ISSN 1974-4110

Università
degli Studi di Urbino
"Carlo Bo"
Facoltà
di Economia
www.econ.uniurb.it

*WP-EMS*
*Working Papers Series in*
*Economics, Mathematics and Statistics*

# "A generalization of Hukuhara difference for interval and fuzzy arithmetic"

- Luciano STEFANINI (University of Urbino)

*WP-EMS # 2008/01*

# A generalization of Hukuhara difference for interval and fuzzy arithmetic

Luciano Stefanini[1]

University of Urbino, Italy. e-mail:luciano.stefanini@uniurb.it

**Abstract.** We propose a generalization of the Hukuhara difference. First, the case of compact convex sets is examined; then, the results are applied to generalize the Hukuhara difference of fuzzy numbers, using their compact and convex level-cuts. Finally, a similar approach is suggested to attempt a generalization of division for real intervals.

## 1 General setting

We consider a metric vector space $\mathbb{X}$ with the induced topology and in particular the space $\mathbb{X} = \mathbb{R}^n$, $n \geq 1$, of real vectors equipped with standard addition and scalar multiplication operations. Following Diamond and Kloeden (see [3]), denote by $\mathcal{K}(\mathbb{X})$ and $\mathcal{K}_C(\mathbb{X})$ the spaces of nonempty compact and compact convex sets of $\mathbb{X}$. Given two subsets $A, B \subseteq \mathbb{X}$ and $k \in \mathbb{R}$, Minkowski addition and scalar multiplication are defined by $A+B = \{a+b|a \in A, b \in B\}$ and $kA = \{ka|a \in A\}$ and it is well known that addition is associative and commutative and with neutral element $\{0\}$. If $k = -1$, scalar multiplication gives the opposite $-A = (-1)A = \{-a|a \in A\}$ but, in general, $A+(-A) \neq \{0\}$, i.e. the opposite of $A$ is not the inverse of $A$ in Minkowski addition (unless $A = \{a\}$ is a singleton). Minkowski difference is $A - B = A + (-1)B = \{a - b|a \in A, b \in B\}$. A first implication of this fact is that, in general, even if it true that $(A + C = B + C) \iff A = B$, addition/subtraction simplification is not valid, i.e. $(A + B) - B \neq A$.

To partially overcome this situation, Hukuhara [4] introduced the following H-difference:

$$A \ominus B = C \iff A = B + C \tag{1}$$

and an important property of $\ominus$ is that $A \ominus A = \{0\}, \forall A \in \mathbb{R}^n$ and $(A+B) \ominus B = A$, $\forall A, B \in \mathbb{R}^n$; H-difference is unique, but a necessary condition for $A \ominus B$ to exist is that $A$ contains a translate $\{c\} + B$ of $B$. In general, $A - B \neq A \ominus B$.

From an algebraic point of view, the difference of two sets $A$ and $B$ may be interpreted both in terms of addition as in (1) or in terms of negative addition, i.e.

$$A \boxminus B = C \iff B = A + (-1)C \tag{2}$$

where $(-1)C$ is the opposite set of $C$. Conditions (1) and (2) are compatible each other and this suggests a generalization of Hukuhara difference:

**Definition 1.** *Let $A, B \in \mathcal{K}(\mathbb{X})$; we define the generalized difference of $A$ and $B$ as the set $C \in \mathcal{K}(\mathbb{X})$ such that*

$$A \ominus_g B = C \iff \begin{cases} (i) \;\; A = B + C \\ or\ (ii) \;\; B = A + (-1)C \end{cases} . \tag{3}$$

**Proposition 1.** *(Unicity of $A \ominus_g B$)*
*If $C = A \ominus_g B$ exists, it is unique and if also $A \ominus B$ exists then $A \ominus_g B = A \ominus B$.*

*Proof. If $C = A \ominus_g B$ exists in case $(i)$, we obtain $C = A \ominus B$ which is unique. Suppose that case $(ii)$ is satisfied for $C$ and $D$, i.e. $B = A + (-1)C$ and $B = A + (-1)D$; then $A + (-1)C = A + (-1)D \Longrightarrow (-1)C = (-1)D \Longrightarrow C = D$. If case $(i)$ is satisfied for $C$ and case $(ii)$ is satisfied for $D$, i.e. $A = B + C$ and $B = A + (-1)D$, then $B = B + C + (-1)D \Longrightarrow \{0\} = C - D$ and this is possible only if $C = D = \{c\}$ is a singleton.*

The generalized Hukuhara difference $A \ominus_g B$ will be called the gH-difference of $A$ and $B$.

*Remark 1.* A necessary condition for $A \ominus_g B$ to exist is that either $A$ contains a translate of $B$ (as for $A \ominus B$) or $B$ contains a translate of $A$. In fact, for any given $c \in C$, we get $B + \{c\} \subseteq A$ from $(i)$ or $A + \{-c\} \subseteq B$ from $(ii)$.

*Remark 2.* It is possible that $A = B + C$ and $B = A + (-1)C$ hold simultaneously; in this case, $A$ and $B$ translate into each other and $C$ is a singleton. In fact, $A = B + C$ implies $B + \{c\} \subseteq A \; \forall c \in C$ and $B = A + (-1)C$ implies $A - \{c\} \subseteq B$ $\forall c \in C$ i.e. $A \subseteq B + \{c\}$; it follows that $A = B + \{c\}$ and $B = A + \{-c\}$. On the other hand, if $c', c'' \in C$ then $A = B + \{c'\} = B + \{c''\}$ and this requires $c' = c''$.

*Remark 3.* If $A \ominus_g B$ exists, then $B \ominus_g A$ exists and $B \ominus_g A = -(A \ominus_g B)$.

**Proposition 2.** *If $A \ominus_g B$ exists, it has the following properties:*
*1) $A \ominus_g A = \{0\}$;*
*2) $(A + B) \ominus_g B = A$;*
*3) If $A \ominus_g B$ exists then also $(-B) \ominus_g (-A)$ does and $-(A \ominus_g B) = (-B) \ominus_g (-A)$;*
*4) $(A - B) + B = C \iff A - B = C \ominus_g B$;*
*5) In general, $B - A = A - B$ does not imply $A = B$; but $(A \ominus_g B) = (B \ominus_g A) = C$ if and only if $C = \{0\}$ and $A = B$;*
*6) If $B \ominus_g A$ exists then either $A + (B \ominus_g A) = B$ or $B - (B \ominus_g A) = A$ and both equalities hold if and only if $B \ominus_g A$ is a singleton set.*

*Proof. Properties 1 and 5 are immediate. To prove 2) if $C = (A + B) \ominus_g B$ then either $A + B = C + B$ or $B = (A + B) + (-1)C = B + (A + (-1)C)$; in the first case it follows that $C = A$, in the second case $A + (-1)C = \{0\}$ and $A$ and $C$ are singleton sets so $A = C$. To prove the firat part of 3) let $C = A \ominus_g B$ i.e.*

*$A = B + C$ or $B = A + (-1)C$, then $-A = -B + (-C)$ or $-B = -A - (-C)$ and this means $(-B) \ominus_g (-A) = -C$; the second part is immediate. To see the first part of 5) consider for example the unidimensional case $A = [a^-, a^+]$, $B = [b^-, b^+]$; equality $A - B = B - A$ is valid if $a^- + a^+ = b^- + b^+$ and this does not require $A = B$ (unless $A$ and $B$ are singletons). For the second part of 5), from $(A \ominus_g B) = (B \ominus_g A) = C$, considering the four combinations derived from (3), one of the following four case is valid: ($A = B + C$ and $B = A + C$) or ($A = B + C$ and $A = B - C$) or ($B = A + (-1)C$ and $B = A + C$) or ($B = A + (-1)C$ and $A = B + (-1)C$); in all of them we deduce $C = \{0\}$. To see 6), consider that if $(B \ominus_g A)$ exists in the sense of $(i)$ the first equality is valid and if it exists in the sense of $(ii)$ the second one is valid.*

If $\mathbb{X} = \mathbb{R}^n$, $n \geq 1$ is the real $n-$dimensional vector space with internal product $\langle x, y\rangle$ and corresponding norm $||x|| = \sqrt{\langle x, x\rangle}$, we denote by $\mathcal{K}^n$ and $\mathcal{K}^n_C$ the spaces of (nonempty) compact and compact convex sets of $\mathbb{R}^n$, respectively. If $A \subseteq \mathbb{R}^n$ and $\mathcal{S}^{n-1} = \{u | u \in \mathbb{R}^n, ||u|| = 1\}$ is the unit sphere, the support function associated to $A$ is

$$s_A : \mathbb{R}^n \longrightarrow \mathbb{R} \text{ defined by}$$
$$s_A(u) = \sup\{\langle u, a\rangle \,| a \in A\},\ u \in \mathbb{R}^n.$$

If $A \neq \emptyset$ is compact, then $s_A(u) \in \mathbb{R}$, $\forall u \in \mathcal{S}^{n-1}$. The following properties are well known (see e.g. [3] or [5]):

- Any function $s : \mathbb{R}^n \longrightarrow \mathbb{R}$ which is continuous, positively homogeneous $s(tu) = ts(u),)$, $\forall t \geq 0$, $\forall u \in \mathbb{R}^n$ and subadditive $s(u' + u'') \leq s(u') + s(u'')$, $\forall u', u'' \in \mathbb{R}^n$ is a support function of a compact convex set; the restriction $\widehat{s}$ of $s$ to $\mathcal{S}^{n-1}$ is such that $\widehat{s}(\frac{u}{||u||}) = \frac{1}{||u||}s(u)$, $\forall u \in \mathbb{R}^n$, $u \neq 0$ and we can consider $s$ restricted to $\mathcal{S}^{n-1}$. It also follows that $s : \mathcal{S}^{n-1} \longrightarrow \mathbb{R}$ is a convex function.
- If $A \in \mathcal{K}^n_C$ is a compact convex set, then it is characterized by its support function and

$$A = \{x \in \mathbb{R}^n |\, \langle u, x\rangle \leq s_A(u),\ \forall u \in \mathbb{R}^n\} = \{x \in \mathbb{R}^n |\, \langle u, x\rangle \leq s_A(u),\ \forall u \in \mathcal{S}^{n-1}\}$$

- For $A, B \in \mathcal{K}^n_C$ and $\forall u \in \mathcal{S}^{n-1}$ we have $s_{\{0\}}(u) = 0$ and

$$A \subseteq B \Longrightarrow s_A(u) \leq s_B(u);\ A = B \iff s_A = s_B,$$
$$s_{kA}(u) = ks_A(u),\ \forall k \geq 0;\ \ s_{kA+hB}(u) = s_{kA}(u) + s_{hB}(u),\ \forall k, h \geq 0$$

  and in particular

$$s_{A+B}(u) = s_A(u) + s_B(u);$$

- If $s_A$ is the support function of $A \in \mathcal{K}^n_C$ and $s_{-A}$ is the support function of $-A \in \mathcal{K}^n_C$, then $\forall u \in \mathcal{S}^{n-1}$, $s_{-A}(u) = s_A(-u)$;

– If $\upsilon$ is a measure on $\mathbb{R}^n$ such that $\upsilon(\mathcal{S}^{n-1}) = \int_{\mathcal{S}^{n-1}} \upsilon(du) = 1$, a distance is defined by

$$\rho_2(A,B) = ||s_A - s_B|| = \left( n \int_{\mathcal{S}^{n-1}} [s_A(u) - s_B(u)]^2 \upsilon(du) \right)^{\frac{1}{2}};$$

– The Steiner point of $A \in \mathcal{K}_C^n$ is defined by $\sigma_A = n \int_{\mathcal{S}^{n-1}} u s_A(u) \upsilon(du)$ and $\sigma_A \in A$.

We can express the generalized Hukuhara difference (gH-difference) of compact convex sets $A, B \in \mathcal{K}_C^n$ by the use of the support functions. Consider $A, B, C \in \mathcal{K}_C^n$ with $C = A \ominus_g B$ as defined in (3); let $s_A$, $s_B$, $s_C$ and $s_{(-1)C}$ be the support functions of $A$, $B$, $C$, and $(-1)C$ respectively. In case $(i)$ we have $s_A = s_B + s_C$ and in case $(ii)$ we have $s_B = s_A + s_{(-1)C}$. So, $\forall u \in \mathcal{S}^{n-1}$

$$s_C(u) = \left\langle \begin{array}{ll} s_A(u) - s_B(u) & \text{in case } (i) \\ s_B(-u) - s_A(-u) & \text{in case } (ii) \end{array} \right.$$

i.e.

$$s_C(u) = \left\langle \begin{array}{ll} s_A(u) - s_B(u) & \text{in case } (i) \\ s_{(-1)B}(u) - s_{(-1)A}(u) & \text{in case } (ii) \end{array} \right. . \tag{4}$$

Now, $s_C$ in (4) is a correct support function if it is continuous, positively homogeneous and subadditive and this requires that, in the corresponding cases $(i)$ and $(ii)$, $s_A - s_B$ and/or $s_{-B} - s_{-A}$ be support functions, assuming that $s_A$ and $s_B$ are.

Consider $s_1 = s_A - s_B$ and $s_2 = s_B - s_A$. Continuity of $s_1$ and $s_2$ is obvious. To see their positive homogeneity let $t \geq 0$; we have $s_1(tu) = s_A(tu) - s_B(tu) = ts_A(u) - ts_B(u) = ts_1(u)$ and similarly for $s_2$. But $s_1$ and/or $s_2$ may fail to be subadditive and the following four cases, related to the definition of gH-difference, are possible.

**Proposition 3.** *Let $s_A$ and $s_B$ be the support functions of $A, B \in \mathcal{K}_C^n$ and consider $s_1 = s_A - s_B$, $s_2 = s_B - s_A$; the following four cases apply:*
*1. If $s_1$ and $s_2$ are both subadditive, then $A \ominus_g B$ exists; (i) and (ii) are satisfied simultaneously and $A \ominus_g B = \{c\}$;*
*2. If $s_1$ is subadditive and $s_2$ is not, then $C = A \ominus_g B$ exists, (i) is satisfied and $s_C = s_A - s_B$;*
*3. If $s_1$ is not subadditive and $s_2$ is, then $C = A \ominus_g B$ exists, (ii) is satisfied and $s_C = s_{-B} - s_{-A}$;*
*4. If $s_1$ and $s_2$ are both not subadditive, then $A \ominus_g B$ does not exist.*

*Proof. In case 1. subadditivity of $s_1$ and $s_2$ means that, $\forall u', u'' \in \mathcal{S}^{n-1}$*

$s_1 : s_A(u' + u'') - s_B(u' + u'') \leq s_A(u') + s_A(u'') - s_B(u') - s_B(u'')$ *and*
$s_2 : s_B(u' + u'') - s_A(u' + u'') \leq s_B(u') + s_B(u'') - s_A(u') - s_A(u'')$;

*it follows that*

$$s_A(u'+u'') - s_A(u') - s_A(u'') \le s_B(u'+u'') - s_B(u') - s_B(u'') \text{ and}$$
$$s_B(u'+u'') - s_B(u') - s_B(u'') \le s_A(u'+u'') - s_A(u') - s_A(u'')$$

*so that equality holds*

$$s_B(u'+u'') - s_A(u'+u'') = s_B(u') + s_B(u'') - s_A(u') - s_A(u'').$$

*Taking* $u' = -u'' = u$ *produces,* $\forall u \in \mathcal{S}^{n-1}$, $s_B(u) + s_B(-u) = s_A(u) + s_A(-u)$ *i.e.* $s_B(u)+s_{-B}(u) = s_A(u)+s_{-A}(u)$ *i.e.* $s_{B-B}(u) = s_{A-A}(u)$ *and* $B-B = A-A$ *(*$A$ *and* $B$ *translate into each other); it follows that* $\exists c \in \mathbb{R}^n$ *such that* $A = B+\{c\}$ *and* $B = A + \{-c\}$ *so that* $A \ominus_g B = \{c\}$*.*
*In case 2. we have that, being* $s_1$ *a support function it characterizes a nonempty set* $C \in \mathcal{K}^n_C$ *and* $s_C(u) = s_1(u) = s_A(u)-s_B(u)$, $\forall u \in \mathcal{S}^{n-1}$*; then* $s_A = s_B+s_C = s_{B+C}$ *and* $A = B + C$ *from which (i) is satisfied.*
*In case 3. we have that* $s_2$ *the support function of a nonempty set* $D \in \mathcal{K}^n_C$ *and* $s_D(u) = s_B(u) - s_A(u)$, $\forall u \in \mathcal{S}^{n-1}$ *so that* $s_B = s_A + s_D = s_{A-D}$ *and* $B = A + D$*. Defining* $C = (-1)D$ *(or* $D = (-1)C$ *) we obtain* $C \in \mathcal{K}^n_C$ *with* $s_C(u) = s_{-D}(u) = s_D(-u) = s_B(-u) - s_A(-u) = s_{-B}(u) - s_{-A}(u)$ *and (ii) is satisfied.*
*In case 4. there is no* $C \in \mathcal{K}^n_C$ *such that* $A = B + C$ *(otherwise* $s_1 = s_A - s_B$ *is a support function) and there is no* $D \in \mathcal{K}^n_C$ *such that* $B = A + D$ *(otherwise* $s_2 = s_B - s_A$ *is a support function); it follows that (i) and (ii) cannot be satisfied and* $A \ominus_g B$ *does not exist.*

**Proposition 4.** *If* $C = A \ominus_g B$ *exists, then* $||C|| = \rho_2(A, B)$ *and the Steiner points satisfy* $\sigma_C = \sigma_A - \sigma_B$*.*

*Proof.* *In fact* $\rho_2(A, B) = ||s_A - s_B||$ *and, if* $A \ominus_g B$ *exists, then either* $s_C = s_A - s_B$ *or* $s_C = s_{-B} - s_{-A}$*; but* $||s_A - s_B|| = ||s_{-A} - s_{-B}||$ *as, changing variable* $u$ *into* $-v$ *and recalling that* $s_{-A}(u) = s_A(-u)$*, we have*

$$\begin{aligned} ||s_{-A} - s_{-B}|| &= \int_{\mathcal{S}^{n-1}} [s_{-A}(u) - s_{-B}(u)]^2 \upsilon(du) \\ &= \int_{\mathcal{S}^{n-1}} [s_A(-u) - s_B(-u)]^2 \upsilon(du) \\ &= \int_{\mathcal{S}^{n-1}} [s_A(v) - s_B(v)]^2 \upsilon(-dv) \ = ||s_A - s_B||. \end{aligned} \tag{5}$$

*For the Steiner points, we proceed in a similar manner:*

$$\sigma_C = \begin{cases} n \int_{\mathcal{S}^{n-1}} u[s_A(u) - s_B(u)]\upsilon(du), \text{ or} \\ n \int_{\mathcal{S}^{n-1}} u[s_{-B}(u) - s_{-A}(u)]\upsilon(du) = n \int_{\mathcal{S}^{n-1}} v[s_A(v) - s_B(v)]\upsilon(-dv) \end{cases} \tag{6}$$

*and the result follows from the additivity of the integral.*

## 2 The case of compact intervals in $\mathbb{R}^n$

In this section we consider the gH-difference of compact intervals in $\mathbb{R}^n$. If $n = 1$, i.e. for unidimensional compact intervals, the gH-difference always exists. In fact, let $A = [a^-, a^+]$ and $B = [b^-, b^+]$ be two intervals; the gH-difference is

$$[a^-, a^+] \ominus_g [b^-, b^+] = [c^-, c^+] \iff \begin{cases} (i) \begin{cases} a^- = b^- + c^- \\ a^+ = b^+ + c^+ \end{cases} \\ \text{or } (ii) \begin{cases} b^- = a^- - c^+ \\ b^+ = a^+ - c^- \end{cases} \end{cases}$$

so that $[a^-, a^+] \ominus_g [b^-, b^+] = [c^-, c^+]$ is always defined by

$$c^- = \min\{a^- - b^-, a^+ - b^+\}\ ,\ \ c^+ = \max\{a^- - b^-, a^+ - b^+\}$$

i.e.

$$[a, b] \ominus_g [c, d] = [\min\{a - c, b - d\}, \max\{a - c, b - d\}].$$

Conditions $(i)$ and $(ii)$ are satisfied simultaneously if and only if the two intervals have the same length and $c^- = c^+$. Also, the result is $\{0\}$ if and only if $a^- = b^-$ and $a^+ = b^+$.

Two simple examples on real compact intervals illustrate the generalization (from [3], p. 8); $[-1, 1] \ominus [-1, 0] = [0, 1]$ as in fact *(i)* is $[-1, 0] + [0, 1] = [-1, 1]$ but $[0, 0] \ominus_g [0, 1] = [-1, 0]$ and $[0, 1] \ominus_g [-\frac{1}{2}, 1] = [0, \frac{1}{2}]$ satisfy *(ii)*.

Of interest are the symmetric intervals $A = [-a, a]$ and $B = [-b, b]$ with $a, b \geq 0$; it is well known that Minkowski operations with symmetric intervals are such that $A - B = B - A = A + B$ and, in particular, $A - A = A + A = 2A$. We have $[-a, a] \ominus_g [-b, b] = [-|a - b|, |a - b|]$.

As $\mathcal{S}^0 = \{-1, 1\}$ and the support functions satisfy $s_A(-1) = -a^-$, $s_A(1) = a^+$, $s_B(-1) = -b^-$, $s_B(1) = b^+$, the same results as before can be deduced by definition (4).

*Remark 4.* An alternative representation of an interval $A = [a^-, a^+]$ is by the use of the midpoint $\widehat{a} = \frac{a^- + a^+}{2}$ and the (semi)width $\overline{a} = \frac{a^+ - a^-}{2}$ and we can write $A = (\widehat{a}, \overline{a})$, $\overline{a} \geq 0$, so that $a^- = \widehat{a} - \overline{a}$ and $a^+ = \widehat{a} + \overline{a}$. If $B = (\widehat{b}, \overline{b})$, $\overline{b} \geq 0$ is a second interval, the Minkowski addition is $A + B = (\widehat{a} + \widehat{b}, \overline{a} + \overline{b})$ and the gH-difference is obtained by $A \ominus_g B = (\widehat{a} - \widehat{b}, |\overline{a} - \overline{b}|)$. We see immediately that $A \ominus_g A = \{0\}$, $A = B \iff A \ominus_g B = \{0\}$, $(A + B) \ominus_g B = A$, but $A + (B \ominus_g A) = B$ only if $\overline{a} \leq \overline{b}$.

Let now $A = \times_{i=1}^n A_i$ and $B = \times_{i=1}^n B_i$ where $A_i = [a_i^-, a_i^+]$, $B_i = [b_i^-, b_i^+]$ are real compact intervals ($\times_{i=1}^n$ denotes the cartesian product).

In general, considering $D = \times_{i=1}^n (A_i \ominus_g B_i)$, we may have $A \ominus_g B \neq D$ e.g. $A \ominus_g B$ may not exist as for the example $A_1 = [3, 6]$, $A_2 = [2, 6]$, $B_1 = [5, 10]$, $B_2 = [7, 9]$ for which $(A_1 \ominus_g B_1) = [-4, -2]$, $(A_2 \ominus_g B_2) = [-5, -3]$, $D = [-4, -2] \times [-5, -3]$ and $B + D = [1, 8] \times [2, 6] \neq A$, $A + (-1)D = [5, 10] \times [5, 11] \neq B$.

But if $A \ominus_g B$ exists, then equality will hold. In fact, consider the support function of $A$ (and similarly for $B$), defined by

$$s_A(u) = \max_x \{\langle u, x\rangle \,|a_i^- \leq x_i \leq a_i^+\},\ u \in \mathcal{S}^{n-1}; \tag{7}$$

it can be obtained simply by $s_A(u) = \sum\limits_{u_i>0} u_i a_i^+ + \sum\limits_{u_i<0} u_i a_i^-$ as the box-constrained maxima of the linear objective functions $\langle u, x\rangle$ above are attained at vertices $\widehat{x}(u) = (\widehat{x}_1(u), ..., \widehat{x}_i(u), ..., \widehat{x}_n(u))$ of $A$, i.e. $\widehat{x}_i(u) \in \{a_i^-, a_i^+\}$, $i = 1, 2, ..., n$. Then

$$s_A(u) - s_B(u) = \sum_{u_i>0} u_i(a_i^+ - b_i^+) + \sum_{u_i<0} u_i(a_i^- - b_i^-) \tag{8}$$

and, being $s_{-A}(u) = s_A(-u) = -\sum\limits_{u_i<0} u_i a_i^+ - \sum\limits_{u_i>0} u_i a_i^-$,

$$s_{-B}(u) - s_{-A}(u) = \sum_{u_i>0} u_i(a_i^- - b_i^-) + \sum_{u_i<0} u_i(a_i^+ - b_i^+). \tag{9}$$

From the relations above, we deduce that

$$A \ominus_g B = C \iff \begin{cases} (i) \begin{cases} C = \times_{i=1}^n [a_i^- - b_i^-, a_i^+ - b_i^+] \\ \text{provided that } a_i^- - b_i^- \leq a_i^+ - b_i^+,\ \forall i \end{cases} \\ \text{or } (ii) \begin{cases} C = \times_{i=1}^n [a_i^+ - b_i^+, a_i^- - b_i^-] \\ \text{provided that } a_i^- - b_i^- \geq a_i^+ - b_i^+,\ \forall i \end{cases} \end{cases}$$

and the gH-difference $A \ominus_g B$ exists if and only if one of the two conditions are satisfied:

$$\begin{array}{l} \text{case } (i)\ \ a_i^- - b_i^- \leq a_i^+ - b_i^+,\ i = 1, 2, ..., n \\ \text{case } (ii)\ a_i^- - b_i^- \geq a_i^+ - b_i^+,\ i = 1, 2, ..., n \end{array}$$

**Examples**:

*1. case (i)*: $A_1 = [5, 10]$, $A_2 = [1, 3]$, $B_1 = [3, 6]$, $B_2 = [2, 3]$ for which $(A_1 \ominus_g B_1) = [2, 4]$, $(A_2 \ominus_g B_2) = [-1, 0]$ and $A \ominus_g B = C = [2, 4] \times [-1, 0]$ exists with $B + C = A$, $A + (-1)C \neq B$.

*2. case (ii)*: $A_1 = [3, 6]$, $A_2 = [2, 3]$, $B_1 = [5, 10]$, $B_2 = [1, 3]$ for which $(A_1 \ominus_g B_1) = [-4, -2]$, $(A_2 \ominus_g B_2) = [0, 1]$ and $A \ominus_g B = C = [-4, -2] \times [0, 1]$ exists with $B + C \neq A$, $A + (-1)C = B$.

*3. case (i) + (ii)*: $A_1 = [3, 6]$, $A_2 = [2, 3]$, $B_1 = [5, 8]$, $B_2 = [3, 4]$ for which $(A_1 \ominus_g B_1) = [-2, -2] = \{-2\}$, $(A_2 \ominus_g B_2) = [-1, -1] = \{-1\}$ and $A \ominus_g B = C = \{(-2, -1)\}$ exists with $B + C = A$ and $A + (-1)C = B$.

We end this section with a comment on the simple interval equation

$$A + X = B \tag{10}$$

where $A = [a^-, a^+]$, $B = [b^-, b^+]$ are given intervals and $X = [x^-, x^+]$ is an interval to be determined satisfying (10). We have seen that, for unidimensional intervals, the gH-difference always exists. Denote by $l(A) = a^+ - a^-$ the length of interval $A$. It is well known from classical interval arithmetic that an interval

$X$ satisfying (10) exists only if $l(B) \geq l(A)$ (in Minkowski arithmetic we have $l(A+X) \geq \max\{l(A), l(X)\}$); in fact, no $X$ exists with $x^- \leq x^+$ if $l(B) < l(A)$ and we cannot solve (10) unless we interpret it as $B - X = A$. If we do so, we get

$$\text{case } l(B) \leq l(A) : \begin{cases} a^- + x^- = b^- \\ a^+ + x^+ = b^+ \end{cases} \text{ i.e. } \begin{array}{l} x^- = b^- - a^- \\ x^- = b^- - a^- \end{array}$$
$$\text{case } l(B) \geq l(A) : \begin{cases} b^- - x^+ = a^- \\ b^+ - x^- = a^+ \end{cases} \text{ i.e. } \begin{array}{l} x^- = b^+ - a^+ \\ x^+ = b^- - a^- \end{array}.$$

We then obtain that $X = B \ominus_g A$ is the unique solution to (10) and it always exists, i.e.

**Proposition 5.** *Let $A, B \in \mathcal{K}_C(\mathbb{R})$; the gH-difference $X = B \ominus_g A$ always exists and either $A + (B \ominus_g A) = B$ or $B - (B \ominus_g A) = A$.*

From property 6) of Proposition 7, a similar result is true for equation $A + X = B$ with $A, B \in \mathcal{K}_C(\mathbb{R}^n)$ but for $n > 1$ the gH-difference may non exist.

## 3 gH-difference of fuzzy numbers

A general *fuzzy set* over a given set (or space) $\mathbb{X}$ of elements (the universe) is usually defined by its membership function $\mu : \mathbb{X} \longrightarrow \mathbb{T} \subseteq [0,1]$ and a fuzzy (sub)set $u$ of $\mathbb{X}$ is uniquely characterized by the pairs $(x, \mu_u(x))$ for each $x \in \mathbb{X}$; the value $\mu_u(x) \in [0,1]$ is the membership grade of $x$ to the fuzzy set $u$. We will consider particular fuzzy sets, called *fuzzy numbers*, defined over $\mathbb{X} = \mathbb{R}$ having a particular form of the membership function. Let $\mu_u$ be the membership function of a fuzzy set $u$ over $\mathbb{X}$. The support of $u$ is the (crisp) subset of points of $\mathbb{X}$ at which the membership grade $\mu_u(x)$ is positive: $supp(u) = \{x | x \in \mathbb{X}, \mu_u(x) > 0\}$. For $\alpha \in ]0,1]$, the $\alpha-$level cut of $u$ (or simply the $\alpha - cut$) is defined by $[u]_\alpha = \{x | x \in \mathbb{X}, \mu_u(x) \geq \alpha\}$ and for $\alpha = 0$ (or $\alpha \to +0$) by the closure of the support $[u]_0 = cl\{x | x \in \mathbb{X}, \mu_u(x) > 0\}$.

A well-known property of the $level - cuts$ is $[u]_\alpha \subseteq [u]_\beta$ for $\alpha > \beta$ (i.e. they are nested).

A particular class of fuzzy sets $u$ is when the support is a convex set and the membership function is quasi-concave i.e. $\mu_u((1-t)x' + tx'') \geq \min\{\mu_u(x'), \mu_u(x'')\}$ for every $x', x'' \in supp(u)$ and $t \in [0,1]$. Equivalently, $\mu_u$ is quasi-concave if the level sets $[u]_\alpha$ are convex sets for all $\alpha \in [0,1]$. A third property of the fuzzy numbers is that the level-cuts $[u]_\alpha$ are closed sets for all $\alpha \in [0,1]$.

By using these properties, the space $\mathcal{F}$ of (real unidimensional) fuzzy numbers is structured by an addition and a scalar multiplication, defined either by the level sets or, equivalently, by the Zadeh extension principle. Let $u, v \in \mathcal{F}$ have membership functions $\mu_u$, $\mu_v$ and $\alpha - cuts$ $[u]_\alpha$, $[v]_\alpha$, $\alpha \in [0,1]$ respectively. The addition $u + v \in \mathcal{F}$ and the scalar multiplication $ku \in \mathcal{F}$ have level cuts

$$[u+v]_\alpha = [u]_\alpha + [v]_\alpha = \{x + y | x \in [u]_\alpha, \ y \in [v]_\alpha\} \tag{11}$$
$$[ku]_\alpha = k[u]_\alpha = \{kx | x \in [u]_\alpha\}. \tag{12}$$

In the fuzzy or in the interval arithmetic contexts, equation $u = v + w$ is not equivalent to $w = u - v = u + (-1)v$ or to $v = u - w = u + (-1)w$ and this has motivated the introduction of the following Hukuhara difference ([3], [5]). The generalized Hukuhara difference is (implicitly) used by Bede and Gal (see [1]) in their definition of generalized differentiability of a fuzzy-valued function.

**Definition 2.** *Given $u, v \in \mathcal{F}$, the H-difference is defined by $u \ominus v = w \Longleftrightarrow u = v + w$; if $u \ominus v$ exists, it is unique and its $\alpha-cuts$ are $[u \ominus v]_\alpha = [u^-_\alpha - v^-_\alpha, u^+_\alpha - v^+_\alpha]$. Clearly, $u \ominus u = \{0\}$.*

The Hukuhara difference is also motivated by the problem of inverting the addition: if $x, y$ are crisp numbers then $(x + y) - y = x$ but this is not true if $x, y$ are fuzzy. It is possible to see that (see [2]), if $u$ and $v$ are fuzzy numbers (and not in general fuzzy sets), then $(u + v) \ominus v = u$ i.e. the H-difference inverts the addition of fuzzy numbers.

The gH-difference for fuzzy numbers can be defined as follows:

**Definition 3.** *Given $u, v \in \mathcal{F}$, the gH-difference is the fuzzy number $w$, if it exists, such that*

$$u \ominus_g v = w \Longleftrightarrow \begin{cases} \quad (i) \;\; u = v + w \\ or\; (ii) \; v = u + (-1)w \end{cases} . \tag{13}$$

*If $u \ominus_g v$ exists, its $\alpha - cuts$ are given by $[u \ominus_g v]_\alpha = [\min\{u^-_\alpha - v^-_\alpha, u^+_\alpha - v^+_\alpha\}, \max\{u^-_\alpha - v^-_\alpha, u^+_\alpha - v^+_\alpha\}]$ and $u \ominus v = u \ominus_g v$ if $u \ominus v$ exists. If (i) and (ii) are satisfied simultaneously, then $w$ is a crisp number. Also, $u \ominus_g u = u \ominus u = \{0\}$.*

A definition of $w = u \ominus_g v$ for multidimensional fuzzy numbers can be obtained in terms of support functions in a way similar to (4)

$$s_w(p;\alpha) = \left\langle \begin{array}{ll} s_u(p;\alpha) - s_v(p;\alpha) & \text{in case } (i) \\ s_{(-1)v}(p;\alpha) - s_{(-1)u}(p;\alpha) & \text{in case } (ii) \end{array} \right. , \; \alpha \in [0,1] \tag{14}$$

where, for a fuzzy number $u$, the support functions are considered for each $\alpha - cut$ and defined to characterize the (compact) $\alpha - cuts$ $[u]_\alpha$:

$$\begin{gathered} s_u \;:\; \mathbb{R}^n \times [0,1] \longrightarrow \mathbb{R} \text{ defined by} \\ s_u(p;\alpha) = \sup\{\langle p, x\rangle \,|x \in [u]_\alpha\} \text{ for each } p \in \mathbb{R}^n,\ \alpha \in [0,1]. \end{gathered}$$

In the unidimensional fuzzy numbers, the conditions for the definition of $w = u \ominus_g v$ are

$$[w]_\alpha = [w^-_\alpha, w^+_\alpha] = [u]_\alpha \ominus_g [v]_\alpha \colon \begin{cases} w^-_\alpha = \min\{u^-_\alpha - v^-_\alpha, u^+_\alpha - v^+_\alpha\} \\ w^+_\alpha = \max\{u^-_\alpha - v^-_\alpha, u^+_\alpha - v^+_\alpha\} \end{cases} . \tag{15}$$

provided that $w^-_\alpha$ is nondecreasing, $w^+_\alpha$ is nonincreasing and $w^-_\alpha \leq w^+_\alpha$.

If $u \ominus_g v$ is a proper fuzzy number, it has the same properties illustrated in section 1. for intervals.

**Proposition 6.** *If $u \ominus_g v$ exists, it is unique and has the following properties:*
*1) $u \ominus_g u = 0$;*
*2) $(u+v) \ominus_g v = u$;*
*3) If $u \ominus_g v$ exists then also $(-v) \ominus_g (-u)$ does and $\{0\} \ominus_g (u \ominus_g v) = (-v) \ominus_g (-u)$;*
*4) $(u-v)+v = w \iff u-v = w \ominus_g v$;*
*5) $(u \ominus_g v) = (v \ominus_g u) = w$ if and only if ($w = \{0\}$ and $u = v$);*
*6) If $v \ominus_g u$ exists then either $u + (v \ominus_g u) = u$ or $v - (v \ominus_g u) = u$ and if both equalities hold then $v \ominus_g u$ is a crisp set.*

If the gH-differences $[u]_\alpha \ominus_g [v]_\alpha$ do not define a proper fuzzy number, we can use the nested property and obtain a proper fuzzy number by

$$[u \widetilde{\ominus}_g v]_\alpha := \bigcup_{\beta \geq \alpha} ([u]_\beta \ominus_g [v]_\beta); \tag{16}$$

As each gH-difference $[u]_\beta \ominus_g [v]_\beta$ exists for $\beta \in [0,1]$ and (16) defines a proper fuzzy number, it follows that $u \widetilde{\ominus}_g v$ can be considered as a generalization of Hukuhara difference for fuzzy numbers, existing for any $u, v$. A second possibility for a gH-difference of fuzzy numbers may be obtained following a suggestion by Kloeden and Diamond ([3]) and defining $z = u \widetilde{\ominus}_g v$ to be the fuzzy number whose $\alpha - cuts$ are as near as possible to the gH-differences $[u]_\alpha \ominus_g [v]_\alpha$, for example by minimizing the functional ($\omega_\alpha \geq 0$ and $\gamma_\alpha \geq 0$ are weighting functions)

$$G(z|u,v) = \int_0^1 (\omega_\alpha \left[z_\alpha^- - (u \ominus_g v)_\alpha^-\right]^2 + \gamma_\alpha \left[z_\alpha^+ - (u \ominus_g v)_\alpha^+\right]^2) d\alpha$$

such that $z_\alpha^- \uparrow$, $z_\alpha^+ \downarrow$, $z_\alpha^- \leq z_\alpha^+$ $\forall \alpha \in [0,1]$.

A discretized version of $G(z|u,v)$ can be obtained by choosing a partition $0 = \alpha_0 < \alpha_1 < ... < \alpha_N = 1$ of $[0,1]$ and defining the discretized $G(z|u,v)$ as

$$G_N(z|u,v) = \sum_{i=0}^N \omega_i \left[z_i^- - (u \ominus_g v)_i^-\right]^2 + \gamma_i \left[z_i^+ - (u \ominus_g v)_i^+\right]^2;$$

we minimize $G_N(z|u,v)$ with the given data $(u \ominus_g v)_i^- = \min\{u_{\alpha_i}^- - v_{\alpha_i}^-, u_{\alpha_i}^+ - v_{\alpha_i}^+\}$ and $(u \ominus_g v)_i^+ = \max\{u_{\alpha_i}^- - v_{\alpha_i}^-, u_{\alpha_i}^+ - v_{\alpha_i}^+\}$, subject to the constraints $z_0^- \leq z_1^- \leq ... \leq z_N^- \leq z_N^+ \leq z_{N-1}^+ \leq ... \leq z_0^+$. We obtain a linearly constrained least squares minimization of the form

$$\min_{z \in \mathbb{R}^{2N+2}} (z-w)^T D^2 (z-w) \text{ s.t. } Ez \geq 0$$

where $z = (z_0^-, z_1^-, ..., z_N^-, z_N^+, z_{N-1}^+, ..., z_0^+)$, $w_i^- = (u \ominus_g v)_i^-$, $w_i^+ = (u \ominus_g v)_i^+$, $w = (w_0^-, w_1^-, ..., w_N^-, w_N^+, w_{N-1}^+, ..., w_0^+)$, $D = diag\{\sqrt{\omega_0}, ..., \sqrt{\omega_N}, \sqrt{\gamma_N}, ..., \sqrt{\gamma_0}\}$ and $E$ is the $(N, N+1)$ matrix

$$E = \begin{bmatrix} -1 & 1 & 0 & \ldots & \ldots & 0 \\ 0 & -1 & 1 & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & \ldots & -1 & 1 \end{bmatrix}$$

which can be solved by standard efficient procedures (see the classical book [6], ch. 23). If, at solution $z^*$, we have $z^* = w$, then we obtain the gH-difference as defined in (13).

## 4 Generalized division

An idea silmilar to the gH-difference can be used to introduce a division of real intervals and fuzzy numbers. We consider here only the case of real compact intervals $A = [a^-, a^+]$ and $B = [b^-, b^+]$ with $b^- > 0$ or $b^+ < 0$ (i.e. $0 \notin B$).

The interval $C = [c^-, c^+]$ defining the multiplication $C = AB$ is given by

$$c^- = \min\{a^-b^-, a^-b^+, a^+b^-, a^+b^+\}\ ,\ \ c^+ = \max\{a^-b^-, a^-b^+, a^+b^-, a^+b^+\}$$

and the multiplicative "inverse" (it is not the inverse in the algebraic sense) of an interval $B$ is defined by $B^{-1} = [\frac{1}{b^+}, \frac{1}{b^-}]$; we define the generalized division (g-division) $\div_g$ as follows:

$$A \div_g B = C \iff \begin{cases} \quad (i) \ \ A = BC \\ \text{or } (ii) \ B = AC^{-1} \end{cases} .$$

If both cases *(i)* and *(ii)* are valid, we have $CC^{-1} = C^{-1}C = \{1\}$, i.e. $C = \{\widehat{c}\}$, $C^{-1} = \{\frac{1}{\widehat{c}}\}$ with $\widehat{c} \neq 0$. It is easy to see that $A \div_g B$ always exists and is unique for given $A = [a^-, a^+]$ and $B = [b^-, b^+]$ with $0 \notin B$. It is easy to see that it can be obtained by the following rules:

Case 1. If ($a^- \leq a^+ < 0$ and $b^- \leq b^+ < 0$) or ($0 < a^- \leq a^+$ and $0 < b^- \leq b^+$) then

$$c^- = \min\{\frac{a^-}{b^-}, \frac{a^+}{b^+}\} \geq 0,\ \ c^+ = \max\{\frac{a^-}{b^-}, \frac{a^+}{b^+}\} \geq 0;$$

Case 2. If ($a^- \leq a^+ < 0$ and $0 < b^- \leq b^+$) or ($0 < a^- \leq a^+$ and $b^- \leq b^+ < 0$) then

$$c^- = \min\{\frac{a^-}{b^+}, \frac{a^+}{b^-}\} \leq 0,\ \ c^+ = \max\{\frac{a^-}{b^+}, \frac{a^+}{b^-}\} \leq 0;$$

Case 3. If ($a^- \leq 0$, $a^+ \geq 0$ and $b^- \leq b^+ < 0$) then

$$c^- = \frac{a^-}{b^-} \leq 0,\ \ c^+ = \frac{a^+}{b^-} \geq 0;$$

Case 4. If ($a^- \leq 0$, $a^+ \geq 0$ and $0 < b^- \leq b^+$) then

$$c^- = \frac{a^-}{b^+} \leq 0,\ \ c^+ = \frac{a^+}{b^+} \geq 0.$$

*Remark 5.* If $0 \in ]b^-, b^+[$ the g-division is undefined; for intervals $B = [0, b^+]$ or $B = [b^-, 0]$ the division is possible but obtaining unbounded results $C$ of the form $C = ]-\infty, c^+]$ or $C = [c^-, +\infty[$: we work with $B = [\varepsilon, b^+]$ or $B = [b^-, \varepsilon]$ and we obtain the result by the limit for $\varepsilon \longrightarrow 0^+$. Example: for $[-2, -1] \div_g [0, 3]$ we consider $[-2, -1] \div_g [\varepsilon, 3] = [c_\varepsilon^-, c_\varepsilon^+]$ with (case 2.) $c_\varepsilon^- = \min\{\frac{-2}{3}, \frac{-1}{\varepsilon}\}$ and $c_\varepsilon^+ = \max\{\frac{-2}{\varepsilon}, \frac{-1}{3}\}$ and obtain the result $C = [-\infty, -\frac{1}{3}]$ at the limit $\varepsilon \longrightarrow 0^+$.

The following properties are immediate.

**Proposition 7.** *For any $A = [a^-, a^+]$ and $B = [b^-, b^+]$ with $0 \notin B$, we have (here 1 is the same as $\{1\}$):*

*1.* $B \div_g B = 1$, $B \div_g B^{-1} = \{b^- b^+\}$ *(=* $\{\widehat{b}^2\}$ *if* $b^- = b^+ = \widehat{b}$*);*

*2.* $(AB) \div_g B = A$*;*

*3.* $1 \div_g B = B^{-1}$ *and* $1 \div_g B^{-1} = B$.

In the case of fuzzy numbers $u, v \in \mathcal{F}$ having membership functions $\mu_u$, $\mu_v$ and $\alpha - cuts$ $[u]_\alpha = [u_\alpha^-, u_\alpha^+]$, $[v]_\alpha = [v_\alpha^-, v_\alpha^+]$, $0 \notin [v]_\alpha \ \forall \alpha \in [0, 1]$, the *g-division* $\div_g$ can be defined as the operation that calculates the fuzzy number $w = u \div_g v \in \mathcal{F}$ having level cuts $[w]_\alpha = [w_\alpha^-, w_\alpha^+]$ (here $[w]_\alpha^{-1} = [\frac{1}{w_\alpha^+}, \frac{1}{w_\alpha^-}]$):

$$[u]_\alpha \div_g [v]_\alpha = [w]_\alpha \iff \begin{cases} \quad (i) \;\; [u]_\alpha = [v]_\alpha [w]_\alpha \\ \text{or } (ii) \; [v]_\alpha = [u]_\alpha [w]_\alpha^{-1} \end{cases},$$

provided that $w$ is a proper fuzzy number.

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