Two-Dimensional Risk-Neutral Valuation Relationships for the Pricing of Options.

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Abstract

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The Black-Scholes model is based on a one-parameter pricing kernel with constant elasticity. Theoretical and empirical results suggest declining elasticity and, hence, a pricing kernel with at least two parameters. We price European-style options on assets whose probability distributions have two unknown parameters. We assume a pricing kernel which also has two unknown parameters. When certain conditions are met, a two-dimensional risk-neutral valuation relationship exists for the pricing of these options: i.e. the relationship between the price of the option and the prices of the underlying asset and one other option on the asset is the same as it would be under risk neutrality. In this class of models, the price of the underlying asset and that of one other option take the place of the unknown parameters.

1 Introduction

There are two distinct approaches to option pricing that have been employed in the literature. In the first approach, assuming that the price of the underlying asset follows a given process in a dynamically complete market, the options are priced by no arbitrage. The classic proof in Black and Scholes (1973) uses this approach. In the second approach, assuming that the terminal probability distribution of the underlying asset is of a specific form and that a stochastic discount factor (pricing kernel) exists with specific properties, European-style options can again be priced. This second, 'pricing kernel approach', used originally by Rubinstein (1976) and Brennan (1979), directly establishes what has become known as a risk-neutral valuation relationship between the option price and the price of the underlying asset. However, as the empirical limitations of the Black-Scholes model became apparent, there has been a widespread search for option pricing models yielding better empirical predictions. Numerous papers have worked on generalizing from the Geometric Brownian Motion assumption, using the dynamically complete market approach. Rather less attention has been paid to generalizing the alternative pricing kernel approach.

In this paper, we extend the pricing kernel approach to provide a new class of option pricing models. Given certain restrictions on the form of the pricing kernel and the probability distrbution, two unknown parameters of the probability distribution and two unknown parameters of the pricing kernel can be replaced by two prices: the price of the underlying asset and the price of an at-the-money option on the asset. For this class of models, a two-dimensional risk-neutral valuation relationship exists relating the option price to these two other prices. If a one-dimensional risk-neutral valuation relationship exists, the relationship between the option price and the underlying asset price is the same as it would be under risk neutrality. Similarly, if a two-dimensional risk-neutral valuation relationship exists, then the relationship between the option price and the prices of the underlying asset and the at-the-money option on the asset is the same as it would be under risk neutrality.

One great advantage of the Black-Scholes model is that it provides a functional relationship between an option price and the underlying asset's price which is independent of preferences. In this case, if S is the underlying asset price and C_K is the price of a European-style call option on the asset with strike price K, then $C_K = C_K(S)$ and the function $C_K(S)$ is independent of preferences. Here, we show that there may exist other risk-neutral valuation relationships which could be useful in valuing options. In particular, we derive conditions under which a two-dimensional risk-neutral valuation relationship exists, linking an option price to the underlying asset price and the price of an at-the-money option on the asset. In this case, if C_A denotes the price of an at-the-money option on the asset, then $C_K = C_K(S, C_A)$, where the function $C_K(S, C_A)$ is independent of preferences.

The use of option prices to calibrate the risk-neutral density is well established in the literature. Jackwerth (2000), Rosenberg and Engle (2002) and Liu et.al. (2005) use option prices at various strike prices to estimate risk-neutral densities. Chernov and Ghysels (2000) and Pan (2000) employ time series of stock and option prices to estimate jointly stock price and volatility processes. However, none of these papers use options to price other options via a risk-neutral valuation relationship as we do here. In a related paper, Camara and Simkins (2006) consider options on a stock which is subject to bankruptcy risk. They assume a representative investor economy where the investor has constant relative risk averse utility. They obtain a preference free option price, but only in the case where the bankruptcy risk and aggregate consumption are uncorrelated. In the general case, they require an estimate of the coefficient of relative risk aversion. A possible application of our two-dimensional risk-neutral valuation relationship could apply in this case, where one option price is used to back out the coefficient of relative risk aversion. However, Camara and Simkins do not address this issue.

The conditions for a risk-neutral valuation relationship (RNVR) to exist are restrictive. For example, for options on a lognormally distributed asset, a one-dimensional RNVR exists if and only if the pricing kernel has constant elasticity. Similarly, in the case of a two-dimensional RNVR, strong restrictions are required on the form of the pricing kernel and the probability distribution. The advantage of pricing with RNVRs is that this approach avoids the necessity of estimating preference parameters and the associated risk premia. An alternative approach, which does not restrict the form of the pricing kernel so strongly, estimates the risk-neutral density directly, using a series of option prices with different strike prices. The advantage of this approach is a possible increase in accuracy, since the conditions for a RNVR are restrictive and may not hold. However, the disadvantage is that more option prices are needed for estimation. Also, these prices may be unreliable due to thin trading of out-of-the-money or in-the-money options.

Recent articles have emphasised the role of the pricing kernel in the pricing of options and our model is closely related to this work. Given the price of the underlying asset, the price of a call (or a put) option with strike price K is constrained by the possible shapes of the pricing kernel function. Bernardo and Ledoit (2000) and Cochrane and Saa-Requejo (2000) both use the pricing kernel approach to derive bounds for option prices. Cochrane and Saa-Requejo (2000) show that the option price can be bounded by limiting the variance of the pricing kernel. In similar vein, Bernardo and Ledoit (2000) show that the option price can be bounded by limiting the convexity of the pricing kernel. In related work, Franke, Stapleton and Subrahmanyam (1999) show that European-style call options have higher prices when the pricing kernel has declining, as compared to constant, elasticity. We take a similar pricing kernel approach. However, rather than deriving option pricing bounds, we look for *precise* option prices that exist using RNVRs.

As the analysis of Heston (1993) makes clear and earlier work by Brennan (1979) and Rubinstein (1976) discovered, constant elasticity of the pricing kernel with respect to the asset price is a sufficient condition for the Black-Scholes RNVR to obtain for options on a lognormally distributed asset price. Hence a generalization of the Black-Scholes model involves a less rigorous condition on the pricing kernel. Here we show that if the pricing kernel has non-constant elasticity but is determined by two parameters, it is possible to generalize the Black-Scholes and similar models, by employing a two-dimensional RNVR. As Heston emphasizes, the essence of the Black-Scholes RNVR is that both a preference parameter and a probability distribution parameter drop out of the option pricing relationship. These parameters are replaced by the price of the underlying asset. In our two-dimensional extension, two preference parameters and two probability distribution parameters drop out of the option pricing relationship. These are replaced by the price of the underlying asset and the price of one option on the asset. Given these prices, our option pricing formula is independent of preferences, i.e. it is a risk-neutral valuation relationship. Hence it potentially applies across a broad range of possible underlying assets.

There is a large body of literature documenting the empirical rejection of the Black-Scholes model for a variety of underlying assets.[1] For example, one recent study of the volatility of the S&P stock index by Corrado and Miller (2003) finds that the implied volatility of the index is between two and four percent above estimates of the historical volatility. Further, there is considerable documentation of the phenomenon that the risk-neutral distribution has fat tails in the case of many assets. Many attempts have been made to extend the Black-Scholes model by modifying the assumption that the asset follows a geometric Brownian motion.[2] The difference in our model, is that we relax both the probability density function and the pricing kernel assumptions underlying the traditional risk-neutral valuation relationship. This leads us to the two-dimensional risk-neutral valuation relationship.

In empirical work which is closely related to our own, Jackwerth (2000) and Rosenberg and Engle (2002) develop methods for estimating the pricing kernel (and hence the marginal utility function of the representative investor) using observed index option prices together with the empirical distribution of the index. Both papers find that the pricing kernel displays declining elasticity for low index levels, then turns even negative in some higher index range (i.e. aggregate relative risk aversion is negative in this range) and finally increases with higher index levels, returning to positive values. Using a GARCH-model, Barone-Adesi, Engel and Mancini (2004) find declining elasticity almost everywhere, with

[1] For a recent summary of the evidence, see for example Jackwerth (2004).

[2] For empirical tests of many of these see Dumas, Fleming and Whaley (1998).

elasticity staying positive everywhere. Ait-Sahalia and Lo (2000) conclude that constant elasticity is misspecified and declining elasticity appears to be more realistic. However, little is known on elasticity for index values more than 15 percent above or below the current future price. Overall, the empirical findings appear to support declining elasticity.

On the theoretical side, there is strong support for declining elasticity of the pricing kernel. Benninga and Mayshar (2000) showed in a rational expectations model of an economy in which each investor has constant relative risk aversion but the level of risk aversion differs across investors, that the pricing kernel has declining elasticity with respect to aggregate wealth. This result also holds if some investors have declining relative risk aversion. It is even likely to hold when some investors have increasing relative risk aversion. Hence there is strong theoretical and empirical support for declining elasticity of the pricing kernel. This suggests a generalization of the one-dimensional RNVR.

As noted above, any generalization of the one-dimensional RNVRs of Black and Scholes (1973) and others must be related to the work on option price bounds. Ryan (2000) considers the bounds on an option price given the price of the underlying asset and the price of one other option on the asset. Ryan finds much tighter bounds for the option price, given these two other prices, than the bounds that exist given only the underlying asset price. Our two-dimensional RNVR-based option price can be thought of as the unique price within the Ryan bounds.

Before proceeding with our analysis, we now relate to the large volume of literature on the pricing of options in a dynamically complete market. Following the original paper by Cox, Ross and Rubinstein (1979), these models assume that the asset price follows a given process, which allows the computation of 'pseudo' probabilities or state prices. These are often referred to as 'risk-neutral' probabilities. They are then used to price the option as if investors used these as probabilities and were risk neutral. These papers also derive the option price process. This general methodology differs from that used in this paper, where we consider, as in Rubinstein (1976), Brennan (1979), Camara (1999) and Schroder (2004), the current pricing of options and the associated risk-neutral probabilities over the whole interval from now until the maturity date of the option. We do not model the option price process and the risk-neutral probability process, which can only be done in a dynamic market setting.

The outline of the article is as follows. In section 2, we review the traditional (one-dimensional) RNVR and discuss its limitations. We also derive *necessary* and sufficient conditions for the existence of this relationship. In section 3, we extend the idea of a RNVR to two dimensions. Section 4 looks at the special case where the underlying asset has a generalized lognormal distribution. Here we derive an option pricing formula which is a generalization of the Black-Scholes formula for the value of a European-style option. In Section 5, we illustrate option prices using a numerical example. In section 6, we summarise our conclusions.

2 One-Dimensional Risk-Neutral Valuation Relationships

In this section, we review the conditions under which a one-dimensional RNVR exists for the valuation of a contingent claim. We consider the valuation of European-style contingent claims, with maturity T, paying $c_T(x)$, which depend on a non-dividend paying underlying asset, whose payoff is x. We assume a no-arbitrage economy in which there exists a positive stochastic discount factor, ψ, that prices all assets. The price of the contingent claim, at time $t = 0$, is:

$$c = e^{-rT} E[c_T(x)\psi], \tag{1}$$

where e^{-rT} is the price of a risk-free zero-coupon bond paying \$1 at date T. For convenience, we can define a variable $\phi(x) = E(\psi|x)$. Then, we can write:

$$\begin{aligned} c &= e^{-rT} E[c_T(x)E(\psi|x)] \\ &= e^{-rT} E[c_T(x)\phi(x)]. \end{aligned} \tag{2}$$

We will refer to this asset specific (forward) pricing function $\phi(x)$ simply as 'the pricing kernel'. This is unambiguous, since throughout our analysis we are pricing claims on a particular underlying asset with payoff x. The pricing kernel, defined in this way, has the property $E[\phi(x)] = 1$. The contingent claim could, for example, be a call option with strike price K. In this case, we denote the price as C_K. The price of the underlying asset is the price of a call with strike price $K = 0$, hence this price is $S = C_{K=0}$. In this case,

$$S = e^{-rT} E[x\phi(x)]. \tag{3}$$

The above analysis can be embedded in an equilibrium pricing framework, as shown by Brennan (1979).[3] Brennan assumes a one-period, representative investor economy. In this economy, $\psi = \psi(w)$ is the 'relative marginal utility of wealth' of the representative investor. Brennan terms $\phi(x)$ the 'conditional expected relative marginal utility function' or more briefly as the 'conditional marginal utility function'.

First, we define the concept of a one-dimensional risk-neutral valuation relationship for contingent claims. Following Brennan(1979), we say that a one-dimensional RNVR exists

[3] Equation (2) above is equivalent to equation (5), page 57, in Brennan (1979).

for the valuation of contingent claims on an asset if the relationship between the price of the claim in (1) and the price of the asset in (3) is the same as it would be under risk neutrality. If a one-dimensional RNVR exists, it follows that the formula for the price of the contingent claim can be written as a function of S, and investor preference parameters do not enter the formula. In the literature, well known one-dimensional RNVRs are the Black-Scholes formula for options on assets with log-normal distributions and the Brennan (1979) formula for the price of an option on a normally distributed asset. Other one-dimensional RNVRs applying to assets following a displaced diffusion process and to options on multiple assets have been established by Rubinstein (1983), Stapleton and Subrahmanyam (1984), and Camara (2003).

However, the set of known one-dimensional RNVRs is limited both by the type of probability density function assumed and by the form of the pricing kernel required for the existence of a RNVR, given the probability distribution. This has led Heston (1993) to propose a generalization (of the set of one-dimensional RNVRs). Heston assumes that the pricing kernel $\phi(x)$ depends on some preference parameter, γ. If this is the case, we write the pricing kernel as $\phi(x) = \phi(x;\gamma)$. He also assumes that the probability density function (PDF) $f(x)$ depends on some parameter q. If this is the case, we write the PDF as $f(x) = f(x;q)$. From here on we will refer to $f(x)$ as the 'objective' PDF. Heston proposes a set of contingent claims formulae, where the price of the claim is independent of the preference parameter γ and of the parameter of the PDF, q. Such an *independence* relationship exists for contingent claims on an asset, if the relationship between the price of the claim and the price of the asset is independent of the preference parameter γ and of the PDF parameter q.

Heston (1993) establishes necessary and sufficient conditions for the existence of an independence relationship. He shows: given the pricing kernel $\phi(x;\gamma)$ and a probability density function $f(x;q)$, the pricing of contingent claims is independent of q and γ, if and only if the pricing kernel and the probability density function have the forms:

$$\phi(x;\gamma) = b(\gamma)h(x)e^{\gamma k(x)} \tag{4}$$

and

$$f(x;q) = a(q)g(x)e^{qk(x)}, \tag{5}$$

where $h(x)$ is not dependent on γ and $g(x)$ is not dependent on q. Clearly, the set of these independence relationships includes the set of one-dimensional RNVRs. This must be the case, since the parameter representing risk aversion drops out in the case of a one-dimensional RNVR.

We now follow Heston's general approach and investigate the set of possible one-dimensional RNVRs. We can characterize the one-dimensional RNVR as follows. Since, under risk neutrality the time-0 forward price of the asset *would* be the expected value of x, under the risk-neutral PDF with parameter q_0,

$$Se^{rT} = \int x f(x; q_0) dx, \tag{6}$$

and since $f(x;q)$ is of the form (5) we can solve (6) for the parameter $q_0 = q_0(S)$, which we call the 'risk neutral' value q. Then, a one-dimensional RNVR exists for the value of contingent claims if and only if

$$ce^{rT} = \int c_T(x) f(x; q_0) dx, \tag{7}$$

for all claims. That is, the forward price of the contingent claim has to be given by the expected value of the claim using the risk-neutral density $f(x;q_0)$, where q_0 is found from solving equation (6), given the forward price of the asset . This characterization provides us with an operational definition of a one-dimensional RNVR. We now derive conditions for the existence of a one-dimensional RNVR. Since the Heston conditions (4) and (5) have to hold for a one-dimensional RNVR to exist, we can write the forward price of the asset as

$$\begin{aligned} Se^{rT} &= \int x\phi(x) f(x; q) dx \\ &= \int x b(\gamma) a(q) h(x) g(x) e^{(\gamma+q)k(x)} dx. \end{aligned}$$

This equation can be rewritten as (6) if and only if $h(x) = 1$. Hence, a necessary and sufficient condition for a one-dimensional RNVR is that the Heston-conditions hold together with $h(x) = 1$. This proves Proposition 1, which provides the complete set of one-dimensional RNVRs.

Proposition 1 *[Necessary and Sufficient Conditions for a one-Dimensional Risk-Neutral Valuation Relationship]*

Let $k(x)$ be a monotonic function. Then there exists a one-dimensional RNVR for contingent claims on x, if and only if the pricing kernel $\phi(x;\gamma)$ is of the form

$$\phi(x; \gamma) = b(\gamma) e^{\gamma k(x)},\ b(\gamma) > 0, \tag{8}$$

and the objective probability density function of the underlying asset, x, is of the form

$$f(x; q) = a(q) g(x) e^{qk(x)},\ a(q)g(x) > 0, \tag{5}$$

Proposition 1 characterizes the set of possible one-dimensional RNVRs that can exist for the pricing of European-style contingent claims. We now investigate how large this set is. From (8) and (5), we obtain the risk-neutral probability distribution

$$\begin{aligned}\hat{f}(x) &= a(q)g(x)b(\gamma)e^{(q+\gamma)k(x)} \\ &= \hat{a}(q+\gamma)g(x)e^{(q+\gamma)k(x)},\end{aligned} \tag{9}$$

where $\hat{a}(q+\gamma)$ is a parameter which ensures that the probability mass equals one. The function $k(x)$ is common to both the probability density function of the asset and to the pricing kernel. In Rubinstein (1976), $k(x) = \ln x$, in Brennan (1979), $k(x) = x$, and in Camara (1999), $k(x) = \ln(x-a)$. In all three cases, $k(x)$ is normally distributed and conditions are found that satisfy Proposition 1. It is the fact that $k(x)$ is common that restricts the set of possible one-dimensional RNVRs. This commonality implies that in the risk-neutral probability density function only the sum of q and γ matters, not their individual values. It follows that these two parameters can be replaced by one observable price. Hence, if the sum $q+\gamma \equiv q_0$ is given, the pricing of the contingent claims is the same if $\gamma = \gamma_1$ and $q = q_1$ or if $\gamma = 0$ (risk neutrality) and $q = q_0$.

Proposition 1 also shows the limitations on the set of possible one-dimensional RNVRs. Suppose, for example, that we wish to price an option on a lognormal asset, where $k(x) = \ln x$. Then a one-dimensional RNVR exists if and only if $\phi(x) = b(\gamma)x^{\gamma}$. In this case the pricing kernel has elasticity: $\eta(x) = -\frac{\partial \ln \phi(x)}{\partial \ln x} = -\gamma$, a constant. However, if we suspect that the pricing kernel has declining elasticity with respect to x, then it follows that no one-dimensional RNVR exists. In order to price options in such economies, we need to generalize the concept of the RNVR. This is the task of the following section.

3 Two-Dimensional Risk-Neutral Valuation Relationships: General Results

In this section we assume that the pricing kernel $\phi(x)$ has non-constant elasticity with respect to x. First, we motivate this assumption in an equilibrium setting. Then, we derive conditions for a two-dimensional RNVR to exist.

We already noted that constant elasticity of $\psi(w)$ is consistent with an economy where the representative investor has wealth w and a constant relative risk averse utility function. In Franke, Stapleton and Subrahmanyam (1999) (FSS), an economy with a declining relative risk averse utility function is investigated. We now assume a similar economy, where the stochastic discount factor $\psi(w)$ has declining elasticity.

In order to generalize from the set of one-dimensional RNVRs, we consider again the stochastic discount factor, $\psi(w)$ in an equilibrium setting, as in Brennan (1979). The elasticity of $\psi(w)$ with respect to wealth is

$$-\frac{\partial \ln \ \psi(w)}{\partial \ln \ w} = -\frac{u''(w)}{u'(w)}w = r(w),$$

where $r(w)$ is the relative risk aversion of the representative investor. Then, if the representative investor has power utility, the elasticity of $\psi(w)$ is a constant.

Now consider the asset specific pricing kernel, $\phi(x)$. In the special case where x and w are joint-lognormal, the work of Brennan (1979) and Rubinstein (1976) establishes that the asset specific pricing kernel, $\phi(x)$ also has constant elasticity in this economy. However, FSS show that the weaker assumption, that x and w are log-linearly related, i.e.

$$\ln \ w = a + \beta \ln \ x + \varepsilon, \ \beta \neq 0,$$

with ε being distributed independently of x, is sufficient to establish the same result.[4] FSS, Theorem 4 establishes that the elasticity of the pricing kernel is

$$\eta(x) = \beta E\left[r(w)\frac{u'(w)}{E[u'(w)|x]}|x\right].$$

Hence, if $r(w)$ is a constant $\overline{r}$, then $\eta(x) = \beta\overline{r}$ is also constant.

Power utility of the representative investor is a strong assumption for the following reasons. First, although this function may be realistic for some investors, it is highly unlikely to apply to all investors. Second, in a important paper, Benninga and Mayshar (2000) show that even if all individual agents have constant relative risk averse utility, but the level of relative risk aversion differs between agents, the representative agent acts as if he has declining relative risk averse utility. We therefore now consider the case where the representative agent has non-constant relative risk aversion. We first establish the following result which generalizes Theorem 4 in FSS regarding the elasticity of the pricing kernel.

[4] Log-linearity is a less strong assumption for the relationship between the market wealth w and the asset payoff x than joint-lognormality. However, it is not necessary in order to find an explicit expression for the elasticity, $\eta(x)$, as shown below.

Assume that aggregate wealth w and the asset payoff x are related by

$$\ln\ w = h(\ln\ x) + \varepsilon, \tag{10}$$

where ε and $\ln\ x$ are independent, and $h(.)$ is an arbitrary twice differentiable function. Then, if $r(w)$ is the relative risk aversion of the representative investor, the elasticity of the asset specific pricing kernel is

$$\eta(x) = \frac{\partial h(\ln\ x)}{\partial \ln\ x} E\left[r(w)\frac{u'(w)}{E[u'(w)|x]}|x\right]. \tag{11}$$

The proof follows directly from FSS, Theorem 4, with β being replaced by $\frac{\partial h(\ln\ x)}{\partial \ln\ x}$. From (11) we derive a corollary which gives a sufficient condition for declining elasticity of the pricing kernel:

Suppose that the relative risk aversion of the representative investor $r(w)$ is declining in w. Then a sufficient condition for the elasticity of the asset specific pricing kernel, $\phi(x)$, to decline in x is that $\frac{\partial^2 \ln\ w}{\partial (\ln\ x)^2} \leq 0$.

Proof: Let $E[\ ] = E\left[r(w)\frac{u'(w)}{E[u'(w)|x]}|x\right]$. Then differentiating $\eta(x)$ in equation (11) we have

$$\eta'(x) = \frac{\partial^2 \ln\ w}{\partial (\ln\ x)^2} E[\ ] + \frac{\partial h(\ln\ x)}{\partial \ln\ x}\frac{\partial E[\ ]}{\partial \ln\ x}.$$

The second term is negative, as shown in the proof of FSS, Corollary 3. The first term is negative if $\frac{\partial^2 h(\ln\ x)}{\partial (\ln\ x)^2} = \frac{\partial^2 \ln\ w}{\partial (\ln\ x)^2} \leq 0$. □

The condition $\frac{\partial^2 \ln\ w}{\partial (\ln\ x)^2} \leq 0$ is fairly weak. It says that the growth in market wealth relative to the growth in the asset value does not increase with the asset value. Hence, there is good reason to believe that declining relative risk aversion of the representative investor translates into declining elasticity of the pricing kernel.

In the following analysis we assume that the asset specific pricing kernel has non-constant elasticity and generalize the one-dimensional RNVR to a two-dimensional RNVR. In the numerical example given in section 5, we assume that the elasticity of the asset-specific pricing kernel is declining.

The set of one-dimensional risk-neutral valuation relationships is severely restricted, as shown by the implications of Proposition 1 above. Hence, it is not surprising that observed option prices deviate from those predicted by one-dimensional RNVRs. In order to price contingent claims in less restricted economies, where the pricing kernel is more complex, we need to relax the conditions imposed on the pricing kernel and on the asset's probability distribution function. For example, how can we price an option on an asset with a generalized lognormal distribution, if the pricing kernel has non-constant elasticity? One possible answer lies in a generalization of the one-dimensional RNVR to a two-dimensional RNVR. In this relationship we make use of the underlying asset price and the price of one option on the asset. Formally, we say that a *two-dimensional risk-neutral valuation relationship* exists for the pricing of contingent claims on an asset, if the relationship between the price of a claim and the two prices (of the underlying asset and one other claim on the asset) is the same as it would be under risk neutrality.

We now establish conditions for the existence of such a two-dimensional RNVR:

Proposition 2 *[Necessary and Sufficient Conditions for a Two-Dimensional-Risk-Neutral Valuation Relationship]*

Assume that $k_1(x)$ and $k_2(x)$ are monotonic functions. Then, a two-dimensional risk-neutral valuation relationship exists for the valuation of contingent claims on x, if and only if the pricing kernel and the objective probability density function have the forms:

$$\phi(x) = b(\gamma_1, \gamma_2)e^{\gamma_1 k_1(x)+\gamma_2 k_2(x)},\ b(\gamma_1, \gamma_2) > 0 \tag{12}$$

and

$$f(x) = a(q_1, q_2)g(x)e^{q_1 k_1(x)+q_2 k_2(x)},\ a(q_1, q_2)g(x) > 0 \tag{13}$$

where $g(x)$ is not dependent on q_1 and q_2.

Proof:

To show sufficiency, assume we know the price of the underlying asset, S, and of one other claim, c_a. If the pricing kernel and the objective probability density function have the above form, then the risk-neutral PDF is:

$$\begin{aligned} \hat{f}(x) &= \phi(x)f(x) \\ &= a(q_1, q_2)b(\gamma_1, \gamma_2)g(x)e^{(\gamma_1+q_1)k_1(x)+(\gamma_2+q_2)k_2(x)} \\ &\equiv \hat{a}(q_1+\gamma_1, q_2+\gamma_2)g(x)e^{(\gamma_1+q_1)k_1(x)+(\gamma_2+q_2)k_2(x)}. \end{aligned} \tag{14}$$

Given the forward prices:

$$Se^{rT} = \int_0^{+\infty} x\phi(x)f(x)dx,$$

$$c_a e^{rT} = \int_0^{+\infty} c_a(x)\phi(x)f(x)dx,$$

we can solve for $\gamma_1 + q_1 = q_{1,0}(S, c_a)$ and $\gamma_2 + q_2 = q_{2,0}(S, c_a)$. Hence, the risk-neutral PDF depends on the prices S and c_a and is independent of the unknown preference parameters γ_1 and γ_2 and the unknown probability density function parameters q_1 and q_2. Necessity is established in the appendix.□

The intuition behind Proposition 2 is similar to that behind Proposition 1. Here, the pricing kernel depends on two parameters γ_1 and γ_2, so that the elasticity of the pricing kernel is not constant. The levels of these parameters determine two parameters of the probability density function of the asset, q_1 and q_2. A two-dimensional RNVR exists if there are two functions, $k_1(x)$ and $k_2(x)$ which are common to the pricing kernel and the probability density function. Then, in the risk-neutral probability density function, $k_1(x)$ is multiplied by $q_{1,0} = q_1 + \gamma_1$ and $k_2(x)$ is multiplied by $q_{2,0} = q_2 + \gamma_2$. Then, $q_{1,0}$ and $q_{2,0}$ can be replaced by functions of the underlying asset price and the price of one option on the asset.

Some care is required in interpreting the two-dimensional RNVR. Although the relationship is the same as it would be under risk neutrality, it is not a RNVR in the conventional sense. It is a valuation relationship that requires both the price of the underlying asset and the *price of a contingent claim on the asset* in order to price any other claim. The second claim required could be an at-the-money call option, for example. If a two-dimensional RNVR exists, any option can be priced if we know the price of the underlying asset and the price of the at-the-money call option. Given that a two-dimensional RNVR holds, two observable prices replace four unknown parameters, γ_1, γ_2, q_1, and q_2. The reason is that in the risk-neutral probability density function only the sums $(q_1 + \gamma_1)$ and $(q_2 + \gamma_2)$ matter.[5]

4 A Two-Dimensional RNVR: The Generalized Lognormal Case

An important special case of the two-dimensional RNVR is the example where the underlying asset has a generalized lognormal distribution with two unknown parameters and the pricing kernel has two unknown parameters. In this case, the two-dimensional risk-neutral valuation relationship leads to a straightforward generalization of the Black-Scholes formula for the price of a European-style call option.

[5]Proposition 2 is extendable to three or more dimensions, as shown in Appendix 7.3. These extensions might be useful if more option prices are available and neither a one-dimensional nor a two-dimensional RNVR exists.

First, we assume that the underlying asset payoff has a generalized lognormal probability density function of the form:

$$f(x) = a(q_1, q_2) g(x) e^{q_1 \ln x + q_2 k_2(x)}, \tag{15}$$

with

$$g(x) = \frac{e^{-(\ln x)^2/2\sigma^2}}{x}.$$

Note that, if $q_2 = 0$, $f(x)$ is lognormal with

$$a(q_1) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\mu^2/2\sigma^2}.$$

However, if $q_2 k_2'(x) < 0$, the distribution departs from lognormality and has a fat left tail.[6]

Also, assume that the pricing kernel is of the form:

$$\phi(x) = b(\gamma_1, \gamma_2) e^{\gamma_1 \ln x + \gamma_2 k_2(x)}.$$

Note that for $\gamma_1 \neq 0$ and $\gamma_2 \neq 0$, we know from Proposition 1 that a one-dimensional RNVR does not exist.

The price of the underlying asset, the at-the-money call option, c_a, and the call option with strike price K, are given by the equations:

$$S = e^{-rT} \int_0^{+\infty} x \hat{f}(x) dx, \tag{16}$$

$$c_a = e^{-rT} \int_0^{+\infty} c_a(x) \hat{f}(x) dx, \tag{17}$$

$$c_K = e^{-rT} \int_0^{+\infty} c_K(x) \hat{f}(x) dx, \tag{18}$$

where the risk-neutral probability density function $\hat{f}(x)$ given by (14) depends only on the parameters $q_{1,0}$ and $q_{2,0}$. Hence we can solve (16) and (17) for these parameters and then use (18) to price the remaining options. Since $q_{1,0}$ determines $\hat{\mu}$, in the following we replace $q_{1,0}$ with $\hat{\mu}$, where $q_{1,0} = \hat{\mu}/\sigma^2$ and $\hat{\mu}$ is the mean of the lognormal distribution $n(\ln x, \hat{\mu}, \sigma)$.

[6] See below, figures 1 and 2 for examples of the distribution shape.

In the appendix[7], we show that the value of a call option with strike price K is given by the 'generalized Black-Scholes' equation:

$$c_K = S\left[1 - \frac{G(K, \hat{\mu}+\sigma^2, \sigma, q_{2,0})}{G(\hat{\mu}+\sigma^2, \sigma, q_{2,0})}\right] - Ke^{-rT}\left[1 - \frac{G(K, \hat{\mu}, \sigma, q_{2,0})}{G(\hat{\mu}, \sigma, q_{2,0})}\right], \tag{19}$$

where

$$\begin{aligned} G(\hat{\mu}, \sigma, q_{2,0}) &\equiv \int_{-\infty}^{\infty} e^{q_{2,0}k_2(x)} n(\ln x, \hat{\mu}, \sigma) d\ln x, \\ G(K, \hat{\mu}, \sigma, q_{2,0}) &\equiv \int_{-\infty}^{\ln k} e^{q_{2,0}k_2(x)} n(\ln x, \hat{\mu}, \sigma) d\ln x. \end{aligned}$$

$G(\hat{\mu}+\sigma^2, \sigma, q_{2,0})$ and $G(K, \hat{\mu}+\sigma^2, \sigma, q_{2,0})$ are defined in the same manner replacing $\hat{\mu}$ by $\hat{\mu}+\sigma^2$.

If $q_2 = 0$, we have a one-dimensional RNVR and the general formula (19) reduces to the Black-Scholes formula for the price of the call option, replacing $\hat{\mu}$ by $[\ln S + rT - \sigma^2/2]$. To obtain this, solve equation (16) for the risk-neutral mean. We have $G(\hat{\mu}, \sigma, 0) = 1$ and

$$G(K, \hat{\mu}, \sigma, 0) = N\left(\frac{\ln K - \hat{\mu}}{\sigma}\right).$$

Also, $G(\hat{\mu}+\sigma^2, \sigma, 0) = 1$ and

$$1 - G(K, \hat{\mu}+\sigma^2, \sigma, 0) = N\left(\frac{\hat{\mu} - \ln K + \sigma^2}{\sigma}\right).$$

We now illustrate this two-dimensional RNVR, in the following section, with a calibrated numerical example.

5 A Numerical Example of the Two-Dimensional RNVR

In this section we illustrate the two-dimensional RNVR in the case of the generalized log-normal distribution, using a numerical example. First, we assume that $k_1(x) = \ln x$ as in

[7]The proof in the appendix derives the value of a call option for the n-dimensional case. Equation (19) gives the formula for the special case where $n = 2$.

section 4, above. The justification for this assumption is that we are looking here for a generalization of the Black-Scholes formula. Second, we assume that

$$k_2(x) = \frac{1}{(x^\alpha + \varepsilon)},$$

where $\alpha > 0$ and $\varepsilon > 0$. Given $\gamma_1 < 0$, $\gamma_2 > 0$, this function has the properties required for positive, declining elasticity of the pricing kernel. ε is a small positive constant which assures integrability.

For illustrative purposes we assume the asset has a mean payoff equal to 1. We assume that the options have one year to maturity and that the forward price of the underlying asset is $x_0 = 0.9400$, representing a 6% risk premium. Note however, that the option prices in the model do not depend on this assumption regarding the true mean, which is in fact irrelevant, just as it is in the Black-Scholes model. For convenience, we also assume a risk-free interest rate equal to zero.

We calibrate the example to the case of an option on a stock price index. We assume that the volatility of the underlying asset is 20%, which approximates to current estimates of the historical volatility of the typical index. However, there is some evidence that the implied volatility for at-the-money options on the index exceeds the historical volatility. In the examples below, we assume that this 'excess volatility' is in the range of 2% to 4%, which is in line with recent estimates. For example, Corrado and Miller (2003) provide estimates of realised volatility and implied volatility for the S&P 100 index for the periods 1988-94 and 1995-2002. In the earlier period the mean realised volatility was 14.75% and the mean implied volatility was 17.84%, an excess volatility of 3.09%. In the later period the mean realised volatility was 21.11% and the mean implied volatility was 24.01%, an excess volatility of 2.90%.

We proceed as follows:

1. First, we use the Black-Scholes model to price the at-the-money option, given a forward price of $Se^{rT} = 0.94$, with the assumed excess volatility of either 2% or 4%.

2. Next, we solve the pair of equations, (16), (17) for values of $q_{1,0}$ and $q_{2,0}$, given the true volatility of 20%.

3. Then, using $q_{1,0}$ and $q_{2,0}$, we solve for option prices in (18), for a range of strike prices, K, given the true volatility.

4. Finally, we invert the Black-Scholes formula, using these prices, in order to report implied volatilities.

In Table 1, we show the option prices that result from these calculations, for the case where $\alpha = 8$ and $\varepsilon = 0.001$. In the table we assume that the excess volatility of the at-the-money option is either 2% or 4%. These two cases are then compared with the Black-Scholes prices.

Solving for the values of $q_{1,0}$ and $q_{2,0}$ in the case of 2% excess volatility yields $q_{1,0} = -1.4185$ and $q_{2,0} = 0.0092$. In the case of 4% excess volatility, the corresponding values are $q_{1,0} = -0.9306$ and $q_{2,0} = 0.0118$. In Figures 1 and 2, we plot the risk-neutral densities for these two cases and compare them in each case with the lognormal density. In both cases the resulting distribution is fat-tailed, especially at the lefthand, low side. When $q_{1,0} = -1.4185$, $q_{2,0} = 0.0092$, the skewness is SK = 0.3137, which compares with 0.6143 for the lognormal distribution. The kurtosis is KU = 3.6244 which compares with 3.6784 for the lognormal distribution. The corresponding values in the case where $q_{1,0} = -0.9306$, $q_{2,0} = 0.0118$, are SK = 0.0342 and KU = 3.5732.[8] Using these parameter values and valuing the options with strike prices ranging from 0.5 to 1.6, yields the call prices shown in column 3 and 5 of Table 1. The implied volatilities are shown in brackets. The Black-Scholes prices are shown in the final column. The results show a 'smile' in the implied volatility function with a shape that is similar to that documented in studies of index option prices. This is illustrated in Figures 3 and 4. It shows volatilities that first decline relatively steeply, especially in the case of 4% excess volatility, and then flatten out.

We further illustrate the two-dimensional RNVR by pricing a given option at different asset forward prices. The option has a strike price of $K = 1.1$. In Table 2, we show prices computed by recalibrating the model so that at each asset forward price, the at-the-money option has an implied volatility of 20%, 22%, and 24% respectively. We then use the model in equation (19) to price the option with strike price $K = 1.1$. The two-dimensional nature of the pricing relationship is illustrated in Figure 5. In this case the option forward prices, for a given forward price, are in a range whose maximum is the price in the 24% at-the-money case and whose minimum is the Black-Scholes value at an implied volatility of 20%.

6 Conclusions

The idea of a risk-neutral valuation relationship is at the core of option pricing theory. However, as shown by Proposition 1, the set of one-dimensional RNVRs is quite limited.

[8] Since the lognormal PDF is skewed to the right, the fat tail on the lefthand side of the generalized lognormal PDF reduces the skewness. It may be surprising that the kurtosis of the generalized lognormal PDF is lower than for the lognormal PDF. For the generalized lognormal PDF, both the second and fourth moments are higher than for the lognormal PDF, but the kurtosis, being a ratio of the two moments, is smaller.

Essentially, the pricing kernel may depend on one preference parameter only. For example, if the underlying asset has a log-normal distribution, a necessary condition for a one-dimensional RNVR is that the pricing kernel has constant elasticity. However, empirical and theoretical research suggests that the pricing kernel has declining elasticity. When looking for less restrictive option pricing models, a more promising approach therefore is to look for a two-dimensional RNVR, which can be used to value options if the underlying asset price, and in addition the price of one other contingent claim on the asset, are known. In the case of a two-dimensional RNVR the pricing kernel may depend on two preference parameters. The numerical examples illustrate that these results can be used to price options on assets with 'fat-tailed' distributions and generate volatility smiles.

In the case of the Black-Scholes model, a valid procedure is to form a probability distribution for the underlying asset price which is constructed 'as if' the world was risk neutral. This distribution is then used to value options using the assumption of risk neutrality. In the case of our two-dimensional extension the procedure is the same. First we construct a distribution for the underlying asset which is consistent, in a risk-neutral world, with the observed prices of the asset and of one option on the asset. We then proceed to value all options on the asset under the assumption of risk neutrality. These are then the correct option prices, if the conditions for a two-dimensional RNVR are met.

Although we have concentrated on examples extending the Black-Scholes RNVR to account for non-constant elasticity of the pricing kernel, a similar methodology could be applied to generalize the Brennan (1979) normal distribution option pricing model. Here we would assume non-constant *absolute* risk aversion of the pricing kernel. Similarly, extensions are possible for the Rubinstein (1983) displaced diffusion model.

The two-dimensional RNVR relates a contingent claim price to any two other contingent claim prices on the same underlying asset. This analysis admits to a further extension. We could price claims using an n-dimensional RNVR. However, as the number of claims used in the pricing increases, we are in effect merely describing the shape of the pricing kernel in more detail, as in some of the empirical studies quoted in the introduction. The key to option pricing is to price contingent claims with as few pieces of information as possible. Therefore, the analysis here has suggested extending the pricing of claims from the rather unrealistic search for one-dimensional RNVRs of the Black-Scholes type, to the less restrictive and hopefully more available set of two-dimensional RNVRs.

The key advantage of the two-dimensional RNVR, over other methodologies which use option prices to estimate the risk-neutral density, is that only one option price is required, in addition to the underlying asset price. Also, the functional relationship between the price of an option with strike price K, and the asset price and the price of the at-the-money option, is preference free, as in the case of traditional one-dimensional RNVRs.

7 Appendix

7.1 Proof of Proposition 2

First, we establish necessary and sufficient conditions for a two-dimensional *independence* relationship. Extending the idea in Heston (1993), we say that a two-dimensional independence relationship exists for the pricing of contingent claims on an asset, if the relationship between the price of the claim and the prices of the underlying asset and one other contingent claim on the asset is independent of two preference parameters γ_1 and γ_2 and two probability density function parameters q_1 and q_2.

Lemma 1 *[Necessary and Sufficient Conditions for a Two-Dimensional Independence Relationship]*

Assume that $k_1(x)$ and $k_2(x)$ are monotonic functions, then a two-dimensional independence relationship exists if and only if the pricing kernel and the objective probability density function have the forms

$$\phi(x) = b(\gamma_1, \gamma_2)h(x)e^{\gamma_1 k_1(x)+\gamma_2 k_2(x)},\ b(\gamma_1, \gamma_2)h(x) > 0 \tag{20}$$

and

$$f(x) = a(q_1, q_2)g(x)e^{q_1 k_1(x)+q_2 k_2(x)},\ a(q_1, q_2)g(x) > 0,$$

where $h(x)$ is not dependent on γ_1 and γ_2, and $g(x)$ is not dependent on q_1 and q_2.

Proof

Sufficiency

If the pricing kernel and the objective probability density function have the above forms, then the risk neutral PDF:

$$\hat{f}(x) = f(x)\phi(x)$$

depends on the sums $q_{1,0} = q_1 + \gamma_1$ and $q_{2,0} = q_2 + \gamma_2$ and not on the individual parameter values. $q_{1,0}$ and $q_{2,0}$ are functions of the underlying asset price and the price of one other contigent claim. Hence, any contingent claim can be priced independently of the individual parameter values.

Necessity

Since the time 0-price of any contingent claim paying $c(x)$, given by

$$ce^{rT} = \int f(x)\phi(x)c(x)dx, \tag{21}$$

is independent of γ_1 and γ_2, differentiating with respect to $\gamma_i,\ i = 1, 2$, we have

$$\int c(x)\frac{\partial}{\partial\gamma_i}[f(x)\phi(x)]dx = 0,\ i = 1, 2.$$

Now, since this must hold for any contingent claim, it follows that

$$\frac{\partial}{\partial\gamma_i}[f(x)\phi(x)] = 0,\ i = 1, 2, \forall x.$$

Rewriting in logarithms, it then follows that

$$\frac{\partial \ln f(x)}{\partial q_1}\frac{\partial q_1}{\partial\gamma_i} + \frac{\partial \ln f(x)}{\partial q_2}\frac{\partial q_2}{\partial\gamma_i} + \frac{\partial \ln\phi(x)}{\partial\gamma_i} = 0,\ i = 1, 2. \tag{22}$$

Now consider given values of the parameters $q_i = q_i^*,\ i = 1, 2$. Writing $k_i(x) \equiv \frac{\partial \ln f(x)}{\partial q_i}|q_i = q_i^*$,

we obtain

$$\frac{\partial \ln\phi(x)}{\partial\gamma_1} = -[k_1(x)a_1 + k_2(x)a_2], \tag{23}$$

$$\frac{\partial \ln\phi(x)}{\partial\gamma_2} = -[k_1(x)b_1 + k_2(x)b_2], \tag{24}$$

where

$$a_i = \frac{\partial q_i}{\partial\gamma_1}|q_i = q_i^*,\ i = 1, 2,$$

$$b_i = \frac{\partial q_i}{\partial\gamma_2}|q_i = q_i^*.\ i = 1, 2.$$

From (23), it follows that

$$\ln\phi(x) = a_0(x) - k_1(x)\int a_1 d\gamma_1 - k_2(x)\int a_2 d\gamma_1, \tag{25}$$

where $a_0(x)$ is independent of γ_1. Differentiating (25) w.r.t. γ_2, we have

$$\frac{\partial \ln\phi(x)}{\partial\gamma_2} = \frac{\partial a_0(x)}{\partial\gamma_2} - k_1(x)\frac{\partial}{\partial\gamma_2}\int a_1 d\gamma_1 - k_2(x)\frac{\partial}{\partial\gamma_2}\int a_2 d\gamma_1.$$

Comparing this with (24), we obtain

$$a_0(x) = a_{00}(x) + a_{01}k_1(x) + a_{02}k_2(x),$$

where $a_{00}(x)$ is independent of γ_1 and γ_2.

From the above equation and (25), we conclude that $\phi(x)$ can be written as

$$\phi(x) = B_0 h(x) \exp\{B_1 k_1(x) + B_2 k_2(x)\},$$

where $h(x)$ is independent of γ_1 and γ_2. We can view B_1 and B_2 as two preference parameters. Thus $\phi(x)$ has the following form

$$\phi(x) = b(\gamma_1, \gamma_2) h(x) \exp\{\gamma_1 k_1(x) + \gamma_2 k_2(x)\}.$$

Substituting this into (22), we obtain

$$\frac{\partial \ln f(x)}{\partial q_1}\frac{\partial q_1}{\partial \gamma_i} + \frac{\partial \ln f(x)}{\partial q_2}\frac{\partial q_2}{\partial \gamma_i} + k_i(x) + \frac{\partial b}{\partial \gamma_i} = 0,\ i = 1, 2.$$

Since γ_1 and γ_2 are two independent parameters, $k_1(x)$ and $k_2(x)$ must be linearly independent. Hence we must have

$$\frac{\partial q_1}{\partial \gamma_1}\frac{\partial q_2}{\partial \gamma_2} - \frac{\partial q_1}{\partial \gamma_2}\frac{\partial q_2}{\partial \gamma_1} \neq 0. \tag{26}$$

This implies that we can write (at least for a neighborhood)

$$\gamma_i = \gamma_i(Se^{rT}, ce^{rT}; q_1, q_2),\ i = 1, 2.$$

The last three equations imply that

$$\frac{\partial \ln f(x)}{\partial q_i} = u_i k_1(x) + v_i k_2(x) + w_i,\ i = 1, 2.$$

It follows that

$$f(x) = \beta_0 g(x) \exp\{\beta_1 k_1(x) + \beta_2 k_2(x)\},$$

where $g(x)$ is independent of q_1 and q_2. We can view β_1 and β_2 as two parameters; thus $f(x)$ has the following form

$$f(x) = a(q_1, q_2) g(x) \exp\{q_1 k_1(x) + q_2 k_2(x)\}.\ \ \square$$

The lemma can now be used to establish necessity in Proposition 2. Since the set of two-dimensional RNVRs is a subset of the set of two-dimensional independence valuation relationships, it is necessary that the pricing kernel and PDF have the forms stated in Lemma 1. Now suppose that $h_2(x)$ is not a constant. Then $h_2(x)$ is dependent upon preference parameters other than γ_1 and γ_2. Also, the risk-neutral PDF would depend on these parameters, ruling out a two-dimensional RNVR. Hence $h_2(x)$ must be a constant and without loss of generality, $h_2(x) = 1$. This proves necessity in Proposition 2.

7.2 An n-Dimensional Risk-Neutral Valuation Relationship

Corollary 1 *Let $k_i(x)$ be a monotonic function, $i = 1, ..., n$. Given the pricing kernel $\phi(x) = \phi(x : \gamma_1, \gamma_2, ..., \gamma_n)$ and an objective PDF $f(x) = f(x : q_1, q_2, ..., q_n)$, the relationship between the price of a contingent claim on an asset and the prices of any n other contingent claims on the asset is independent of $(\gamma_1,\ \gamma_2, ..., \gamma_n)$ and $(q_1, q_2, ..., q_n)$ if and only if the pricing kernel and the objective PDF have the form*

$$\phi(x) = b(\gamma_1, \gamma_2, ..., \gamma_n)e^{\gamma_1 k_1(x)+\gamma_2 k_2(x)+...+\gamma_n k_n(x)},\ b(\gamma_1, \gamma_2, ..., \gamma_n) > 0$$

and

$$f(x) = a(q_1, q_2, ..., q_n)g(x)e^{q_1 k_1(x)+q_2 k_2(x)+...+q_n k_n(x)},\ a(q_1, q_2, ..., q_n)g(x) > 0. \qquad (27)$$

Proof

The proof follows the same lines as the proof of Proposition 2.

7.3 Derivation of the n-Dimensional RNVR Call Option Price for a Generalised Lognormal Distribution.

We assume that the risk-neutral PDF has the form:

$$\hat{f}(\ln x) = a(\hat{\mu}, q_{2,0}, q_{3,0}, ..., q_{n,0})e^{\sum_{j=2}^{n} q_{j,0}k_j(x)}n(\ln x, \hat{\mu}, \sigma).$$

First we define:

$$\begin{aligned} G(\hat{\mu}, \sigma, q_{2,0}, ..., q_{n,0}) &= \int_{-\infty}^{\infty} e^{\sum_{j=2}^{n} q_{j,0}k_j(x)}n(\ln x, \hat{\mu}, \sigma)d\ln x, \\ G(K, \hat{\mu}, \sigma, q_{2,0}, ..., q_{n,0}) &= \int_{-\infty}^{\ln K} e^{\sum_{j=2}^{n} q_{j,0}k_j(x)}n(\ln x, \hat{\mu}, \sigma)d\ln x. \end{aligned}$$

It follows that

$$a(\hat{\mu}, q_{2,0}, ..., q_{n,0}) = \frac{1}{G(\hat{\mu}, \sigma, q_{2,0}, ..., q_{n,0})}.$$

The forward price of the call option with strike price K is then given by

$$\begin{aligned} c(K)e^{rT} &= \int_{\ln K}^{\infty} xa(\hat{\mu}, q_{2,0}, ..., q_{n,0})e^{\sum_{j=2}^{n} q_{j,0}k_j(x)}n(\ln x, \hat{\mu}, \sigma)d\ln x \\ &- K\int_{\ln K}^{\infty} a(\hat{\mu}, q_{2,0}, ..., q_{n,0})e^{\sum_{j=2}^{n} q_{j,0}k_j(x)}n(\ln x, \hat{\mu}, \sigma)d\ln x. \end{aligned}$$

Note that, from Rubinstein (1976),

$$xn(\ln x, \hat{\mu}, \sigma) = e^{\hat{\mu}+\frac{1}{2}\sigma^2}n(\ln x, \hat{\mu}+\sigma^2, \sigma).$$

Hence,

$$\begin{aligned} a(\hat{\mu}, q_{2,0}, ..., q_{n,0}) & \int_{\ln K}^{\infty} xe^{\sum_{j=2}^{n} q_{j,0}k_j(x)}n(\ln x, \hat{\mu}, \sigma)d\ln x \\ = & \quad a(\hat{\mu}, q_{2,0}, ..., q_{n,0})e^{\hat{\mu}+\frac{1}{2}\sigma^2}\int_{\ln K}^{\infty} e^{\sum_{j=2}^{n} q_{j,0}k_j(x)}n(\ln x, \hat{\mu}+\sigma^2, \sigma)d\ln x. \end{aligned}$$

Now, using the definitions above, the forward price of the call option is

$$\begin{aligned} c(K)e^{rT} &= e^{\hat{\mu}+\frac{1}{2}\sigma^2}\frac{a(\hat{\mu}, q_{2,0}, ..., q_{n,0})}{a(\hat{\mu}+\sigma^2, q_{2,0}, ..., q_{n,0})}\left[1 - \frac{G(K, \hat{\mu}+\sigma^2, \sigma, q_{2,0}, ..., q_{n,0})}{G(\hat{\mu}+\sigma^2, \sigma, q_{2,0}, ..., q_{n,0})}\right] \\ &- K\left[1 - \frac{G(K, \hat{\mu}, \sigma, q_{2,0}, ..., q_{n,0})}{G(\hat{\mu}, \sigma, q_{2,0}, ..., q_{n,0})}\right] \end{aligned}$$

Note that the stock is the call option with a strike price of 0. Hence from the above equation, the price of the stock, S, is given by

$$S = e^{-rT} e^{\hat{\mu}+\frac{1}{2}\sigma^2} \frac{a(\hat{\mu}, q_{2,0}, ..., q_{n,0})}{a(\hat{\mu}+\sigma^2, q_{2,0}, ..., q_{n,0})}.$$

Hence the price of the option is

$$c(K) = S\left[1 - \frac{G(K, \hat{\mu}+\sigma^2, \sigma, q_{2,0}, ..., q_{n,0})}{G(\hat{\mu}+\sigma^2, \sigma, q_{2,0}, ..., q_{n,0})}\right] - Ke^{-rT}\left[1 - \frac{G(K, \hat{\mu}, \sigma, q_{2,0}, ..., q_{n,0})}{G(\hat{\mu}, \sigma, q_{2,0}, ..., q_{n,0})}\right].$$

Taking the special case where $n = 2$, then gives equation (19) in the text.

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Table 1 : Two-Dimensional RNVR: Call Option Forward Prices

Strike K	K/x_0	$c_a(22\%)$	σ_I	$c_a(24\%)$	σ_I	$c_a(20\%)$	σ_I
0.50	0.53	0.4413	(30.1)	0.4442	(36.5)	0.4400	(20)
0.60	0.64	0.3442	(27.3)	0.3498	(33.0)	0.3406	(20)
0.70	0.74	0.2514	(24.7)	0.2593	(29.2)	0.2450	(20)
0.80	0.85	0.1687	(23.2)	0.1776	(26.3)	0.1606	(20)
0.90	0.96	0.1030	(22.2)	0.1111	(24.5)	0.0950	(20)
0.94	1.00	0.0823	(22.0)	0.0898	(24.0)	0.0749	(20)
1.00	1.06	0.0572	(21.7)	0.0635	(23.4)	0.0509	(20)
1.10	1.17	0.0292	(21.4)	0.0333	(22.7)	0.0250	(20)
1.20	1.28	0.0138	(21.2)	0.0162	(22.3)	0.0114	(20)
1.30	1.38	0.0061	(21.0)	0.0074	(21.9)	0.0049	(20)
1.40	1.49	0.0026	(20.9)	0.0032	(21.7)	0.0020	(20)
1.50	1.60	0.0011	(20.8)	0.0013	(21.5)	0.0008	(20)
1.60	1.70	0.0004	(20.7)	0.0005	(21.4)	0.0003	(20)

1. Column 1 and 2 respectively show the strike price and the strike price ratio (given a forward price of the asset of $x_0 = 0.9400$). Columns 3, 5 and 7 show the forward prices of one-year maturity call options, for the cases where the implied volatilities of the at-the-money option are 24%, 22%, and 20% respectively. Implied volatilities are shown in brackets. In the last case, prices are those from the Black-Scholes model.

2. Forward prices are computed as expectations under the risk-neutral probability density function. The risk-neutral density is

$$\hat{f}(x) = a_0 e^{q_{2,0} \frac{1}{(x^\alpha + \varepsilon)}} n(\ln x; \hat{\mu}, \sigma)$$

   with $\alpha = 8$, $\varepsilon = 0.001$ and $\sigma = 0.20$.

3. The model is calibrated so that the at-the-money option, with strike price $K = 0.9400$, sells at a Black-Scholes implied volatility of 24%, 22%, or 20% respectively. This determines the values of $q_{1,0}$ and $q_{2,0}$. For example, given an implied volatility of 22%, we have $q_{1,0} = -1.4185$ and $q_{2,0} = 0.0092$.

Table 2 : Two-Dimensional RNVR:
Call Option Forward Price and Asset Forward Price
Strike Price $K = 1.1$

Asset price	$\sigma_{I,a} = 20\%$	$\sigma_{I,a} = 22\%$	$\sigma_{I,a} = 24\%$
0.8	0.0044	0.0057	0.0070
0.9	0.0164	0.0196	0.0228
1.0	0.0429	0.0488	0.0546
1.1	0.0876	0.0964	0.1051
1.2	0.1501	0.1614	0.1727
1.3	0.2269	0.2403	0.2534

1. Column 1 shows the underlying asset's forward price. Columns 2, 3 and 4 show the forward prices of the call option, with strike price $K = 1.1$, given that the implied volatilities of the at-the-money options are, $\sigma_{I,a} = 20\%$, 22%, and 24% respectively. In the first case, prices are those from the Black-Scholes model.

2. The risk-neutral probability density function is approximated as

$$\hat{f}(x) = a_0 e^{q_{2,0} \frac{1}{(x^\alpha + \varepsilon)}} n(\ln x; \hat{\mu}, \sigma)$$

   with $\alpha = 8$, $\varepsilon = 0.001$ and $\sigma = 0.20$.

3. For each $\sigma_{I,a}$, the model is calibrated for each asset price, so that the at-the-money option sells at the given Black-Scholes implied volatility.

Notes for Figures

1. In Figure 1, the lognormal density is based on $\mu = 0.94$ and $\sigma = 0.2$.
2. In Figure 2, the lognormal density is based on $\mu = 0.94$ and $\sigma = 0.2$.
3. Figure 3 plots the implied volatilities shown in column 4 of Table 1.
4. Figure 4 plots the implied volatilities shown in column 6 of Table 1.
5. Figure 5 plots the prices of call options shown in columns 2, 3, and 4 of Table 2. The lower/ middle/ upper curves show prices when the excess volatility of the at-the-money option is 0%, 2%, and 4% respectively.

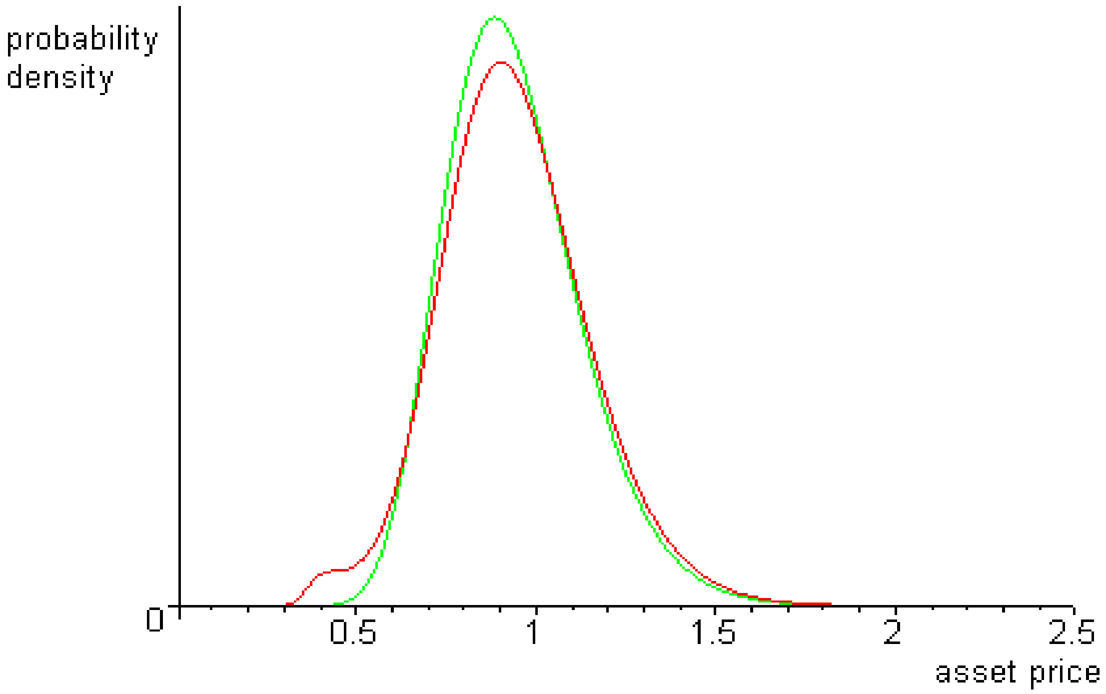

Figure 1: Risk Neutral and Lognormal Probability Density; Excess volatility 2%

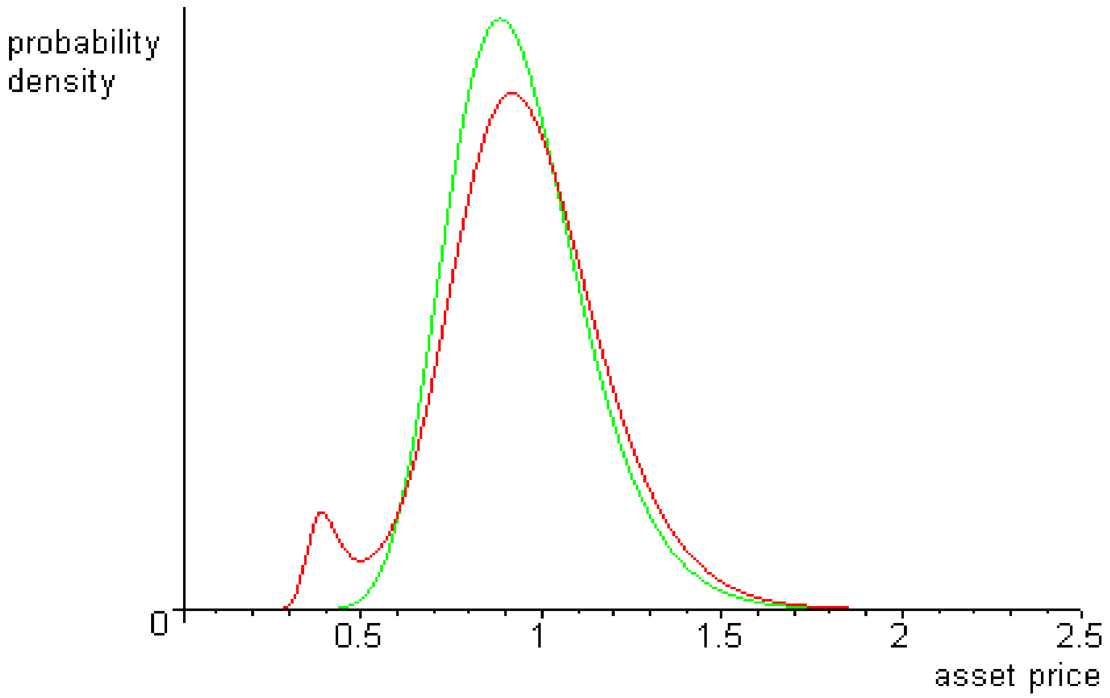

Figure 2: Risk Neutral and Lognormal Probability Density; Excess volatility 4%

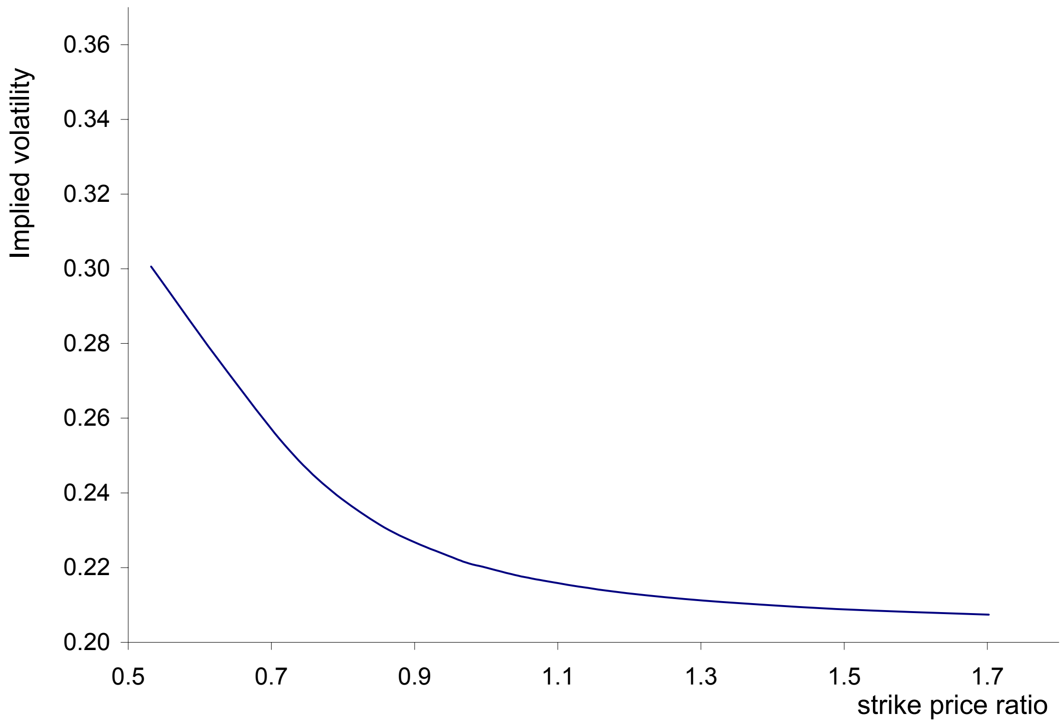

Figure 3: Implied volatility smile, excess volatility = 2%; sigma = 0,20

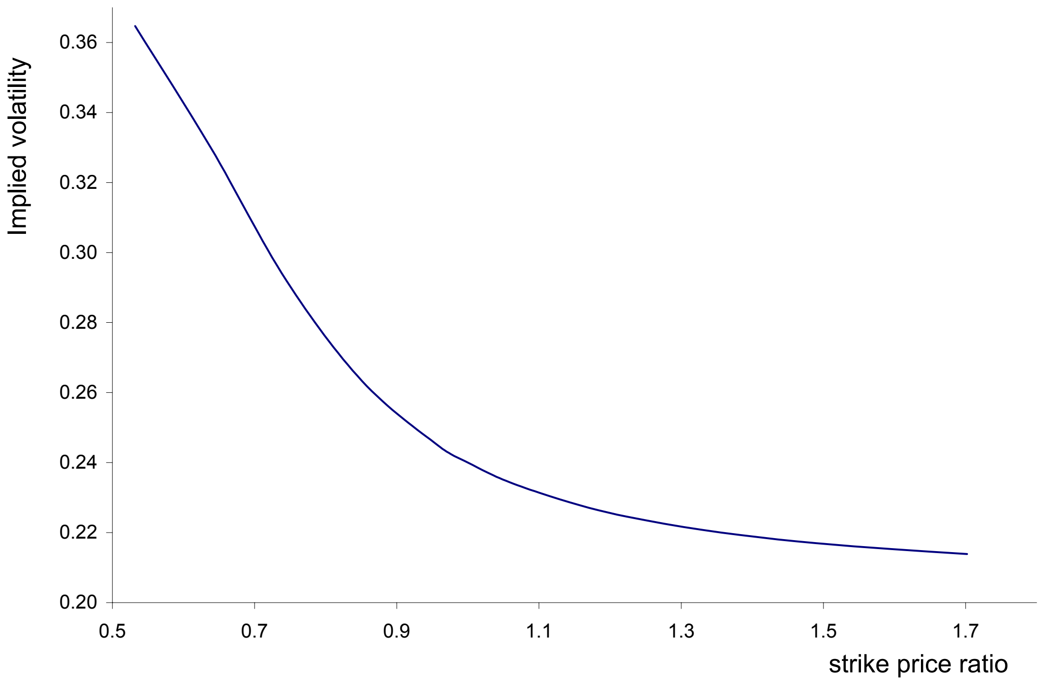

Figure 4: Implied volatility smile, excess volatility = 4%; sigma = 0,20

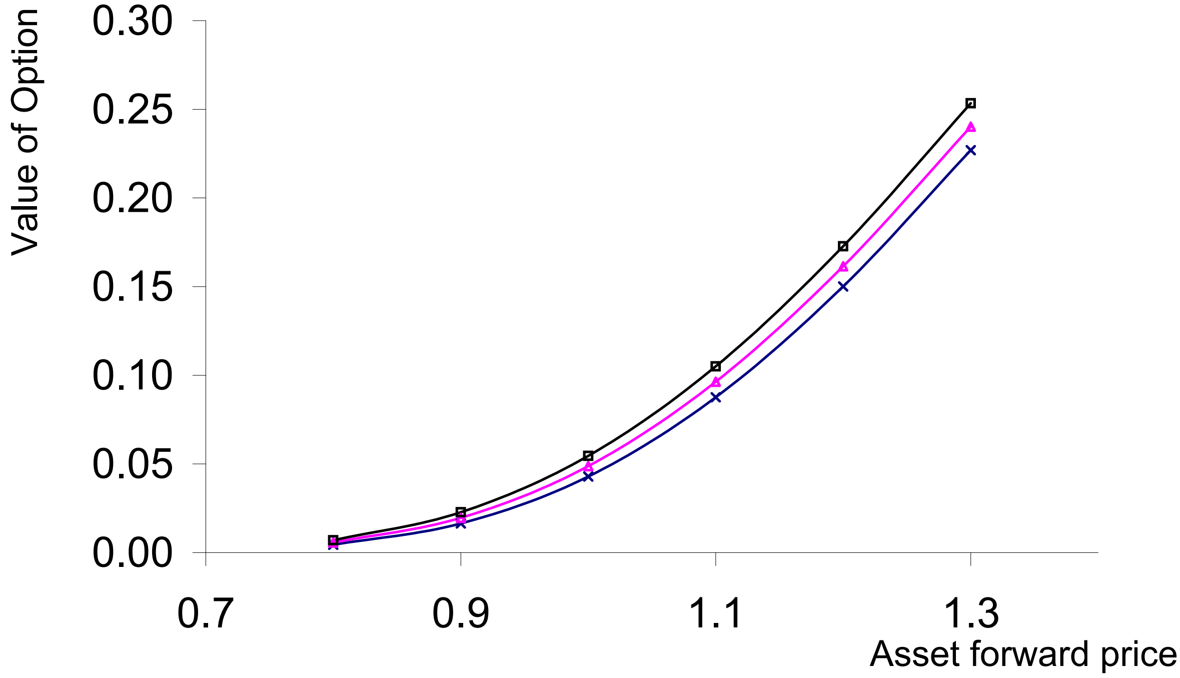

Figure 5: A two-dimensional RNVR, call option v. asset price