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"An LU-fuzzy Calculator for the Basic Fuzzy Calculus"

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An LU-fuzzy calculator for the basic fuzzy calculus

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Abstract

The LU-model for fuzzy numbers has been introduced in [4] and applied to fuzzy calculus in [9]; in this paper we build an LU-fuzzy calculator, in order to explain the use of the LU-fuzzy representation and to show the advantage of the parametrization. The calculator produces the basic fuzzy calculus: the arithmetic operations (scalar multiplication, addition, subtraction, multiplication, division) and the fuzzy extension of many univariate functions (power with integer positive or negative exponent, exponential, logarithm, general power function with numeric or fuzzy exponent, sin, arcsin, cos, arccos, tan, arctan, square root, Gaussian and standard Gaussian functions, hyperbolic sinh, cosh, tanh and inverses, erf error function and complementary erfc error function, cumulative standard normal distribution). The use of the calculator is illustrated.

1 Introduction

The arithmetic operations on fuzzy numbers are usually approached either by the use of the extension principle (in the domain of the membership function) or by the interval arithmetics (in the domain of the α -cuts).

The exact analytical fuzzy mathematics dates back from the early eighties and are outlined by Dubois and Prade (see [1]); the same authors have introduced the well known L-R model and the corresponding formulas for the fuzzy operations (see [2]).

Very recent literature on fuzzy numbers is rich of contributions on the fuzzy arithmetic operations and the use of simple formulas to approximate them; an extensive survey and bibliography on fuzzy intervals is in [3].

We suggest in [4] the use of monotonic splines to approximate the fuzzy numbers, using several interpolation forms (monotonic rational interpolators and mixed cubic-exponential interpolator) and we derive a procedure to control

the locations of the nodes so that the error of the approximation is controlled by the possible insertion of additional nodes into the piecewise interpolation.

We see that, with only a few nodes, our approximations of fuzzy calculus maintain accurate results. The parametric LU representation of the fuzzy numbers allows a set of possible shapes (types of membership functions) that seems to be much wider than the well-known L-R framework.

The paper is organized as follows: section 2 contains a brief description of the fuzzy calculus; in section 3 we describe the LU-fuzzy model and in section 4 we describe the detailed algorithms which implement the LU-fuzzy extension principle. Section 5 contains the description of the LU-fuzzy calculator.

2 Basic fuzzy calculus

We adopt the so called *a – cut* setting for the definition of a fuzzy number:

Definition 1 *A continuous fuzzy number (or interval) u is any pair (u^-, u^+) of functions $u^\pm : [0, 1] \rightarrow \mathbb{R}$ satisfying the following conditions:*

(i) $u^- : \alpha \rightarrow u_\alpha^- \in \mathbb{R}$ is a bounded monotonic increasing (non decreasing) continuous function $\forall \alpha \in [0, 1]$;

(ii) $u^+ : \alpha \rightarrow u_\alpha^+ \in \mathbb{R}$ is a bounded monotonic decreasing (non increasing) continuous function $\forall \alpha \in [0, 1]$;

(iii) $u_\alpha^- \leq u_\alpha^+ \forall \alpha \in [0, 1]$.

If $u_1^- < u_1^+$ we have a fuzzy interval and if $u_1^- = u_1^+$ we have a fuzzy number.

The notation:

$$u_\alpha = [u_\alpha^-, u_\alpha^+]$$

denotes explicitly the α – cuts of u . We will also refer to u^- and u^+ as the left (lower) and the right (upper) branches on u , respectively.

If $u = (u^-, u^+)$ and $v = (v^-, v^+)$ are two given fuzzy numbers, the arithmetic operations are defined as follows:

Definition 2 (*Addition*)

$$u + v = (u^- + v^-, u^+ + v^+)$$

or, in terms of α – cuts , for $\alpha \in [0, 1]$:

$$(u + v)_\alpha = [u_\alpha^- + v_\alpha^-, u_\alpha^+ + v_\alpha^+] \quad (1)$$

Definition 3 (*Scalar Multiplication*)

For given $k \in \mathbb{R}$

$$\begin{cases} ku = (ku^-, ku^+) & \text{if } k > 0 \\ ku = (ku^+, ku^-) & \text{if } k < 0 \end{cases}$$

or, for $\alpha \in [0, 1]$:

$$(ku)_\alpha = [\min \{ku_\alpha^-, ku_\alpha^+\}, \max \{ku_\alpha^-, ku_\alpha^+\}] \quad (2)$$

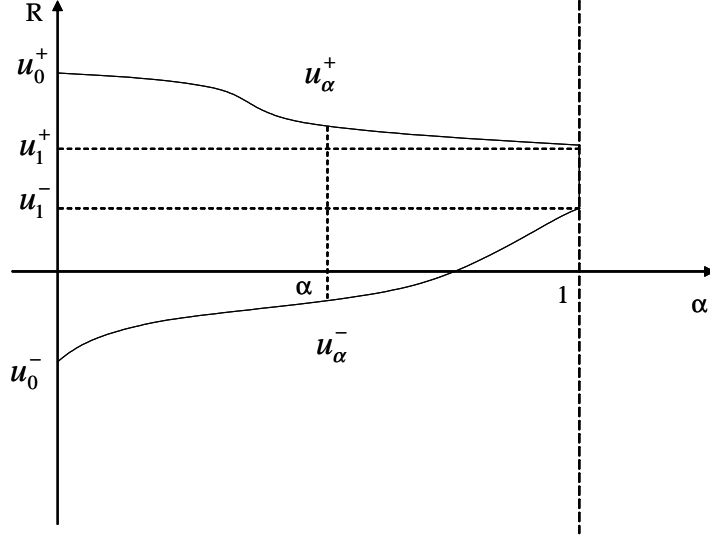


Figure 1: The α - cut representation of a general fuzzy interval.

In particular, if $k = -1$, we obtain:

$$-u = (-u^+, -u^-)$$

with the α - cuts :

$$(-u)_\alpha = [-u_\alpha^+, -u_\alpha^-] , \alpha \in [0, 1]$$

Definition 4 (Subtraction)

$$u - v = u + (-v) = (u^- - v^+, u^+ - v^-)$$

or, in terms of the α - cuts :

$$(u - v)_\alpha = [u_\alpha^- - v_\alpha^+, u_\alpha^+ - v_\alpha^-] , \alpha \in [0, 1] \quad (3)$$

Definition 5 (Multiplication)

$$uv = ((uv)^-, (uv)^+)$$

where, $\forall \alpha \in [0, 1]$:

$$\begin{cases} (uv)_\alpha^- = \min \{u_\alpha^- v_\alpha^-, u_\alpha^- v_\alpha^+, u_\alpha^+ v_\alpha^-, u_\alpha^+ v_\alpha^+\} \\ (uv)_\alpha^+ = \max \{u_\alpha^- v_\alpha^-, u_\alpha^- v_\alpha^+, u_\alpha^+ v_\alpha^-, u_\alpha^+ v_\alpha^+\} \end{cases} \quad (4)$$

Definition 6 (*Division*)

If $0 \notin [v_0^-, v_0^+]$:

$$\frac{u}{v} = \left(\left(\frac{u}{v} \right)^-, \left(\frac{u}{v} \right)^+ \right)$$

where, $\forall \alpha \in [0, 1]$:

$$\begin{cases} \left(\frac{u}{v} \right)_\alpha^- = \min \left\{ \frac{u_\alpha^-}{v_\alpha^-}, \frac{u_\alpha^-}{v_\alpha^+}, \frac{u_\alpha^+}{v_\alpha^-}, \frac{u_\alpha^+}{v_\alpha^+} \right\} \\ \left(\frac{u}{v} \right)_\alpha^+ = \max \left\{ \frac{u_\alpha^-}{v_\alpha^-}, \frac{u_\alpha^-}{v_\alpha^+}, \frac{u_\alpha^+}{v_\alpha^-}, \frac{u_\alpha^+}{v_\alpha^+} \right\} \end{cases} \quad (5)$$

We denote by \mathbb{F} the set of the fuzzy numbers/intervals.

3 Basic LU-fuzzy calculus

The parametric LU representation of a fuzzy number is defined on a decomposition of the interval $[0, 1]$

$$0 = \alpha_0 < \alpha_1 < \dots < \alpha_{i-1} < \alpha_i < \dots < \alpha_N = 1$$

for both the lower $u^-(\alpha)$ and the upper $u^+(\alpha)$ branches of the fuzzy numbers involved. In each of the N subintervals

$$I_i = [\alpha_{i-1}, \alpha_i] \quad , \quad i = 1, 2, \dots, N$$

the values of the two functions

$$\begin{aligned} u^-(\alpha_{i-1}) &= u_{0,i}^- \quad , \quad u^+(\alpha_{i-1}) = u_{0,i}^+ \\ u^-(\alpha_i) &= u_{1,i}^- \quad , \quad u^+(\alpha_i) = u_{1,i}^+ \end{aligned}$$

and of their first derivatives

$$\begin{aligned} u'^-(\alpha_{i-1}) &= d_{0,i}^- \quad , \quad u'^+(\alpha_{i-1}) = d_{0,i}^+ \\ u'^-(\alpha_i) &= d_{1,i}^- \quad , \quad u'^+(\alpha_i) = d_{1,i}^+ \end{aligned}$$

are assumed to be known; we are interested in families of monotonic functions that satisfy the above eight Hermite-type conditions for each subinterval I_i . In general, by the use of the following transformation of each subinterval I_i into the standard $[0, 1]$ interval,

$$t_\alpha = \frac{\alpha - \alpha_{i-1}}{\alpha_i - \alpha_{i-1}}, \quad \alpha \in I_i, \quad (6)$$

we can determine each piece independently and obtain the general left-continuous LU-fuzzy numbers.

Globally continuous or more regular $C^{(1)}$ fuzzy numbers can be obtained directly from the data if the following conditions are met for the values:

$$u_{1,i}^- = u_{0,i+1}^- \quad , \quad u_{1,i}^+ = u_{0,i+1}^+ \quad \text{for } i = 1, 2, \dots, N - 1$$

and possibly for the slopes

$$d_{1,i}^- = d_{0,i+1}^- \quad , \quad d_{1,i}^+ = d_{0,i+1}^+ \quad \text{for } i = 1, 2, \dots, N-1.$$

Let $p_i(t)$ denotes the approximation of u_α on a generic subinterval I_i of the α -decomposition, with the transformation $t = \frac{\alpha - \alpha_{i-1}}{\alpha_i - \alpha_{i-1}}$, so that each subinterval is re-mapped to the standard interval $[0, 1]$ by

$$\begin{aligned} p_i(t) &= u(\alpha_{i-1} + t(\alpha_i - \alpha_{i-1})) \\ p_i'(t) &= u'(\alpha_{i-1} + t(\alpha_i - \alpha_{i-1}))(\alpha_i - \alpha_{i-1}). \end{aligned} \quad (7)$$

For simplicity of notation, we omit the subscript i and we refer to $I_i = [0, 1]$ i.e. to the two-point Hermite interpolation problem of determining the monotonic function $p(t)$, $t \in [0, 1]$, such that

$$p(0) = u_0 \quad , \quad p'(0) = d_0 \quad , \quad p(1) = u_1 \quad , \quad p'(1) = d_1. \quad (8)$$

If $u_0 \leq u_1$ (i.e. the data are increasing) then $d_0 \geq 0$ and $d_1 \geq 0$ are required and, if $u_0 \geq u_1$ (i.e. the data are decreasing) then $d_0 \leq 0$ and $d_1 \leq 0$ are required. In particular, $u_0 = u_1 \Leftrightarrow d_0 = d_1 = 0$.

We will use the notation (m, n) -rational to mean the ratio $\frac{P(t)}{Q(t)}$ of an m -degree polynomial $P(t)$ to an n -degree polynomial $Q(t)$.

3.1 Quadratic/quadratic rational spline

The Delbourgo and Gregory (2,2)-rational monotonic spline has the following form:

$$p(t) = \begin{cases} \frac{P(t)}{Q(t)} & \text{if } u_1 \neq u_0 \\ u_0 & \text{if } u_1 = u_0 \end{cases},$$

where

$$\begin{aligned} P(t) &= (u_1 - u_0) u_1 t^2 + (u_0 d_1 + u_1 d_0) t(1-t) + (u_1 - u_0) u_0 (1-t)^2 \\ Q(t) &= (u_1 - u_0) t^2 + (d_1 + d_0) t(1-t) + (u_1 - u_0) (1-t)^2. \end{aligned} \quad (9)$$

Without any additional parameters, the function above satisfies the Hermite interpolation conditions at the points $t = 0$ and $t = 1$.

3.2 Cubic/linear rational spline

The Shrivastava and Joseph (3,1)-rational monotonic spline is given by $p(t) = \frac{P(t)}{Q(t)}$, where

$$\begin{aligned} P(t) &= v u_0 (1-t)^3 + w u_1 t^3 + [(2v+w) u_0 + v d_0] t(1-t)^2 \\ &\quad + [(v+2w) u_1 - w d_1] t^2 (1-t) \\ Q(t) &= v + (w-v) t \end{aligned} \quad (10)$$

with $v, w > 0$ and $w \geq v$. If $v = w$ we obtain the ordinary cubic spline.

A choice of the tension parameters v and w that guarantees the global monotonicity of p on $[0, 1]$ is

$$\frac{v}{w} \geq \frac{d_1 - u_1 + u_0}{u_1 - u_0 - d_0}$$

so that a simple choice may be, for example, $v = 1$ and

$$w = \max \left\{ \frac{u_1 - u_0 - d_0}{d_1 - (u_1 - u_0)}, \frac{1}{1 - \epsilon} \right\} \geq 1$$

where ϵ is a nonnegative small number, say $\epsilon \in [0, .1]$.

3.3 Cubic/quadratic rational spline

The (3,2)-rational form is suggested by Gregory

$$\begin{aligned} p(t) &= \frac{P(t)}{Q(t)} \text{ with} & (11) \\ P(t) &= u_0(1-t)^3 + (wu_0 + d_0)t(1-t)^2 + (wu_1 - d_1)t^2(1-t) + u_1t^3 \\ Q(t) &= 1 + t(1-t)(w-3) \end{aligned}$$

and a choice for the tension parameter $w \geq 0$ to have global monotonicity is $w = \frac{d_0 + d_1}{u_1 - u_0}$, obtaining the ordinary cubic spline if $w = 3$.

3.4 Cubic/cubic rational spline

The (3,3)-rational spline has been proposed by Sarfraz

$$\begin{aligned} p(t) &= \frac{P(t)}{Q(t)} \text{ with} & (12) \\ P(t) &= u_0(1-t)^3 + (vu_0 + d_0)t(1-t)^2 + (vu_1 - d_1)t^2(1-t) + u_1t^3 \\ Q(t) &= (1-t)^3 + vt(1-t)^2 + wt^2(1-t) + t^3 \end{aligned}$$

with $v = r \frac{d_0 + d_1}{u_1 - u_0}$ and $w = s \frac{d_0 + d_1}{u_1 - u_0}$, $r, s \geq 1$. If $v = w$ then (12) becomes (11).

3.5 Mixed cubic-exponential interpolation

The monotonic Hermite-type interpolator is based on a mixed cubic-exponential spline; in its simpler form, it is given by

$$\begin{aligned} p(t) &= u_0 + (u_1 - u_0 - \frac{d_0 + d_1}{a})t^2(3 - 2t) + \\ &\quad + \frac{d_0}{a} - \frac{d_0}{a}(1-t)^a + \frac{d_1}{a}t^a \end{aligned}$$

where $a = 1 + w \geq \frac{d_0+d_1}{u_1-u_0} \geq 0$ to have monotonicity. We use $w = \frac{d_0+d_1}{u_1-u_0} \geq 0$ or, to work with integer exponents, $w = \text{int}(\frac{d_0+d_1}{u_1-u_0})$.

The linear case (i.e. triangular fuzzy numbers) is obtained by putting $d_0 = d_1 = u_1 - u_0$ and $a = 3$: it is easy to see that the model becomes $p(t) = u_0 + d_0 t$. If the data are quadratic, i.e. $d_0 + d_1 = 2(u_1 - u_0)$, then $a = 3$ and the model becomes quadratic, $p(t) = u_0 + d_0 t + (d_1 - u_1 + u_0)t^2$.

3.6 Parametrization and fuzzy operations

If the slopes d_0 and d_1 are not available, we can proceed by choosing them such that, for a given positive integer n , $d_0 + d_1 = n(u_1 - u_0)$. In particular, if we fix an integer n and a parameter $\beta \in [0, 1]$, we can select (provided that $u_1 - u_0 \neq 0$)

$$\begin{cases} d_0 = n(1 - \beta)(u_1 - u_0) \\ d_1 = n\beta(u_1 - u_0) \end{cases} \quad (13)$$

so that $d_0 + d_1 = n(u_1 - u_0)$ and $a = n + 1$. If $\beta = 0$ or $\beta = 1$ the model gives two extreme shapes having $d_0 = 0$ or $d_1 = 0$.

Using one of the previous forms to represent the lower and the upper branches of the fuzzy number $u = (u^-, u^+)$ we can write the general form of the representation

$$\begin{aligned} u &= (u_{0,i}^-, d_{0,i}^-, u_{1,i}^-, d_{1,i}^-; u_{0,i}^+, d_{0,i}^+, u_{1,i}^+, d_{1,i}^+)_{i=1,\dots,N} \\ &\Updownarrow \\ u_\alpha &= [p_i(t_\alpha; u_{0,i}^-, \tilde{d}_{0,i}^-, u_{1,i}^-, \tilde{d}_{1,i}^-), p_i(t_\alpha; u_{0,i}^+, \tilde{d}_{0,i}^+, u_{1,i}^+, \tilde{d}_{1,i}^+)]_{i=1,2,\dots,N} \end{aligned} \quad (14)$$

where the functions $p_i(t_\alpha; u_{0,i}^-, \tilde{d}_{0,i}^-, u_{1,i}^-, \tilde{d}_{1,i}^-)$ and $p_i(t_\alpha; u_{0,i}^+, \tilde{d}_{0,i}^+, u_{1,i}^+, \tilde{d}_{1,i}^+)$ are obtained by monotonic models, with $\tilde{d}_{k,i} = d_{k,i}(\alpha_i - \alpha_{i-1})$, $i = 0, 1$ and $t_\alpha = \frac{\alpha - \alpha_{i-1}}{\alpha_i - \alpha_{i-1}}$ for $\alpha \in [\alpha_{i-1}, \alpha_i]$. For $N \geq 1$ we have a total of $8N$ parameters $u_{0,1}^- \leq u_{1,1}^- \leq u_{0,2}^- \leq u_{1,2}^- \leq \dots \leq u_{0,N}^- \leq u_{1,N}^-$, $d_{k,i}^- \geq 0$ defining the increasing lower branch u_α^- and $u_{0,1}^+ \geq u_{1,1}^+ \geq u_{0,2}^+ \geq u_{1,2}^+ \geq \dots \geq u_{0,N}^+ \geq u_{1,N}^+$, $d_{k,i}^+ \leq 0$ defining the decreasing upper branch u_α^+ (obviously, also $u_{1,N}^- \leq u_{1,N}^+$ is required).

A simplification of (14) can be obtained by requiring continuous or differentiable branches; in the first case, $u_{1,i}^- = u_{0,i+1}^-$ and $u_{1,i}^+ = u_{0,i+1}^+$ for $i = 1, 2, \dots, N-1$ while, to have differentiability, also the conditions $d_{k,i}^- = d_{k,i+1}^-$, $d_{k,i}^+ = d_{k,i+1}^+$ are required. For the two cases we then have $6N + 2$ or $4N + 4$ parameters, respectively.

In the strictly monotonic case, the membership function $\mu(x)$ of the LU-fuzzy number u , given by the equations $\mu(u_\alpha^-) = \mu(u_\alpha^+) = \alpha$ for $\alpha \in [0, 1]$, is the following.

$$\begin{aligned} \mu(x) &= 0 \text{ if } x \notin [u_0^-, u_0^+] \\ &= \alpha \text{ if } x = u_\alpha^- \text{ or } x = u_\alpha^+ \text{ for a given } \alpha \in [0, 1[\\ &= 1 \text{ if } x \in [u_1^-, u_1^+]. \end{aligned} \quad (15)$$

In particular, corresponding to the nodes of the α -decomposition,

$$\mu(u_i^-) = \mu(u_i^+) = \alpha_i \quad \text{for } i = 0, 1, \dots, N$$

and, for the differentiable case (the general piecewise differentiable case is similar)

$$\mu'(u_i^-) = \frac{1}{d_i^-}, \quad \mu'(u_i^+) = \frac{1}{d_i^+} \quad \text{for } i = 0, 1, \dots, N.$$

In the applications to fuzzy calculus described in [9] we consider only the differentiable case, for which we use the representation:

$$u = (u_i^-, d_i^-, u_i^+, d_i^+)_{i=0,1,\dots,N} \quad (16)$$

with the data

$$u_0^- \leq u_1^- \leq \dots \leq u_N^- \leq u_N^+ \leq u_{N-1}^+ \leq \dots \leq u_0^+ \quad (17)$$

and the slopes

$$d_i^- \geq 0, d_i^+ \leq 0. \quad (18)$$

By the Lower-Upper representations we can define corresponding spaces of fuzzy numbers, on which the standard operations and metrics can be introduced by the use of the standard fuzzy calculus.

Denote by $\mathbb{F}_N = \{u \mid u = (u_{0,i}^-, d_{0,i}^-, u_{1,i}^-, d_{1,i}^-; u_{0,i}^+, d_{0,i}^+, u_{1,i}^+, d_{1,i}^+)_{i=1,\dots,N}\}$ or, in the differentiable case, $\widehat{\mathbb{F}}_N = \{u \mid u = (u_i^-, d_i^-, u_i^+, d_i^+)_{i=0,1,\dots,N}\}$ the set of LU-fuzzy numbers. $\widehat{\mathbb{F}}_N$ is a $4(N+1)$ -dimensional space.

Given two LU-fuzzy numbers

$$u = (u_i^-, d_i^-, u_i^+, d_i^+)_{i=0,1,\dots,N} \quad \text{and} \quad v = (v_i^-, e_i^-, v_i^+, e_i^+)_{i=0,1,\dots,N}$$

the arithmetic operators associated to the LU representation can be obtained easily.

The *addition* is defined by:

$$u + v = (u_i^- + v_i^-, d_i^- + e_i^-, u_i^+ + v_i^+, d_i^+ + e_i^+)_{i=0,1,\dots,N}.$$

The *scalar multiplication* is defined as follows:

if $k \geq 0$ then

$$ku = (ku_i^-, kd_i^-, ku_i^+, kd_i^+)_{i=0,1,\dots,N};$$

if $k < 0$ then

$$ku = (ku_i^+, kd_i^+, ku_i^-, kd_i^-)_{i=0,1,\dots,N}.$$

In particular, if $k = -1$, we have

$$-u = (-u_i^+, -d_i^+, -u_i^-, -d_i^-)_{i=0,1,\dots,N}$$

and the *subtraction* is defined by

$$u - v = u + (-v).$$

For the fuzzy *multiplication* we introduce an easy to implement algorithm, based of the applications of the exact fuzzy multiplication at the nodes of the α -subdivision; define

$$(uv)_i^- = \min\{u_i^- v_i^-, u_i^- v_i^+, u_i^+ v_i^-, u_i^+ v_i^+\} \quad (19)$$

$$(uv)_i^+ = \max\{u_i^- v_i^-, u_i^- v_i^+, u_i^+ v_i^-, u_i^+ v_i^+\} \quad (20)$$

and set the following:

$$uv = (w_i^-, f_i^-, w_i^+, f_i^+)_{i=0,1,\dots,N}$$

To implement the multiplication we can proceed as follows: let (p_i^-, q_i^-) be the pair associated to the combination of superscripts + and - giving the minimum $(uv)_i^-$ in (19), and similarly let (p_i^+, q_i^+) the pair associated to the combination of + and - giving the maximum $(uv)_i^+$ in (20), then we obtain:

$$\begin{aligned} w_i^- &= u_i^{p_i^-} v_i^{q_i^-} \quad \text{and} \quad w_i^+ = u_i^{p_i^+} v_i^{q_i^+} \\ f_i^- &= d_i^{p_i^-} v_i^{q_i^-} + u_i^{p_i^-} e_i^{q_i^-} \quad \text{and} \quad f_i^+ = d_i^{p_i^+} v_i^{q_i^+} + u_i^{p_i^+} e_i^{q_i^+} \end{aligned}$$

where we use the product derivative rule to obtain the new slopes.

Analogous formulas can be deduced for the *division*:

$$u/v = (z_i^-, g_i^-, z_i^+, g_i^+)_{i=0,1,\dots,N}$$

$$\begin{aligned} (u/v)_i^- &= \min\{u_i^-/v_i^-, u_i^-/v_i^+, u_i^+/v_i^-, u_i^+/v_i^+\} \quad \text{and} \\ (u/v)_i^+ &= \max\{u_i^-/v_i^-, u_i^-/v_i^+, u_i^+/v_i^-, u_i^+/v_i^+\}. \end{aligned} \quad (21)$$

Let (r_i^-, s_i^-) be the pair associated to the combination of + and - giving the minimum in $(u/v)_i^-$ and similarly let (r_i^+, s_i^+) be the pair associated to the combination of + and - giving the maximum in $(u/v)_i^+$, then it follows:

$$\begin{aligned} z_i^- &= u_i^{r_i^-} / v_i^{s_i^-} \quad \text{and} \quad z_i^+ = u_i^{r_i^+} / v_i^{s_i^+} \\ g_i^- &= (d_i^{r_i^-} v_i^{s_i^-} - u_i^{r_i^-} e_i^{s_i^-}) / (v_i^{s_i^-})^2 \quad \text{and} \quad g_i^+ = (d_i^{r_i^+} v_i^{s_i^+} - u_i^{r_i^+} e_i^{s_i^+}) / (v_i^{s_i^+})^2. \end{aligned}$$

We note explicitly that the scalar multiplication is always reproduced exactly in all the models for all $\alpha \in [0, 1]$ but, in general, this is not true for the addition as the sum of rational or mixed functions is not a rational or a mixed function of the same orders.

As pointed out by the results of the experimentation reported in [4], the operations above are exact at the nodes α_i of the representation and have very small global errors on $[0, 1]$. Further, it is easy to control the error by introducing additional nodes into the representation or by using a sufficiently high number of nodes with $\max\{\alpha_i - \alpha_{i-1}\}$ sufficiently small. To control the error of the approximation, we can proceed by increasing the number $N + 1$ of points; a possible strategy is to double the number of points by using $N = 2^K$ and by moving automatically to $N = 2^{K+1}$ if a better precision is necessary.

For all the computations, as in [4], the illustrated algorithms are immediate to implement and also a standard spreadsheet can be used. The results in [4] of the parametric operators have shown that both the rational (12) and the mixed (??) models perform very well, with a percentage average error for a single multiplication and division of the order of 0.1%.

4 Fuzzy extension of univariate functions

In the general form the LU-representation can be written as:

$$X = (\alpha_i, x_i^-, \delta x_i^-, x_i^+, \delta x_i^+)_{i=0,1,\dots,N}$$

where the nodes α_i of the representation are shown explicitly (the symbol δ is used to denote the slopes).

The fuzzy extension of a single (real) variable (differentiable) function $f : \mathbb{R} \rightarrow \mathbb{R}$ to a fuzzy argument $u_\alpha = [u_\alpha^-, u_\alpha^+]$ has α -cuts

$$f(u)_\alpha = [\min\{f(x) \mid x \in u_\alpha\}, \max\{f(x) \mid x \in u_\alpha\}]. \quad (22)$$

Note that if f is monotonic increasing we obtain $f(u)_\alpha = [f(u_\alpha^-), f(u_\alpha^+)]$ while, if f is monotonic decreasing, $f(u)_\alpha = [f(u_\alpha^+), f(u_\alpha^-)]$.

For this specific case, we introduce a notation similar to the one used for multiplication or division: let $p_\alpha^-, p_\alpha^+ \in \{-, +\}$ be defined as follows

$$p_\alpha^- = \begin{cases} - & \text{if } \min\{f(u_\alpha^-), f(u_\alpha^+)\} = f(u_\alpha^-) \\ + & \text{if } \min\{f(u_\alpha^-), f(u_\alpha^+)\} = f(u_\alpha^+) \end{cases}$$

$$p_\alpha^+ = \begin{cases} - & \text{if } \max\{f(u_\alpha^-), f(u_\alpha^+)\} = f(u_\alpha^-) \\ + & \text{if } \max\{f(u_\alpha^-), f(u_\alpha^+)\} = f(u_\alpha^+) \end{cases}$$

We simplify $p_{\alpha_i}^\pm \equiv p_i^\pm$, $i = 0, 1, \dots, N$, in the points of the α -decomposition. So, we have $f(u)_\alpha^- = f(u_\alpha^{p_\alpha^-})$ and $f(u)_\alpha^+ = f(u_\alpha^{p_\alpha^+})$. If X is the LU-fuzzy number $X = (x_i^-, D_i^-, x_i^+, D_i^+)_{i=0,1,\dots,N}$ then its image $f(X)$ is

$$f(X) = (\alpha_i, f(x_i^{p_i^-}), f'(x_i^{p_i^-})D_i^{p_i^-}, f(x_i^{p_i^+}), f'(x_i^{p_i^+})D_i^{p_i^+})_{i=0,1,\dots,N}.$$

The case of the fuzzy extension of a nonmonotonic function is handled in a similar way. In this case, the points where the minimum and the maximum

values (22) are taken, can be internal to the interval u_α or coincident with one of the extremal values. In the last case, the α -cut of the extension is obtained as described above. Let

$$f(X) = (\alpha_i, f_i^-, \delta f_i^-, f_i^+, \delta f_i^+)_{i=0,1,\dots,N}$$

denote the fuzzy extension in the general case and suppose that the function f is differentiable. If the minimum value, defining the lower branch $f(u)_i^-$, is taken at an internal point $x_i^- \in]u_i^-, u_i^+]$, then

$$f_i^- = f(x_i^-) \quad \text{and} \quad \delta f_i^- = 0;$$

if the maximum value, defining the upper branch $f(u)_i^+$, is taken at an internal point $x_i^+ \in]u_i^-, u_i^+]$, then

$$f_i^+ = f(x_i^+) \quad \text{and} \quad \delta f_i^+ = 0.$$

The details for the fuzzy extensions of the basic elementary functions and some special functions of particular interest are given in the rest of this section.

In some computations, we require a big constant to represent infinity (positive or negative): we denote it as BIG and in our computations we use BIG=1000.

4.1 Fuzzy extension of $X \rightarrow X^n$

(the input are the fuzzy X and a positive integer $n \geq 2$).

Denote

$$Y = X^n$$

within the LU-representation framework

$$Y = (\alpha_i, Y_i^-, \delta Y_i^-, Y_i^+, \delta Y_i^+)_{i=0,1,\dots,N}$$

case 1. $n = 3, 5, \dots$ (odd)

For each $i = 0, 1, \dots, N$

$$\begin{cases} Y_i^- = (X_i^-)^n \\ \delta Y_i^- = n (X_i^-)^{n-1} \delta X_i^- \\ Y_i^+ = (X_i^+)^n \\ \delta Y_i^+ = n (X_i^+)^{n-1} \delta X_i^+ \end{cases}$$

case 2. $n = 2, 4, \dots$ (even)
 For each $i = 0, 1, \dots, N$

$$\left\{ \begin{array}{l} \text{If } x_i^- \geq 0 \text{ then } \left\{ \begin{array}{l} Y_i^- = (X_i^-)^n \\ Y_i^+ = (X_i^+)^n \\ \delta Y_i^- = n (X_i^-)^{n-1} \delta X_i^- \\ \delta Y_i^+ = n (X_i^+)^{n-1} \delta X_i^+ \end{array} \right. \\ \\ \text{else if } x_i^+ \leq 0 \text{ then } \left\{ \begin{array}{l} Y_i^- = (X_i^+)^n \\ Y_i^+ = (X_i^-)^n \\ \delta Y_i^- = n (X_i^+)^{n-1} \delta X_i^+ \\ \delta Y_i^+ = n (X_i^-)^{n-1} \delta X_i^- \end{array} \right. \\ \\ \text{else } \left\{ \begin{array}{l} Y_i^- = 0 \\ \delta Y_i^- = 0 \\ \text{if } |x_i^-| \leq |x_i^+| \left\{ \begin{array}{l} \text{then } \left\{ \begin{array}{l} Y_i^+ = (X_i^+)^n \\ \delta Y_i^+ = n (X_i^+)^{n-1} \delta X_i^+ \end{array} \right. \\ \\ \text{else } \left\{ \begin{array}{l} Y_i^+ = (X_i^-)^n \\ \delta Y_i^+ = n (X_i^-)^{n-1} \delta X_i^- \end{array} \right. \end{array} \right. \end{array} \right. \end{array} \right.$$

4.2 Fuzzy extension of $X \rightarrow X^{-n} = \frac{1}{X^n}$

(the input are the non-zero fuzzy X and a positive integer $n \geq 1$). Validity test can be stated as follows: X is non-zero if either $X_0^- > 0$ or $X_0^+ < 0$; in other words it means that $0 \notin [x_0^-, x_0^+]$

Denote

$$Y = (\alpha_i, Y_i^-, \delta Y_i^-, Y_i^+, \delta Y_i^+)_{i=0,1,\dots,N}$$

with

$$Y = X^{-n}$$

case 1. $n = 1, 3, 5, \dots$ (odd)
 For each $i = 0, 1, \dots, N$

$$\left\{ \begin{array}{l} Y_i^- = \frac{1}{(X_i^+)^n} \\ \delta Y_i^- = -\frac{n}{(X_i^+)^{n+1}} \delta X_i^+ \\ Y_i^+ = \frac{1}{(X_i^-)^n} \\ \delta Y_i^+ = -\frac{n}{(X_i^-)^{n+1}} \delta X_i^- \end{array} \right.$$

case 2. $n = 2, 4, \dots$ (even)
 For each $i = 0, 1, \dots, N$

$$\left\{ \begin{array}{l} \text{If } x_i^- > 0 \text{ then} \\ \text{If } x_i^+ < 0 \text{ then} \end{array} \right. \left\{ \begin{array}{l} Y_i^- = \frac{1}{(X_i^+)^n} \\ Y_i^+ = \frac{1}{(X_i^-)^n} \\ \delta Y_i^- = -\frac{n}{(X_i^+)^{n+1}} \delta X_i^+ \\ \delta Y_i^+ = -\frac{n}{(X_i^-)^{n+1}} \delta X_i^- \\ Y_i^- = \frac{1}{(X_i^-)^n} \\ Y_i^+ = \frac{1}{(X_i^+)^n} \\ \delta Y_i^- = -\frac{n}{(X_i^-)^{n+1}} \delta X_i^- \\ \delta Y_i^+ = -\frac{n}{(X_i^+)^{n+1}} \delta X_i^+ \end{array} \right.$$

4.3 Fuzzy extension of $X \rightarrow \exp(X)$

(the input is the fuzzy X).

Validity test can be stated as follows: X is non-zero if either $X_0^- > 0$ or $X_0^+ < 0$; in other words it means that $0 \notin [x_0^-, x_0^+]$. Let

$$Y = \exp(X)$$

For each $i = 0, 1, \dots, N$

$$\left\{ \begin{array}{l} Y_i^- = \exp(X_i^-) \\ Y_i^+ = \exp(X_i^+) \\ \delta Y_i^- = (\exp(X_i^-)) (\delta X_i^-) \\ \delta Y_i^+ = (\exp(X_i^+)) (\delta X_i^+) \end{array} \right.$$

4.4 Fuzzy extension of $X \rightarrow \ln(X)$

(the input is the positive fuzzy X). Validity test: $X_0^- > 0$. Let

$$Y = \ln(X)$$

For each $i = 0, 1, \dots, N$

$$\left\{ \begin{array}{l} Y_i^- = \ln(X_i^-) \\ Y_i^+ = \ln(X_i^+) \\ \delta Y_i^- = \frac{\delta X_i^-}{X_i^-} \\ \delta Y_i^+ = \frac{\delta X_i^+}{X_i^+} \end{array} \right.$$

4.5 Fuzzy extension of $(X, Y) \rightarrow X^Y$

(the input are the positive fuzzy X and the fuzzy Y).

We use the definition

$$X^Y = \exp(Y \ln((X)))$$

and we compute

$$Z = X^Y$$

by the sequence of operations:

$$\begin{array}{ll} (i) \text{ (natural logarithm)} & Z \longleftarrow \ln(X) \\ (ii) \text{ (standard multiplication)} & Z \longleftarrow YZ \\ (iii) \text{ (exponential)} & Z \longleftarrow \exp(Z) \end{array}$$

4.6 Fuzzy extension of $X \rightarrow \sin(X)$

on the invertibility domain (the input is the fuzzy X with support in $[-\frac{\pi}{2}, \frac{\pi}{2}]$).

Validity test: $-\frac{\pi}{2} \leq X_0^- \leq X_0^+ \leq \frac{\pi}{2}$

Let

$$Y = \sin(X)$$

For each $i = 0, 1, \dots, N$

$$\begin{cases} Y_i^- = \sin(X_i^-) \\ Y_i^+ = \sin(X_i^+) \\ \delta Y_i^- = \cos(X_i^-) \delta X_i^- \\ \delta Y_i^+ = \cos(X_i^+) \delta X_i^+ \end{cases}$$

4.7 Fuzzy extension of $X \rightarrow \arcsin(X)$

(the input is the fuzzy X with support in $[-1, 1]$).

Validity test: $-1 \leq X_0^- \leq X_0^+ \leq 1$

Let

$$Y = \arcsin(X)$$

For each $i = 0, 1, \dots, N$, (if necessary, use the constant BIG)

$$\begin{cases} Y_i^- = \arcsin(X_i^-) \\ Y_i^+ = \arcsin(X_i^+) \\ \delta Y_i^- = \begin{cases} +\text{BIG} & \text{if } X_i^- = -1 \text{ or } X_i^- = +1 \\ \frac{1}{\sqrt{1-X_i^-}} \delta X_i^- & \text{if } -1 < X_i^- < 1 \end{cases} \\ \delta Y_i^+ = \begin{cases} -\text{BIG} & \text{if } X_i^+ = -1 \text{ or } X_i^+ = +1 \\ \frac{1}{\sqrt{1-X_i^+}} \delta X_i^+ & \text{if } -1 < X_i^+ < 1 \end{cases} \end{cases}$$

4.8 Fuzzy extension of $X \rightarrow \cos(X)$

on the invertibility domain (the input is the fuzzy X with support in $[0, \pi]$).

Validity test: $0 \leq X_0^- \leq X_0^+ \leq \pi$

Let

$$Y = \cos(X)$$

For each $i = 0, 1, \dots, N$

$$\begin{cases} Y_i^- = \cos(X_i^+) \\ Y_i^+ = \cos(X_i^-) \\ \delta Y_i^- = -\sin(X_i^+) \delta X_i^+ \\ \delta Y_i^+ = -\sin(X_i^-) \delta X_i^- \end{cases}$$

4.9 Fuzzy extension of $X \rightarrow \arccos(X)$

(the input is the fuzzy X with support in $[-1, 1]$).

Validity test: $-1 \leq X_0^- \leq X_0^+ \leq 1$

Let

$$Y = \arccos(X)$$

For each $i = 0, 1, \dots, N$,

$$\begin{cases} Y_i^- = \arccos(X_i^+) \\ Y_i^+ = \arccos(X_i^-) \\ \delta Y_i^- = \begin{cases} \text{BIG} & \text{if } X_i^+ = -1 \text{ or } X_i^+ = +1 \\ \frac{-\delta X_i^+}{\sqrt{1-X_i^+}} & \text{if } -1 < X_i^+ < 1 \end{cases} \\ \delta Y_i^+ = \begin{cases} -\text{BIG} & \text{if } X_i^- = -1 \text{ or } X_i^- = +1 \\ \frac{-\delta X_i^-}{\sqrt{1-X_i^-}} & \text{if } -1 < X_i^- < 1 \end{cases} \end{cases}$$

4.10 Fuzzy extension of $X \rightarrow \tan(X)$

(the input is the fuzzy X with support in $]-\frac{\pi}{2}, \frac{\pi}{2}[$).

Validity test: $-\frac{\pi}{2} < X_0^- \leq X_0^+ < \frac{\pi}{2}$

Let

$$Y = \tan(X)$$

For each $i = 0, 1, \dots, N$

$$\begin{cases} Y_i^- = \tan(X_i^-) \\ Y_i^+ = \tan(X_i^+) \\ \delta Y_i^- = (1 + \tan^2(X_i^-)) \delta X_i^- \\ \delta Y_i^+ = (1 + \tan^2(X_i^+)) \delta X_i^+ \end{cases}$$

4.11 Fuzzy extension of $X \rightarrow \arctan(X)$

(the input is the fuzzy X with support in \mathbb{R}).

Let

$$Y = \arctan(X)$$

For each $i = 0, 1, \dots, N$,

$$\begin{cases} Y_i^- = \arctan(X_i^-) \\ Y_i^+ = \arctan(X_i^+) \\ \delta Y_i^- = \frac{\delta X_i^-}{1+(X_i^-)^2} \\ \delta Y_i^+ = \frac{\delta X_i^+}{1+(X_i^+)^2} \end{cases}$$

4.12 Fuzzy extension of $X \rightarrow \sqrt{X}$

(the input is the non negative fuzzy X).

Validity test: $X_0^- \geq 0$

Let

$$Y = \sqrt{X}$$

For each $i = 0, 1, \dots, N$

$$\begin{cases} Y_i^- = \sqrt{X_i^-} \\ Y_i^+ = \sqrt{X_i^+} \\ \delta Y_i^- = \frac{\delta X_i^-}{2\sqrt{X_i^-}} = \begin{cases} \text{BIG} & \text{if } X_i^- = 0 \\ \frac{\delta X_i^-}{2Y_i^-} & \text{if } X_i^- > 0 \end{cases} \\ \delta Y_i^+ = \frac{\delta X_i^+}{2\sqrt{X_i^+}} = \begin{cases} \text{BIG} & \text{if } X_i^+ = 0 \\ \frac{\delta X_i^+}{2Y_i^+} & \text{if } X_i^+ > 0 \end{cases} \end{cases}$$

4.13 Fuzzy extension of $X \rightarrow X^\alpha$

(the input are the positive fuzzy X and the real α).

We use the definition

$$X^\alpha = \exp(\alpha \ln(X))$$

and we compute

$$Y = X^\alpha$$

by the sequence of three steps:

$$\begin{array}{lll} (i) \text{ (logarithm)} & Y & \leftarrow \ln(X) \\ (ii) \text{ (scalar multiplication)} & Y & \leftarrow \alpha Y \\ (iii) \text{ (exponential)} & Y & \leftarrow \exp(Y) \end{array}$$

4.14 Fuzzy extension of $X \rightarrow e^{-X^2}$

(the input is the fuzzy X).

The computation

$$Y = \exp(-X^2)$$

can be done by following three steps:

$$\begin{array}{lll}
\text{(square power function)} & (i) Y & = X^2 \\
\text{(opposite)} & (ii) Y & \leftarrow -Y \\
\text{(exponential)} & (iii) Y & \leftarrow \exp(Y)
\end{array}$$

4.15 Fuzzy extension of the Gauss standard function

$$X \rightarrow \frac{1}{\sqrt{2\pi}} e^{-\frac{X^2}{2}}$$

(the input is the fuzzy X).

The computation can be done by following four steps:

$$\begin{array}{lll}
\text{(square power function)} & (i) Y & = X^2 \\
\text{(negative scalar multiplication)} & (ii) Y & \leftarrow -\frac{1}{2}Y \\
\text{(exponential)} & (iii) Y & \leftarrow \exp(Y) \\
\text{(scalar multiplication)} & (iv) Y & \leftarrow \frac{1}{\sqrt{2\pi}}Y
\end{array}$$

4.16 Fuzzy extension of the hyperbolic sinusoidal function

$$X \rightarrow \sinh(X)$$

(the input is the fuzzy X).

Let

$$Y = \sinh(X) = \frac{e^X - e^{-X}}{2}$$

For each $i = 0, 1, \dots, N$

$$\begin{cases}
Y_i^- = \left(e^{X_i^-} - e^{-X_i^-} \right) / 2 \\
Y_i^+ = \left(e^{X_i^+} - e^{-X_i^+} \right) / 2 \\
\delta Y_i^- = \delta X_i^- \left(e^{X_i^-} - e^{-X_i^-} \right) / 2 \\
\delta Y_i^+ = \delta X_i^+ \left(e^{X_i^+} - e^{-X_i^+} \right) / 2
\end{cases}$$

4.17 Fuzzy extension of the hyperbolic cosinusoidal function

$$X \rightarrow \cosh(X)$$

(the input is the fuzzy X).

Let

$$Y = \cosh(X) = \frac{e^X + e^{-X}}{2}$$

For each $i = 0, 1, \dots, N$:

$$\begin{array}{l}
\text{if } X_i^+ \leq 0 \text{ then } \left\{ \begin{array}{l} Y_i^- = \cosh(X_i^+) \\ Y_i^+ = \cosh(X_i^-) \\ \delta Y_i^- = \delta X_i^+ \sinh(X_i^+) \\ \delta Y_i^+ = \delta X_i^- \sinh(X_i^-) \end{array} \right. \\
\text{else if } X_i^+ \leq 0 \text{ then } \left\{ \begin{array}{l} Y_i^- = \cosh(X_i^-) \\ Y_i^+ = \cosh(X_i^+) \\ \delta Y_i^- = \delta X_i^- \sinh(X_i^-) \\ \delta Y_i^+ = \delta X_i^+ \sinh(X_i^+) \end{array} \right. \\
\text{else } \left\{ \begin{array}{l} Y_i^- = 1, \delta Y_i^- = 0 \\ \text{if } \text{abs}(X_i^-) \geq \text{abs}(X_i^+) \text{ then } \left\{ \begin{array}{l} Y_i^+ = \cosh(X_i^-) \\ \delta Y_i^+ = \delta X_i^- \sinh(X_i^-) \end{array} \right. \\ \text{else } \left\{ \begin{array}{l} Y_i^+ = \cosh(X_i^+) \\ \delta Y_i^+ = \delta X_i^+ \sinh(X_i^+) \end{array} \right. \end{array} \right.
\end{array}$$

4.18 Fuzzy extension of the hyperbolic tangentoid function

$X \rightarrow \tanh(X)$
(the input is the fuzzy X).

Let

$$\begin{aligned}
Y &= \tanh(X) \\
&\text{and} \\
\tanh(x) &= \frac{\sinh(x)}{\cosh(x)} = \frac{e^x - e^{-x}}{e^x + e^{-x}}
\end{aligned}$$

For each $i = 0, 1, \dots, N$

$$\left\{ \begin{array}{l} Y_i^- = \tanh(X_i^-) \\ Y_i^+ = \tanh(X_i^+) \\ \delta Y_i^- = \delta X_i^- / \cosh^2(X_i^-) \\ \delta Y_i^+ = \delta X_i^+ / \cosh^2(X_i^+) \end{array} \right.$$

4.19 Fuzzy extension of the inverse hyperbolic sinusoidal function

$X \rightarrow \sinh^{-1}(X)$.
(the input is the fuzzy X).

Note that

$$\begin{aligned}
\sinh^{-1}(x) &= \ln(x + \sqrt{1 + x^2}) \\
\frac{d}{dx} \sinh^{-1}(x) &= \frac{1}{\sqrt{1 + x^2}}
\end{aligned}$$

For each $i = 0, 1, \dots, N$

$$\begin{cases} Y_i^- = \sinh^{-1}(X_i^-) \\ Y_i^+ = \sinh^{-1}(X_i^+) \\ \delta Y_i^- = \delta X_i^- / \sqrt{1 + (X_i^-)^2} \\ \delta Y_i^+ = \delta X_i^+ / \sqrt{1 + (X_i^+)^2} \end{cases}$$

4.20 Fuzzy extension of the inverse hyperbolic cosinusoidal function

$X \rightarrow \cosh^{-1}(X)$.

(the input is the fuzzy x , with support ≥ 1 and the validity test is $x_0 \geq 1$).

Note that

$$\begin{aligned} \cosh^{-1}(x) &= \ln(x + \sqrt{x^2 - 1}), \quad x \geq 1 \\ \frac{d}{dx} \cosh^{-1}(x) &= \frac{1}{\sqrt{x^2 - 1}} \end{aligned}$$

For each $i = 0, 1, \dots, N$

$$\begin{cases} Y_i^- = \cosh^{-1}(X_i^-) \\ Y_i^+ = \cosh^{-1}(X_i^+) \\ \delta Y_i^- = \delta X_i^- / \sqrt{(X_i^-)^2 - 1} \quad (\text{if } X_i^- = 1 \text{ then } \delta Y_i^- = \text{BIG}) \\ \delta Y_i^+ = \delta X_i^+ / \sqrt{(X_i^+)^2 - 1} \quad (\text{if } X_i^+ = 1 \text{ then } \delta Y_i^+ = -\text{BIG}) \end{cases}$$

4.21 Fuzzy extension of the inverse hyperbolic tangentoid function

$X \rightarrow \tanh^{-1}(X)$.

(the input is the fuzzy x , with support in $] -1, 1[$ and the validity test is $-1 < x_0^- \leq x_0^+ < 1$).

Note that

$$\begin{aligned} \tanh^{-1}(x) &= \frac{1}{2} \ln\left(\frac{1+x}{1-x}\right), \quad x \in] -1, 1[\\ \frac{d}{dx} \tanh^{-1}(x) &= \frac{1}{1-x^2} \end{aligned}$$

For each $i = 0, 1, \dots, N$

$$\begin{cases} Y_i^- = \tan^{-1}(X_i^-) \\ Y_i^+ = \tan^{-1}(X_i^+) \\ \delta Y_i^- = \delta X_i^- / (1 - (X_i^-)^2) \\ \delta Y_i^+ = \delta X_i^+ / (1 - (X_i^+)^2) \end{cases}$$

4.22 Fuzzy extension of the *erf* and *erfc* error functions

$$\begin{aligned}
 x \rightarrow \operatorname{erf}(x) &= \frac{2}{\sqrt{\pi}} \int_0^x \exp(-t^2) dt = && \text{(increasing)} \\
 &= 1 - \frac{2}{\sqrt{\pi}} \int_x^{+\infty} \exp(-t^2) dt = \\
 &= 1 - \operatorname{erfc}(x)
 \end{aligned}$$

with

$$\operatorname{erfc}(x) = \frac{2}{\pi} \int_x^{+\infty} \exp(-t^2) dt \quad \text{(decreasing)}$$

For the *erfc* function we use the following approximation, with a fractional error less than 1.2×10^{-7}

$$\begin{aligned}
 z &= \operatorname{abs}(x) \\
 t &= \frac{1}{1 + \frac{1}{2}z}
 \end{aligned}$$

and

$$\operatorname{erfc} = \begin{cases} t \exp(-z^2 + p(t)) & \text{if } x \geq 0 \\ 2 - t \exp(-z^2 + p(t)) & \text{if } x < 0 \end{cases}$$

where (Korner rule)

$$p(t) = a_0 + t(a_1 + t(a_2 + t(a_3 + t(a_4 + t(a_5 + t(a_6 + t(a_7 + t(a_8 + ta_9))))))))))$$

and the coefficients assume the following values

$$\begin{aligned}
 a_0 &= -1.26551223 & a_5 &= 0.27886807 \\
 a_1 &= 1.00002368 & a_6 &= -1.13520398 \\
 a_2 &= 0.37409196 & a_7 &= 1.48851587 \\
 a_3 &= 0.09678418 & a_8 &= -0.82215223 \\
 a_4 &= -0.18628806 & a_9 &= 0.17087277
 \end{aligned}$$

Let

$$Y = \operatorname{erf}(X)$$

For each $i = 0, 1, \dots, N$

$$\begin{cases} Y_i^- = \operatorname{erf}(X_i^-) \\ Y_i^+ = \operatorname{erf}(X_i^+) \\ \delta Y_i^- = \delta X_i^- \frac{2}{\sqrt{\pi}} \exp(-X_i^-)^2 \\ \delta Y_i^+ = \delta X_i^+ \frac{2}{\sqrt{\pi}} \exp(-X_i^+)^2 \end{cases}$$

Let now

$$Y = \text{erf } c(X)$$

For each $i = 0, 1, \dots, N$

$$\begin{cases} Y_i^- = \text{erf } c(X_i^+) \\ Y_i^+ = \text{erf } c(X_i^-) \\ \delta Y_i^- = -\delta X_i^+ \frac{2}{\sqrt{\pi}} \exp(-X_i^+)^2 \\ \delta Y_i^+ = -\delta X_i^- \frac{2}{\sqrt{\pi}} \exp(-X_i^-)^2 \end{cases}$$

4.23 Fuzzy extension of the cumulative standard normal distribution function

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x \exp\left(-\frac{t^2}{2}\right) dt$$

The function $\Phi(x)$, $x \in \mathbb{R}$, can be approximated by the following procedure:

$$\Phi(x) = \begin{cases} \frac{1}{2} \left(1 + \text{erf}\left(\frac{x}{\sqrt{2}}\right)\right) & \text{if } x \geq 0 \\ \frac{1}{2} \left(1 - \text{erf}\left(-\frac{x}{\sqrt{2}}\right)\right) & \text{if } x < 0 \end{cases}$$

Let

$$Y = \Phi(X) \quad \text{with } X \text{ fuzzy}$$

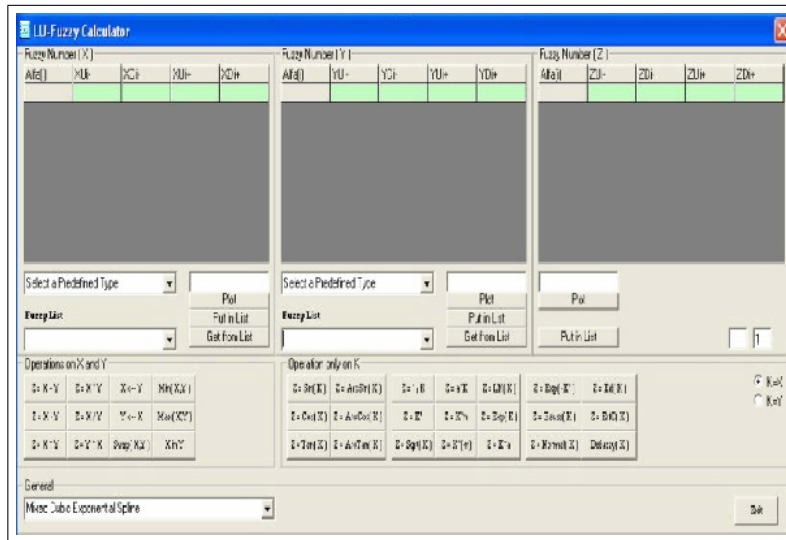
For each $i = 0, 1, \dots, N$

$$\begin{cases} Y_i^- = \Phi(X_i^-) \\ Y_i^+ = \Phi(X_i^+) \\ \delta Y_i^- = \delta X_i^- \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-X_i^-^2}{2}\right) \\ \delta Y_i^+ = \delta X_i^+ \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-X_i^+^2}{2}\right) \end{cases}$$

5 Implementation of the LU-fuzzy calculator

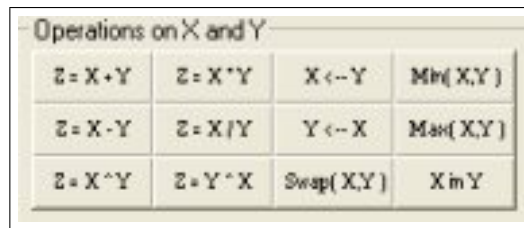
To implement the LU-fuzzy calculator, we have written a windows-based frame similar to a standard hand-calculator.

Figure 2 shows a panoramic view of the calculator; from left to right we can see the grids of the fuzzy numbers X , Y and Z . Z is the result of the operations while X and/or Y are the operands. For each element $u \in \{X, Y, Z\}$ the grid contains the values α_i , u_i^- , δu_i^- , u_i^+ and δu_i^+ respectively. To start the calculations, we have implemented a set of predefined types, including triangular, trapezoidal, exponential, gamma, ect. For a given type, it is possible to define the number N of subintervals ($N + 1$ points) in the uniform α -decomposition.



General window of the LU-fuzzy calculator

The calculations are performed by clicking the button of the corresponding operation. The left group of buttons involves the binary operations (see figure 3.)



Binary operations and assignments

The second group of operators (see figure 4.) require the assignment of either X or Y to the temporary K (see figure 5) and operate on K itself putting the result into Z.

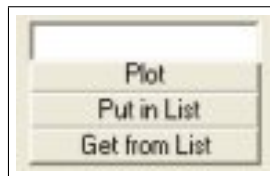


Extension of univariate functions



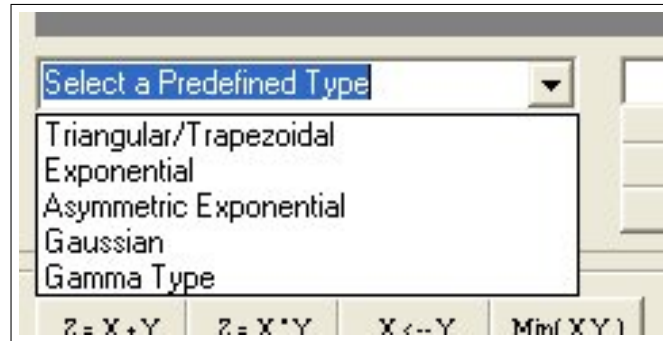
Selection of the argument for fuzzy extension functions

It is possible to save a given (X, Y or Z) temporary result into a stored list (Put in List button), by assigning a name to it; a saved fuzzy number can be reloaded either in X or Y for further use (Get from List button). The Plot button (see figure 6.) opens a popup window with the graph of the membership function of the corresponding fuzzy number.



Buttons to Plot, Put into List and Get from saved List

Figure 7. illustrates how to select a fuzzy number from a list of predefined types.



Selection of predefined types of fuzzy numbers

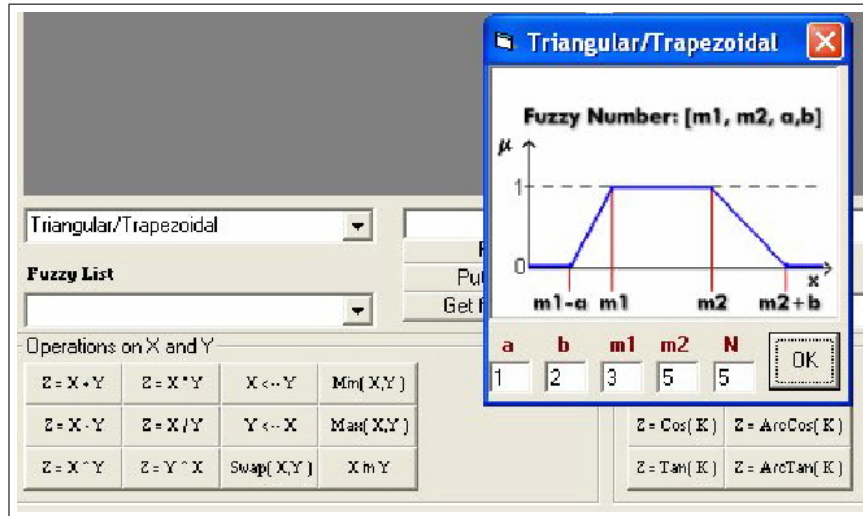
To obtain the graphs or other representations, one of the models described in section 3. can be selected (figure 8.).



Choosing the monotonic spline model

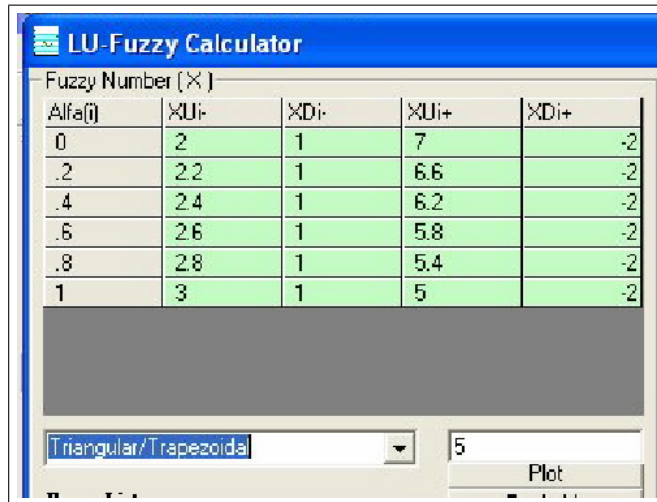
We illustrate an example to show how the calculator works.

1. First (see figure 9.) select a trapezoidal fuzzy number and set to 5 the number of subintervals in the α -decomposition (the higher N the higher the precision in the calculations); the maximal value of N is 100; typical values are 2, 5 or 10.



Example for trapezoidal fuzzy number

If the selection is loaded into the X-area, the corresponding grid appears as in figure 10.



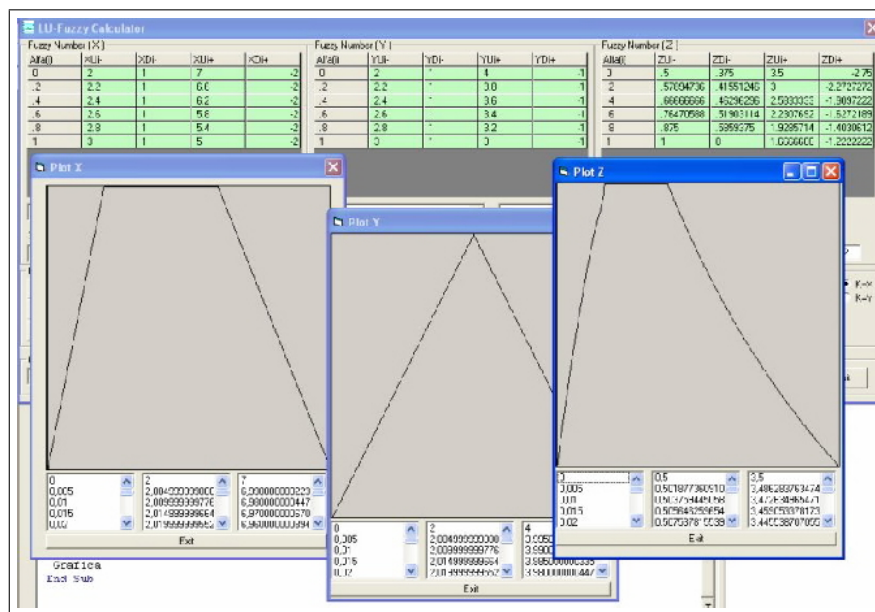
Assigne the trapezoidal fuzzy number to X

2. A second fuzzy number is loaded into Y and the button corresponding to the operation $Z=X/Y$ is activated. The Z-grid is calculated by the rules of the LU-fuzzy calculus (see figure 11.).

Fuzzy Number (X)					Fuzzy Number (Y)					Fuzzy Number (Z)				
A(x)	XL-	x(0)	XJ+	X(+)	A(y)	YL-	y(0)	YJ+	Y(+)	A(z)	ZL-	z(0)	ZJ+	Z(+)
0	2	1	7	0	0	2	1	4	0	0	5	3.75	3.5	2.75
.2	2.2	1	6.6	0	.2	2.2	1	3.9	0	.2	.5704736	.4151246	3	2.272272
.4	2.4	1	6.2	0	.4	2.4	1	3.6	0	.4	.6666666	.46296296	2.533333	-1.307222
.6	2.6	1	5.6	0	.6	2.6	1	3.4	0	.6	.76470588	.5193714	2.230769	-1.527288
.8	2.8	1	5.4	0	.8	2.8	1	3.2	0	.8	.875	.589275	1.928714	-1.403612
1	3	1	5	0	1	3	1	3	0	1	1	0	1.636666	-1.222222

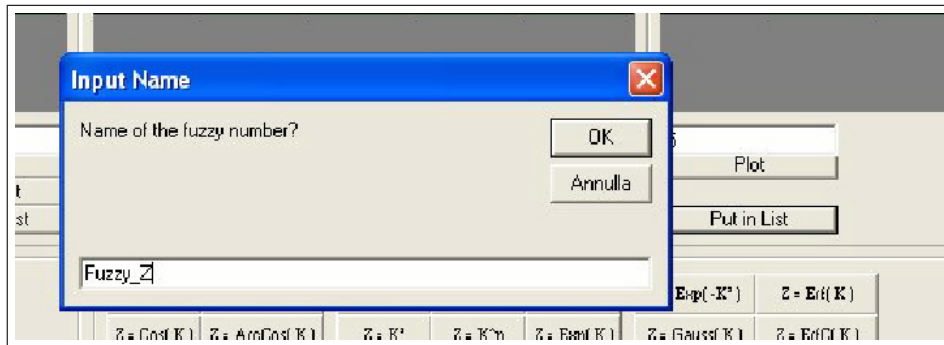
An example of a fuzzy division: $Z=X/Y$

3. To see the graphical representation of X, Y and/or Z, click the corresponding Plot button and the popup windows appear (figure 12.).



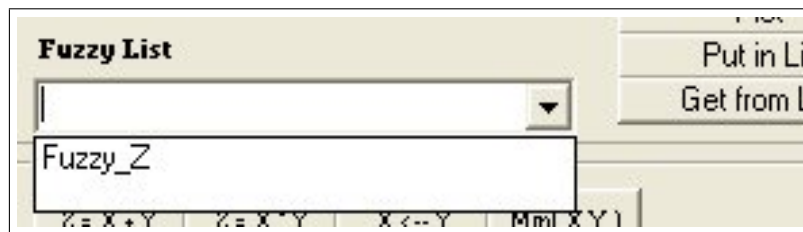
Plotting the operands and the result of an operation

4. Now, we save the result Z of the previous operation and we call Fuzzy_Z its name (figure 13.).

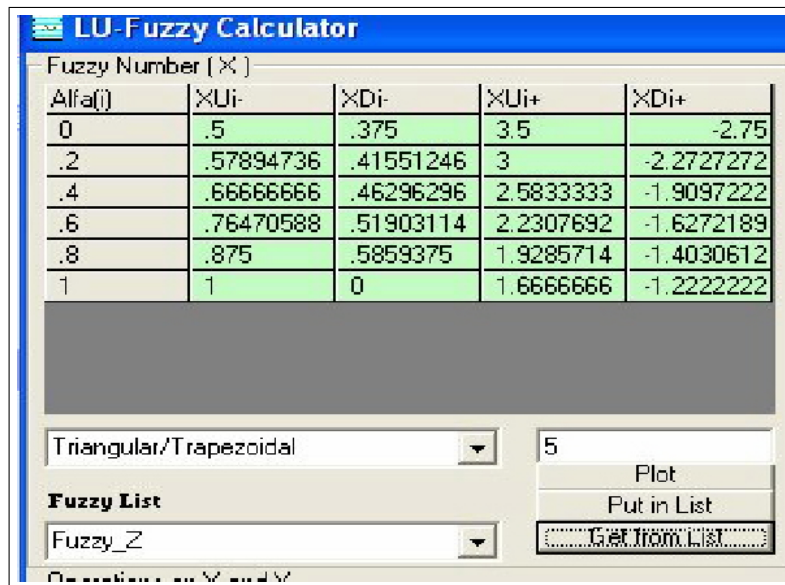


Saving of an intermediate result for further use

5. Now we load the saved Fuzzy_Z into the X-area, by getting it from the list of saved elements (figures 14. and 15.).

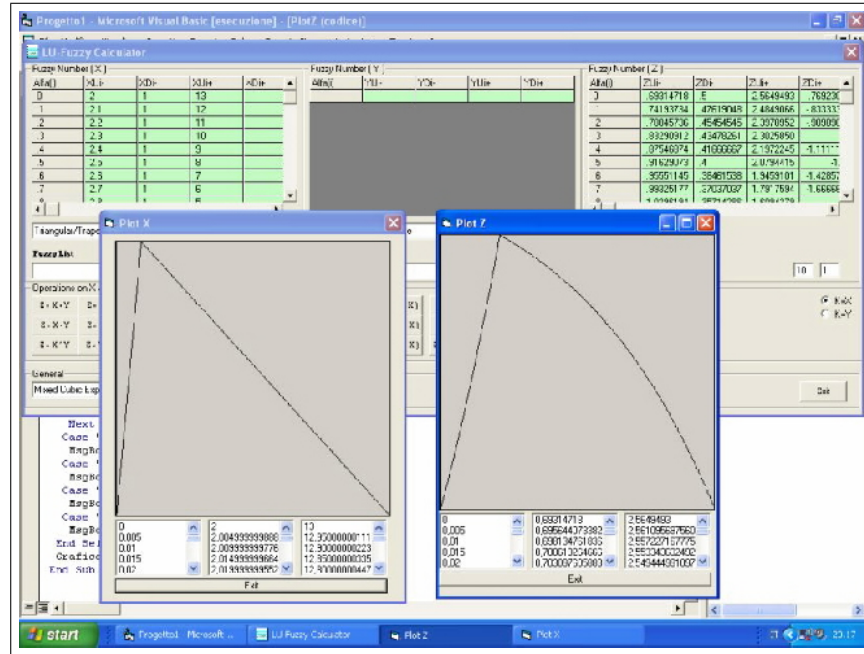


Loading of a saved intermediate fuzzy number into X area



Result of loading a saved fuzzy number into the X area

6. If we try to apply the fuzzy extension of the log function to a fuzzy number X, we first select the assignment K=X and then we click on the Z=log(K) button and on the Plot buttons of X and Z (figure 16.).



Calculating and plotting X and Z=log(X)

6 Example of Use: Black-Sholes option pricing

In this final section we illustrate the use of the calculator for the fuzzification of Black-Sholes formula in the valuation of a european put or call option.

Standard Black Scholes (B-S) formula for the European put/call options (without dividends)

- S_t : (current) stock price at time $t \in [0, T]$ FUZZY
 - T : time to maturity CRISP
 - K : exercise price (strike price) FUZZY
 - r : interest rate (continuously compounded), FUZZY
 - σ : (standard deviation) volatility, FUZZY
- then

FUZZY price C_t of (European) call option at time t is given by:

$$C_t = S_t \Phi(D_1(S_t, K, r, \sigma, T - t)) - K e^{-r(T-t)} \Phi(D_2(S_t, K, r, \sigma, T - t))$$

FUZZY price P_t of put option at time t (with the same expiry date T and strike price K) is given by:

$$P_t = S_t [\Phi(D_1(S_t, K, r, \sigma, T - t)) - 1] - K e^{-r(T-t)} [\Phi(D_2(S_t, K, r, \sigma, T - t)) - 1].$$

where

$\Phi(x)$: cumulative (increasing) standard normal function and

$$D_1 = \frac{\ln\left(\frac{S}{K}\right) + rt}{\sigma\sqrt{t}} + \frac{\sigma}{2}\sqrt{t},$$

$$D_2 = \frac{\ln\left(\frac{S}{K}\right) + rt}{\sigma\sqrt{t}} - \frac{\sigma}{2}\sqrt{t}.$$

Here, we apply exact extension principle for the calculation of D_1 and D_2 by extending the two dimensional functions $(x, y) \rightarrow \frac{x}{y} \pm \frac{y}{2}$ where $x = \ln\left(\frac{S}{K}\right) + rt$ and $y = \sigma\sqrt{t}$.

The probability density function for the option value C_t is given by (log-normal type):

$$D(C_t) = \frac{a}{2\pi} e^{-\frac{1}{2}b^2} \text{ where}$$

$$a = \frac{e^{r(T-t)}}{\sigma(T-t)(C_t e^{r(T-t)} + S_t)} \text{ and}$$

$$b = \frac{\ln\left(\frac{C_t e^{r(T-t)} + k}{S}\right) - \left(r - \frac{\sigma^2}{2}\right)(T-t)}{\sqrt{\sigma(T-t)}}.$$

Also the fuzzy versions of the GREEKS can be easily calculated.

The results obtained by the LU-fuzzy decomposition with $N = 10$ are exact up to 8 decimal places.

The data are from H.-C. WU, Computers and Operations Research, 2004.

$K = 30$, $T = 0.25$, $t = 0$ and the fuzzy numbers are all triangular and symmetric:

$$S_0 = \langle 32, 33, 34 \rangle,$$

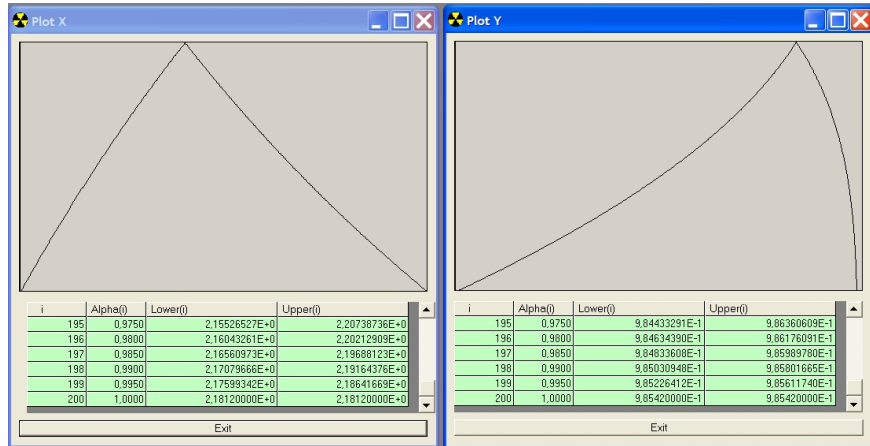
$$r = \langle 0.048, 0.05, 0.052 \rangle \text{ and}$$

$$\text{modified to } r_{LU} = (0.048, 0.1, 0.12, -0.002; 0.05, 0.005, 0.05, -0.01)$$

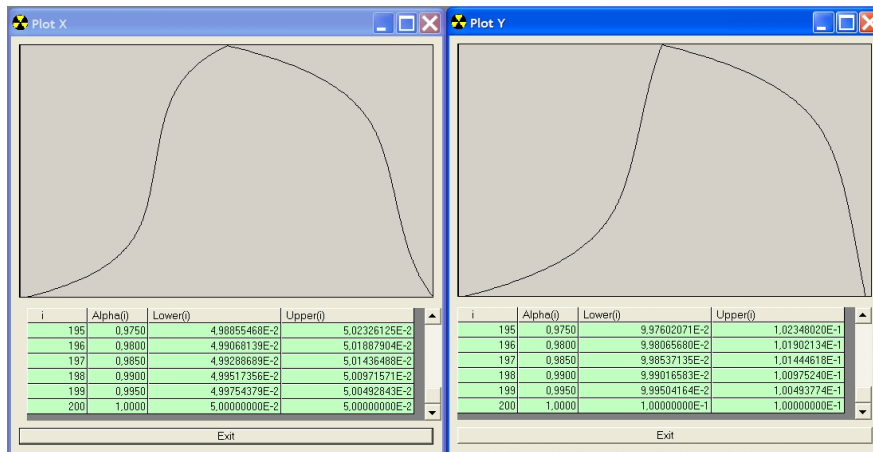
$$\sigma = \langle 0.08, 0.1, 0.12 \rangle \text{ and}$$

$$\text{modified to } \sigma_{LU} = (0.08, 0.1, 0.12, -0.005; 0.1, 0.01, 0.1, -0, 1)$$

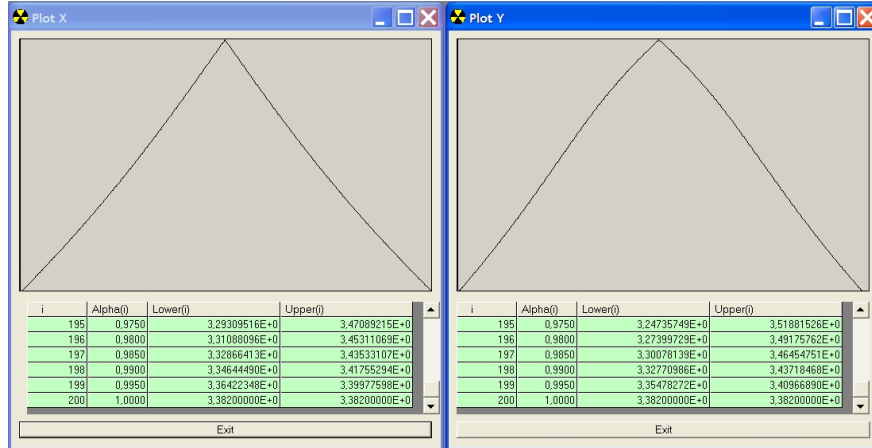
Example Plot $X = D_1$, Plot $Y = \Phi(D_1)$



Modified r and σ to LU: Plot $X = r_{LU}$, Plot $Y = \sigma_{LU}$



Plot $X = C_0$, Plot $Y = ModifiedC_0$



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