)dt + z_t^1\,dw_t \quad , \quad t\in[0,T] \\ p(T) &= \xi. \end{aligned} \tag{3.1}$$

In general this BSDE will not have a solution. A way out of this difficulty is proposed in [11] without making use of BDE-techniques and in a special situation of "partial information" in [8], so our approach relates results in [4], [6] to BSDE-techniques. The idea is to add an orthogonal martingale term which is orthogonal to $\int_0^t z_s^1\,dw_s$. With the $(\mathcal{F}_t, P)$-martingale

$$\begin{aligned} dN_t &= \sqrt{1-\rho_t^2}\,dw_t^1 - \rho_t\,dw_t^2 \\ N_0 &= 0 \end{aligned}$$

for $t\in[0,T]$ such an orthogonal term is given by

$$\int_0^t z_s^2\,dN_s.$$

So instead of (3.1) we consider

(3.2)
$$\begin{aligned} dp\,(t) &= \left(r\,(t)\;p\,(t) + \theta\,(t)\;z_t^1\right)dt + z_t^1\,dw_t + z_t^2\,dN_t \quad , \quad t \in [0,T] \\ p\,(T) &= \xi. \end{aligned}$$

Remark 2. This procedure has been known in a control-theoretical setting for quite a long time. As the BSDE under consideration may be seen as the formal adjoint of a trivial control problem [9], a similar procedure is already described in [1].

One way to see that this BSDE has a (unique) solution (p, z^1, z^2) follows the following argument: Let $\eta \in L^2_{\mathcal{F}_T}(\Omega, \mathbb{R})$, then η may be represented as

$$\eta = E\,(\eta) + \int_0^T \psi_s^1 dw_s^1 + \int_0^T \psi_s^2 dw_s^2$$

where $\psi^1, \psi^2 \in L^2_{\mathcal{F}}(0, T; \mathbb{R})$. With

$$\begin{aligned} \gamma &= \frac{1}{\rho}\psi^1 - \frac{1-\rho^2}{\rho}\psi^1 + \sqrt{1-\rho^2}\psi^2 \\ \beta &= \sqrt{1-\rho^2}\psi^1 - \rho\,\psi^2 \end{aligned}$$

and an easy calculation we derive

$$\eta = E\,(\eta) + \int_0^T \gamma_s\,dw_s + \int_0^T \beta_s\,dN_s$$

with $\gamma, \beta \in L^2_{\mathcal{F}}(0, T; \mathbb{R})$. This shows that any square integrable $\mathcal{F}_T = \sigma\,(w^1, w^2)$-measurable random variable is representable with respect to the orthogonal martingales (w, N). This implies that the BSDE (3.2) has a unique solution (p, z^1, z^2) (See, for example, [12] for an existence and uniqueness theorem for BSDEs.)

Now we compute a pricing system u from a Feynman-Kac formula in order to represent

$$p(t) = u\,(t, P_1(t), y(t)).$$

The observation of S_T given $\mathcal{F}_t$ is given by

$$\begin{aligned} dy_t &= \left(r(t)y_t + \theta(t)q_t^1\right)dt + q_t^1 dw_t + q_t^2 dN_t \\ y_T &= S_T. \end{aligned} \tag{3.3}$$

For $\xi = g(P_1(T), S_T) = p(T)$ we can compute $p(t) = u\,(t, P_1(t), y(t))$ with Itô's-formula (with indices at u indicating partial derivatives of u).

For simplicity we omit the notation of the dependence of the functions and processes on t:

$$\begin{aligned}
dp &= u_t dt + u_x dP_1 + u_y dy \\
&\quad + \frac{1}{2}\left(u_{xx} d\left\langle P_1, P_1\right\rangle + u_{xy} d\left\langle P_1, y\right\rangle + u_{yx} d\left\langle y, P_1\right\rangle + u_{yy} d\left\langle y, y\right\rangle\right) \\
&= u_t dt + u_x P_1 b dt + u_x P_1 \sigma dw + u_y\left(ry + \theta q^1\right) dt + u_y q^1 dw + u_y q^2 dN \\
&\quad + \frac{1}{2} u_{xx}\left(P_1\right)^2 \sigma^2 dt + \frac{1}{2} u_{xy} P_1 \sigma q^1 dt \\
&\quad + \frac{1}{2} u_{yx} P_1 \sigma q^1 dt + \frac{1}{2} u_{yy}\left(q^1\right)^2 dt + \frac{1}{2} u_{yy}\left(q^2\right)^2 dt \\
&= u_t dt \\
&\quad + \left[u_x P_1 b + u_y\left(ry + \theta q^1\right) + \frac{1}{2} u_{xx}\left(P_1\right)^2 \sigma^2 + u_{xy} P_1 \sigma q^1\right] dt \\
&\quad + \left[\frac{1}{2} u_{yy}\left(q^1\right)^2 + \frac{1}{2} u_{yy}\left(q^2\right)^2\right] dt \\
&\quad + \left[u_x P_1 \sigma + u_y q^1\right] dw + u_y q^2 dN
\end{aligned}$$

This must be equal to

$$dp = \left(r\,p + \theta\, z^1\right) dt + z^1\, dw + z^2\, dN,$$

so we derive

$$\begin{aligned}
z^1 &= u_x P_1 \sigma + u_y q^1 \\
z^2 &= u_y q^2 .
\end{aligned}$$

Thus we end up with the following partial differential equation for u

$$\begin{aligned}
ru + \theta u_x x \sigma + \theta u_y q^1 &= u_t + u_x x b + u_y\left(ry + \theta q^1\right) \\
&\quad + \frac{1}{2} u_{xx} x^2 \sigma^2 + u_{xy} x \sigma q^1 + \frac{1}{2} u_{yy}\left(\left(q^1\right)^2 + \left(q^2\right)^2\right) \\
u(T, x, y) &= g(x, y).
\end{aligned}$$

or

$$ru - u_t - r(u_x x + u_y y) - \tfrac{1}{2}\left(u_{xx} x^2 \sigma^2 + u_{yy}\left(\left(q^1\right)^2 + \left(q^2\right)^2\right)\right) - u_{xy} x \sigma q^1 = 0$$
$$u(T, x, y) = g(x, y)$$

as a generalized stochastic Black Scholes formula.

4. The control problem

Now we go back to the mean variance hedging problem:

$$J(x, \pi) := \frac{1}{2} E\left[x^{x,\pi}(T) - \xi\right]^2 = \min_{\pi}! \tag{4.1}$$

To derive the optimal portfolio π we follow in identical steps the proof in [7]. So we have

Theorem 3. *The optimal portfolio of the hedging problem (4.1) is*

$$\pi^*(t) = -\theta(t)\sigma(t)^{-1}\left(x^*(t) - p(t)\right) + \sigma(t)^{-1} z_t^1 \quad , \qquad 0 \le t \le T, \tag{4.2}$$

where (p, z^1, z^2) *is the solution of*

(4.3)
$$\begin{aligned} dp(t) &= \left(r(t)p(t) + \theta(t)z_t^1\right)dt + z_t^1 dw_t + z_t^2 dN_t \quad , \qquad 0 \le t \le T, \\ p(T) &= \xi = g(P_1(T), S_T) \end{aligned}$$

Proof. To apply [7] to our wealth process we write this process as

$$\begin{aligned} dx(t) &= (r(t)\,x(t) + (b(t) - r(t))\,\pi(t))\,dt + \sigma(t)\,\pi(t)\,dw_t + 0 \cdot \pi(t)\,dN_t \\ x(0) &= x > 0. \end{aligned}$$

So here the volatility matrix from [7] $\widetilde{\sigma}$, is

$$\widetilde{\sigma}(t) = (\sigma(t), 0)\,.$$

If we take account of this the theorem is proved by applying Theorem 5.1 in [7].

As already noted in [7] the optimal portfolio is divided in two parts. The term on the right hand side is known as the *replicating portfolio* [12]. The other term is some kind of a Merton portfolio [5]. So we first try to hedge the claim, the rest of the money is invested in a Merton type portfolio.

In this framework it is also possible to consider the generalization in [7]. Instead of having only terminal costs we add now running costs. Here we introduce also weights on terminal and running costs. So the hedging problem with terminal and running costs is:

(4.4)
$$J(x,\pi) := \frac{1}{2}\nu E\left[x^{x,\pi}(T) - \xi\right]^2 + \frac{1}{2}\lambda E\left[\int_0^T |x^{x,\pi}(t) - E(\xi|\mathcal{F}_t)|^2 dt\right] = \min_{\pi}!$$

for given $\nu > 0$, $\lambda \ge 0$.

It is important that $\nu \neq 0$ as in the following we want to satisfy the conditions in (4.5). Now we have

Theorem 4. *The optimal portfolio of the hedging problem (4.4) is*

$$\pi^*(t) = -\theta(t)\sigma(t)^{-1}\left(x^*(t) - p(t)\right) + \sigma(t)^{-1} z_t^1 \quad , \qquad 0 \le t \le T,$$

where (p, z^1, z^2) *is, for* $0 \le t \le T$*, the solution of*

$$\begin{aligned} dp(t) &= \left(r(t)p(t) + \theta(t)z_t^1 + \lambda \frac{p(t) - E(\xi|\mathcal{F}_t)}{P(t)} \right) dt + z_t^1 dw_t + z_t^2 dN_t \\ p(T) &= \xi = g(P_1(T), S_T) \end{aligned}$$

and $P(t)$ *is the solution of*

$$\begin{aligned} \dot{P}(t) + 2r(t)P(t) - \theta(t)^2 P(t) + \lambda &= 0 \quad , t \in [0,T]\,, \\ P(T) &= \nu \\ P(t) &> 0 \quad \textit{for all } t \in [0,T] \end{aligned} \tag{4.5}$$

with explicit solution

$$P(t) = \nu \exp\left(-\int_t^T \theta(u)^2 - 2r(u)du \right) + \lambda \int_t^T \exp\left(-\int_t^s \theta(u)^2 - 2r(u)du \right) ds.$$

5. An explicit example

Let us assume that all coefficients r, b, σ, μ, α are constant, and $g\,(P_1(T), S_T) = S_T$. Then we set

$$\begin{aligned} y_t &= S_t \exp\left(-\int_t^T (\alpha\theta\rho_s - \mu + r)\, ds \right) \quad , \qquad t \in [0,T]\,, \\ q_t^1 &= \alpha\rho_t y_t \\ q_t^2 &= \alpha\sqrt{1-\rho_t^2}\, y_t. \end{aligned}$$

Using

$$\begin{aligned} dw_t^1 &= \left(\frac{1}{\rho_t} - \frac{(1-\rho_t^2)}{\rho_t} \right) dw_t + \sqrt{1-\rho_t^2}\, dN_t \\ &= \rho_t dw_t + \sqrt{1-\rho_t^2}\, dN_t \end{aligned}$$

to compute

$$
\begin{aligned}
dy_t &= dS_t \cdot \exp\left(-\int_t^T (\alpha\theta\rho_s - \mu + r)\, ds\right) + S_t \cdot d\exp\left(-\int_t^T (\alpha\theta\rho_s - \mu + r)\, ds\right) \\
&= S_t \left(\mu\, dt + \alpha\, dw_t^1\right) \exp\left(-\int_t^T (\alpha\theta\rho_s - \mu + r)\, ds\right) \\
&\quad + S_t \left(\alpha\theta\rho_t - \mu + r\right) \exp\left(-\int_t^T (\alpha\theta\rho_s - \mu + r)\, ds\right) dt \\
&\qquad (5.1) \\
&= r y_t dt + \underbrace{\alpha\theta\rho_t y_t}_{=\theta q_t^1} dt + \underbrace{\alpha\rho_t y_t}_{=q_t^1} dw_t + \underbrace{\alpha\sqrt{1-\rho_t^2}\, y_t}_{=q_t^2} dN_t \\
&\qquad (5.2) \\
&= \left(r y_t + \theta q_t^1\right) dt + q_t^1 dw_t + q_t^2 dN_t \\
y_T &= S_T
\end{aligned}
$$

we see that the settings for (y_t), (q_t^1), (q_t^2) solve the BSDE (3.3) for (y_t). As furthermore $dp(t) = dy_t$ and $p(T) = y_T$ we have $p(t) = y_t$ for all $t \in [0,T]$. Now we compare the coefficients in (4.3) and (5.2) and derive

$$
\begin{aligned}
z_t^1 &= q_t^1 = \alpha\rho_t y_t \\
z_t^2 &= q_t^2 = \alpha\sqrt{1-\rho_t^2}\, y_t.
\end{aligned}
$$

Finally we find the solution for the optimal portfolio from equation (4.2)

$$
\pi^*(t) = -\theta\sigma^{-1}\left(x^*(t) - p(t)\right) + \sigma^{-1}\alpha\rho_t\, p(t) \quad , \qquad 0 \le t \le T,
$$

or in terms of the observation process

$$
\pi^*(t) = -\theta\sigma^{-1}\left(x^*(t) - p(t)\right) + \sigma^{-1}\alpha\rho_t\, y(t) \quad , \qquad 0 \le t \le T.
$$

Hence

$$
p(t) = S_t \exp\left(-\int_t^T (\alpha\theta\rho_s - \mu + r)\, ds\right) \text{ for } t \in [0,T].
$$

This completely solves the problem and explicitly shows the dependence of the price on the drift of S_t.

6. Conclusion

The problem of hedging a nontradable claim was brought to our attention recently by Mark Davis. In this paper we tried to go into this problem under the aspect of being able to find a complete solution.So many extensions are possible for instance by introducing techniques from filtering theory to describe the observation process (y_t) in a more general setting. This will be described in a forthcoming paper.

In [10] a similar problem will be solved: Roughly speaking there the observation process together with some information about the market structure is given and the properties of the resulting price process are examined. This gives some new insight into the asset prices.

References

[1] Bismut, Jean-Michel: Conjugate convex functions in optimal stochastic control, J.Math.Anal.Appl. 1973, Vol 44, 384-404

[2] Duffie, Darrell; Richardson, Henry R.: Mean-variance hedging in continuous time. The Annals of Applied Probability 1991, Vol. 1, No. 1, 1-15

[3] El Karoui, Nicole; Quenez, M.C.: Nonlinear pricing theory and backward stochastic differential equations, Lect. Notes Math., Vol. 1656, Springer-Verlag, Berlin, 1997, pp. 191-246.

[4] Elliott, Robert J.; Föllmer, Hans: Orthogonal martingale representation, working paper UALTA, Edmonton, (1991)

[5] Fleming, Wendell H.; Rishel, Raymond W.: Deterministic and Stochastic Optimal Control. Springer-Verlag 1975

[6] Föllmer, Hans; Schweizer, Martin: Hedging of contingent claims under incomplete information, in Appl. stoch. Analysisi, ed M.H.A.Davis, R.J.Elliott, Stochastics Monographs 5, 389-414, Gordon&Breach (1991)

[7] Kohlmann, Michael; Zhou, Xun Yu: Relationship between Backward Stochastic Differential Equations and Stochastic Controls: An LQ Approach, to appear in SICON (1999)

[8] Kohlmann, Michael; Zhou, Xun Yu: The Informed and Uninformed Agent's Price of a Contingent Claim, working paper (1998)

[9] Kohlmann, Michael: Reflected forward backward stochastic differential equations andcontingent claims, in Control of distributed parameter and stochastic systems, ed. S. Chen, X. Li, J. Yong, X.Y. Zhou, Kluwer Acad. Publishers (1999)

[10] Lüders, Erik; Peisl, Bernhard: On the relationship of information processes and asset price processes, CoFE working paper, in preparation

[11] Schweizer, Martin: Mean-variance hedging for general claims The Annals of Applied Probability, 1992, Vol. 2, No. 1, 171-179

[12] Yong, Jiongmin; Zhou, Xun Yu: Stochastic Controls: Hamiltonian Systems and HJB Equations. Springer-Verlag 1999

University of Konstanz, Department of Mathematics and Statistics
E-mail address: `michael.kohlmann(bernhard.peisl)@uni-konstanz.de`