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"An Overview of Coalition & Network Formation Models for Economic Applications"

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An Overview of Coalitions and Networks
Formation Models for Economic Applications

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Abstract

This paper presents synthetically some recent developments in the theory of coalition and network formation. For this purpose, some major equilibrium concepts recently introduced to model the formation of coalition structures and networks among players are briefly reviewed and discussed. A few economic applications are also illustrated to give a flavour of the type of predictions such models are able to provide.

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1 Introduction

Very often in social life individuals take decisions within groups (households, friendships, firms, trade unions, local jurisdictions, etc.). Since von Neumann and Morgenstern's (1944) seminal work on game theory, the problem of the formation of coalitions has been a highly debated topic among game theorists.[1] However, during this seminal stage and for a long period afterward, the study of coalition formation was almost entirely conducted within the framework of *games in characteristic form* (*cooperative games*) which proved not entirely suited in games with externalities, i.e. virtually all games with genuine interaction among players. Only in recent years, a widespread literature on what is currently known as *noncooperative coalition formation* or *endogenous coalition formation* has come into the scene with the explicit purpose to represent the process of formation of coalitions of agents and hence modelling a number of relevant economic and social phenomena.[2] Moreover, following this theoretical and applied literature on coalitions, the seminal paper by Jackson and Wolinsky (1996) opened the way to a new stream of contributions using *networks* (graphs) to model the formation of links among individuals[3]

Throughout these brief notes, I survey non exhaustively some relevant contributions of this wide literature, with the main aim to provide an overview of some modelling tools for economic applications. For this purpose, some basic guidelines to the application of coalition formation in economics are presented using as primitives the games in strategic form.[4] As far as economic applications are concerned, most of the examples presented here mainly focus, for convenience, on a restricted number of I.O. topics, as cartel formation, horizontal merger and R&D alliances.

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[1] Von Neumann & Morgenstern's (1944) *stable set* and Aumann and Maschler's (1964) *bargaining set*, among the others, were solution concepts primarily designed to solve simultaneously the formation of a coalition structure and the allocation of the coalitional payoff among the members of each coalition (see also Greenberg (1994) and Bloch (1997)).

[2] Extensive surveys of the coalition formation literature are contained in Greenberg (1994), Bloch (1997, 2003), Yi (1997, 2003) and Ray and Vohra (1997).

[3] Excellent surveys for networks are contained in Dutta and Jackson (2003) and in Jackson (2003, 2005a, 2005b. 2007).

[4] Some of the results presented here are also contained in Currarini and Marini (2006).

2   Coalitions

2.1   Cooperatives Games with Externalities

Since von Neumann and Morgenstern (1944), a wide number of papers have developed solution concepts specific to games with coalitions of players. This literature, known as *cooperative games* literature, made initially a predominant use of the *characteristic function* as a way to represent the worth of a coalition of players.

Definition 1. *A cooperative game with transferable utility (TU cooperative game) can be defined as a pair (N, v), where $N = \{1, 2, .., i, ..n\}$ is the set of players and $v : 2^N \to R_+$ is a real valued function (characteristic function) representing the worth of every feasible coalition.*[5]

The value $v(S)$ can be interpreted as the maximal aggregate amount of utility members of coalition S can achieve by coordinating their strategies. However, in strategic environments players' payoffs are defined on the strategies of all players and the worth (or value) of a group of players cannot be defined independently of the groups (or coalitions) formed by external players $(N \backslash S)$. Hence, to obtain $v(S)$ from a strategic situation we need first to define an underlying strategic form game.

Definition 2. *A strategic form game is a triple $G = \left(N, (X_i, u_i)_{i \in N}\right)$, in which for each $i \in N$, X_i is the set of strategies with generic element x_i, and $u_i : X_1 \times ... \times X_n \to R_+$ is player's i payoff function.*

Moreover, henceforth we restrict the action space of each coalition $S \subset N$ to $X_S \equiv \prod_{i \in S} X_i$. Let, also, $v(S) = \sum_{i \in S} u_i(x)$, for $x \in X_N \equiv \prod_{i \in N} X_i$.[6]

Example 1. *Two-player prisoner's dilemma.*

	A	**B**
A	$3, 3$	$1, 4$
B	$4, 1$	$2, 2$

Therefore, $v(N) = 6$ and $v(\{i\}) = \begin{cases} 4 \text{ if } x_j = A \\ 2 \text{ if } x_j = B \end{cases}$ for $j \neq i$.

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[5]Here we mainly deal with *games with transferable utility*. In games without transferable utility, the worth of a coalition associates with each coalition a players' utility frontier (a vector of utilities).

[6]See Section 2.5 for an economic explanation of these restrictions.

The cooperative allocation $(3,3)$ *can be considered stable only if every player is expected to react with strategy* B *to a deviation of the other player from the cooperative strategy* A.

The above example shows that in order to define the worth of a coalition of players, it is required a specific assumption on the behaviour of the remaining players.

2.1.1 $\alpha-$ and β-Characteristic Functions

The concepts of α and β core, formally studied by Aumann (1967), are based on von Neumann and Morgenstern's (1944) early proposal of representing the worth of a coalition as the minmax or maxmin aggregate payoff that it can guarantee its members in the underlying strategic form game.

Accordingly, the characteristic function $v(S)$ in games with externalities can be obtained assuming that outside players act to minimize the payoff of every deviating coalition $S \subset N$. In this minimax formulation, if members of S move second, the obtained characteristic function,

$$(1) \qquad v_\beta(S) = \min_{x_{N \setminus S}} \max_{x_S} \sum_{i \in S} u_i(x_S, x_{N \setminus S}),$$

denoted β-characteristic function, represents what members in S *cannot be prevented from getting.* Alternatively, if members of S move first, we have

$$(2) \qquad v_\alpha(S) = \max_{x_S} \min_{x_{N \setminus S}} \sum_{i \in S} u_i(x_S, x_{N \setminus S}),$$

denoted α-characteristic function, which represents what members in S *can guarantee themselves,* when they expect a retaliatory behaviour from the complement coalition $N \setminus S$.[7]

When the underlying strategic form game G is zero-sum, (1) and (2) coincide. In non-zero sum games they can differ and, usually, $v_\alpha(S) < v_\beta(S)$ for all $S \subset N$.

However, α and β characteristic functions express an irrational behaviour of coalitions of players, acting *as if* they expected their rivals to minimize their payoff. Although appealing because immune from any *ad hoc* assumption on the reaction of the outside players (indeed, their minimizing behavior is here not meant to represent the expectation of S but rather as a mathematical way to determine the lower bound of S's aggregate payoff), still this approach

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[7]Note that players outside S are treated as one coalition, so the implicit assumption here is that players in $N \setminus S$ stick together after S departure from the grand coalition N.

has important drawbacks: deviating coalitions are too heavily penalized, while outside players often end up bearing an extremely high cost in their attempt to hurt deviators. Moreover, the little profitability of coalitional objections usually yield very large set of solutions (e.g., large cores).

2.1.2 Nash Behaviour among Coalitions

Another way to define the characteristic function in games with externalities is to assume that in the event of a deviation from N, a coalition S plays *à la* Nash with the remaining players.[8]

Although appealing, such a modelling strategy requires some specific assumptions on the coalition structure formed by remaining players $N\backslash S$ once a coalition S has deviated from N.

Following the Hart and Kurtz's (1983) coalition formation game, two extreme predictions can be assumed on the behaviour of remaining players.

Under the so called γ-assumption,[9] when a coalition deviates from N, the remaining players split up in singletons; under the δ-assumption, players in $N\backslash S$ stick together as a unique coalition.[10]

Therefore, the obtained characteristic functions can be defined as follows:

$$(3) \qquad v_\gamma(S) = \sum_{i \in S} u_i\left(\overline{x}_S, \{\overline{x}_j\}_{j \in N\backslash S}\right)$$

where $\overline{x}$ is a strategy profile such that, for all $S \subset N$, $\overline{x}_S \in X_S$ and $\forall j \in N\backslash S$, $\overline{x}_j \in X_j$

$$\overline{x}_S = \underset{x_S \in X_S}{\arg\max} \sum_{i \in S} u_i\left(x_S, \{\overline{x}_j\}_{j \in N\backslash S}\right)$$

$$\overline{x}_j = \underset{x_j \in X_j}{\arg\max}\ u_j\left(\overline{x}_S, \{\overline{x}_k\}_{k \in (N\backslash S)\backslash\{j\}}, x_j\right).$$

Moreover,

$$(4) \qquad v_\delta(S) = \sum_{i \in S} u_i\left(\overline{x}_S, \overline{x}_{N\backslash S}\right)$$

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[8] This way to define the worth of a coalition in as a non-cooperative equilibrium payoff of a game played between coalitions was firstly proposed by Ichiishi (1983).

[9] Hurt and Kurz's (1983) Γ game is indeed a strategic coalition formation game with fixed payoff division, in which the strategies consist of the choice of a coalition. Despite the different nature of the two games, there is an analogy concerning the coalition structure induced by a deviation from the grand coalition.

[10] See Chander and Tulkens (1997) and Carraro and Siniscalco (1993) for applications of this approach.

where,

$$\overline{x}_S = \underset{x_S \in X_S}{\arg\max} \sum_{i \in S} u_i \left(x_S, \overline{x}_{N \backslash S} \right)$$

and

$$\overline{x}_{N \backslash S} = \underset{x_{N \backslash S} \in X_{N \backslash S}}{\arg\max} u_j \left(\overline{x}_S, x_{N \backslash S} \right).$$

In both cases, for (3) and (4) to be well defined, the Nash equilibrium of the strategic form game played among coalitions must be unique. Usually, $v_\alpha(S) < v_\beta(S) < v_\delta(S)$ for all $S \subset N$.

2.1.3   Timing and the Characteristic Function

It is also conceivable to modify the γ- or δ-assumption (coalitions playing simultaneously à la Nash in the event of a deviation from the grand coalition) reintroducing the temporal structure typical of the α and β-core.[11]

When a deviating coalition S moves first under the γ-assumption, members of S choose a coordinated strategy as leaders, thus anticipating the reaction of the players in $N \backslash S$, who simultaneously choose their best response as singletons. The strategy profile associated to the deviation of a coalition S is the Stackelberg equilibrium of the game in which S is the leader and players in $N \backslash S$ are, individually, the followers. We can indicate this strategy profile as a $\tilde{x}(S) = (\tilde{x}_S, x_j(\tilde{x}_S))$ such that

$$(5) \qquad \tilde{x}_S = \underset{x_S \in X_S}{\arg\max} \sum_{i \in S} u_i \left(x_S, \{x_j(x_S)\}_{j \in N \backslash S} \right)$$

and, $\forall j \in N \backslash S$,

$$(6) \qquad x_j(x_S) = \underset{x_j \in X_j}{\arg\max} u_j \left(x_S, \{x_k(x_S)\}_{k \in (N \backslash S) \backslash \{j\}}, \ x_j \right).$$

Sufficient condition for the existence of $\tilde{x}(S)$ can be provided. Assume that $G(N \backslash S, x_S)$, the restriction of the game G to the set of players $N \backslash S$ given the fixed profile x_S, possesses a unique Nash Equilibrium for every $S \subset N$ and $x_S \in X_S$, where X_S is assumed compact. Let also each player's payoff be continuous in each player's strategy. Thus, by the closedness of the Nash equilibrium correspondence (see, for instance, Fudemberg and Tirole (1991)), members of S maximize a continuous function over a compact set (condition (5)) and, by

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[11]See Currarini & Marini (2003) for details.

Weiestrass Theorem, a maximum exists. As a consequence, for every $S \subset N$, there exists a Stackelberg equilibrium $\tilde{x}(S)$.

We can thus define the characteristic function $v_\lambda(S)$ as follows:

$$(7) \qquad v_\lambda(S) = \sum_{i \in S} u_i \left(\widetilde{x}_S, \{x_j(\widetilde{x}_S)\}_{j \in N \setminus S} \right).$$

Obviously, $v_\lambda(S) \geq v_\gamma(S)$. In a similar way, the γ-assumption can be modified by assuming that a deviating coalition S plays as follower against all remaining players in $N \setminus S$ acting as singleton leaders. Obviously, the same can be done under the δ-assumption.

2.1.4 The Core in Games with Externalities

We can test the various conversions of $v(S)$ introduced above by examining the different predictions obtained using the *core* of (N, v).

We first define an *imputation* for (N, v) as a vector $\mathbf{z} \in \mathbf{R}_+^n$ such that $\sum_{i \in N} z_i \leq v(N)$ (feasibility) and $z_i \geq v(i)$ (individual rationality) for all $i \in N$.

Definition 3. *The core of a TU cooperative game (N, v) is the set of imputations $z \in R_+^n$ such that $\sum_{i \in S} z_i \geq v(S)$ for all $S \subseteq N$.*

Given that coalitional payoffs are obtained from an underlying strategic form game, the core can also be defined in terms of strategies, as follows.

Definition 4. *The joint strategy $x \in X_N$ is core-stable if there is no coalition $S \subset N$ such that $v(S) > \sum_{i \in S} u_i(x)$.*

Example 2. (Merger in a linear Cournot oligopoly). *Consider three firms $N = \{1, 2, 3\}$ with linear technology competing à la Cournot in a linear demand market. Let the demand parameters a and b and the marginal cost c, be selected in such a way that interior Nash equilibria for all coalition structures exist. The set of all possible coalitions of the N players is $\mathbb{N} = (\{1, 2, 3\}, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{\varnothing\})$. By definition, $v(\{\varnothing\}) = 0$. Note that if all firms merge, they obtain the monopoly payoff $v(\{1, 2, 3\}) = \frac{A}{4}$, where $A = (a - c)^2/b$, independently of the assumptions made on the characteristic function. These assumptions matters for the worth of intermediate coalitions. Under the α- and β-assumptions, if either one single firm or two firms leave the grand coalition N, remaining firms can play a minimizing strategy in such a way that, for every $S \subset N$, $v_\alpha(S) = v_\beta(S) = 0$.*

In this case, the core coincides with all individually rational Pareto efficient payoffs, i.e. all points weakly included in the set of coordinates, $\mathbf{Z} = \left[\left(\frac{A}{8}, \frac{A}{16}, \frac{A}{16} \right), \left(\frac{A}{16}, \frac{A}{8}, \frac{A}{16} \right), \left(\frac{A}{16}, \frac{A}{16}, \frac{A}{8} \right) \right]$. Under the γ-assumption, we know that when, say firms 1 and 2, jointly leave the merger, a simultaneous duopoly game is played between the coalition $\{1, 2\}$ and firm $\{3\}$. Hence, $v_\gamma(\{1, 2\}) = \frac{A}{9}$. Similarly for all other couples of firms deviating from N. When instead a single firm i leaves N, a triopoly game is played, with symmetric payoffs $v_\gamma(\{i\}) = \frac{A}{16}$ (these payoffs are obtained from the general expression $v_\gamma(S) = \frac{A}{(n-s+2)^2}$ expressing firms' profits in a n-firm oligopoly). In this case, since intermediate coalitions made of two players do not give each firm more than their individually rational payoff, the core under the γ-assumption coincides with the core under the α- and β-assumptions. We know from Salant et al. (1982) model of merger in oligopoly, that $v_\gamma(S) > \sum_{i \in S} v_\gamma(\{i\})$ only for $|S| > 0, 8 |N|$. This means that in the merger game the core under the γ-assumption shrinks with respect to the core under the α- and β-assumptions only for $n \geq 5$. Under the δ-assumption, when a single firm leaves N, a simultaneous duopoly game is played between the firm $\{i\}$ and the remaining firms $N \setminus \{i\}$ acting as a single coalition. As a result, $v_\delta(\{i\}) = \frac{A}{9}$, which is greater than $\frac{A}{12}$, the maximum payoff at least one firm will obtain in the grand coalition. Therefore, under the δ-assumption, the core is empty. Finally, note that since under the λ-assumption every single firm playing as leader obtains $v_\lambda(\{i\}) = \frac{A}{12}$, in such a case the core is unique and contains only the equal split imputation $\mathbf{z} = \left(\frac{A}{12}, \frac{A}{12}, \frac{A}{12} \right)$ [see **Figure 1** and **2**].

2.2   Coalitional Equilibria in Strategic Form Games

2.2.1   Strong Nash Equilibrium

In the core 'approach' described above, players can sign *binding agreements*.[12] When this assumption is relaxed, a Nash approach to coalitional deviations becomes more appropriate. The concept of equilibrium proposed by Aumann (1959), denoted *strong Nash equilibrium*, extends the Nash equilibrium to every coalitional deviation. Accordingly, a strong Nash equilibrium is defined as a strategy profile that no group of players can profitably object, given that remaining players are expected not to change their strategies.

Definition 5 *A strategy profile $\hat{x} \in X_N$ for G is a strong Nash equilibrium (SNE) if*

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[12]More specifically, a coalition can change its strategy only by deviating from the grand coalition and it cannot change strategy while remaining in the grand coalition.

there exists no $S \subseteq N$ and $x_S \in X_S$ such that

$$
\begin{aligned}
u_i(x_S, \hat{x}_{N\setminus S}) &\geq u_i(\hat{x}) \ \ \forall i \in S; \\
u_h(x_S, \hat{x}_{N\setminus S}) &> u_h(\hat{x}) \ \text{ for some } h \in S.
\end{aligned}
$$

Obviously, all SNE of G are both Nash Equilibria and Pareto Efficient; in addition they satisfy the Nash stability requirement for each possible coalition. As a result, SNE fails to exist in many economic problems, and in particular, whenever Nash Equilibria fail to be Pareto Efficient.

For the three players merger game of Example 1, the set of SNE is empty. This is because the symmetric strategy profile $\mathbf{x} = \left(\frac{(a-c)}{6b}, \frac{(a-c)}{6b}, \frac{(a-c)}{6b} \right)$ yielding a Pareto efficient allocation, is not a Nash Equilibrium.

2.2.2 Coalition-proof Nash Equilibrium

To soften the existence problem of the SNE, a refinement was proposed by Bernheim, Peleg and Whinston (1987) and named *coalition-proof Nash Equilibrium* (CPNE). Differently from the SNE, here a restriction is imposed on coalional deviations that have to be self-enforcing, i.e., not further improvable by subcoalitions of players.

Definition 6. *A coalition-proof Nash equilibrium (**CPNE**) $x^* \in X_N$ is defined inductively with respect to the number of players n in the game:* (i) *If $n = 1$, then $x_1^* \in X_1$ is a CPNE if and only if $u_1(x^*) \geq u_1(x_1)$ for any $x_1 \in X_1$, $x_1 \neq x_1^*$.* (ii) *Let $n > 1$. Assume that the coalition-proof Nash equilibria have been defined for games with fewer than n players.* (a) *For any game G with n players, $x^* \in X_N$ is a self-enforcing strategy profile if, for all $S \subsetneq N$, x_S^* is a CPNE of the reduced game $G|_{x_{-S}^*}$.* (b) *Profile x^* is a CPNE of G if it is a self-enforcing strategy profile and there is no other self-enforcing strategy profile $x \in X_N$ such that $u_i(x_i, x_{N\setminus i}^*) \geq u_i(x^*)$ for all $i \in N$ and $u_i(x_i, x_{N\setminus i}^*) > u_i(x^*)$ for some $i \in N$.*

For the three players merger game of Example 1, the symmetric Nash strategy profile at which the three firms play independently $x = \left(\frac{(a-c)}{4b}, \frac{(a-c)}{4b}, \frac{(a-c)}{4b} \right)$ is a CPNE, since coalitional deviations made by two or three players are not self-enforcing.

2.3 Cooperative Games with Coalition Structures

According to the original spirit of von Neumann and Morgenstern (1944), "the purpose of game theory is to determine everything can be said about coalitions between players, compensation between partners in every coalition, mergers or fight between coalitions" (p.240). To introduce the topic of competition among coalitions, a framework different from which used by traditional cooperative games is required. The first required step is to extend the game (N, v) to a game with a coalition structure $\pi = (S_1, S_2, ..., S_m)$, i.e., a partition of players N such that for all $S_h, S_j \in \pi$, $S_h \cap S_j = \varnothing$ and $\bigcup_{k=1,..,m} S_k = N$. The second step is to define the worth to every coalition belonging to a given coalition structure. Finally, a relevant issue is which coalition structure can be considered stable.

In their seminal contribution, Aumann and Drèze (1974) extends the solution concepts of cooperative game theory to games with exogenous coalition structures. In every $\pi \in \Pi(N)$, the set of all partitions of the N players, each coalition is allowed to distribute its members only its own worth $v(S_k)$, here assumed equal to the Shapley value defined for every given coalition structure $\pi \in \Pi$.[13] However, the above restriction has been criticized as inadequate for all models in which "the raison d'etre for a coalition S to form is that its members try to receive more than $v(S)$ - the worth of S." (Greenberg, 1994, p.1313). A part from this criticism, the most commonly used stability concept within this framework is the *coalition structure core*.

Definition 7. *Let (N, v) be a cooperative game. The coalition structure $\pi \in \Pi(N)$ is stable if its core is nonempty, i.e., if there exists a feasible payoff $z \in Z(\pi)$ such that, for every $S_k \in \pi$, $z_k \geq v(S_k)$. The game (N, v) has a coalition structure core if there exists at least one partition that is stable.*

2.3.1 The Partition Function Approach

The presence of externalities among coalitions of players calls for a more encompassing approach than that offered by a cooperative games in characteristic function form. For this purpose, in a seminal paper Thrall and Lucas (1963) introduce the games in *partition function form*.

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[13] The Shapley value is defined as $\phi(N, v) = \sum_{S \subset N} q(s) \Delta_i(s)$, where $q(s) = \frac{(s-1)!(n-s)!}{n!}$ and $\Delta_i(s) = v(S) - v(S \backslash \{i\}$ is the marginal contribution of player i to any coalition S in the game (N, v). Therefore, the Shapley value of player i represents the weighted sum of his marginal contribution to all coalitions he can join.

Definition 8. *A TU game in partition function form can be defined as a triple (N, π, w), where $\pi = \{S_1, S_2, ..., S_m\}$ is a partition of players N and $w(S, \pi) : 2^N \times \Pi \to \mathbf{R}$ is a mapping that assigns to each coalition S embedded in a given partition $\pi \in \Pi(N)$ a real number (a worth).*

In this way, the authors can define the value of every non-empty coalition S of N as

$$v(S) = \min_{\{\pi | S \in \pi\}} w(S_k, \pi),$$

where this minimum is over all partitions π which contain S as a distinct coalition. This approach constitutes a generalization of the cooperative game (N, v) and the two games coincides when the worth of a coalition is independent of the coalitions formed by the other players. When coaltions' payoffs are not independent, some assumptions are still required to model the behaviour of coalitions with respect to 'rival' coalitions. Since Ichiishi (1983), the modern theory of coalition formation adopt the view that coalitions cooperate inside and compete *à la* Nash with the other coalitions.

2.3.2 The Valutions Approach

Since the games in partition function are hard to handle and often pose technical difficulties, many recent contributions impose a **fixed allocation rule** distributing the worth of a coalition to all its members.

Such a fixed sharing rule gives rise to a per-member payoff (*valuation*) mapping coalition structures into vectors of individual payoffs.

Definition 9. *A game in valuation form can be defined as a triple (N, π, v_i), where $\pi = \{S_1, S_2, ..., S_m\}$ is a partition of players N and $v_i(S, \pi) : 2^N \times \Pi \to \mathbf{R}^{|S|}$ is a mapping that assigns to each individual belonging to a coalition S embedded in a given partition $\pi \in \Pi$ a real number (a valuation).*

Definition 10. *A coalition structure π is core stable if there not exists a coalition S and a coalition structure π' such that for $S \in \pi'$ and for all $i \in S$, $v_i(S, \pi') > v_i(S, \pi)$.*[14]

In the merger game of Example 1, the set of all feasible partitions is

$$\Pi = ((\{1, 2, 3\}), (\{1, 2\}, \{3\}), (\{1, 3\}, \{2\}), (\{2, 3\}, \{1\}), (\{1\}, \{2\}, \{3\})),$$

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[14]Analogous concept of $\alpha, \beta, \gamma, \delta, \lambda$- core stability can be defined for games in valuation form.

and the grand coalition is a core-stable coalition structure under the valuations $v_i^\alpha, v_i^\beta, v_i^\gamma$ and v_i^λ. It is not core-stable under the valuation v_i^δ.

2.4 Noncooperative Games of Coalition Formation

Most recent approaches have looked at the process of coalition formation as a strategy in a well defined game of coalition formation (see Bloch, 1997, 2003 and Yi, 2003 for surveys). Within this new stream of literature, usually indicated as *noncooperative theory of coalition formation* (or *endogenous coalition formation*), the work by Hurt and Kurz (1985) represents a seminal contribution. Most recent contributions along these lines include Bloch (1995, 1996), Ray and Vohra (1997, 1999) and Yi (1997). In all these works, cooperation is modelled as a two stage process: at the first stage players form coalitions, while at the second stage formed coalitions interact in a well defined strategic setting. This process is formally described by a coalition formation game, in which a given rule of coalition formation maps players' announcements of coalitions into a well defined coalition structure, which in turns determines the equilibrium strategies chosen by players at the second stage.

A basic difference among the various models lies on the timing assumed for the coalition formation game, which can either be *simultaneous* (Hurt & Kurz (1982), Ray & Vohra (1994), Yi (1997)) or *sequential* (Bloch (1994), Ray & Vohra (1995)).

2.4.1 Hurt & Kurz's Games of Coalition Formation

Hurt and Kurz (1983) were among the first to study games of coalition formation with a valuation in order to identify stable coalition structures.[15] As valuation, Hurt & Kurz adopt a general version of Owen value for TU games (Owen, 1977), i.e. a Shapley value with prior coalition structures, that they call *Coalitional Shapley value*, assigning to every coalition structure a payoff vector $\varphi(\pi)$ in R^N, such that (by the efficiency axiom) $\sum_{i \in N} \varphi(\pi) = v(N)$. Given this valuation, the game of coalition formation is modelled as a game in which each player $i \in N$ announces a coalition $S \ni i$ to which he would like to belong; for each profile $\sigma = (S_1, S_2, ..., S_n)$ of announcements, a partition $\pi(\sigma)$ of N is assumed to be induced on the system. The rule according to which $\pi(\sigma)$ originates from σ is obviously a crucial issue for the prediction of which coalitions will emerge in equilibrium.

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[15]Another seminal contribution is Shenoy (1979).

Hurt and Kurz's game Γ predicts that a coalition emerges if and only if all its members have declared it (from which the name of "unanimity rule" also used to describe this game). Formally:

$$\pi(\sigma) = \{S_i(\sigma) : i \in N\}$$

where

$$S_i(\sigma) = \begin{cases} S_i & if \quad S_i = S_j \quad \text{for all } j \in S_i \\ \{i\} & \text{otherwise.} \end{cases}.$$

Their game Δ predicts instead that a coalition emerges if and only if all its members have declare the same coalition S (which may, in general, differs from S). Formally:

$$\pi(\sigma) = \{S \subset N : i, j \in S \text{ if and only if } S_i = S_j\}.$$

Note that the two rules of formation of coalitions are "exclusive" in the sense that each player of a forming coalition has announced a list of its members. Moreover, in the gamma-game this list has to be approved unanimously by all coalition members. Once introduced these two games of coalition formation, a *stable coalition structure* for the game Γ (Δ) can be defined as a partition induced by a Strong Nash Equilibrium strategy profile of these games.

Definition 10. *The partition π is a γ-stable (δ-stable) coalition structure if $\pi = \pi(\sigma^*)$ for some σ^* with the following property: there exists no $S \subseteq N$ and $\sigma_S \in \Sigma_S$ such that*

$$v_i(\sigma_S, \sigma^*_{N\backslash S}) \geqslant v_i(\sigma^*), \; for \; all \; i \in S$$

$$and$$

$$v_h(\sigma_S, \sigma^*_{N\backslash S}) > v_h(\sigma^*), \; for \; some \; h \in S.$$

It can be seen that the two rules generate different partitions after a deviation by a coalition: in the Γ-game, remaining players split up in singletons; in the Δ-game, they stick together.

Example 2: $N = \{1, 2, 3\}$. $\sigma_1 = \{1, 2, 3\}, \sigma_2 = \{1, 2, 3\}, \sigma_3 = \{3\}$.

$$\pi^\gamma(\sigma) = (\{1\}, \{2\}, \{3\}),$$

$$\pi^\delta(\sigma) = (\{1, 2\}, \{3\}).$$

In the recent literature on endogenous coalition formation, the coalition formation game by Hurt and Kurz is usually modelled as a first stage of a game in which, at the second stage

formed coalitions interact in some underlying strategic setting. The coalition formation rules are used to derive a valuation v_i mapping from the set of all players' announcements Σ into the set of real numbers. The payoff functions v_i are obtained by associating with each partition $\pi = \{S_1, S_2, ..., S_m\}$ a game in strategic form played by coalitions

$$G(\pi) = (\{1, 2, ..., m\}, (X_{S_1}, X_{S_2}, ..., X_{S_m}), (U_{S_1}, U_{S_2}, ..., U_{S_m})),$$

in which X_{S_k} is the strategy set of coalition S_k and $U_{S_k} : \Pi_{k=1}^m X_{S_k} \to R_+$ is the payoff function of coalition S_k, for all $k = 1, 2, ..., m$. The game $G(\pi)$ describes the interaction of coalitions after π has formed as a result of players announcements in Γ or Δ.

The Nash equilibrium of the game $G(\pi)$ (assumed unique) gives the payoff of each coalition in π; within coalitions, a fix distribution rule yields the payoffs of individual members.

Following our previous assumptions (see section 1.2) we can derived the game $G(\pi)$ from the strategic form game G by assuming that $X_{S_k} = \prod_{i \in S_k} S_k$ and $U_{S_k} = \sum_{i \in S_k} u_i$, for every coalition $S_k \in \pi$. We can also assume $u_i = \frac{U_{S_k}}{|S_k|}$ as per capita payoff function of members of S_k. Therefore, using Example 1, for the Γ game, $u_i(x^*(\{1, 2, 3\})) = \frac{A}{12}$ for $i = 1, 2, 3$, $u_i(x^*(\{i, j\}, \{k\})) = u_j(x^*(\{i, j\}, \{k\})) = \frac{A}{18}$, $u_k(x^*(\{i, j\}, \{k\})) = \frac{A}{9}$, and $u_i(x^*(\{i\}, \{j\}, \{k\})) = \frac{A}{16}$. $i = 1, 2, 3$. Therefore, the grand coalition is the only stable coalition structure of the Γ-game of coalition formation. For the Δ-game, there are no stable coalition structures.

If we extend the merger game to n firms, we know that the payoff of each firm $i \in S \subset N$ when all remaining firms split up in singletons, is given by:

$$v_i^\gamma(x(\pi(\sigma'))) = \frac{(a-c)^2}{s(n-s+2)^2},$$

where $n \equiv |N|$, $s \equiv |S|$ and $\sigma' = \left(\{S\}_{i \in S}, \{N\}_{i \in N \setminus S}\right)$. The grand coalition, induced by the profile $\sigma^* = \left(\{N\}_{i \in N}\right)$, is a stable coalition structure of the Γ-game of coalition formation, if

$$v_i^\gamma(x(\pi(\sigma^*))) = \frac{(a-c)^2}{4n} \geq v_i^\gamma(x(\pi(\sigma'))) = \frac{(a-c)^2}{s(n-s+2)^2}$$

The condition above is usually verified for every $s \leq n$. Therefore, the stability of the grand coalition for the Γ-merger game holds also for a n-firm oligopoly.

2.4.2 Sequential Games of Coalition Formation

Bloch (1996,1997) introduces a sequential coalition-formation game with infinite horizon in which, as in Hurt and Kurz's (1988) Γ-game, a coalition forms if and only if all its members have agreed to form the same coalition. The sequence of moves of the coalition formation game is organized as follows. At the beginning, the first player (according to a given ordering) makes a proposal for a coalition to form. Then, the player on his list with the smallest index accepts or rejects his proposal. If he accepts, it is the turn of the following player on the list to accept or reject. If all players on the list accept first player's proposal, the coalition is formed and the remaining players continue the coalition formation game, starting with the player with the smallest index who thus makes a proposal to remaining players. If any of the players has rejected first player's proposal, the player who first rejected the proposal starts proposing another coalition. Once a coalition forms it cannot break apart or merge with another player or a coalition of players. Bloch (1996) shows that this game yields the same stationary subgame perfect equilibrium coalition structure as a much simpler "size-announcement game", in which the first player announces the size of his coalition and the first s_1 players accept; then player i_{s+1} proposes a size s_2 coalition and this is formed and so on, until the last player is reached [see **Figure 3** and **4**]. This equivalence is basically due to the *ex ante* symmetry of players. It can also shown that this size-announcement game possesses a generically unique subgame perfect equilibrium coalition structure.

If we extend the merger game of Example 1 to $n > 2$ firms, the unique subgame perfect equilibrium coalition structure of Bloch's (1996) sequential game of coalition formation is a coalition structure $\pi = (\{S\}, \{j\}_{j \in N \setminus S})$, with $s = |S|$ equal to the first integer following $\left(2n + 3 - \sqrt{4n+5}\right)/2$.[16] The explanation is as follows. We know that when a merger of size s is formed in a Cournot market, the equal-split payoff of each firm $i \in S$ in the merger is $u_i \left(x^*((\{S\}, \{j\}_{j \in N \setminus S})\right) = A/s(n-s+2)^2$ which is greater than the usual Cournot profit $u_i \left(x^*(\{i\}_{i \in N \setminus S})\right) = A/(n+1)^2$ only for $s > \left(2n + 3 - \sqrt{4n+5}\right)/2$. When a merger of size s is in place, each independent firm outside the merger earns a higher profit than that of the members of the merger, equal to $u_j \left(x^*((\{S\}, \{j\}_{j \in N \setminus S})\right) = A(n-s+2)^2$. Therefore, in the sequential game of coalition formation, the first firms choose to remain independent and free-ride on the merger formed by subsequent firms. When the number of remaining firms is exactly equal to the minimal profitable merger size $s = \left(2n + 3 - \sqrt{4n+5}\right)/2$, they will

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[16] We know (see Salant *et al.*,1983) that $\left(2n + 3 - \sqrt{4n+5}\right)/2 \simeq 0.8n$.

choose to merge, as it is no longer profitable to remain independent.

2.4.3 Equilibrium Binding Agreement

Ray and Vohra (1997) propose a different stability concept. In this solution concept, players start from some coalition structure and are only allowed to break coalitions to smaller ones. The deviations can be unilateral or multilateral (i.e., several players can deviate together). The deviators take into account future deviations, both by members of their own coalitions and by members of other coalitions. Deviations to finer partitions must be credible, i.e. stable themselves, and therefore the nature of the definition is recursive. We can start with a partition π and we can denote by $B(\pi)$ all coalition structures that are finer than π. A coalition $\pi' \in B(\pi)$ can be induced from π if π' is formed by breaking a coalition in π. A coalition S is a perpetrator if it can induce $\pi' \in B(\pi)$ from π. Obviously, S is a subcoalition of a coalition in π. Denote the finest coalition structure, such that $|S| = 1$ for all S, by π_0. There are no deviations allowed from π_0 and therefore π_0 is by definition stable. Recursively, suppose that for some π, all stable coalitions were defined for all $\pi' \in B(\pi)$, i.e., for all coalition structures finer than π. Now, we can say that a strategy profile associated to a coalition structure $x(\pi)$ is *sequentially blocked* by $x(\pi')$ for $\pi' \in B(\pi)$ if i) there exists a sequence $\{x(\pi_1), x(\pi_2), ..., x(\pi_m)\}$ with $x(\pi_1) = x(\pi)$ and $x(\pi_m) = x(\pi')$; ii) for every $j = 2, ..., m$, there is a deviator S_j that induces π_j from π_{j-1}; iii) $x(\pi')$ is stable; iv) π_j is not stable for any $x(\pi_j)$ and $1 < j < m$; v) $u_i(x(\pi_0) > u_i(x(\pi_{j-1}))$ for all $i \in S_j$ and $j = 2, ..., m$.

Definition 11. $x(\pi)$ *is an equilibrium binding agreement if there is no* $x(\pi')$ *for* $\pi' \in B(\pi)$ *that sequentially blocks* $x(\pi)$.

Applying the Equilibrium Binding Agreement to the Example 1, we obtain that, apart from $x(\pi_0)$, with $\pi_0 = (\{1\}, \{2\}, \{3\})$, which is by definition stable, also the grand coalition strategy profile $x(\pi)$ with $\pi = \{1, 2, 3\}$ is an equilibrium binding agreement. For the n-firm merger game, Ray and Vohra's show that there is a cyclical pattern, in which, depending on n, the grand coalition can or not be a stable coalition structure. For $n = 3, 4, 5$ it is stable, but not for $n = 6, 7, 8$. For $n = 9$ is again stable and so on, with a rather unpredictable pattern. "The grand coalition survives if there exist 'large' zones of instability in intermediate coalition structures." (Ray & Vohra, 1997, p.73).

2.5 Some Guidelines to Coalition Formation in Economic Applications

In order to compare and interpret the main predictions that endogenous coalition formation theories obtain in some classical economic problems, it can be useful to use a very simple setup in which the equal sharing rule within each coalition is not assumed but it is obtained through some symmetry assumptions imposed on the strategic form game describing the economic problem at hand.[17] Once some basic assumptions are imposed on the strategic form games underlying the games of coalition formation, the main economic applications can be divided in a few categories: 1) games with positive or negative players-externalities; 2) games with actions that are strategic complements or substitutes; 3) games with or without coalition-synergies. According to these three features, we usually have a much clearer picture of the predictions which can be expected from the different concepts of coalitional stability illustrated above and, in particular, of the stability of the grand coalition.

We start imposing some symmetry requirements on the strategic form game G.

Assumption 1. *(Symmetric Players)*: $X_i = X \subset R$ *is compact and convex for all* $i \in N$. *Moreover, for all* $x \in X_N$ *and all permutations* $p : N \to N$:

$$u_{p(i)}\left(x_{p(1)}, ..., x_{p(n)}\right) = u_i\left(x_1, ..., x_n\right).$$

Assumption 2. *(Monotone Player-externalities)*. *One of the following two cases must hold for* $u_i(x) : X_N \to R$ *assumed quasi-concave:*

1. *Positive player-externalities:* $u_i(x)$ *strictly increasing in* $x_{N \setminus i}$ *for all* i *and all* $x \in X_N$;

2. *Negative player-externalities:* $u_i(x)$ *strictly decreasing in* $x_{N \setminus i}$ *for all* i *and all* $x \in X_N$.

Assumption 1 requires that all players have the same strategy set, and that players payoff functions are symmetric, by this meaning that any switch of strategies between players induces a corresponding switch of payoffs. Assumption 2 requires that the cross effect on payoffs of a change of strategy have the same sign for all players and for all strategy profiles.

Lemma 1. For all $S \subseteq N$, $\widetilde{x}_S \in \arg\max_{x_S \in X_S} \sum_{i \in S} u_i(x_S, x_{N \setminus S})$ implies $\widetilde{x}_i = \widetilde{x}_j$ for all $i, j \in S$ and all $x_{N \setminus S} \in X_{N \setminus S}$.

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[17] I follow here Currarini and Marini (2006).

Proof. See Appendix. ∎

An important implication of Lemma 1 is that all players belonging to a given coalition $S \subseteq N$ will play the same maximizing strategy and then will obtain the same payoff. We can thus obtain a game in valuation form from a game in partition function form without imposing a fixed allocation rule.

The next lemma expresses the fact that in every coalition structure π, at the Nash equilibrium played by coalitions, when players-externalities are positive (negative), being a member of bigger rather than a smaller coalition is convenient only when each member of S plays a strategy that is lower (higher) than that played by each member of a smaller coalition.

Lemma 2. *Let Assumptions 1 and 2 hold. Then, for every $S \in \pi$ and $T \in \pi$, with $|T| \geq |S|$:*

i) Under Positive Player-externalities, $u_s(x(\pi)) \geq u_t(x(\pi))$ if and only if $x_s \leq x_t$;

ii) Under Negative Player-externalities, $u_s(x(\pi)) \geq u_t(x(\pi))$ if and only if $x_s \geq x_t$.

Proof. See Appendix. ∎

Finally, we can use a well known classification of all economic models in two classes: 1) games in which players' actions are *strategic complements*; 2) games in which players' actions are *strategic substitutes*.[18]

One basic feature of games with actions that are strategic complements is that players' payoffs exhibits increasing differences on $X_{N.}$.

Definition 12. *The payoff function u_i exhibits increasing differences on X_N if for all S, $x_S \in X_S$, $x'_S \in X_S$, $x_{N \setminus S} \in X_{N \setminus S}$ and $x'_{N \setminus S} \in X_{N \setminus S}$ such that $x'_S > x_S$ and $x'_{N \setminus S} > x_{N \setminus S}$ we have*

$$u_i\left(x'_S, x'_{N \setminus S}\right) - u_i\left(x_S, x'_{N \setminus S}\right) \geq u_i\left(x'_S, x_{N \setminus S}\right) - u_i\left(x_S, x_{N \setminus S}\right).$$

This feature is typical of games, as price oligopoly models with differentiated goods, for which players' best-replies are upward-sloping. For these games, we can prove the following.

Lemma 3. *Let assumptions 1-2 hold, and let u_i have increasing differences on X_N, for all $i \in N$. Then, for every $S \in \pi$ and $T \in \pi$, with $|T| \geq |S|$: i) Positive Player-externalities imply $x_s \leq x_t$; ii) Negative Player-externalities imply $x_s \geq x_t$.*

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[18]See for this definition Bulow *et al.* (1985)

Proof. See Appendix. ∎

Suppose now to have a game with actions that are strategic substitutes. This is the case of Cournot oligopoly and many other economic models. Suppose also that a boundary on the slope of reaction maps is imposed by the following contraction assumption.

Assumption 3. *(contraction)* Let $S \in \pi$. Then, there exists a $c < 1$ such that for all $x_{N \setminus S}$ and $x'_{N \setminus S}$:

$$\left\| f_s \left(x_{N \setminus S} \right) - f_{t_1} \left(x'_{N \setminus S} \right) \right\| \leq c \left\| x_{N \setminus S} - x'_{N \setminus S} \right\|.$$

where $\|.\|$ denotes the Euclidian norm in the space R^{n-s}

Lemma 4. *Let assumptions 1-3 hold. Then, for every $S \in \pi$ and $T \in \pi$, with $|T| \geq |S|$: i) Positive Player-externalities imply $x_s \leq x_t$; ii) Negative Player-externalities imply $x_s \geq x_t$.*

Proof. See Currarini and Marini (2006). ∎

Using all lemmata presented above we are now able to compare the valuation of players belonging to different coalitions in a given coalition structure and then, to a certain extent, the profitability of deviations. However, the above analysis is limited to games in which forming a coalition does not enlarge the set of strategy available to its members and does not modify the way payoffs within a coalition originate from the strategies chosen by players in N. In fact, as assumed at the beginning of the paper, the action space of each coalition $S \subset N$ is restricted to $X_S \equiv \prod_{i \in S} X_i$. Moreover $v(S, \pi) = \sum_{i \in S} u_i(x(\pi))$. The only advantage for players to form coalitions is to coordinate their strategies in order to obtain a coalitional efficient outcome. This approach encompasses many well known *games without synergies*, such as Cournot and Bertrand merger or cartel formation and public good and environmental games, but rules out an important driving force of coalition formation, i.e. the exploitation of synergies, typically arising for instance in R&D alliances or mergers among firms yielding some sort of economies of scales.

Within this framework, we can present the following result.

Proposition 1. *Let assumptions 1-2 hold, and let u_i have increasing differences on X_N, for all $i \in N$. Then the grand coalition N is a stable coalition structure in the game of coalition formation Γ derived from the game in strategic form G.*

Proof. By Lemma 3, positive externalities imply that for all π, at $x(\pi)$ larger coalitions

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choose larger strategies than smaller coalitions, while the opposite holds under negative externalities, and then $v_i(S, \pi) \geq v_i(T, \pi)$ for all $S, T \in \pi$ with $|T| \geq |S|$. This directly implies the stability of the grand coalition in Γ. To provide a sketch of that proof, we note that any coalitional deviation from the strategy profile σ^* yielding the grand coalition induces a coalition structure in which all members outside the deviating coalitions appear as singleton. Since these players are weakly better off than any of the deviating members, and since all players were receiving the same payoff at σ^*, a strict improvement of the deviating coalition would contradict the efficiency of the outcome induced by the grand coalition. $\blacksquare$

In games with increasing differences, players strategies are strategic complements, and best replies are therefore positively sloped. The stability of the efficient coalition structure $\pi^* = \{N\}$ in this class of games can be intuitively explained as follows. In games with positive externalities, a deviation of a coalition $S \subset N$ will typically be associated with a lower level of S's members' strategies with respect to the efficient profile $x(\pi^*)$, and with a higher level in games with negative externalities (see lemma 3 and 4 above). If strategies are the quantity of produced public good or prices (positive player-externalities), S will try to free ride on non members by reducing its production or its price; if strategies are emissions of pollutant or quantities (negative player-externalities), $\acute{S}$ will try to emit or produce more and take advantage of non members' lower emissions or quantities. The extent to which these deviations will be profitable ultimately depend on the reaction of non members. In the case of positive externalities, S will benefit from an increase of non members' production levels or prices; however, strategic complementarity implies that the decrease of S's production levels or prices will be followed by a decrease of the produced levels or prices of non members. Similarly, the increase of S's pollutant emissions or quantities will induce higher pollution or quantity levels by non members. Free riding is therefore little profitable in these games.

From the above discussion, it is clear that deviations can be profitable only if best reply functions are negatively sloped, that is, strategies must be substitutes in G. However, the above discussion suggests that some "degree" of substitutability may still be compatible with stability. Indeed, if S's decrease in the production of public good is followed by a moderate increase in the produced level of non members, S may still not find it profitable to deviate from the efficient profile induced by π^*. Therefore, if the absolute value of the slope of the reaction maps is bounded above by 1, the stability result of proposition 1 extends to games with strategic substitutes.

Proposition 2. *Let assumptions 1-3 hold. The grand coalition N is a stable coalition structure in the game of coalition formation Γ derived from the game in strategic form G.*

Moreover, we can extend the results of proposition 1 and 2 to *games with negative coalition-externalities.*[19]

Definition 13. *A game in valuation form $v_i(K, \pi)$ exhibits positive (negative) coalition-externalities if, for any coalition structure π and a coalition $S \in \pi$,*

$$v_i(S, \pi') > (<) \, v_i(S, \pi)$$

where π' is obtained by π by merging coalitions in $\pi \backslash S$.

It is clear from the above definition, that under negative coalition-externalities, $v_i^\gamma(x(\pi(\sigma'))) < v_i^\delta(x(\pi(\sigma')))$, where $\sigma' = \left(\{S\}_{i \in S}, \{N\}_{i \in N \backslash S} \right)$, just because $\pi^\gamma(\sigma') = \left(\{S\}, \{j\}_{j \in N \backslash S} \right)$ and $\pi^\delta(\sigma') = (\{S\}, \{N \backslash S\})$. The following propositions exploits this fact.

Proposition 3. *Let assumptions 1-2 hold, and let u_i have increasing differences on X_N, for all $i \in N$. Let also the game $G(\pi)$ exhibits negative coalition-externalities. Then the grand coalition N is a stable coalition structure in the game of coalition formation Δ derived from the game in strategic form G.*

Proposition 4. *Let assumptions 1-3 hold. Let also the game $G(\pi)$ exhibits negative coalition-externalities. Then the grand coalition N is a stable coalition structure in the game of coalition formation Δ derived from the game in strategic form G.*

A comparison of the above results, obtained for Hurt and Kurz's (1985) games of coalition formation, with the other solution concepts can be mentioned. It can be shown (see Yi, 1997) that for all games without synergies in which - as in the merger example - players prefer to stay as singletons to free-ride on a forming coalition - Bloch's (1996) sequential game of coalition formation gives rise to equilibrium coalition structures formed by one coalition and a fringe of coalition acting as singletons. Moreover, even in a linear oligopoly merger game,

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[19]See Bloch (1997) or Yi (2003) for such a definition. There is not a clear relationship between games with positive (or negative) *player-externalities* and games with positive (or negative) *coalition-externalities*. However, for most well known games without synergies, both positive-player externalities (PPE) plus strategic complement actions (SC) as well as negative-player externalities (NPE) plus strategic substitute actions (SS) yield games with positive coalition-externalities. These are the cases of merger or cartel games in quantity oligopolies (NPE+SS), merger or cartel games in price oligopolies (PPE+SC) and public goods (PPE+SS) or environmental games (NPE+SS). Similarly, we can obtain Negative Coalition-Externalities in a game by associating NPE and SC as in a cartel game in which goods are complements and then the game exhibits SC.

Ray and Vohra's (1997) Equilibrium Binding Agreement may or may not support the grand coalition as a stable coalition structure, depending on the number of firms in the market.

When the game G is a *game with synergies*, a classification of the possible results. becomes even more complex. To give an illustration, we can introduce a simple form of synergy by assuming, as in Bloch's (1995) and Yi's (1997) R&D alliance models, that when firms coordinate their action and create a R&D alliance, they pool their research assets in such a way to reduce the cost of each firm in proportion to the number of firms cooperating in the project.[20] Let the producing cost of firms participating to a R&D alliance of s firms be $c(x_i, s_i) = (c + 1 - s_i) x_i$, where s_i is the cardinality of the alliance containing firm i. Let also $a > c \geq n$. As shown by Yi (1997), at the unique Nash equilibrium associated with every coalition structure π, the profit of each firm in a coalition of size s_i is given by:

$$v_i^\gamma (x(\pi)) = \frac{\left(a - (n+1)(c + 1 - s_i) + \sum_{j=1}^k s_j (c + 1 - s_j) \right)^2}{(n+1)^2},$$

When $\pi = \pi^\gamma (\sigma')$, symmetry can be used to reduce the above expression to:

$$v_i^\gamma (\pi(\sigma')) = \frac{(a - (n - s_i + 1)(c + 1 - s_i) + (n - s_i) c)^2}{(n+1)^2}.$$

Straightforward manipulations show that the deviation of a coalition S from the grand coalition in the game Γ is always profitable whenever:

$$s_i > -\frac{1}{2} n + c - \frac{1}{2} \sqrt{(n^2 - 4(nc - c^2) - 8(a - c - 1)}.$$

To give an example, for $n = 8$, the deviation of a group of six firms ($s_i = 6$) induces a per firm payoff of $v_i (\pi(\sigma')) = \frac{(a - c + 15)^2}{81}$ higher than firms' payoff in the grand coalition $v_i (\pi(\sigma^*)) = \frac{(a - c + 7)^2}{81}$. Therefore, it becomes more difficult to predict the stable coalition structures in Hurt and Kurz's Γ and Δ games. In the sequential games of coalition formation (Bloch, 1996 and Ray & Vohra 1999) for a linear Cournot oligopoly in which firms can form reducing-cost alliances, and each firm's $i \in S$ bears a marginal cost

$$c_i = \gamma - \theta s$$

where s is the size of the alliance to which firm i's belongs, the equilibrium profit of each firm $i \in S$ is:

$$v_i (\pi) = \left[\frac{1 - \gamma}{n + 1} + \theta s_i - \frac{\sum_{j \neq i} s_i^2}{(n+1)} \right].$$

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[20]This is usually classified as a game with negative coalition-externalities (see Yi, 1997, 2003).

Therefore, the formation of alliances induces negative externalities on outsiders, just because an alliance reduces marginal costs of participants and make them more aggressive in the market. Moreover, members of larger alliance have higher profits and then, if membership is open, all firms wants to belong to the association (Bloch, 1996, 2005). In the game of sequential coalition formation, anticipating that remaining players will form an association of size $(n - s)$, the first s players optimally decide to admit $s^* = \frac{3n+1}{4}$ and the unique equilibrium coalition structure results in the formation of two associations of unequal size $\pi^* = \left(\left\{ \frac{3n+1}{4} \right\}, \left\{ \frac{n-1}{4} \right\} \right)$.

3 Networks

3.1 Notation

We follow here the standard notation applied to networks.[21] A nondirected network (N, g) describes a system of reciprocal relationships between individuals in a set $N = \{1, 2, ..n\}$, as friendships, information flows and many others. Individuals are nodes in the graph g and links represent bilateral relationship between individuals.[22] It is common to refer directly to g as a network (omitting the set of players). The notation $ij \in g$ indicates that i and j are linked in network g. Therefore, a network g is just a list of which pairs of individuals are linked to each other. The set of all possible links between the players in N is denoted by $g^N = \{ij | i, j \in N, i \neq j\}$. Thus $G = \{g \subset g^N\}$ is the set of all possible networks on N, and g^N is denoted as the *complete network*. To give an example, for $N = \{1, 2, 3\}$, $g = \{12, 13\}$ is the network with links between individuals 1 and 2 and 1 and 3, but with no link between player 2 and 3. The complete network is $g^N = \{12, 23, 13\}$. The network obtained by adding link ij to a network g is denoted by $g + ij$, while the network obtained by deleting a link ij from a network g is denoted $g - ij$. A *path* in g between individuals i and j is a sequence of players $i = i_1, i_2, ...i_K = j$ with $K \geq 2$ such that $i_k i_{k+1} \in g$ for each $k \in \{1, 2, ..., K - 1\}$. Individuals who are not connected by a path are in different *components* C of g; those who are connected by a path are in the same component. Therefore, the components of a network are the distinct connected subgraphs of a network. The set of all component can be indicated as $C(g)$. Therefore, $g = \bigcup_{g' \in C(g)} g'$. Let also indicate with $N(g)$ the players who have at

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[21] See, for instance, Jakcson and Wolinski (1996), Jackson (2003) and van den Noweland (2005).

[22] Here both individuals engadged in a relationship have to give their consent for the link to form. If the relationship is unilateral (as in advertising) the appropriate model is a *directed network*. Also, here the intensity of a link is assumed constant.

least one link in network g.

3.2 Value Functions and Allocation Rules

It is possible to define a value function assigning to each network a worth.

Definition 14. *A value function for a network is a function* $v : G \to \mathbf{R}$.

Let us denote V the set of all possible value functions. In some applications $v(g) = \sum_i u_i(g)$ where $u_i : G \to \mathbf{R}$. A network $g \in G$ is defined (strongly) *efficient* if $v(g) \geq v(g')$ for all $g' \in G$. If the value is transferable across players, this coincides with Pareto-efficiency.[23] Since the network is finite, it always exists an efficient network.

Another relevant modelling feature is the way in which the value of a network is distributed among the individuals forming the network.

Definition 15. *An allocation rule is a function* $Y : G \times V \to \mathbf{R}^N$.

Thus, $Y_i(g, v)$ is the payoff obtained by every player $i \in N(g)$ under the value function v.

Some important properties of the value functions v and of the allocation rules Y are listed below.[24]

(1) Component Additivity. *A value function* v *is component additive if* $v(g) = \sum_{g' \in C(g)} v(g')$ *for all* $g \in G$.

This property requires that the value of the network equals the sum of the value of its components. This means that the value of one component is independent of the structure of the other components.

When an allocation rule Y distributes all the value accruing to one component to its members, it is *component balanced*.[25]

(2) Component Balance. An allocation rule is component balanced if for any compo-

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[23] A network g is Pareto efficient (PE) with respect to a value v and an allocation rule Y if there not exists any $g' \in G$ such that $Y_i(g', v) \geq Y_i(g, v)$ with strict inequality for some i. Note that if a network is PE with respect to v and Y for all possible allocation rules Y, it is (strong) efficient (see Jackson 2003).

[24] See Jackson and Wolinsky (1996) and Jackson (2005a) for details.

[25] An allocation rule is *balanced* if $\sum_i Y_i(g, v) = v(g)$ for all v and g.

nent additive v, for every $g \in G$ and $g' \in C(g)$

$$\sum_{i \in N(g')} Y_i(g', v) = v(g').$$

(3) Fairness (Equal Bargaining Power). *An allocation rule Y satisfies fairness if, for any component additive v and for every $g \in G$,*

$$Y_i(g) - Y_i(g - ij) = Y_j(g) - Y_j(g - ij).$$

This property implies that under Y every i and j gain equally from the existence of their link when compared to their payoffs in absence of this link. If we take a permutation of the players $p : N \rightarrow N$ we can define the same network with permuted individuals as $g^p = \{ij \mid i = p(h), j = p(k), hk \in g\}$ and, if $v^p(g^p) = v(g)$, we say that the value function is *anonymous*.

(4) Anonymity. *An allocation rule Y is anonymous if for any permutation p of the N players, $Y_{p(i)}(g^p, v^p) = Y_i(v, g)$.*

A strong symmetry assumption on the allocation rule Y requires that for all anonymous $v \in V$, $g \in G$ and permutations p such that $g^p = g$, $Y_{p(i)}(g^p, v^p) = Y_i(g, v)$ (*equal treatment of equals*).

When compared to the characteristic function of cooperative games (see Section 1.1), here a value function v is sensitive not only to the number of players connected (in a component of g) but also to the specific architecture in which they are connected. However, v can be restricted to depend only on the number of players connected in a coalition. In a seminal contribution, Myerson (1977) starts with a TU cooperative game (N, v) and overlaps a communication network g to such a framework. Myerson (1977) associates a "graph-restricted value" $v^g : 2^N \rightarrow \mathbf{R}$, assigning to each coalition S a value equal to the sum of worth generated by the connected components of players in S. Formally, players in S have links in $g(S) = \{ij \in g \mid i \in S, j \in S\}$ and this induces a partition of S into subsets of players $S(g)$ that are connected in S by g. Thus, $v^g(S) = \sum_{g' \in C^S(g)} v(g')$ for every $S \subset N$, where $C^S(g)$ indicates the set of components induced by g involving players belonging to coalition S. This value assumes that players in S can coordinate their action only within their own components.[26] Two assumptions underline this value: i) there are no externalities between different components of a network; ii) what matters for the worth v^g is only the worth of the

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[26] This implies a component balanced allocation rule Y.

coalition of players which are in a component, not the type of connections existing within the coalition. Within this framework, Myerson characterizes a specific *allocation rule* (known as Myerson value) distributing the payoffs among individuals, and shows that under two axioms - fairness and component additivity - the unique allocation rule satisfying these properties is the Shapley value of the graph-restricted game (N, v^g):

$$Y_i(g, v^g) = \sum_{S \subset N \setminus \{i\}} \frac{|S|! \, (|N| - 1 - |S|)!}{|N|!} (v^g(S \cup \{i\}) - v^g(S)).$$

3.3 Networks Formation Games

3.3.1 Networks Formation in Extensive Form

Aumann and Myerson (1988) propose an extensive form game to model the endogenous formation of cooperation structures. In their approach, which involves a sequential formation of links among players, bilateral negotiations take place in some predetermined order. Firstly, an exogenous rule determines the sequential order in which pairs of players negotiate to form a link. A link is formed if and only if both players agree and, once formed, cannot be broken. The game is one of perfect information and each player knows the entire history of links accepted or rejected at any time of the game. Once all links between pairs of players have formed, single players can still form links. Once all players have decided, the process stops and the network g forms and the payoff is assigned according to the Myerson value, i.e., the Shapley value of the restricted game (N, v^g). Stable cooperative structure are considered only those associated with subgame perfect equilibria of the game.

Example 3.[27] *Suppose a TU majority game with $N = \{1, 2, 3\}$ and $v(S) = 1$ if $|S| \geq 2$ and $v(S) = 0$ otherwise. If the exogenous rule specifies the following order of pairs: $\{1, 2\}, \{1, 3\}, \{2, 3\}$. The structure $\{1, 2\}$ is the only cooperation structure supported by a subgame perfect equilibrium of the game. Neither player 1 nor player 2 have an interest to form a link with player 3, provided that the other player has not formed a link with 3. So, using backward induction, if at the final stage $\{2, 3\}$ has formed, at stage 2 also $\{1, 3\}$ forms and player 1 obtains a lower payoff than in a coalition with only player 2. Thus, at stage 1 player 1 forms a link with player 2 and the latter accepts. No other links are formed at the following stages.*

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[27] This example is taken from Dutta, van den Noweland & Tijs (1995).

It is possible that a subgame Nash equilibrium of the Aumann and Myerson's network formation game in extensive form does not support the formation of the complete network even for superadditive games. Moreover, no general results are known for the existence of stable complete networks even for symmetric convex games.[28]

3.3.2 Networks Formation in Strategic Form

Myerson (1991) suggests a noncooperative game of network formation in strategic form.[29] For each player $i \in N$ a strategy $\sigma_i \in \Sigma_i$ is given by the set of players with whom she want to form a link, i.e. $\Sigma_i = (S \mid S \subseteq N \setminus \{i\})$. Given a n-tuple of strategies $\sigma \in \Sigma_1 \times \Sigma_2 \times ... \times \Sigma_n$ a link ij is formed if and only if $j \in \sigma_i$ and $i \in \sigma_j$. Denoting the formed (undirected) network $g(\sigma)$, the payoff of each player is given by $Y_i(v, g(\sigma))$ for every $\sigma \in \Sigma_N$. A strategy profile σ is a Nash equilibrium of the Myerson's *linking game* if and only if, for all player i and all strategies $\sigma' \in \Sigma_i$,

$$Y_i(v, g(\sigma)) \geq Y_i(v, g(\sigma'_i, \sigma_{-i})).$$

We can also define a network g *Nash stable* with respect to a value function v and an allocation rule Y, if there exists a pure strategy Nash equilibrium σ such that $g = g(\sigma)$.

The concept of Nash equilibrium applied to the network formation game appears a too weak notion of equilibrium, due to the bilateral nature of links. The empty network (a g with no links) is always Nash stable for any v and Y. Moreover, all networks in which there is a gain in forming additional links but no convenience to sever existing links are also Nash stable. Refinements of the Nash equilibrium concept for the network formation process have been proposed. The *pairwise stability* introduced by Jackson and Wolinsky (1996) plays a prominent role in the recent developments of the analysis of networks formation.

3.3.3 Pairwise Stability

We should expect that in a stable network players do not benefit by altering the structure of the network. Accordingly, Jackson and Wolinsky (1996) defines a notion of network stability denoted *pairwise stability*.

Definition 16. *A network g is pairwise stable with respect to the allocation rule Y and value function v if*

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[28]See, for a survey of this approach, van den Noweland (2005).
[29]This game is also analyzed by Quin (1993) and Dutta, van den Noweland & Tijs (1995).

(i) for all $ij \in g$, $Y_i(g, v) \geq Y_i(g - ij, v)$ and $Y_j(g, v) \geq Y_j(g - ij, v)$, and

(ii) for all $ij \notin g$, if $Y_i(g + ij, v) > Y_i(g, v)$ then $Y_j(g + ij, v) < Y_j(g, v)$.

As shown by Jackson and Watts (1998), a network is pairwise stable if and only if it has no *improving path* emanating from it. An improving path *is* a sequence of networks $\{g_1, g_2, ..., g_K\}$ where each network g_k is defeated by a subsequent (adjacent) network g_{k+1}, i.e., $Y_i(g_{k+1}, v) > Y_i(g_{k+1}, v)$ for $g_{k+1} = g_k - ij$ or $Y_i(g_{k+1}, v) \geq Y_i(g_{k+1}, v)$ and $Y_j(g_{k+1}, v) \geq Y_j(g_{k+1}, v)$ for $g_{k+1} = g_k + ij$, with at least one inequality holding strictly. Thus, if there not exists any pairwise stable network , then it must exists at least one cycle, i.e. an improving path $\{g_1, g_2, ..., g_K\}$ with $g_1 = g_k$. Jackson and Wolinsky (1996) show that the existence of pairwise stable networks is always ensured for certain allocation rules. They prove that under the *egalitarian* and the *component-wise egalitarian rules*[30] pairwise stable *networks* always exists. In particular, under the egalitarian rule, any efficient network is pairwise stable. Under the component-wise allocation rule, a pairwise stable network can always be found. This can be done for component additive v by finding components C that maximize the payoffs of its players, and then continuing this process for the remaining players $N \backslash N(C)$. The network formed by all these components is pairwise stable. Another allocation rule with strong existence properties is the Myerson value. As shown by Jackson and Wolinsky (1996), under Myerson's allocation rule there always exists a pairwise network for every value function $v \in V$. Moreover, all improving paths emanating from any network lead to pairwise stable networks, i.e. there are no cycles under the Myerson value allocation rule.[31]

However, as it is shown by Jackson and Wolinsky and by Jackson (2003), there exists a tension between efficiency and stability whenever the allocation rule Y is component balanced and anonymous, in the sense that there does not exists an allocation rule with such properties that for all $v \in V$ yields an efficient network that is pairwise stable. In what follows we report the illustrative example by Jackson and Wolinsky (1996) known as the connection model.

Example 4 - (The Connection Model-Jackson and Wolinsky 1996). *This is a model dealing with social communication between individuals. Links among individuals allows them to communicate directly, but also indirectly with those individuals to whom their*

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[30] The *egalitarian allocation rule* Y^e is such that $Y_i^e(g, v) = \frac{v(n)}{n}$ for all i and g. The component-wise allocation rule Y^{ce} is an egalitarian rule respecting component balance, i.e., such that $Y_i^{ce}(g, v) = \frac{v(C)}{|N(C)|}$ when $N(C)$, the set of players in component C is non empty and $Y_i^{ce}(g, v) = 0$ otherwise. See Jackson and Wolinsky (1996) and Jackson (2003) for details.

[31] See also Jackson (2003) for details.

adjacent individuals are linked, and so on. To form a link is costly, but yields also a benefit depending on the distance t_{ij} among individuals, defined as the number of links in the shortest path between i and j ($t_{ij} = \infty$ when there is no path between i and j). Defining w_{ij} the value of individual j to individual i, the utility of each player from network g is

$$u_i(g) = w_{ii} + \sum_{j \neq i} \delta^{t_{ij}} w_{ij} - \sum_{j:ij \in g} c_{ij},$$

*where $0 < \delta < 1$ is a parameter expressing the value for i of the proximity of j: less distant links are more valuable than more distant ones. Let also $v(g) = \sum_{i \in N} u_i(g)$. In the symmetric case, with $c_{ij} = c$, $w_{ij} = 1$ for all $j \neq i$ and $w_{ii} = 0$, we have the following results: i) For $c < \delta - \delta^2$ the unique efficient and pairwise stable network is the complete network g^N; ii) For $\delta - \delta^2 < c < \delta + ((n-2)/2)\delta^2$ the star network with one individual maintaining one link with all other individuals is the only efficient network and this is pairwise stable for $\delta - \delta^2 < c < \delta$; (iii) For $c > \delta$ any non-empty pairwise stable network is such that each player has two links and is inefficient;(iv) For $c > \delta + ((n-2)/2)\delta^2$ the empty network is the only efficient network. Let us show these results for $N = \{1, 2, 3\}$ [see also **Figure 5**] When $c < \delta - \delta^2$, this implies that $\delta^2 < \delta - c$, and every pair of individuals not directly connected would gain by forming a direct link (since $c < \delta$), and this also increases the network value. The value of the complete network is $v(g^N) = 6(\delta - c)$. The value obtained with the star network (only one individual linked to any other) is $v(g^S) = 4\left(\delta + \frac{\delta^2}{2} - c\right)$ and therefore $v(g^N) - v(g^S) = 2\left(\delta - \delta^2 - c\right) > 0$ for $c < \delta - \delta^2$. For g^S to become efficient it is required that $\delta - \delta^2 < c < \delta + \delta^2$, where the right hand side of the inequality ensures that every player who is maintaining only one link receives a positive payoff. The star network g^S becomes the unique pairwise stable network when $\delta - \delta^2 < c < \delta$, since in this case neither peripheral players want to create links nor the player mantaining all links (center of the star) want to sever her links. [32] The critical cost range is $\delta + \frac{\delta^2}{2} > c > \delta$, since in this case the unique pairwise stable network is the empty network, but this is inefficent given that the star network yields a value of $v(g^S) = 4\left(\delta + \frac{\delta^2}{2} - c\right) > 0$. Finally, for $c > \delta + \frac{\delta^2}{2}$ the empty network is the only efficient pairwise stable network.*

Thus, the example above shows that a pairwise stable network can either be inefficient or efficient, depending on the cost range. The tension between efficiency and stability appears here for intermediate levels of the cost.

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[32] For $n > 3$ the encompassing star is not necessarily the unique pairwise network.

3.3.4 Further Refinements of Network Stability Concepts

As in the case of coalition formation, equilibrium concepts immune to coordinated deviations by players are also conceivable for networks (see Dutta and Mutuswami, 1997, Dutta, Tijs and van den Noweland, 1998 and Jackson and van den Noweland 2005). By allowing every subset of players to coordinate their strategies in arbitrary ways yields a strong Nash equilibrium for network formation games. That is, a strategy profile $\sigma \in \Sigma_N$ is a *strong Nash equilibrium* of the network formation game if there not exist a coalition $S \subseteq N$ and a strategy profile $\sigma'_S \in \Sigma_S$ such that

$$Y_i(v, g(\sigma'_S, \sigma_{N \setminus S})) \geq Y_i(v, g(\sigma))$$

with strict inequality for at least one $i \in S$. Hence, a network g is *strongly stable* with respect to a value function v and an allocation rule Y, if there exists a strong Nash equilibrium σ such that $g = g(\sigma)$.

Similarly, an intermediate concept of stability, stronger than pairwise stability and weaker than strong Nash equilibrium, has been proposed (Jackson and Wolinsky, 1996) and denoted *pairwise Nash equilibrium*. This can be defined as a strategy profile $\sigma \in \Sigma_N$ such that, for all player i and all strategies $\sigma'_i \in \Sigma_i$,

$$Y_i(v, g(\sigma'_i, \sigma_{N \setminus i})) \geq Y_i(v, g(\sigma))$$

and there not exists a pair of agents (i, j) such that

$$
\begin{aligned}
Y_i(v, g(\sigma) + ij) &\geq Y_i(v, g(\sigma)), \\
Y_j(v, g(\sigma) + ij) &\geq Y_j(v, g(\sigma))
\end{aligned}
$$

with strict inequality for at least one of the agents. Therefore, a network g is *pairwise Nash stable* with respect to a value function v and an allocation rule Y, if there exists a pairwise Nash equilibrium σ such that $g = g(\sigma)$.[33]

It can be shown that, given a value function v and an allocation rule Y, the set of *strongly stable networks* is weakly included in the *set of pairwise Nash stable networks* and that the latter set coincides with the intersection of *pairwise stable networks* and *Nash stable*

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[33] This equilibrium concept has been adopted in applications by Goyal and Joshi (2003) and Belleflamme and Bloch (2004) and formally studied by Calvo-Armengol and Ilkilic (2004), Ilkilic (2004) and Gillies and Sarangi (2004).

networks.[34] Moreover, the set of pairwise stable networks and the set of Nash stable networks can be completely disjoint even though neither is empty.[35]

In the next section, I briefly illustrate some very simple applications of network formation games to classical I. O. models. These are taken from Bloch (2002), Belleflamme and Bloch (2004) as well as Goyal and Joshi (2003).

3.3.5 Some Economic Applications

Collusive Networks In Bloch (2002) and in Belleflamme and Bloch (2004) it is assumed that firms can sign bilateral market sharing agreements. Initially firms are present on different (geographical) markets. By signing bilateral agreement they commit not to enter each other's market.

If $ij \in g$, firm i withdraws from market j and firm j withdraws from market i. For every network g and given N firms, let $n_i(g)$ be the number of firms in firm i's market. with $n_i(g) = n - d_i(g)$ where $d_i(g)$ is the degree of vertex (firm) i in the network, i.e. the number of its links.

If all firms are identical, firm i's total profit is

$$U_i(g) = u_i(n_i(g)) + \sum_{j, \, ij \notin g} u_i(n_j(g)).$$

With linear demand and zero marginal cost, under Cournot competition we obtain

$$U_i(g) = \frac{a^2}{[n_i(g) + 1]^2} + \sum_{j, \, ij \notin g} \frac{a^2}{[n_j(g) + 1]^2}.$$

If $n \geq 3$, there are exactly two pairwise stable networks, the empty network and the complete network. For $n = 2$, the complete network is the only stable network.

Note that the empty network is stable since for every symmetric firm the benefit to form a link is

$$U_i(g + ij) - U_i(g) = \frac{a^2}{n^2} - 2\frac{a^2}{(n + 1)^2}$$

that, for $n \geq 3$, is negative.

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[34] See, for instance, Jackson and. van den Nouweland, (2005) and Bloch and Jackson (2006).

[35] See Bloch and Jackson (2006) and Bloch and Jackson (2007), for an extensions of these equilibrium concepts to the case in which transfers among players are allowed.

For every incomplete network, $U_i(g) - U_i(g - ij) \geq 0$, requires that

$$\frac{a^2}{[n_i(g) + 1]^2} - \left[\frac{a^2}{[n_i(g) + 2]^2} + \frac{a^2}{[n_j(g) + 1]^2} \right] \geq 0$$

and this holds only for $n_i(g) = n_j(g) = 1$, i.e., when the network is complete.

In this case (see, **figure 5** for the case with 3 firms),

$$U_i(g^N) - U_i(g^N - ij) = \frac{a^2}{4} - \frac{2a^2}{9} > 0.$$

Therefore, it follows that the only nonempty network which is pairwise stable is the complete network.

3.3.6 Bilateral Collaboration among Firms

Bloch (2002) and Goyal and Joshi (2003) consider the formation of bilateral alliances between firms that reduce their marginal cost. as

$$c_i = \gamma - \theta d_i(g)$$

where $d_i(g)$ denotes the degree of vertex i, i.e. the number of bilateral agreements signed by firm i.

Under Cournot competition with linear demand, we have each firm's profit is given by

$$U_i(g) = \left[\frac{a - \gamma}{n + 1} + \theta d_i(g) - \frac{\theta \sum_j d_j(g)}{n + 1} \right]^2.$$

For such a case, the only pairwise stable network turns out to be the complete network g^N (see Goyal and Joshi, 2003). This is because, by signing an agreement, each firm increases its quantity by $\Delta q_i = \frac{n\theta}{n+1}$ and, consequently, its profit. Moreover, when a large fixed cost to form a link is included in the model, Goyal and Joshi show that stable networks possess a specific form, with one complete component and a few singleton firms.

4 Appendix

Lemma 1. For all $S \subseteq N$, $\widetilde{x}_S \in \arg\max_{x_S \in X_S} \sum_{i \in S} u_i(x_S, x_{N \setminus S})$ implies $\widetilde{x}_i = \widetilde{x}_j$ for all $i, j \in S$ and all $x_{N \setminus S} \in X_{N \setminus S}$.

Proof. Suppose $\widetilde{x}_i \neq \widetilde{x}_j$ for some $i, j \in S$. By symmetry we can derive from $\widetilde{x}_S$ a new vector $\widetilde{x}_S^p$ permuting the strategies of players i and j such that

$$\sum_{i \in S} u_i(\widetilde{x}_S^p, x_{N \setminus S}) = \sum_{i \in S} u_i(\widetilde{x}_S, x_{N \setminus S})$$

and hence, by the quasi-concavity of all $u_i(x)$, for all $\lambda \in (0, 1)$ we have that:

$$\sum_{i \in S} u_i(\lambda \widetilde{x}_S^p + (1 - \lambda)\widetilde{x}_S, x_{N \setminus S}) > \sum_{i \in S} u_i(\widetilde{x}_S, x_{N \setminus S}).$$

Since, by convexity of X, the strategy vector $\left(\lambda \widetilde{x}_S^p + (1 - \lambda)\widetilde{x}_S, x_{N \setminus S}\right) \in X_S$, we obtain a contradiction. $\blacksquare$

Lemma 2. *Let Assumptions 1 and 2 hold. Then:*

i) Under Positive Externalities, $u_s(x(\pi)) \geq u_t(x(\pi))$ if and only if $x_s \leq x_t$;

ii) Under Negative Externalities, $u_s(x(\pi)) \geq u_t(x(\pi))$ if and only if $x_s \geq x_t$.

Proof. Given a partition π of N, we consider $S \in \pi$ and $T \in \pi$, with $|T| \geq |S|$. We denote by $x_s \in X$ and by $x_t \in X$ the strategies chosen by each member of S and T at the equilibrium profile $x(\pi)$, respectively. It will be useful to refer to a partition of the coalition T into the disjoint subsets T_1 and T_2 of T, such that $|T_1| = |S|$ (T_2 is, of course, the empty set if $|T| = |S|$). To keep notation simple, we will refer to players payoffs omitting from the argument of payoff functions all the strategies played by players in $N \setminus (T \cup S)$ at the equilibrium profile $x(\pi)$. More precisely, we will use the following notational convention:

$$((x, y), z) \equiv \left((x)_{i \in T_1}, (y)_{i \in T_2}, (z)_{i \in S}, (x_j(\pi))_{j \in N \setminus (T \cup S)}\right)$$

where $(x)_{i \in T_1}$ denotes the joint strategy $x_{T_1} \in X_{T_1}$ in which $x_i = x$ for all $i \in T_1$, and the same notational convention applies to $(y)_{i \in T_2}$ and $(z)_{i \in S}$. It follows that the triplet $((x_t, x_t), x_s)$ identifies the equilibrium profile $x(\pi)$.

We prove the result for the case of positive externalities, starting with the "only if" part. By lemma 1, all members of T get the same payoff at $x(\pi)$. By definition of $x(\pi)$, the profile in which all members of T play x_t maximizes the utility of each member of T, so that

$$(8) \qquad u_t((x_t, x_t) x_s) \geq u_t((x_s, x_s), x_s).$$

Suppose now that $x_s > x_t$. By assumption 1 and 2.1 we have

$$(9) \qquad u_t((x_s, x_s), x_s) = u_{t_i}((x_s, x_s), x_s) = u_s((x_s, x_s), x_s) > u_s((x_t, x_t), x_s).$$

To prove the "if" part, consider coalitions T_1, T_2 and S which, as defined at the beginning of this section, are such that $|T_1| = |S|$ and such that $\{T_1, T_2\}$ forms a partition of T. By definition of $x(\pi)$, the utility of each member of S is maximized by the strategy profile x_S. Using the definition of u_s and of x_s we write:

$$(10) \qquad u_s((x_t, x_t), x_s) \geq u_s((x_t, x_t), x_t).$$

By assumption 2.1, if $x_s \leq x_t$ then

$$(11) \qquad u_s((x_t, x_t), x_t) \geq u_s((x_s, x_t), x_t).$$

Finally, by assumption 1 and the fact that $|T_1| = |S|$, we obtain

$$(12) \qquad u_s((x_s, x_t), x_t) = u_{t_1}((x_t, x_t), x_s) = u_t((x_t, x_t), x_s),$$

implying, together with (11) and (12), that

$$(13) \qquad u_s(x(\pi)) = u_s((x_t, x_t), x_s) \geq u_t((x_t, x_t), x_s) = u_t(x(\pi)).$$

The case of negative externalities can be proved along similar lines. ■

Lemma 3. *Let assumptions 1-2 hold, and let u_i have increasing differences on X_N, for all $i \in N$. Let π, T and S be defined as in Notation 1. Then: i) Positive Externalities imply $x_s \leq x_t$; ii) Negative Externalities imply $x_s \geq x_t$.*

Proof. *i)* Suppose that, contrary to our statement, positive externalities hold and $x_s > x_t$. By increasing differences of u_i for all $i \in N$ (and using the fact that the sum of functions with increasing difference has itself increasing differences), we obtain:

$$(14) \qquad u_s((x_s, x_t), x_s) - u_s((x_s, x_t), x_t) \geq u_s((x_t, x_t), x_s) - u_s((x_t, x_t), x_t).$$

By definition of x_s we also have:

$$(15) \qquad u_s((x_t, x_t), x_s) - u_s((x_t, x_t), x_t) \geq 0.$$

Conditions (14) and (15) directly imply:

$$(16) \qquad u_s((x_s, x_t), x_s) - u_s((x_s, x_t), x_t) \geq 0.$$

Referring again to the partition of T into the disjoint coalitions T_1 and T_2 as defined in Notation 1, an application of the symmetry assumption 1 yields:

$$(17) \qquad \begin{aligned} u_s((x_s, x_t), x_s) &= u_{t_1}((x_s, x_t), x_s); \\ u_s((x_s, x_t), x_t) &= u_{t_1}((x_t, x_t), x_s). \end{aligned}$$

Conditions (16) and (17) imply:

$$u_{t_1}((x_s, x_t), x_s) \geq u_{t_1}((x_t, x_t), x_s). \tag{18}$$

Positive externalities and the assumption that $x_s > x_t$ imply:

$$u_{t_2}((x_s, x_t), x_s) > u_{t_2}((x_t, x_t), x_s). \tag{19}$$

Summing up conditions (18) and (19), and using the definition of T_1 and T_2, we obtain:

$$u_t((x_s, x_t), x_s) > u_t((x_t, x_t), x_s), \tag{20}$$

which contradicts the assumption that x_t maximizes the utility of T given x_s.

The case $ii)$ of negative externalities can be proved along similar lines. ∎

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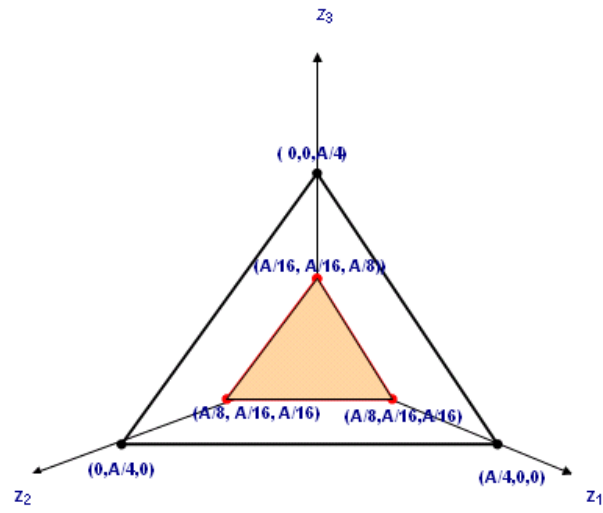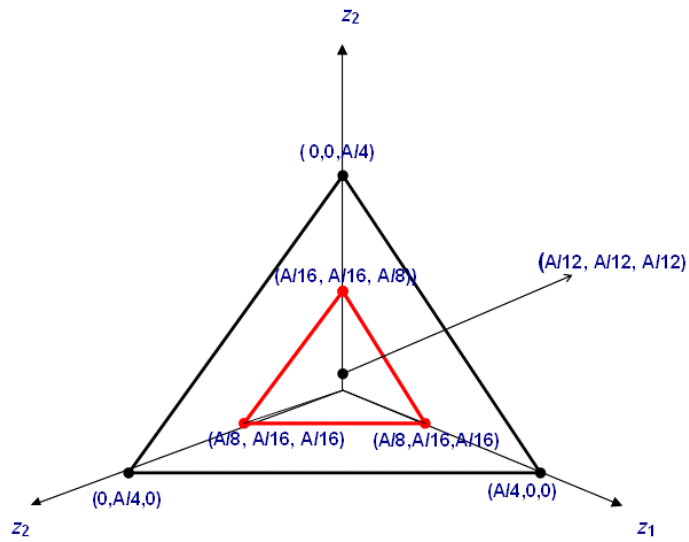
Figure 1

Figure 2
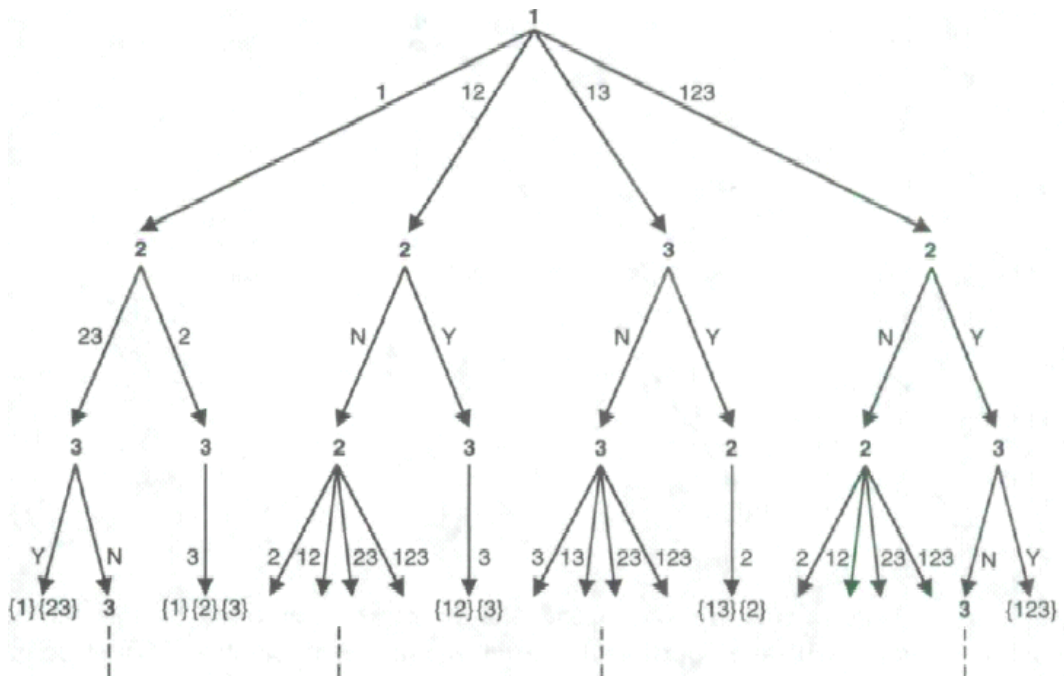
41

THE GAME OF ASSOCIATION FORMATION

Figure 3

THE GAME OF CHOICE OF ASSOCIATION SIZES

Figure 4

42

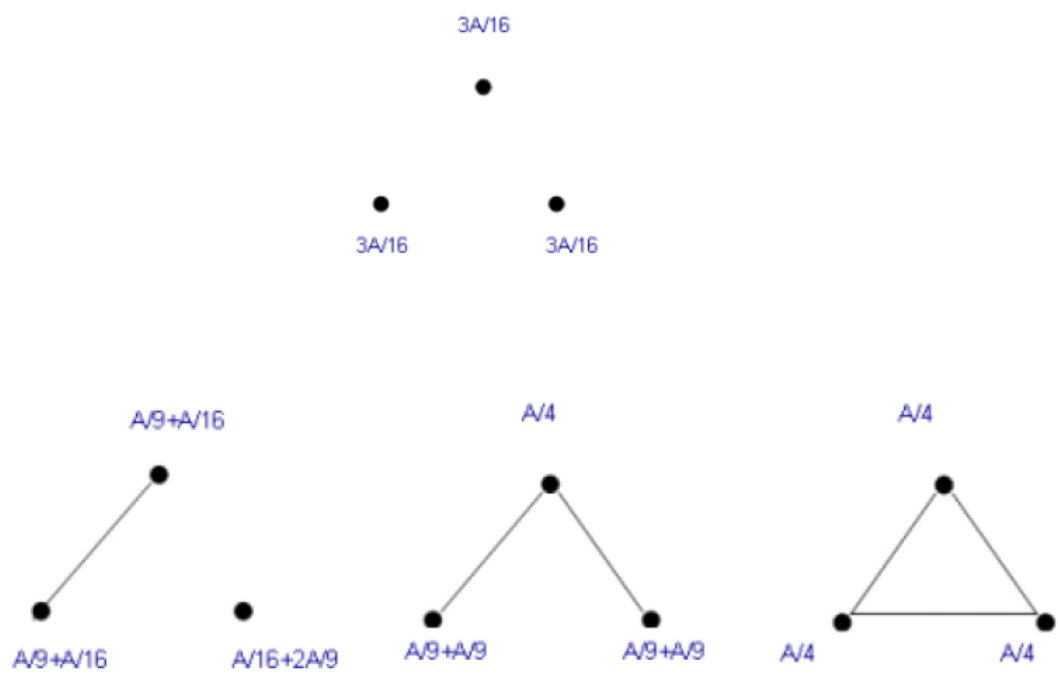**Figure 5**

The figure shows four network diagrams:

Top-left network (empty graph):
$$g^0 = \{\varnothing\}$$
$$u_i(g^0) = 0$$
$$v(g^1) = 0$$

Top-right network (single link 12):
$$g^1 = \{12\}$$
$$u_1(g^1) = u_2(g^1) = \delta - c$$
$$u_3(g^1) = 0$$
$$v(g^1) = 2(\delta - c)$$

Bottom-left network (star with links 12, 13):
$$g^S = \{12, 13\}$$
$$u_1(g^S) = 2(\delta - c)$$
$$u_2(g^S) = u_3(g^S) = \delta + \delta^2 - c$$
$$v(g^S) = 4\delta + 2\delta^2 - 4c$$

Bottom-right network (complete graph with links 12, 13, 23):
$$g^N = \{12, 13, 23\}$$
$$u_i(g^N) = 2(\delta - c)$$
$$v(g^N) = 6(\delta - c)$$

3A/16

3A/16   3A/16

A/9+A/16   A/4   A/4

A/9+A/16   A/16+2A/9   A/9+A/9   A/9+A/9   A/4   A/4

Figure 6