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"Bifurcation Curves in Discontinuous Maps"

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Bifurcation Curves in Discontinuous Maps.[1]

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Abstract

Several discrete-time dynamic models are ultimately expressed in the form of iterated piecewise linear functions, in one- or two- dimensional spaces. In this paper we study a one-dimensional map made up of three linear pieces which are separated by two discontinuity points, motivated by a dynamic model arising in social sciences. Starting from the bifurcation structure associated with one-dimensional maps with only one discontinuity point, we show how this is modified by the introduction of a second discontinuity point, and we give the analytic expressions of the bifurcation curves of the principal tongues (or tongues of first degree), for the family of maps considered, that depends on five parameters.

1 Introduction.

In the recent literature, several papers on dynamic modelling applied to the description of economic and social applications, as well as in engineering applications, ultimately propose discrete-time models which are expressed in the form of iterated piecewise linear (or more generally piecewise smooth) maps, continuous (see e.g. Day 1984, 1992, Hommes 1991, 1995 Hommes and Nusse, 1991, Hommes et al. 1995, Gallegati et al. 2003, Puu and Sushko 2002, 2006, Sushko et al., 2003, 2005, 2006, Gardini et al. 2006a, 2006b, 2008) or discontinuous, with one or more discontinuity points (Puu et al. 2002, 2005, Puu 2007, Sushko et al. 2004, Tramontana et al. 2008). The bifurcations involved in such class of maps are often described in terms of the so called *border-collision bifurcations.* We can classify as *border-collision* any contact between an invariant set of a map with the border of its region of definition. However, such contacts may, or may not, produce a *bifurcation.* The term *border-collision bifurcation* was used for the first time by Nusse

[1] JEL classifications: C02, C62, C63.

and Yorke in 1992 (see also Nusse and Yorke, 1995) and it is now widely used in this context, i.e. for piecewise smooth maps, although the study and description of such border collision bifurcations started several years before those papers. For example, Leonov (1959, 1962) described several bifurcations of that kind, and gave a recursive relation to find the analytic expression of the sequence of bifurcations occurring in a one-dimensional piecewise linear map with one discontinuity point, which is still almost unknown, except for a limited number of researchers among which Mira (1978), Maistrenko et al. 1993, 1995, 1998. In particular, the results obtained by Feigen in 1978, re-proposed in di Bernardo et al. 1999, being almost unknown up to their quotation, that was the first author able to give a clear and simple analysis for *$n-$dimensional piecewise linear continuous maps, with $n > 1$*. These authors already described, using different names and notations, the contact bifurcations now called of border collision after Nusse and Yorke (1992).

These kinds of bifurcations are now widely studied and gave rise to a flourishing literature in the last years, mainly because of their relevant applications in Electrical and Mechanical Engineering. In fact, several papers on piecewise smooth dynamical systems and border collision bifurcations have been motivated by the study of models used to describe particular electrical circuits or systems for the transmission of signals (di Bernardo et al., 1999, Banerjee and Grebogi 1999, Banerjee et al. 2000a, 2000b, Feely et al. 2000, Fournier et al. 2001, Halse et al., 2003, Zhanybai et al. 2003, Avrutin and Schanz 2006, Avrutin et al., 2006, Zhusubaliyev et al. 2006, 2007).

The present work is motivated by some papers dealing with dynamic models in social sciences, in which the models proposed are described by one dimensional maps, piecewise linear or piecewise smooth, with two (or more) discontinuity points, such as the duopoly model in Tramontana et al. (2008) and the model in Bischi and Merlone (2008) (related to the works of Schelling, 1973, 1978). The family of iterated maps considered in the present paper has the form:

$$x' = T(x) = \begin{cases} T_L(x) = m_1 x + (1 - m_1) & \text{if} \quad 0 \leq x < d_1 \\ T_R(x) = m_2 x & \text{if} \quad d_1 < x < d_2 \\ T_3(x) = m_3 x + (1 - m_3) & \text{if} \quad d_2 < x \leq 1 \end{cases} \tag{1}$$

where the parameters satisfy the following conditions:

$$0 < m_i < 1 \quad , \quad i = 1, 2, 3 \quad , \quad 0 < d_1 < d_2 \leq 1 \tag{2}$$

so that T maps the interval $[0, 1]$ into itself. The goal of this paper is to describe the possible bifurcations occurring in the map T in (1) when the parameters vary in the ranges given in (2). As we shall prove later, the slope m_3 is not a relevant parameter, in the sense that whichever is its value inside the interval $(0, 1)$ we obtain the same kind of dynamics, hence we can arbitrarily fix its value, for example setting $m_3 = m_1$. For this reason we shall also consider the following three-pieces piecewise linear map:

$$x' = T_2(x) = \begin{cases} T_L(x) = m_1 x + (1-m_1) & \text{if} \quad 0 \le x < d_1 \ \text{ or } \ d_2 < x \le 1 \\ T_R(x) = m_2 x & \text{if} \quad d_1 < x < d_2 \end{cases} \tag{3}$$

such that the first and the third pieces belong to the same line.

Moreover, in order to understand the bifurcations occurring in this map it is convenient to describe first the bifurcations occurring in the following piecewise linear map with only one discontinuity point:

$$x' = T_1(x) = \begin{cases} T_L(x) = m_1 x + (1-m_1) & \text{if} \quad 0 \le x < d_1 \\ T_R(x) = m_2 x & \text{if} \quad d_1 < x \le 1 \end{cases} \tag{4}$$

keeping the constraints on the parameters as given in (2).

The plan of the paper is the following. In section 2 we show the graphical representation of the standard numerical exploration of the regions, in the space of the parameters, where stable cycles of different periods exist. The main goal of the paper is the analytic computation of the bifurcation conditions that mark the separation between such regions. In order to obtain this we first study, in section 3, the border collision bifurcations of the map with one discontinuity point T_1. On the basis of the results obtained in this case, by using methods already given in the literature, in section 4 we move to the analytic study of the sequences of bifurcations that characterize the creation and destruction of periodic cycles of the map T_2 characterized by the presence of two points of discontinuity. Finally, in section 5, we show how the study of the more general map T, with two discontinuities and three different slopes, can be trivially deduced from the one of the map T_2. Section 6 concludes.

2 Numerical explorations of the existence of periodic cycles

In this section we consider the problem of existence of periodic cycles for the map T. Let us first consider the map with only one discontinuity, $x' = T_1(x)$. As we shall see, the set of bifurcation curves of this map gives a basic structure which is then modified by the introduction of another discontinuity point. In other words, the basic skeleton that gives the conditions of existence of the periodic cycles of the map T_1, created and destroyed by border collision bifurcations, constitutes a benchmark case from which the more involved bifurcation structures of the maps T_2, as well as the more general map T, characterized by the presence of two discontinuities, can be derived.

This property can be easily conjectures even by a quick numerical computation and graphical representations of the regions of existence of stable cycles of different periods. In fact, for any value of the discontinuity point $d_1 \in [0,1]$ the numerically computed two-dimensional bifurcation diagram in the parameter plane (m_1, m_2) has a structure like the one represented Fig.1, where each color corresponds to a cycle of fixed period, according to the numbers reported in the picture. Such colored regions are usually called *periodicity regions* (or *periodicity tongues* due to their shape). Along boundaries that separate two adjacent periodicity tongues a *border-collision* occurs involving the cycles existing inside the regions.Let us consider now the map

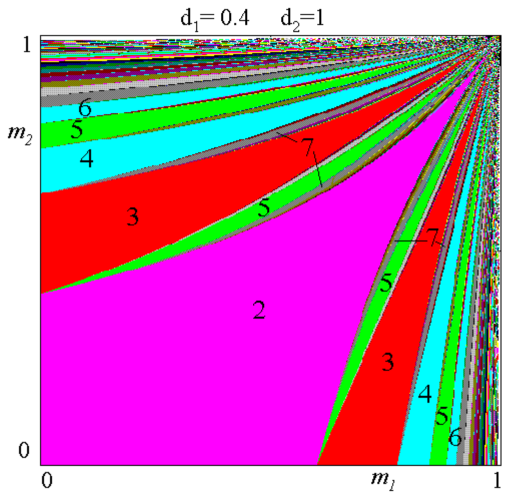

Figure 1: Two-dimensional bifurcation diagram in the plane (m_1, m_2) at $d_1 = 0.4$ and $d_2 = 1$ fixed, of the map T_2, i.e. of the map T_1. Different colors correspond to the existence of cycles of different periods.

T, given in (1), characterized by two points of discontinuity $0 < d_1 < d_2 < 1$. The introduction of the second discontinuity clearly changes the bifurcations which involve the periodic points (of any period) which may still exist. Two examples are shown in Fig.2, at two different values of the pair of discontinuity points d_1 and d_2, and it is immediately clear that the periodicity regions are involved in new kind of bifurcations, again *border-collision* ones. The bifurcation curves associated with the map T_1 can be obtained by using methods already known in the literature, as described in section 3. The main results of this paper concern the explanation of the bifurcation curves shown in Fig.2, whose shape is evidently related to the "skeleton" of periodicity tongues of Fig.1, but there are evident modifications due to the presence of the second discontinuity point. In section 4 we shall describe how to obtain the analytic expression of such bifurcation curves as well.

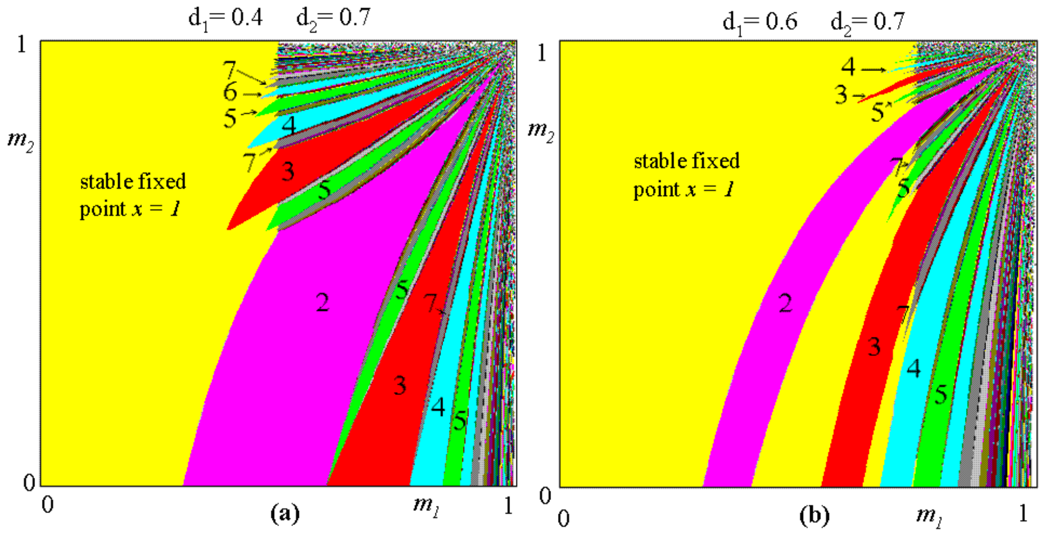

Figure 2: Two-dimensional bifurcation diagrams in the plane (m_1, m_2) of the map T_2. In (a) at $d_1 = 0.4$ and $d_2 = 0.7$. In (b) at $d_1 = 0.6$ and $d_2 = 0.7$.

3 Bifurcation curves of the map T_1.

As recalled in the introduction, the main results of the dynamics of the map $x' = T_1(x)$ are due to Leonov (1959, 1962), also described by Mira (1987), and are quite known nowadays, also due to the recent works of di Bernardo et al. (1999), Banerjee and Grebogi (1999), Avrutin and Schanz (2006). We can also refer to the "word-shifting" technique as defined in Hao-Bai Lin (1989). However, in order to make the paper more self contained, we apply these techniques to our map T_1, because this will then be useful to understand the properties of the map T with two discontinuities.

First of all, it is immediate to see that all the possible cycles of the map T_1 of period $k > 1$ are always stable. In fact, the stability of a $k-$cycle is given by the slope (or eigenvalue) of the function $T_1^k = T_1 \circ ... \circ T_1$ (k times) in the periodic points of the cycle, which are fixed points for the map T_1^k, so that, considering a cycle with p points on the left side of the discontinuity and $(k-p)$ on the right side, the eigenvalue is given by $m_1^p m_2^{(k-p)}$ which, in our assumptions, is always positive and less than 1.

Moreover each $k-$cycle exists in a proper region, called "periodicity region" or "periodicity tongue" (due to the particular shape of these regions). Now let us show, by using the method described by Leonov (1959, 1962), how to obtain the analytical equation of the bifurcation curves that we have seen in Fig.1. Let us consider first the bifurcation curves of the so-called "principal tongues", or "main tongues" (Maistrenko et al. 1993, 1995, 1998, di Bernardo et al. 1999, Banerjee

2000a, Avrutin and Schanz 2006, Avrutin et al. 2006) or "tongues of first level of complexity" (Leonov, 1959, 1962) or "tongues of first degree of complexity" (Mira, 1978, 1987). In the following we shall use "degree" for short. These are the regions of existence of the cycles of period k having one point on one side of the discontinuity point d_1 and $(k-1)$ points on the other side (for any integer $k > 1$). It is plain that we may have one point on the left side and $(k-1)$ points on the right side, or *vice-versa*. To formalize the results it is useful to label the two components of the map $x' = T_1(x)$ as T_L and T_R, as we have already done in the definition (4). Let us start with the conditions to determine a cycle of period k having one point on the left side, L, and $(k-1)$ points on the right side, R.The condition that starts the existence of a

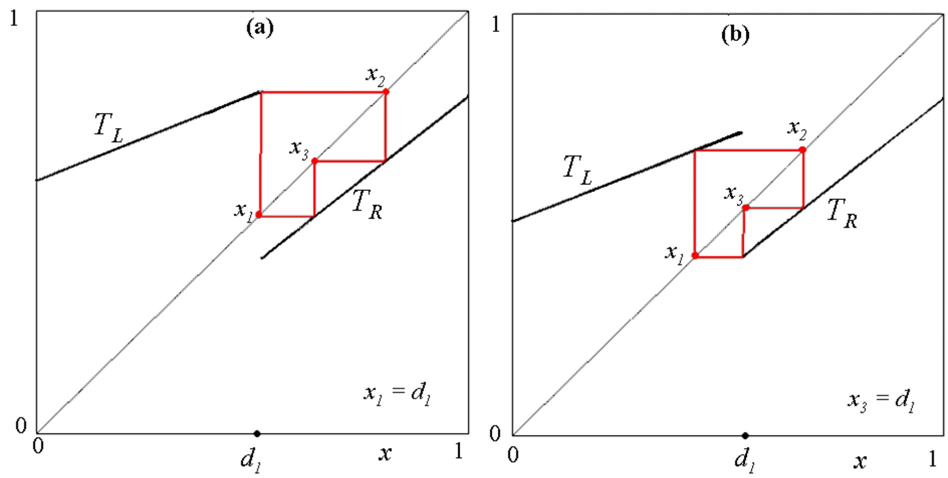

Figure 3: Starting condition in (a) and closing condition in (b) related with the period-3 orbit associated with the symbol sequence LRR.

$k-$cycle, i.e. the bifurcation that marks the creation of a cycle of period k, is that *the critical point $x = d_1$ (the discontinuity point) is a periodic point to which we apply, in the sequence, the maps $T_L, T_R, ..., T_R$.* Figure 3a shows the starting condition related with $k = 3$, i.e. a $3-$cycle, given by $T_R \circ T_R \circ T_L(d_1) = d_1$. Then the $k-$cycle with periodic points $x_1, ..., x_k$, numbered with the first point on the left side ($x_1 < d_1$, $x_i > d_1$ for $i = 2, ..., k$) satisfies $x_2 = T_L(x_1)$, $x_3 = T_R(x_2), ..., x_1 = T_R(x_k)$. In formulae we get:

$$\begin{aligned} p_1 &= d_1 \\ p_2 &= m_1 d_1 + 1 - m_1 \\ p_3 &= m_2(m_1 d_1 + 1 - m_1) \\ &\ldots \\ p_{k+1} &= m_2^{(k-1)}(m_1 d_1 + 1 - m_1) \end{aligned} \tag{5}$$

and a $k-$cycle occurs when $p_{k+1} = p_1$, i.e. the following equality holds:

$$d_1 = m_2^{(k-1)}(m_1 d_1 + 1 - m_1)$$

and rearranging we obtain (for any $k > 1$):

$$\phi_{ka}: \quad m_1 = \frac{m_2^{(k-1)} - d_1}{(1-d_1)m_2^{(k-1)}} \tag{6}$$

This cycle ends to exist when the last point (x_k) merges with the discontinuity point, that is, $x_k = d_1$, which may be stated also as: *the point $x = d_1$ is a periodic point to which we apply, in the sequence, the maps $T_R, T_L, T_R, ..., T_R$.* In the qualitative picture in Fig.3b we show the closing condition related with the $3-$cycle, that is, $T_R \circ T_L \circ T_R(d_1) = d_1$. In general, a $k-$cycle exists until the point $x = d_1$ becomes a periodic point when we apply, in the sequence, the maps $T_R, T_L, T_R, ..., T_R$. Thus we obtain the following expressions:

$$\begin{aligned} p_1 &= d_1 \\ p_2 &= m_2 d_1 \\ p_3 &= m_1 m_2 d_1 + 1 - m_1 \\ p_4 &= m_2(m_1 m_2 d_1 + 1 - m_1) \\ &... \\ p_{k+1} &= m_2^{(k-2)}(m_1 m_2 d_1 + 1 - m_1) \end{aligned} \tag{7}$$

and the condition for a $k-$cycle $p_{k+1} = p_1$ becomes

$$d_1 = m_2^{(k-2)}(m_1 m_2 d_1 + 1 - m_1)$$

which, rearranged, gives (for any $k > 1$):

$$\phi_{kb}: \quad m_1 = \frac{m_2^{(k-2)} - d_1}{(1 - m_2 d_1)m_2^{(k-2)}} \tag{8}$$

Summarizing, the $k-$cycle exists for m_1 in the range

$$m_{1a} := \frac{m_2^{(k-1)} - d_1}{(1-d_1)m_2^{(k-1)}} \leq m_1 \leq m_{1b} := \frac{m_2^{(k-2)} - d_1}{(1 - m_2 d_1)m_2^{(k-2)}}$$

The relations given above can also be used to find the explicit coordinates of the $k-$cycles. Let $(x_1^*, x_2^*, ..., x_k^*)$ be the periodic points of the $k-$cycle, where $x_1^* < d_1$ and $x_i^* > d_1$ for $i > 1$, then from (6) we have:

$$x_1^* = \frac{m_2^{(k-1)}(1 - m_1)}{1 - m_1 m_2^{(k-1)}} \tag{9}$$

and then (10)

$$\begin{aligned}
x_2^* &= T_L(x_1^*) = m_1x_1^* + 1 - m_1 \\
x_3^* &= T_R(x_2^*) = m_2(m_1x_1^* + 1 - m_1) \\
x_4^* &= T_R(x_3^*) = m_2^2(m_1x_1^* + 1 - m_1) \\
&\dots \\
x_k^* &= T_R(x_{k-1}^*) = m_2^{(k-2)}(m_1x_1^* + 1 - m_1)
\end{aligned}$$

Analogously, from the conditions in (7) we get:

$$x_k^* = \frac{m_2^{(k-2)}(1 - m_1)}{1 - m_1m_2^{(k-1)}} \tag{11}$$

Similar arguments apply to find the condition of existence of a $k-$cycle with periodic points $x_1, ..., x_k$ having one point in the R side and $(k-1)$ points in the L side (also called principal orbits) (see in Fig.4 a $3-$cycle of this kind). We consider

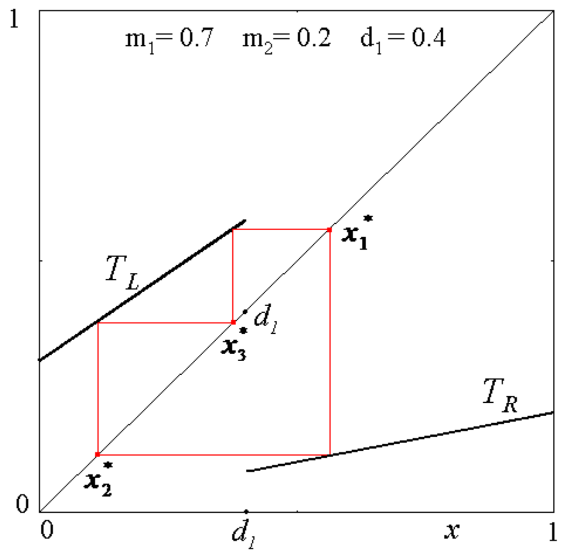

Figure 4: Period-3 cycle with the symbol sequence RLL.

the critical point $x = d_1$, and we apply the maps $T_R, T_L, ..., T_L$, so we get:

$$\begin{aligned}
p_1 &= d_1 \\
p_2 &= m_2d_1 \\
p_3 &= m_1m_2d_1 + 1 - m_1 \\
p_4 &= m_1^2m_2d_1 + m_1(1 - m_1) + (1 - m_1) \\
&\dots \\
p_{k+1} &= m_1^{(k-1)}m_2d_1 + (1 - m_1^{(k-1)})
\end{aligned} \tag{12}$$

and a $k-$cycle occurs when the following equality holds:

$$d_1 = m_1^{(k-1)} m_2 d_1 + (1 - m_1^{(k-1)})$$

Rearranging we obtain (for any $k > 1$):

$$\psi_{ka}: \quad m_2 = \frac{d_1 - 1 + m_1^{(k-1)}}{d_1 m_1^{(k-1)}} \tag{13}$$

The cycle exists until the point $x = d_1$ becomes a periodic point to which we apply, in the sequence, the maps $T_L, T_R, T_L, ..., T_L$. Thus we obtain the following expressions:

$$\begin{aligned} p_1 &= d_1 \\ p_2 &= m_1 d_1 + 1 - m_1 \\ x_3 &= m_2(m_1 d_1 + 1 - m_1) \\ p_4 &= m_1 m_2 (m_1 d_1 + 1 - m_1) + (1 - m_1) \\ &... \\ p_{k+1} &= m_1^{(k-2)} m_2 (m_1 d_1 + 1 - m_1) + (1 - m_1^{(k-2)}) \end{aligned} \tag{14}$$

and the condition for a $k-$cycle $p_{k+1} = p_1$ occurs when the following equality holds:

$$d_1 = m_1^{(k-2)} m_2 (m_1 d_1 + 1 - m_1) + (1 - m_1^{(k-2)})$$

which, rearranged, gives (for any $k > 1$):

$$\psi_{kb}: \quad m_2 = \frac{d_1 - 1 + m_1^{(k-2)}}{m_1^{(k-2)} (m_1 d_1 + 1 - m_1)} \tag{15}$$

Summarizing, the $k-$cycle $(x_1^*, x_2^*, ..., x_k^*)$ where $x_1^* > d_1$ and $x_i^* < d_1$ for $i > 1$ exists for m_2 in the range

$$m_{2a} := \frac{d_1 - 1 + m_1^{(k-1)}}{d_1 m_1^{(k-1)}} \leq m_2 \leq m_{2b} = \frac{d_1 - 1 + m_1^{(k-2)}}{m_1^{(k-2)} (m_1 d_1 + 1 - m_1)}$$

The periodic points of the $k-$cycle, $(x_1^*, x_2^*, ..., x_k^*)$ where $x_1^* > d_1$ and $x_i^* < d_1$ for $i > 1$, can be obtained from the equations in (14)

$$\begin{aligned} x_1^* &= m_1 x_k^* + 1 - m_1 \\ x_2^* &= m_2 (m_1 x_k^* + 1 - m_1) \\ x_3^* &= m_1 m_2 (m_1 x_k^* + 1 - m_1) + (1 - m_1) \\ &... \\ x_{k-1}^* &= m_1^{(k-3)} m_2 (m_1 x_k^* + 1 - m_1) + (1 - m_1^{(k-3)}) \\ x_k^* &= \frac{m_1^{(k-2)} m_2 (1 - m_1) + 1 - m_1^{(k-2)}}{1 - m_2 m_1^{(k-1)}} \end{aligned} \tag{16}$$

Also, from the conditions in (12) we get:

$$x_1^* = \frac{1 - m_1^{(k-1)}}{1 - m_2 m_1^{(k-1)}} \tag{17}$$

The equations given above in (13) and (15) are the analytic expressions of the bifurcation curves at which a "border-collision" bifurcation occurs. That is, fixing a value of m_1, for values of the parameter $m_2 \in (m_{2a}, m_{2b})$ the $k-$cycle exists, and its periodic points are explicitly given in (16). If the slope m_2 is decreased until $m_2 = m_{2a}$, then the periodic point x_1^* collides with the boundary of its region of periodicity (i.e. we have $x_1^* = d_1$) and then it will disappear, that is, it is no longer a periodic point for $m_2 < m_{2a}$. Vice-versa, increasing m_2 until $m_2 = m_{2b}$ the periodic point x_k^* collides with the boundary of its region (i.e. we have $x_k^* = d_1$) and the cycle will disappear: it no longer exists for $m_2 > m_{2b}$.

It is easy to see that for $k = 2$ the formulas in (6) and in (8) and those in (13) and (15) give the same equations (and thus the same bifurcations curves). However they are labelled symmetrically. That is: what is called initial (or starting curve) and final (or closing curve) in the former case, is called final and first, respectively, in the latter. In fact, they are obtained by naming the periodic points as $x_1^* < d_1$ and $x_2^* > d_1$ in the former case, while they are labelled as $x_1^* > d_1$ and $x_2^* < d_1$ in the latter, but clearly we have a unique cycle of period 2, and the border collision curves in the two cases are the same, that is $x_1^* = d_1$ and $x_2^* = d_1$ (although the starting and closing labels are inverted).

However, the 2-cycle is the only exception: for any $k > 2$ we have two different regions of existence of the $k-$cycles. By using the formulas in (13) and (15) with $k = 3, ..., 16$, we obtain all the bifurcation curves of the principal tongues located *above the period-2 tongue* in Fig.5a, and by the formulas in (13) and (15) we get all the bifurcation curves of the principal tongues located *below the period-2 tongue* in Fig.5a. Note that the formulas given in (6) and in (8) and those in (13) and (15) are generic, and hold whichever is the position of the discontinuity point $x = d_1$. A three dimensional view of the bifurcations is shown in Fig.5b. It is worth noticing that following similar arguments it is possible to find also the boundaries of the other bifurcation curves. In fact, besides the regions associated with the "tongues of first degree" (Leonov,1959, 1962, Mira, 1978, 1987) there are infinitely many (countable) periodicity tongues. The simplest well known mechanism to find the periods that characterize these infinite sequence of periodicity tongues is that between any two tongues having periods k_1 and k_2 there exists also a tongue having period k_1+k_2 (see the numbers of the periods indicated in Fig.1 and Fig.5: between the tongues of period 2 and 3 a tongue of period 5 exists, and between 5 and 2 we can see 7, and so on...).

To be more specific, in the description of the periodicity tongues we can associate a number to each region, which may be called "*rotation number*", in order to classify all the periodicity tongues. In this notation a periodic orbit of period k is charac-

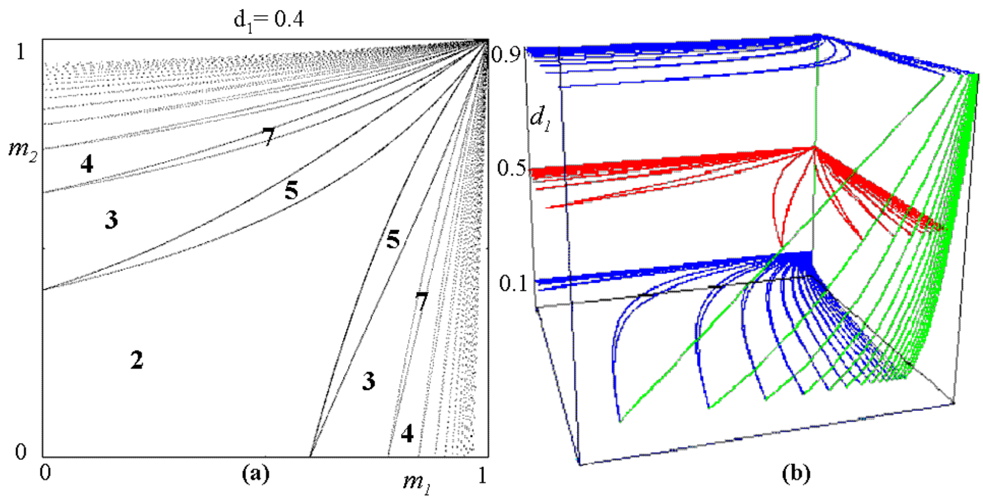

Figure 5: In (a), parameter plane (m_1, m_2) at $d_1 = 0.4$ fixed, bifurcation curves of first degree of complexity for cycles of periods $2, ..., 16$ obtained from the analytical expressions, for the map T_1. In (b) the bifurcation curves are shown in three sections of the three dimensional parameter space (m_1, m_2, d_1) at d_1 fixed, with $d_1 = 0.1$, $d_1 = 0.5$ and $d_1 = 0.9$.

terized not only by the period but also by the number of points in the two branches separated by the discontinuity point (denoted by T_L and T_R respectively). So, we can say that a cycle has a rotation number $\frac{p}{k}$ if a k-cycle has p points on the L side and the other $(k-p)$ on the R side. Then between any pair of periodicity regions associated with the "rotation number" $\frac{p_1}{k_1}$ and $\frac{p_2}{k_2}$ there exists also the periodicity tongue associated with the "rotation number" $\frac{p_1}{k_1} \oplus \frac{p_2}{k_2} = \frac{p_1+p_2}{k_1+k_2}$ (also called "direct sum").

Then, following Leonov (1959, 1962) (see also Mira, 1978, 1987), between any pair of contiguous "*tongues of first degree*", say $\frac{1}{k_1}$ and $\frac{1}{k_1+1}$, we can construct two infinite families of periodicity tongues, called "*tongues of second degree*" by the sequence obtained by adding with the Farey composition rule (see also in Hao-Bai Lin, 1989) $\oplus$ iteratively the first one or the second one, i.e. $\frac{1}{k_1} \oplus \frac{1}{k_1+1} = \frac{2}{2k_1+1}$, $\frac{2}{2k_1+1} \oplus \frac{1}{k_1} = \frac{3}{3k_1+1}$, ...and so on, that is:

$$\frac{n}{nk_1+1} \quad \text{for any } n > 1$$

and $\frac{1}{k_1} \oplus \frac{1}{k_1+1} = \frac{2}{2k_1+1}$, $\frac{2}{2k_1+1} \oplus \frac{1}{k_1+1} = \frac{3}{3k_1+2}$, $\frac{3}{3k_1+2} \oplus \frac{1}{k_1+1} = \frac{4}{4k_1+3}$..., that is:

$$\frac{n}{nk_1+n-1} \quad \text{for any } n > 1$$

which give two sequences of tongues accumulating on the boundary of the two starting tongues.

Clearly, this mechanisms can be repeated: between any pair of contiguous ”*tongues of second degree*”, for example $\frac{n}{nk_1+1}$ and $\frac{n+1}{(n+1)k_1+1}$, we can construct two infinite families of periodicity tongues, called ”*tongues of third degree*” by the sequence obtained by adding with the composition rule $\oplus$ iteratively the first one or the second one. And so on. All the rational numbers are obtained in this way, giving all the infinitely many periodicity tongues.

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Besides the notation used above, called method of the *rotation numbers,* we may also follow a different approach, related with the symbolic sequence associated to a cycle. In this notation, considering the principal tongue of a periodic orbit of period k constituted by one point on the L side and $(k-1)$ on the R side, we associate to the cycle the *symbol sequence* $LR..(k-1\ times).R.$ Then the direct sum of two consecutive cycles is given by:

$$LR..(k-1\ times).R \oplus LR..(k\ times).R = LR..(k-1\ times).RLR..(k\ times).R$$

that is, the two sequences are just put together in file (and indeed this sequence of bifurcations is also called by Mira ”*boxes in files*” in Mira 1987), and the sequence of maps to apply in order to get the cycle are listed from left to right. More generally, it is true that given a periodicity tongue associated with a symbols' sequence σ (consisting of letters L and R, giving the cycle from left to right) and a second one with a symbols' sequence τ, then also the direct sum of the two sequences exists, associated with a periodicity tongue with symbols' sequence $\sigma\tau$:

$$\sigma \oplus \tau = \sigma\tau$$

It is worth to notice that these periodicity tongues never overlap, and this implies that coexistence of different periodic cycles is not possible.

As we shall see in the next section, part of the bifurcation curves described above is also present in the map T_2 with two discontinuity points, whose study is the main goal of this paper.

4 Bifurcation curves of the map T_2.

Now let us introduce a second discontinuity point, by considering the map $x' = T_2(x)$.Our analysis starts from the description of the possible attractors of the map. First of all we notice that for any values of the parameters all the initial conditions taken in the points $x > d_2$ converge to the fixed point $x = 1$. This may be the unique attractor or it may coexist with another attracting $k-$cycle C (with $k > 1$), existing in the interval $0 < x < d_2$. This depends on the values of the four parameters: d_1, d_2, m_1 and m_2. Moreover, when we have coexistence of two attractors, their basins may consist in only one interval, or in the union of two or more intervals.

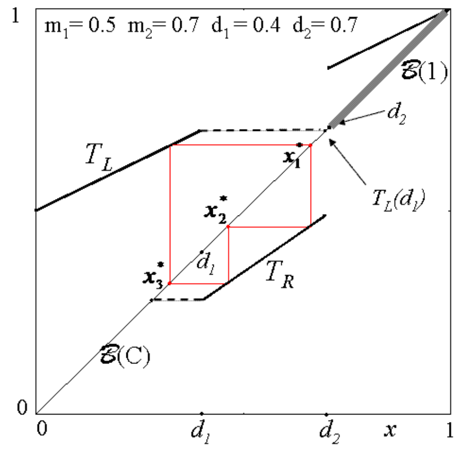

Fig. 6 Coexistence of two attractors, and related connected basins.

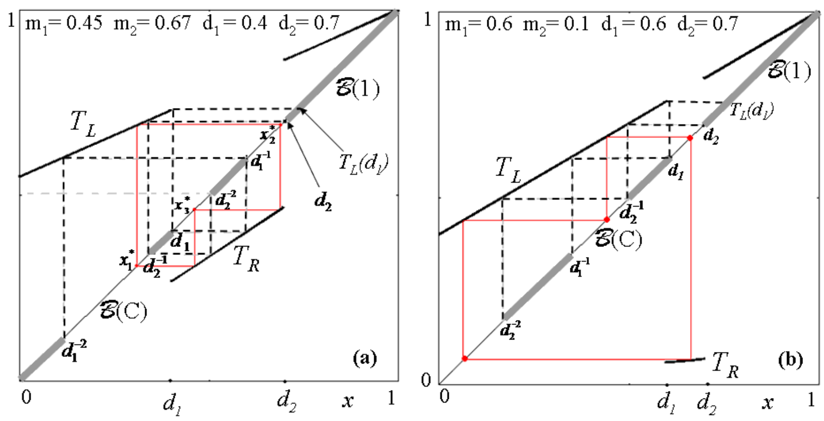

Fig.7 Coexistence of two attractors, and related non connected basins. In (a) $B(1)$ is formed by four disjoint intervals, while it is formed by three disjoint intervals in (b).

In figures 6 and 7 we show some examples. In Fig.6 we see the coexistence of two attractors, the fixed point $x = 1$ and a cycle of period 3 respectively, and each of the two basins of attraction, say $B(1)$ (represented by a thicker line along the diagonal) and $B(C)$, consists in one interval. Instead in Fig.7 we show two cases in which the fixed point $x = 1$ still coexist with a cycle of period 3, but their basins are made up of several disjoint intervals: in Fig.7a $B(1)$ is formed by four disjoint intervals, in Fig.7b by three disjoint intervals. The complementary set in the interval $]0,1[$ belongs to the basin $B(C)$.

Summarizing, here we have only two possibilities:

I) each of the two basins is made up of only one interval, separated by the discontinuity point d_2 (as in the case shown in Fig.6), so that the basin of the attracting cycle is: $B(C)=[0,d_2[$ while the other points converge to the attracting fixed point: $B(\mathit{1})=]d_2,1]$;

II) each of the two basins is made up of disjoint intervals as in the cases shown in Fig.7, separated by the two discontinuity points d_1 and d_2 and their preimages. We can consider as immediate basin of the fixed point $x=1$ the interval $]d_2,1]$, hence we have $B(1)=\cup_{j\geq 1}T_2^{-j}(]d_2,1])$, while the complementary region in $[0,1[$ gives the basin of the $k-$cycle. We can also notice that $B(C)=\cup_{j\geq 1}T_2^{-j}(]d_1,d_2^{-2}[)$.

From the description given above it is clear that the qualitative change in the structure of the basins depends on a *border-collision bifurcation*: as shown in Fig.6, in the simplest case (when the basins are two disjoint intervals) the interval $]T_R(d_1),T_L(d_1)[$ is invariant, and this occurs as long as $T_L(d_1)<d_2$, or, equivalently, as long as d_2 has no rank-1 preimage with T_L^{-1}. When we have $T_L(d_1)>d_2$, i.e. $T_L^{-1}(d_2)<d_1$, then each basin is made up of at least two intervals. At the bifurcation value, $T_L(d_1)=d_2$, we have an invariant set ($]T_R(d_1),T_L(d_1)[$ belonging to the basin of a cycle C) which is merging with a border (of another basin), and this collision produces a qualitative change in the structure of the two basins (note that for the attractors nothing changes: in this description they are not involved in the *border-collision bifurcation*).

So the bifurcation between the two cases (I) and (II) occurs when

$$T_L(d_1)=d_2$$

that is, when $m_1d_1+(1-m_1)=d_2$ or, equivalently, when $m_1=\frac{1-d_2}{1-d_1}$. So, let us define

$$m_1^*=\frac{1-d_2}{1-d_1} \tag{18}$$

then the following proposition is proved:

Proposition 1.

(I) *When $m_1>m_1^*$ then there are two coexisting attractors (the fixed point $x=1$ and a $k-$cycle C for some integer k) and the two basins are made up of only one interval:$B(1)=]d_2,1]$ and $B(C)=[0,d_2[$.*

(II) *When $m_1<m_1^*$ then either the fixed point is the only attractor in the whole interval $[0,1]$ or there are two coexisting attractors (the fixed point and a $k-$cycle C for some integer $k>1$) and the two basins are made up of at least two pieces: $B(1)=\cup_{j\geq 1}T_2^{-j}(]d_2,1])$ and $B(C)=\cup_{j\geq 1}T_2^{-j}(]d_1,d_2[)$.*

As remarked above we have two different dynamic behaviors in our map. In the simplest case (I) in the interval $]d_2,1]$ we have only convergence to the fixed point $x=1$, whereas what occurs in the interval $[0,d_2[$, as a function of the parameters, has been already described in the previous section, because the restriction of the map T_2 to that interval reduces to the map T_1. And in fact, for $m_1 > m_1^*$ the bifurcation diagrams in the plane m_1, m_2 are the same as those in Fig.1 and 5. In the case of two discontinuities the meaning of the colors in Fig.2 is that in the yellow region the fixed point is the only attractor, while in the other regions the fixed point coexists with a $k-$cycle, whose period $k > 1$ is different in the regions with different colors. So we have proved that for $m_1 > m_1^*$ we know the bifurcation curves (those of the principal tongues are given in Section 1), and what is left is to describe the bifurcations occurring for $m_1 < m_1^*$. Moreover, from Fig.2 we may also argue that different kinds of *border-collision bifurcation* occur in the periodicity tongues of the family $LR...R$ (see Fig.2a) with respect to those occurring to the family $RL...L$ (see Fig.2b) however we shall see that the two different bifurcations have something in common, and clearly related with the second discontinuity point.

Consider $m_1 = m_1^*$ and let us decrease m_1 starting in the region of cycles of the family $LR...R$ (i.e. above the periodicity region of the 2-cycle). Then, no curves of type ϕ_{ka} (of the "starting" cycles) can exist for $m_1 < m_1^*$, while if a $k-$cycle already exists then

i) either it ends to exist in the usual way, by border collision with the discontinuity point d_1, that is $x_k^* = d_1$ and this is represented in the portion of curves of the family ϕ_{kb} existing for $m_1 < m_1^*$,

ii) or the cycle ends to exist by border-collision with the basin of attraction of the fixed point, and this occurs when the periodic point of the $k-$cycle closest to d_1 from the left is merging with $T_L^{-1}(d_2)$, that belongs to the boundary of $B(1)$ (see Fig.7a), and from the expression reported in (9) we obtain the condition of the contact bifurcation by using:

$$x_1^* = T_L^{-1}(d_2) \tag{19}$$

that is:

$$\frac{m_2^{(k-1)}(1-m_1)}{1-m_1 m_2^{(k-1)}} = \frac{d_2 - 1 + m_1}{m_1}$$

which, rearranged, may be rewritten as:

$$\phi_{kc}: \quad m_1 = \frac{1-d_2}{1-d_2 m_2^{(k-1)}} \tag{20}$$

Similarly (or better, symmetrically) we can reason for the other family, $RL...L$, of principal tongues. In this case the periodic points of the $k-$cycle $(x_1^*, x_2^*, ..., x_k^*)$ are $x_1^* > d_1$ and $x_i^* < d_1$ for $i > 1$. In particular x_k^* is the one closest to d_1 from the left. Then decreasing m_1 from m_1^* some portion of curves of type ψ_{ka} (of the "starting" cycles) for $m_1 < m_1^*$ can still exist (see Fig.7b), while if a $k-$cycle already exists then for $m_1 < m_1^*$ it cannot end to exist because of the collision of the periodic point x_k^* with the border d_1 (because of the structure of the basins, (see Fig.7b). So that for $m_1 < m_1^*$ no curves of type ψ_{kb} (of the "closing" cycles) can exist. Thus an existing cycle can end to exist only by collision of the periodic point of the cycle closest to d_1 with the point $T_L^{-1}(d_2)$ (i.e. the collision of the cycle with the basin of attraction of the fixed point). For the main tongues in this region the periodic point x_k^* is the one involved in the contact.

$$x_k^* = T_L^{-1}(d_2)$$

Equivalently, we can say that the periodic point on the right of the discontinuity d_1, closest to d_2, has a collision with the discontinuity point d_2. That is, in our case:

$$x_1^* = d_2$$

¿From both conditions we obtain the same bifurcation curve, which may be written as:

$$\psi_{kc}: \quad m_2 = \frac{-1 + d_2 + m_1^{(k-1)}}{d_2 m_1^{(k-1)}} \tag{21}$$

So, the border-collision bifurcation curves existing for the second family for $m_1 < m_1^*$ are the curves ψ_{ka} and ψ_{kc}.

So far we have considered a generic $k-$cycle in a main tongue, and we remark that the result holds also for the 2-cycle: in the family $LR...R$ the bifurcation curve is given by ϕ_{2c} and in the $RL...L$ family the bifurcation curve is given by ψ_{2c}. Which is obvious: for $m_1 < m_1^*$ a $2-$cycle either exists or it cannot appear any longer, and an existing one either ends to exist when $x_2^* = d_2$ or when $x_1^* = d_2$.

We have so proved the following proposition:

Proposition 2.

(I) *When $m_1 > m_1^*$ the border-collision bifurcation curves of the $k-$cycles of the map T_2 are the same as those of the map T_1.*

(II) *When $m_1 < m_1^*$ then an existing cycle may end to exist either by collision with the border d_1 or with the border d_2.*

The explicit analytic expression of the bifurcation curves of the principal tongues (or tongues of first degree) has been determined above, and with similar reasoning also the analytic expressions of the bifurcation curves of second or higher degree of complexity can be determined.

By using the curves of the family ϕ_{kb} and ϕ_{kc} for $k = 2, ..., 16$ we have drown all the bifurcations curves of of the principal tongues of the family $LR...R$ reported in figures.8 and 9, and with the curves of the family ψ_{ka} and ψ_{kc}, $k = 2, ..., 16$, we have drown all the bifurcations curves of the principal tongues of the family $RL...L$ reported in figures 8,9.

In the figures 8b and 9b only a few periodicity tongues are visible, even if an infinite countable set of tongues are nested inside every couple of tongues of first degree. Of course, more and more regions can be seen by proper enlargement of the figures.

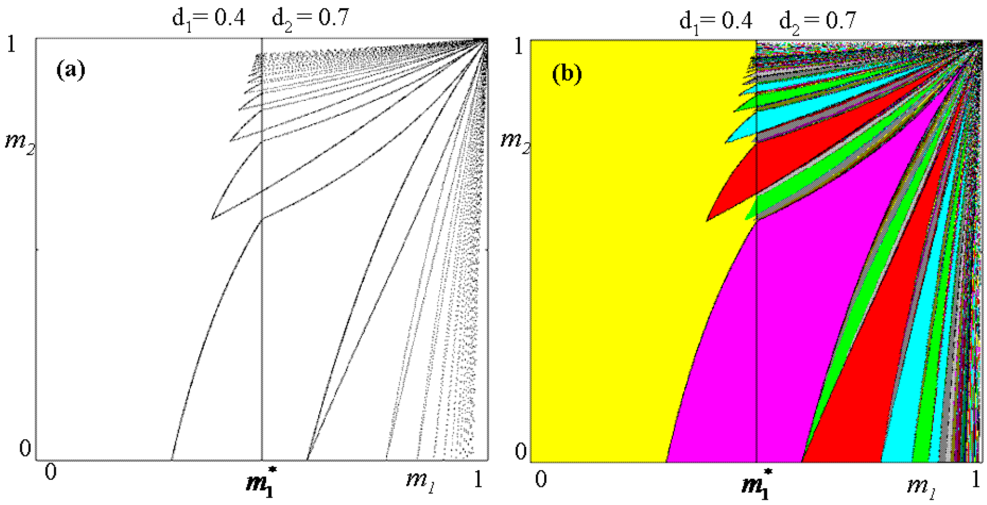

Fig.8 In (a), parameter plane (m_1, m_2) at $d_1 = 0.4$ and $d_2 = 0.7$ fixed, bifurcation curves of first degree of complexity for cycles of periods $2, ..., 16$ obtained from the analytical expressions, for the map T_2. In (b) the same bifurcation curves are shown with colors associated with the different periodicity.

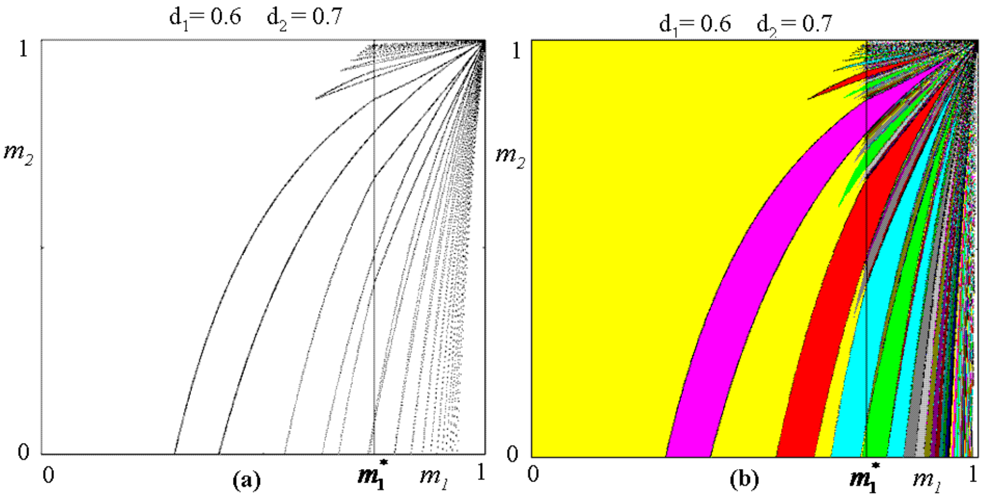

Fig.9 In (a), parameter plane (m_1, m_2) at $d_1 = 0.6$ and $d_2 = 0.7$ fixed, bifurcation curves of first degree of complexity for cycles of periods $2, ..., 16$ obtained from the analytical expressions, for the map T_2. In (b) the same bifurcation curves are shown with colors associated with the different periodicity.

This can be also seen in the one-dimensional bifurcation diagram (or orbit diagram, as it is also called) shown in Fig.10, obtained with the same set of parameters as in Fig.8 but with a fixed value of the parameter $m_2 = 0.5$ and bifurcation parameter m_1 varying in the range $[0.4, 1]$. It clearly shows how infinitely many intervals ("*boxes in files*") associated with the bifurcation of periodic orbits exist, one close to the other.

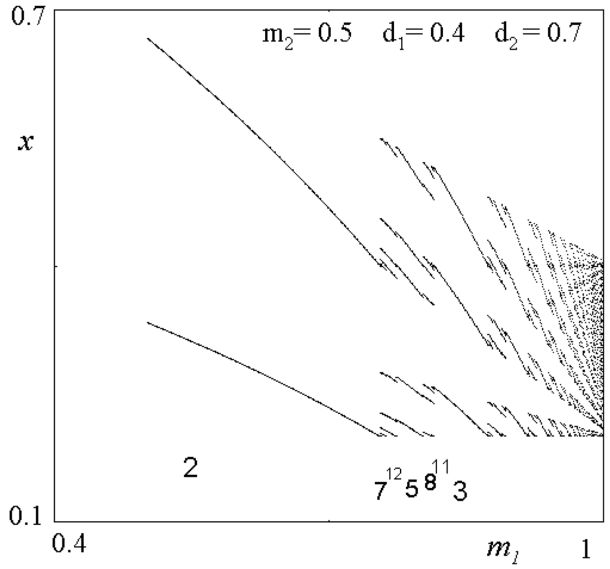

Fig.10 One-dimensional orbit diagram of the map T_2 as a function of m_1 at $m_2 = 0.5$, $d_1 = 0.4$ and $d_2 = 0.7$ fixed.

5 Reduction of the map T to T_2

In this section we show that all the dynamic properties of the map T_3 are independent of the third slope m_3, so that its study is immediately reduced to the one of the map T_2 considered in the previous section. In order to prove this statement, let us first consider the map $x' = T(x)$ with a given slope m_3 in the third branch of the map, and let us start with the description of the possible attractors.

Clearly $x = 1$ is a fixed point of the map T, and it is always locally asymptotically stable, because the slope associated with the third branch of the function is $m_3 \in]0, 1[$. It follows that all the points (initial conditions) $x > d_2$ generate trajectories which are converging to this stable fixed point of T. In other words, the interval $]d_2, 1[$ is the so-called *immediate basin* of the fixed point $x = 1$. As argued in the previous section, this stable fixed point may be the unique attractor or it may coexist with a stable cycle.

In the latter case, its basin of attraction may be made up of the unique interval $]d_2, 1[$ or it may include also other intervals, whose union forms the total basin of attraction. However, whichever is the value of the third slope $m_3 \in]0, 1[$, the dynamic behavior of the map T is determined from the values of the other parameters, as it can be easily deduced from the qualitative picture in Fig.11, where it is quite evident that for any value $m_3 \in (0,1)$ the map has the same coexisting attractors as well as the same basins of attraction. This is a simple consequence of the property noticed above, i.e. that the fixed point $x = 1$ is always stable with immediate basin the interval $]d_2, 1[$. Moreover the presence of a possible coexistent attractor depends only on the values of the other parameters, as well as the possible preimages of the immediate basin. So, the value of the parameter m_3 only influences the rate of convergence to the fixed point $x = 1$.

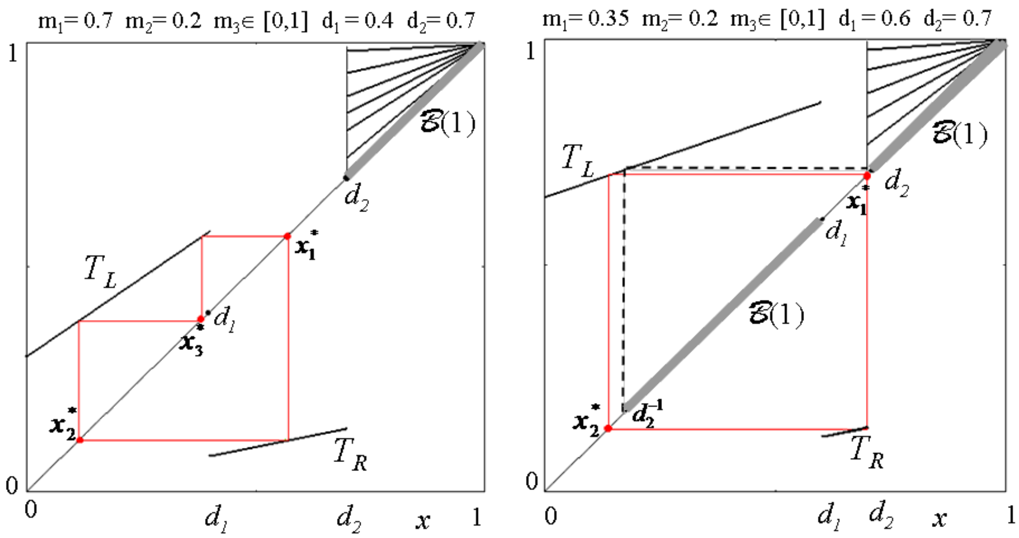

Fig.11 Coexistence of attractors of the map T, with connected basins in (a), and disconnected basins in (b).

6 Conclusions.

The main goal of this paper was the analytic description of the bifurcations occurring in a piecewise linear map $T : [0,1] \to [0,1]$ formed by three portions of different slopes, $m_i \in [0,1]$, $i = 1,2,3$, separated by two discontinuity points $0 \leq d_1 < d_2 \leq 1$. For this piecewise continuous map that depends on these five parameters we have first recalled how some border-collision bifurcations occur in the benchmark case in which $d_2 = 1$, i.e. the map has only one discontinuity point. In this case we have obtained the analytic expression of the border collision bifurcation curves that bound the periodicity tongues of first degree in the plane of parameters (m_1, m_2). On the basis of this result obtained for the simpler benchmark case by using methods already known in the literature, we have then studied the effects, on the structure of the border collision bifurcation curves, induced by the introduction of the second discontinuity point. This was investigated in Section 4 keeping a fixed value of the third slope m_3. Then we have shown that the value of the third slope does not influence the results obtained, as the global dynamic properties of the map T are independent of the values of m_3.

The analytic results obtained in this paper for the construction of the whole structure of the periodicity tongues of first degree constitute a generalization of the studies given in the literature on one-dimensional maps with only one discontinuity point. Moreover, the methods followed in this paper to obtain such analytic expressions are quite general and can be easily generalized to cases with more than two discontinuities and with slopes of different values with respect to the ones considered in this paper.

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