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A NEW MODEL FOR STOCK PRICE MOVEMENTS

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Abstract:

This paper presents a new alternative diffusion model for asset price movements. In contrast to the popular approach of Brownian Motion it proposes Deterministic Diffusion for the modelling of stock price movements. These diffusion processes are a new area of physical research and can be created by the chaotic behaviour of rather simple piecewise linear maps, but can also occur in chaotic deterministic systems like the famous Lorenz system. The motivation for the investigation on Deterministic Diffusion processes as suitable model for the behaviour of stock prices is, that their time series can obey mostly observed stylized facts of real world stock market time series. They can show fat tails of empirical log returns in union with timevarying volatility i.e. heteroscedasticity as well as slowly decaying autocorrelations of squared log returns i.e. long range dependence. These phenomena cannot be explained by a geometric Brownian Motion and have been the largest criticism to the lognormal random walk. In this paper it will be shown that Deterministic Diffusion models can obey those empirical observed stylized facts and the implications of these alternative diffusion processes on economic theory with respect to market efficiency and option pricing are discussed.

Keywords: Deterministic Diffusion, Stock Pricing, Fat Tails, Heteroscedasticity, Long Range Dependence, Option Pricing.

JEL Classiffication: G12

1. Introduction

Despite the popularity of normal distributed log returns to model the movement of stock prices there has been a large discussion in literature whether this model is appropriate or not. A large bulk of empirical research has been brought forward that mainly concludes that stock market returns are not normally and independently distributed and hence do not follow random walks. For an example see [1]. It is also a fact that so far no satisfying timeseries-model exists for the concurrent explanation of common stylized facts of stock market time series. That are: 1.) fat tails, 2.) heteroscedasticity and 3.) long range dependence. Even stochastic models cannot always accommodate all stylized facts at a time sufficiently. For an extensive discussion please refer to [2].

2. Deterministic Diffusion

Firstly Deterministic Diffusion will be defined and secondly simple piecewise linear maps will be presented that generate such time series.

Definition 2.1*(Deterministic Diffusion):*

Deterministic Diffusion is the displacement of a particle X on the real line in time according to a deterministic law:

$$X(t+1) = M(X(t)) \tag{1}$$

Where X(t) denotes the position of the particle at time t and M is a deterministic mapping. A mapping is called expanding if. M´> 1.

Call $\Im$ the family of piece wise linear maps M:
$\mathbb{R} \to \mathbb{R}$ with uniform slope s having the properties:

[1] Any views expressed are solely the views of the author and not those of the firm.

1. M is expanding: $s > 1$.
2. M is lifting: $M(X-n)+n=M(X)$ for any real number X and integer n with $n=\text{int}(X)$[2].
3. M is chaotic i.e. for its Lyapunov Exponent $\lambda = \ln(s)$ holds $\lambda > 0$.

In the context of modelling of economic price time series it is mandatory to be restricted only to diffusion processes with a non negative outcome. Therefore we shall specify another class $\Im_{>0}$ with the same properties like maps of $\Im$ with only positive values permitted.

Call $\Im_{>0}$ the family of piece wise linear maps $M_{>0}$: $\mathbb{R}^+ \rightarrow \mathbb{R}^+$ with uniform slope s having the properties:

1. There exists a map $M \in \Im$ with M>0=M if $\mu M,n > 0$ with $\mu M,n = \min(M([n-1,n]))$, $n \geq 1$.
2. If $\mu M,n < 0$ $M>0(X)= M-\mu M,n$.
3.

The example maps $S \subset \Im$ and $S_{>0} \subset \Im_{>0}$ to be considered in the following are the saw tooth map S(X) and the on $\mathbb{R}^{\cdot}$ restricted saw tooth map $S_{>0}(X)$.

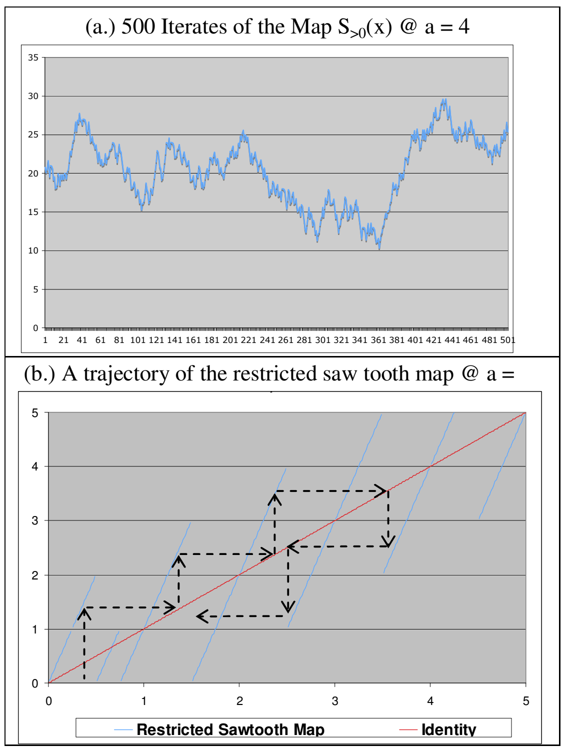


Figure 1. (a.) Iterates of the restricted Saw Tooth Map and (b.) a trajectory shown schematically.

S(X) is defined by:

$$S(X) = \begin{cases} s\left[x-\text{int}(x)\right]+\text{int}(x) \\ \quad if \left\{0 < x-\text{int}(x) < \frac{1}{2}\right\} \\ s\left[x-\text{int}(x)\right]-s+1+\text{int}(x) \\ if \ \left\{\frac{1}{2} < x-\text{int}(x) < 1\right\} \end{cases} \tag{2}$$

[2] n is the smallest integer smaller than X

with s > 2. Note that for $s \in [1,2]$ S is chaotic but its iterates do not leave the unit interval. To yield its analogon $S_{>0}(X)$ in $\Im_{>0}$, S(X) needs to be modified to require S(X) >> 0.

$$S_{>0}(X)=\begin{cases} s\left[x-\text{int}(x)\right]+\text{int}(x) \\ if\ \left\{0<x-\text{int}(x)<\frac{1}{2}\right\} \\ s\left[x-\text{int}(x)\right]-s+1+\text{int}(x) \\ if\left\{\left(\frac{1}{2}<x-\text{int}(x)<1\right)\cup\left(s\le 2+2\text{int}(x)\right)\right\} \\ s\left[x-\text{int}(x)\right]-\frac{1}{2}s \\ if\left\{\left(\frac{1}{2}<x-\text{int}(x)<1\right)\cup\left(s>2+2\text{int}(x)\right)\right\} \end{cases} \tag{3}$$

To illustrate how the dynamics of S and $S_{>0}$ work in Figure 1(a.) a trajectory of the map $S_{>0}$ is shown schematically and as time series. As one can see, the behaviour of the map already resembles the behaviour of stock market prices. There are crashes and booms and periods of only slight movements. A more detailed introduction to Deterministic Diffusion can be found in [3] and [4].

3. Chaotic Stock Pricing

One motivation for the choice of Deterministic Diffusion as model for stock price movements is that it can be reproduced in an extended model framework of Day and Huang [5] presented in the following. The original model did not allow the stock price to diffuse. Therefore it will be enriched to permit the prices to show diffusion. Consider three phenotypes of investors:

1.) α-investors

The so called sophisticated investors have the following demand function:

$$D_\alpha\left(p,p_F^\alpha,d,u\right)=\left(a\left[p_F^\alpha-p\right]\right)\Theta\left(p,d,u\right),a>0$$
$$\Theta\left(p,d,u\right)=\left(p-d+0.01\right)^{-\delta_1}\left(u-p+0.01\right)^{\delta_2} \tag{4}$$

Once the stock price is above the level they assume to be the fundamental price p_F^α according to some public information they want to sell the stock because they expect it to decline towards the fundamental value. On the other hand they buy the stock when it is cheaper than the fundamental price they assume. The strength of their reaction is determined by the parameter a > 0 and the chance that the stock price will fall or rise respectively if it is above or beneath p_F^α is expressed by the chance function Θ(p,d,u).

The chance of a price fall or rise will be judged by α-investors more likely the more the price is distinct from p_F^α. The demand of an investor is bounded by the levels u and d which represent the highest price he or she would sell and the lowest price he or she would buy. When prices are above u the chance of loosing money on a crash is perceived as too high therefore the asset is not bought. If the price is lower than d then the chance that it will ever rise again will be perceived as too little due to the fact that other investors in the market do not seem to be rational enough to allow for a reasonable stock pricing. The parameters δ_1 and δ_2 represent the relative strength of the bottoming or topping price d and u respectively in the chance function.

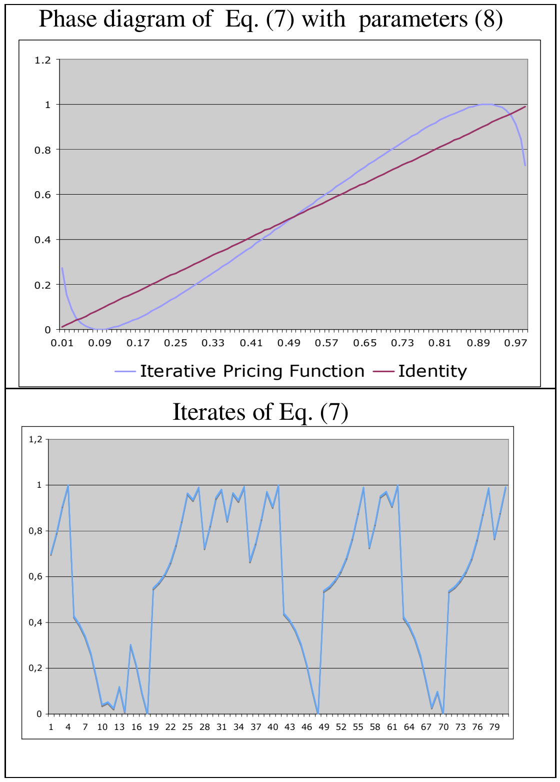


Figure 2. 1 Phase diagram and iterates of the iterative pricing equation (7) with parameters (8).

2.) β-investors

Are the less sophisticated investors. They expect the prices to rise when they are above the fundamental value and they estimate them to decline when they are beneath the fundamental value. One might call them trend-investors. Their demand function is represented as:

$$D_\beta(p, p_F^\beta) = b\left(p - p_F^\beta\right), b > 0 \qquad (5)$$

The parameter b > 0 represents the strength of their demand reacting to the difference between p and p_F^β.

3.) Investors

Simply make the market in the sense that they buy excess supply and sell from their own stock in case of excess demand E(p). In case of excess supply they lower the price and in case of excess demand they rise the price in order to sell or buy not too much from or to their inventory in line to keep their stock on a reasonable average over time. Their demand and price adjustment function θ is given as:

$$D_\gamma\left(p, E(p)\right) = -E(p)$$
$$\theta(p) = p + cE(p) \qquad (6)$$

Where c>0 is the adjustment parameter for the price.

In our simple dynamical model one yields the following iterative price formula Θ:

$$p_{t+1} = \theta(p_t) = p_t + c\left[\frac{\left(a\left[p_F^{\alpha} - p_t\right]\right)\left(p_t - d + 0.01\right)^{-\delta_1}}{\left(u - p_t + 0.01\right)^{-\delta_2} + b\left(p - p_F^{\beta}\right)}\right] \tag{7}$$

In [5] the parameters for a numerical experiment were chosen to be:

$$\begin{aligned} &d = 0, \delta_1 = \delta_2 = 0.5, u = 1, \\ &a = 0.3, b = 0.88\; c = 2, \\ &p_F^{\alpha} = p_F^{\beta} = p_F = 0.5 \end{aligned} \tag{8}$$

Given this parameter setting the systems fixed points of every period (i.e. cycles and equilibrium prices) are unstable and dense in [d , u] thus deterministic chaotic motion is generated intrinsically by the model. See [5].

Figure 2 shows a trajectory and the phase diagram of the price adjustment equation Eq. (1). The trajectories do not look realistic and the price does not diffuse. To improve the model, some modifications will be considered.

Assume that not just three groups of investors are on the market but rather beside γ-Investors N different groups of α- and β-investors α_n, β_n: n=1,2,3,…..,N with the same parameters a, b $\forall$ n and with topping and bottoming prices u_n, d_n satisfying $1<n<N$: $u_n=n*u_1$, $d_1=0$, $d_n=(n-1)*d_2$, and fundamental prices $p^{\alpha}_{F,n}=p^{\beta}_{F,n}=p_{F,n}$ with: $p_{F,n}=(n-1)+p_{F,1}$, $d_1<p_{F,1}<u_1$. Further claim that if $p\in[d_n,u_n]$ $\max(\Theta(p)) > u_n$ and $\min(\Theta(p)) < d_n$ whenever $\Theta(d_n) \gg 0$ and $\Theta(u_n) \ll u_N$ and claim $0 \le \Theta(p) \le u_N$, $\Theta(p) \in \Im_{>0}$ if $N = \infty$, i.e. γ-Investors never make negative prices. Assume $|\Theta'(p)| > 1$ and that at price levels $p\in\,]u_{n-1},d_{n+1}[$ only the group of α_n, β_n, and γ investors commits trading. Finally let the parameters $\delta_1^{\,n}$ and $\delta_2^{\,n}$ of the chance functions of all the α-investors be zero.

On the basis of these assumptions one can construct a chain of chaotic maps that obey Deterministic Diffusion and the iterative pricing formula Θ(p) is a piecewise linear map in p (see Figure 3).

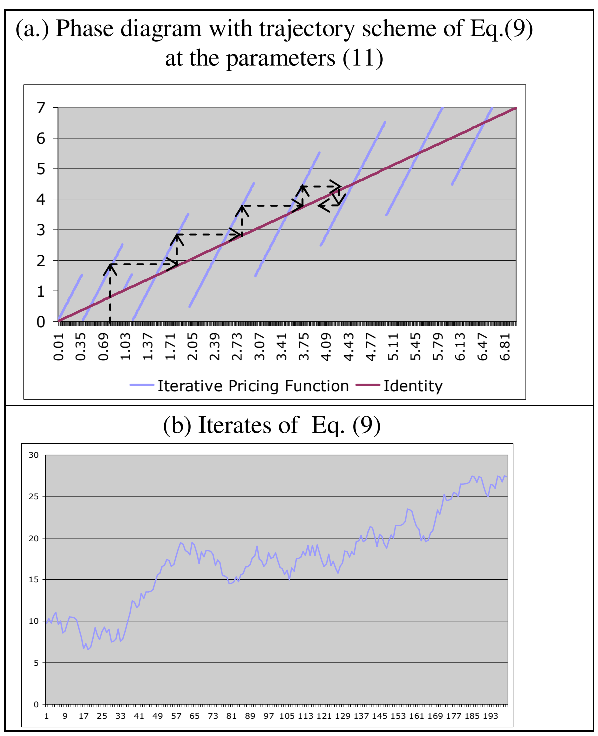


Figure 3. 1 Phase diagram with trajectory scheme and iterates of the iterative pricing equation (9) with parameters (11)

The model can now be expressed as:

$$p_{t+1} = \theta(p_t) = p_t + c\left[a\left(p_F^n - p_t\right) + b\left(p_t - p_F^n\right)\right]$$
$$if \begin{Bmatrix} \left(p_t + c\left[a\left(p_F^n - p_t\right) + b\left(p_t - p_F^n\right)\right] > 0\right) \cup \\ \left(p_t + c\left[a\left(p_F^n - p_t\right) + b\left(p_t - p_F^n\right)\right] < u_N\right) \end{Bmatrix} \tag{9.1}$$

$$p_{t+1} = \theta(p_t) = p_t + c\left[a\left(p_F^n - p_t\right) + b\left(p_t - p_F^n\right)\right]$$
$$-c\left[a\left(p_F^n - d_n\right) + b\left(d_n - p_F^n\right)\right] - d_n \tag{9.2}$$
$$if\left\{ p_t + c\left[a\left(p_F^n - p_t\right) + b\left(p_t - p_F^n\right)\right] < 0\right\}$$

$$p_{t+1} = \theta(p_t) = p_t + c\left[a\left(p_F^n - p_t\right) + b\left(p_t - p_F^n\right)\right]$$
$$-c\left[a\left(p_F^n - u_N\right) + b\left(u_N - p_F^n\right)\right] - u_N \tag{9.3}$$
$$if\left\{ p_t + c\left[a\left(p_F^n - p_t\right) + b\left(p_t - p_F^n\right)\right] > u_N \right\}$$

With the requirement int(p_t) = n-1.

As can be seen, it is possible to derive an expectation driven asset pricing model with only a few assumptions regarding the pricing process and one can yield stochastic looking time series despite the fact are generated by a deterministic process. Note that the model permits rational and irrational behaviour as well as disagreement between investor groups regarding the fundamental value of an asset.

The time series of the model also exhibits patterns of real world stock price time series where sometimes a.) small changes are followed by small changes (no significant news/events) and b.) suddenly large changes are followed subsequently by large changes (stock market crash/boom). These patterns arise when: a.) α- and β-investors of one group n trade with γ-Investors and b.) suddenly Θ(p) > u_n or Θ(p) < d_n and the price gets adjusted so that the next group α- and β-investors involves in trading and this happens for a few subsequent groups in a row. Such behaviour is typical for stock market prices and hence an argument to use deterministic scattering maps to model stock market price movements.

In context of what has been presented we can conclude that expectations and behavioural patterns might drive the price in the context of Deterministic Diffusion and the behaviour of those artificial time series seems to mime the real world very well. The independence assumption of Gaussian white noise seems hence too restrictive and too naive. In general the stock market could presumably be better understood as a deterministic scattering mechanism where one event depends on the previous. Please note that the considered scattering maps can be understood as a simplified model of a poincare′ map of a deterministic system in one of its unstable directions. For details see [4].

4. Stylized facts of stock price time series

In this section the most popular empirically observed stylized facts of stock return distributions that are contradictory to the assumption of Brownian Motion are presented. Each fact gets exemplified with real world data of the German equity index DAX™ and the time series of the model from Section 3.

The stylized facts commonly observed on stock returns and their distributions are:

- **i.** Fat tails
- **ii.** Heteroscedasticity
- **iii.** Long range dependence

iv. Sensitivity to initial conditions

It will be shown that deterministic diffusive processes like the model of Section 3 have similar features and can therefore, in contrast to simple random walks, give better explanation to real world behaviour of stock prices. Subject to the forthcoming analysis in this section were 6800 model iterates and 6832 consecutive daily closing prices of the German equity index DAX™ from 01.01.1976 to 22.02.2003. The parameter values of the model used for all following numerical investigations were:

$$\begin{aligned} &d_1 = 0, u_1 = 1, a = 0.2 + 0.01 * \sqrt{2}, b = 1 + 0.01 * \sqrt{10} \\ &c = 4,25 + 0.0001 * \sqrt{3}, p_F^{\alpha} = p_F^{\beta} = p_F = 0.5 \end{aligned} \tag{11}$$

The choice of the irrational parameter settings Eq. (11) was motivated by the fact that most parameters in the real world should be irrational.

4.1 Fat tails

The tails of a probability distribution describe the probability of the occurrence of extreme events. They are called fat if this probability does not decay exponentially with the magnitude of the event. This feature has been observed frequently in stock market returns.

Definition 4.1.1 *(Fat tailed probability distribution)* A probability distribution P is called fat tailed if the probability of extreme events vanishes by a power law with an exponent α and the following scaling law holds:

$$P\left[|X| > x\right] \approx x^{-\alpha} \quad \alpha > 0 \tag{10}$$

Note that if $\alpha < 2$ the variance and all higher moments of the distribution do not exist. We will examine our model log returns by plotting log[P(|X|>x)] against –log(x). The slope of the regression curve is used as estimate for α. An extreme event was assumed to be at least two standard deviations away from the centre of the distribution. In Figure 4(a). the long term behaviour of the model at parameters Eq. (11) is shown for 2000 iterates. Additionally the volatility of 50 consecutive values is implemented in the same graph.

In Figure 4(b) the log return distribution of the model is compared to a standard normal distribution and finally Figure 4(c) shows log[P(|X|>x)] against –log(x) plot of the model. The slope estimated from this plotting was $\alpha \approx 3.16$. In figure 5(a)-(c) the same analysis with the DAX™ time series is shown. The value of α for the DAX™ time series was estimated to be 4.02. Both time series have fat tails and show strong heteroskedastic behaviour. From this section the reader should have got the first impression how fat tails and volatility clustering of stock returns could be related to a deterministic law generating them.

4.2 Heteroscedasticity

Heteroscedasticity is a common feature observed in stock market time series and also other economic time series. It happens to occur when a lot of large changes abruptly follow a series of moderate changes.

Definition 4.2.1*(Heteroscedasticity)*

Define the m-sample variance estimator at sample point k of a sample of N realizations of a variable $x_1, x_2 \ldots.., x_N$ as:

$$\hat{\sigma}_{m,k}^2 = \frac{1}{m-1} \sum_{i=k}^{m+k} \left(x_i - \hat{\mu}_{m,k} \right) \tag{12}$$

where $$\hat{\mu}_{m,k} = \frac{1}{m} \sum_{i=k}^{m+k} x_i \tag{13}$$

with 1<k<N and n+k<N ∀ k is defined as the m-sample mean estimator at sample point k. And define the sample variance by:

$$\hat{\sigma}_N^2 = \frac{1}{N-1}\sum_{i=1}^{N}\left(x_i - \hat{\mu}_N\right) \tag{14}$$

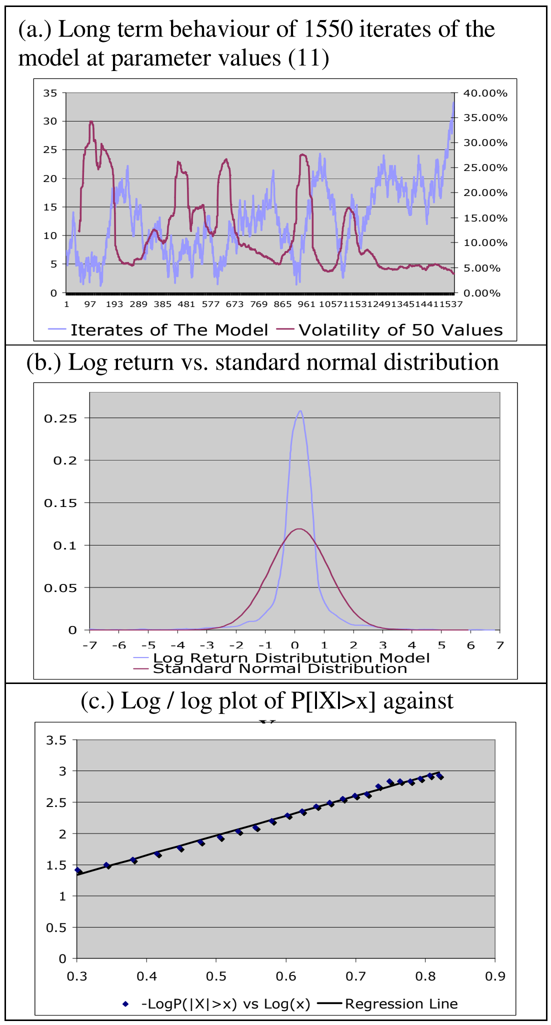
where $\hat{\mu}_N = \frac{1}{N}\sum_{i=1}^{N} x_i$ (15)

is defined as the sample mean estimator of the sample variance.

In this context heteroscedasticity would mean that for N samples there exists significantly many values k so that the m-sample estimators of those sample buckets differ significantly. From Figure 4(a.) and 5(a.) we can see clearly that the model generated price time series and DAX™ should show heteroscedasticity. The word significantly can be given a meaning by a statistical hypothesis test as in [1] and [6].

4.3 Long Range Dependence

One striking feature of Brownian motion is that it has no memory. Thus all realizations are independent from one another in time. As will be shown shortly real world stock market returns and the model returns show exactly the opposite.



(a.) Long term behaviour of 1550 iterates of the model at parameter values (11)

(b.) Log return vs. standard normal distribution

(c.) Log / log plot of P[|X|>x] against

Figure 4: (a.) 1550 iterates of the model at parameter values Eq. (11) plus the volatility of 50 consecutive returns calculated on revolving time intervals. (b.) Comparison of log return distribution of the model against a standard normal distribution. (c.) Plot of log(P(|X|>x)) against log(X) for model returns.

Definition 4.3.1 *(Long range dependence)*

A time dependent process x(t) is said to be long range dependent, if the autocorrelation of its absolute time lagged values raised by any power k >=1 is greater than zero and decays in time by a power law with the rate δ^k.

$$\begin{aligned}
&\forall t > 0 s > 0, k >= 1: \\
&\rho(s,k) = corr\left(\left|x(t)\right|^k, \left|x(t+s)\right|^k\right) > 0\, and \\
&\rho(s,k) \approx corr\left(\left|x(t)\right|^k, \left|x(t+1)\right|^k\right) s^{\delta_k}
\end{aligned} \tag{16}$$

The lower the absolute value of δ^k the less the decay of dependence.

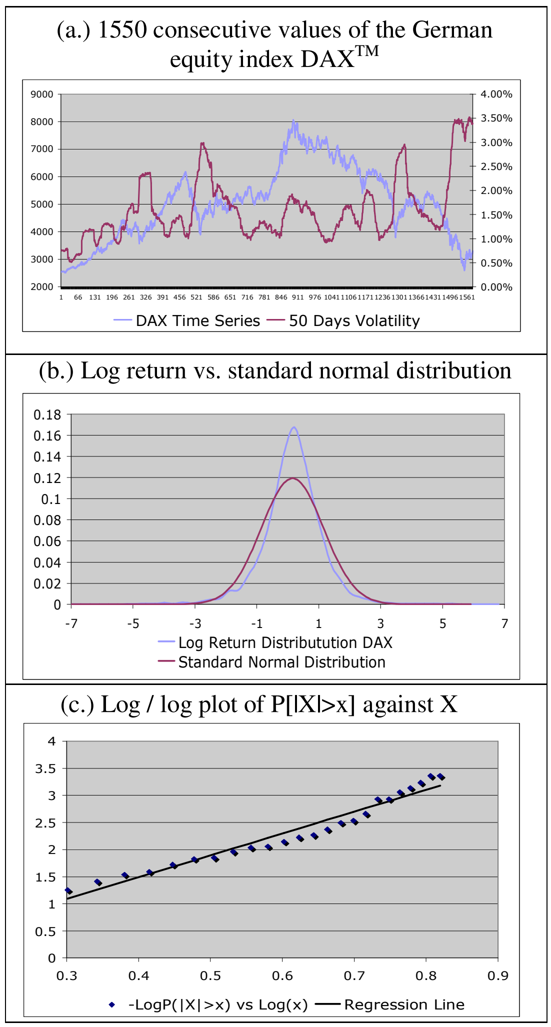


Figure 5. (a) 1550 iterates of the german equity index DAX plus the volatility of 50 consecutive returns calculated on revolving time intervals. (b) Comparison of log return distribution of the DAX™ time series against a standard normal distribution. (c) Plot log[P(|X|>x)] against log(X) for DAX™ returns.

To examine the DAX™ and the model time series of log returns ρ(n,2), n ∈ $\mathbb{R}^+$ was calculated as well as the ordinary autocorrelation function. The results are shown in Figure 6. Both time series show the same qualitative behaviour, but with different quantitative peculiarity.

The model autocorrelations of squared returns start at a very high level and decay very fast where as the real world time series correlations start at a lower level and decay more slowly. For the model series δ_2= -0.5 and for the DAX™ δ_2 = -0.3 got computed showing a faster decay and loss of dependence in the model time series than in the DAX™ returns.

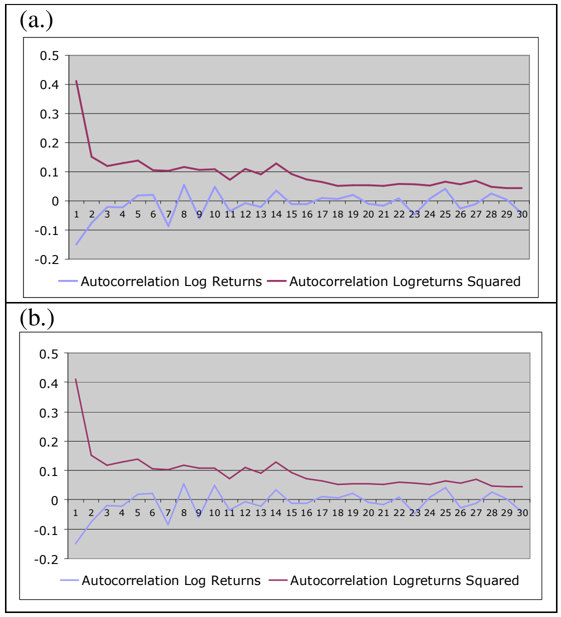


Figure 6. (a.) Autocorrelation Correlation Decay for normal and squared model time series of log returns with δ^2 =-0.5 (b.) Autocorrelation Correlation Decay for normal and squared DAX™ time series of log returns with δ^2 =-0.3.

Another measure of long range dependence or persistence is the Hurst Exponent, denoted in the following with H. It is named after its inventor, the hydrologist Harold Edwin Hurst. He invented it when analysing yearly water run offs of the Nile river. Consider n observations of a variable x: $x_1, x_2, x_3, \ldots, x_n$ and the cumulated values $X_k = x_1 + x_2 + x_3 + \ldots x_k$. The value $X_k - (k/n) X_n$ measures the divergence of the cumulated value of a time series of length k from the rescaled cumulated value of the whole time series. Define the Range R_n as:

$$R_n = \max_{1 \le k \le n}\left(X_k - \frac{k}{n} X_n \right) - \min_{1 \le k \le n}\left(X_k - \frac{k}{n} X_n \right) \tag{17}$$

The empirical Standard deviation is given by:

$$S_n = \sqrt{\frac{1}{n}\sum_{k=1}^{n}\left(x_k - \frac{X_n}{n} \right)^2} \tag{18}$$

Hurst found that for the rescaled range R/S:

$$\left(R/S \right)_n = \frac{R_n}{S_n} \approx cn^H \tag{19}$$

where the values X_k had approximately the same distribution like $n^H X_1$ with H≈0.7, indicating that the Nile River run-offs are not i.i.d. random events, but rather depend on one another persistently. A process with such a scaling behaviour and distributional congru- ency is called statistically self similar.

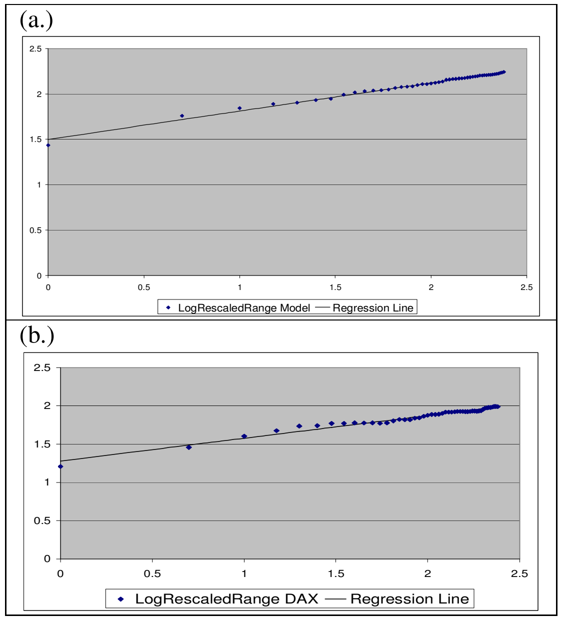


Figure 7. (a.) log/log plot for the R/S scaling of the model time series, slope estimated H= 0.31 (b.) log/log plot for the DAX^{TM} R/S scaling yielding H=0.29.

Definition 4.3.2 *(statistical self similarity)*
A statistical self similar process x is defined

by: $x(\alpha t) \cong x(t)\alpha^{2H}$ (20)

where x(t) is the value of the process after t time steps, $\alpha \in \mathbb{R}^{+}$, H∈ [0,1] is the Hurst Exponent and the operator ≅ means congruency in distribution. If H > 0.5 a process is called persistent, If H < 0.5 a process is called anti-persistent for H=0.5 the process is called stable (since it's stable under addition).

The estimation of H from empirical data is straight forward. One plots log((R/S)n) against log(n). The slope of the regression line holds as estimate for H. Figure 7(a.) and (b.) show the resulting log/log plots for the model time series and the DAX^{TM} time series respectively.

The value H=0.31 was estimated for the model time series and H=0.29 for the DAX^{TM} time series. The DAX^{TM} and the model estimated Hurst Exponent values show clearly that both time series are anti-persistent and long range dependent. To get a better insight in this phenomenon, that should be called Hurst Effect in the following, Figure 8 shows the maximum and the minimum standardized log return of the DAX^{TM} and model time series relative to the time frame that was observed. The extreme events of both time series seem to decay by a power law in time. This is a commonly observed effect (called: mean reversion) in stock market time series. The loss or gain one experiences decreases with time, after periods of large losses new speculators enter the market and after periods of gains, positions get unwound in the course of realizing profits. Another way to look at this phenomenon is to observe the information entropy.

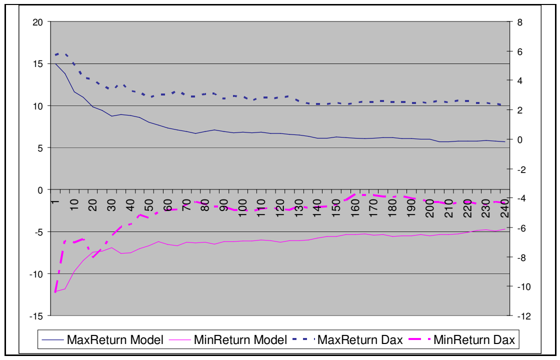


Figure 8: Max and Min standardized log returns of model (primary axis) and DAX™ time series (secondary axis).

Definition 4.3.2*(The Information Entropy)*

The Information Entropy for a random variable x with density h(x) according to Shannon is defined by:

$$\Pi = -\int h(x)\log_2\left(h(x)\right)dx \tag{21}$$

It gives the maximum Information in bits one learns from one outcome of a random variable. Thus the higher the information entropy, the more information produces an experiment. For example consider a coin toss. The information entropy of it is:

$$-2*(1/2)*\log_2(1/2) = -\log_2(1/2)=1 \tag{22}$$

Now consider a skew of the coin toss so that the probability of the one side of the coin turns ¼ and that of the other ¾. The information entropy then is:

$$-(1/4)*\log_2(1/4)- (3/4)*\log_2(3/4) = 0.81. \tag{23}$$

Thus the experiment needs to be repeated more often to get the same information than one outcome of the not skewed coin toss produces. Figure 9. shows the development of the information entropy over different time horizons for the model time series and the DAX™ time series computed for the normalized distributions of their log returns. From observing Figure 9 it is clear, that we cannot be dealing with a self similar stochastic process in both cases, because for such a process the information entropy would be constant in time. Figure 8 underlines this conclusion since constancy in time would also be expected to hold for the min and max log returns. It can also be seen from Figure 9 that the information entropy increases with time, which means, that the riskiness or uncertainty involved in the process decreases since one learns more about the world by one experiment on a longer time horizon.

From the previous observations naturally the question arises how the distribution of the processes may evolve over time. To investigate this issue the normalized distributions for the time buckets 1,20,240 were drawn into one graph shown in Figure 10. Again both time series obey the same peculiarities. On different time scales the distributions diverge from each other showing different shapes. The shape of the distribution of log returns depends on the time horizon, unlike the ones of a stable or self similar stochastic process.

As time progresses, autocorrelation of squares are strictly positive and decay slowly by a power law and risk declines measured in the form of the information entropy and min/max returns. Hurst analysis shows strong anti persistency for the DAX™ and the model. Contrary to a stable or self similar stochastic process the time dependent normalized frequency distributions of the DAX™ and model returns are not alike.

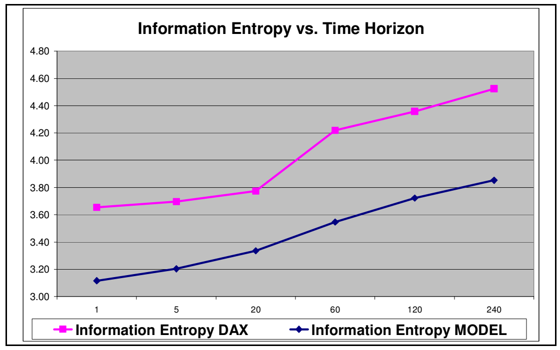


Figure 9. Development of the information entropy of model and DAX time series log returns in relation to time horizon in days/iterations.

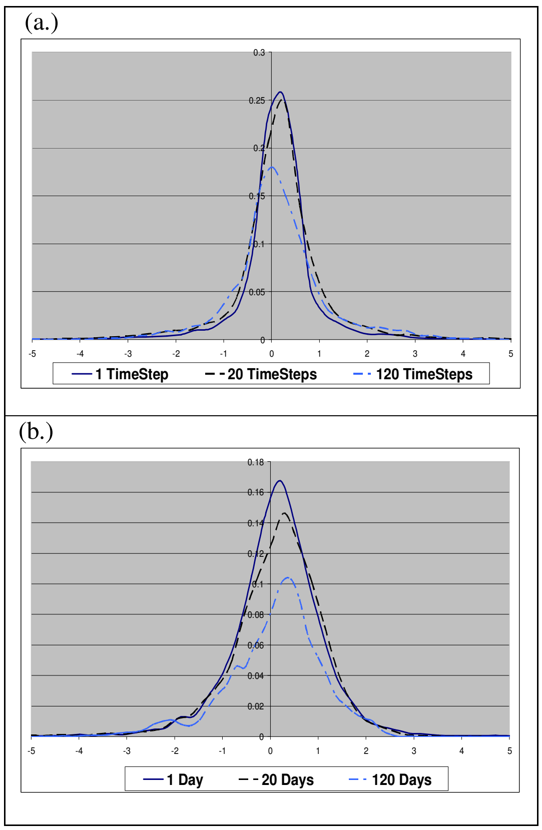


Figure 10. (a.) and (b.) the densities for the (a.) model and (b) DAX™ log returns for different time horizons.

4.4 Sensitivity to initial conditions

Apart from the classical stylized facts the author would like to add this section to make the sections following thereafter more clear. Sensitivity to initial conditions is the property of a dynamic system that describes how it reacts on a small difference in the starting value in the long run. A popular measure of this kind of behaviour is the Lyapunov Exponent. It describes the average exponential expansion rate of a small error in the initial conditions.

Definition 4.1.1 *(Sensitivity to initial conditions)* A dynamical system is said to be sensitive to its initial conditions if a small error δx expands on the average exponential with rate $\lambda > 0$, called Lyapunov Exponent. The formal definition of λ is given by:

$$\lambda = \lim_{t\to\infty} \lim_{\delta x \to 0} \frac{1}{t} \sum_{n=1}^{t} \ln\left(\frac{|\delta x_n|}{|\delta x_0|} \right) \tag{24}$$

where δx_0 is the error in the initial condition of the iterates x(t) of the system and δx_t is the error after t time steps.

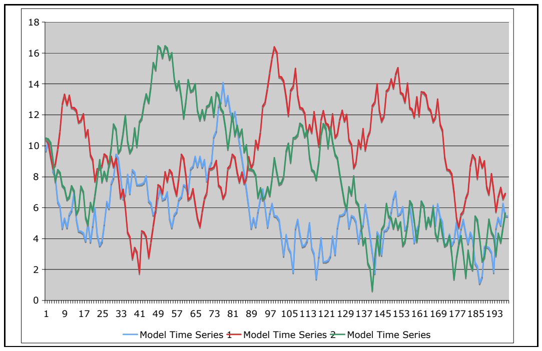


Figure 11 Model time series for initial conditions 9.7; 9.71; 9.72 respectively yielding extremely different trajectories.

Table 1 Results of the sensitivity analysis of the DAX^{TM} and model time series

	Model	DAX
Lyapunov Exponent:	2.76	3.52
Lyapunov Time	3.08	3.46
initial Error	0.01	0.5
System Extend	50	10000

The meaning of this sensitivity is illustrated in Figure 11. Three trajectories of the model all only one hundredth i.e. 0.01 apart from each other in the starting value are shown, resulting in significantly different trajectories.

As measure of sensitivity on initial conditions, the Lyapunov Exponents for the model and the DAX^{TM} have been computed. The following method was applied. Denote x_{NN} the nearest neighbour of the starting value x(0) of a time series in the sense of:

$$|x_{NN} - x(0)| = \inf |x(n) - x(0)| \; n = 1, 2, .., T \tag{25}$$

where T is the length of the time series. Then the following quantity holds as estimate for λ:

$$\tilde{\lambda} = \frac{1}{T - I_{NN}} \sum_{n=1}^{T-I_{NN}} \ln\left(\frac{|x(n) - x(I_{nn} + n)|}{|x_{NN} - x(0)|} \right) \tag{26}$$

where I_{NN} is the time index of the nearest neighbour value x_{NN}.

For the DAX^{TM} time series $\lambda_{DAX} = 3.52$ was obtained and for the model $\lambda_{model} = 4.2$. Thus the model has less forecast ability than the DAX^{TM}. To illustrate this statement consider the Lyapunov time, defined as the maximal time a dynamical system can be forecasted:

$$T_\lambda = -\frac{1}{\lambda} \ln\left(\frac{\delta x_0}{\in} \right) \tag{27}$$

The variable $\in$ denotes the maximal extend of the system. The Lyapunov times of the DAX^{TM} and the model respective were estimated assuming an initial error of 0.5 and 0.01 as well as an extend of 10000 and 50 respectively. The DAX^{TM} time series had a Lyapunov time of $T_\lambda = 3.46$ days where as the model had $T_\lambda = 3.08$ iterates. The results are summarized in Table 1. It turns out, that if we knew the true model only a limited forecast of approx. 3 Days for the DAX™ and the model would be possible.

5. Implications on economic theory

In the following paragraphs the implications of the so far introduced model of Deterministic Diffusion will be discussed and reviewed against classical results of capital markets theory like the Market Efficiency Hypothesis, the CAPM and the Black Scholes option pricing formula.

5.1 Market Efficiency / CAPM

In the beginning the classical Market Efficiency Hypothesis will be recalled and afterwards the Deterministic Diffusion model will be related to it.

Market Efficiency Hypothesis (MEH)

The Markets are:

1. Semi efficient

If all information about price histories is contained in the prices.

2. Efficient

If they are semi efficient and all public information is contained in the prices.

3. Strong efficient

If they are efficient and all non public information is contained in the prices.

In strong efficient markets all prices are assumed to follow random walks of geometric Brownian motion in classical capital market theory, since only the occurrence of new information changes the price, and the price and its history do not contain any information about its future development. The consequence for capital asset pricing is that in equilibrium an asset i is priced accordingly that the expected excess return w.r.t. the risk free interest rate $E[r_i]- r_f$ can be expressed in terms of the expected excess return of a market portfolio M $E[r_m]- r_f$ times a beta factor $\beta_{i,M}$ of a stock. The asset's price therefore reflects the risk premium to be paid to an investor in equilibrium relative to the Market M:

$$E[r_i]-r_f=\beta_{iM}\left[E[r_M]-r_f\right],\ \beta_{iM}=\frac{Cov(r_i,r_M)}{\sigma_M^2} \tag{28}$$

The just stated equation forms the heart of the CAPM (Capital Asset Pricing Model).

There have been numerous articles and empirical investigations on whether the CAPM holds or not. The focus of the following will be rather a theoretical reasoning about the validity of the CAPM in the framework of Deterministic Diffusion.

In an environment of Deterministic Diffusion efficiency is not the general case but rather only one part of the story. Efficiency is present only for certain time frames, when crowd behaviour does not dominate price movements (Small changes are followed by small changes). In contrast if crowd behaviour dominates, as represented by β-Investors, (Large changes are followed by large changes), markets loose their efficiency, since trend movements set on and could be exploited by arbitrageurs. As can be seen from Figure 3(a) such a trend path triggered by succeeding entries or exits of several investor groups into the market is easy to construct in the model framework and should occur frequently. Still the question remains if such trends can be utilized systematically such that sustainable positive gains can be made. I.e. the information about the process presumably itself is hardly priced into a time series.

The MEH doesn't explain how the information that's priced into a stock is generated. If it is generated by a chaotic system and only the present state is responsible for the current price (short term pricing) then it would be no surprise that prices follow an unstable movement on a poincare′ map of this system.

Also the MEH doesn't allow for irrational and speculative investors like the β-Investors in our model. All investors should have the same opinion and information in the MEH and CAPM framework. This definitely does not hold in the real world, especially when it comes to the opinion about the future development of prices. It also should be obvious that limited rationality as well plays a significant role in the real world since the future is mostly unknown and impossible to be forecasted, especially on longer time horizons. Therefore no exact pricing of an asset may be possible and prices have to be revised daily.

To outline the different implications and assumptions in the preceding text the author would like to give a Chaotic Market Hypothesis CMH, that can be understood as modification or extension of the Fractal Market Hypothesis stated earlier by Peters in [8].

Chaotic Market Hypothesis (CMH)

1) Efficiency

Markets in general are not semi efficient in the sense of the MEH. There exist switching regimes between efficient and inefficient markets according to the strength of the action of β-Investors. Information gets incorporated into prices depending on the investor behaviour and sentiment. If it does, it reflects a long memory process of a large deterministic system. Every price is right as long as investors are willing to pay it. (The market is always right). The information about the price development process itself is hard to price. Arbitrage opportunities are rather of theoretical nature.

2) Investor Behaviour

Investors can act rational as well as irrational according to their personality or current sentiment. They have different investment horizons and different opinions about the future development of a price and have limited rationality i.e. limited knowledge about the future and the justified value of assets/shares.

3) Evolution of Prices

Prices diffuse according to deterministic laws, that can be to a certain extend interpreted as random. The diffusion is caused by a deterministic system driven by news and behavioural patterns of investors. The evolution law of prices shall be called "Deterministic Diffusion". It has infinite long memory and is not Markov.

When considering the CMH, the CAPM is only valid in a Deterministic Diffusion environment for short time horizons when markets can be interpreted as random walks, i.e. when markets are calm and efficient and the volatility is constant. There from we conclude that classical asset pricing models give a good understanding of how prices should be, but only capture certain aspects of real world stock market time series.

5.2 Option Pricing

At this point the author would like to sketch how an alternative option pricing model should look like.

In the classical investment theoretical framework the Black Scholes (BS) formula is well established. The major problem of this formula is the assumption of a homoscedastic log normal distribution for the price movements. The most crucial input parameter is the volatility since it is heteroscedastic in real economic time series. Any alternative option pricing model should therefore come up with a formula, that does not need the volatility as an input or can deal with heteroscedasticity. However there have been some alternative models brought up like the Option Pricing in case of fractional Brownian Motion, see [9] or Option Pricing for levy stable processes, see [10],[11]. In case of a fractional Brownian Motion a closed form solution exists.

A fractional Brownian Motion FBM stochastic process B^H is a Gaussian processes $B^H(t) \approx N(\sigma^H(t),\mu^H(t))$ with conditional second moment:

$$\sigma_H^2(t,T) = \left(T^{2H} - t^{2H}\right)\sigma^2 \, ; \, \mu_H(t) = B_H(t) \tag{29}$$

and its values are correlated by the covariance function:

$$E\left[B_H(t)B_H(s)\right] = \frac{1}{2}\left(|s|^{2H} + |t|^{2H} - |t-s|^{2H}\right)\sigma^2 \tag{30}$$

σ can be interpreted as the instantaneous volatility per H weighted unit of time. H is again the Hurst exponent.

The fractional Brownian Motion Black Scholes formula reads:

$$C(t,T,r,S(t),X) = S(t)N(d1) - Xe^{-r(T-t)}N(d2)$$
$$P(t,T,r,S(t),X) = Xe^{-r(T-t)}N(-d1) - S(t)N(-d2) \quad (31)$$

where:

$$d1 = \frac{\ln\left(\frac{S(t)}{K}\right) + r(T-t) + \frac{\sigma^2}{2}\left(T^{2H} - t^{2H}\right)}{\sigma\sqrt{T^{2H} - t^{2H}}} \quad (32)$$

$$d2 = \frac{\ln\left(\frac{S(t)}{K}\right) + r(T-t) - \frac{\sigma^2}{2}\left(T^{2H} - t^{2H}\right)}{\sigma\sqrt{T^{2H} - t^{2H}}} \quad (33)$$

C is the price of a European call option and P is the price of a European put option respectively. T > t is the date after the evaluation date t where the option matures, X is the strike price of the option, r is the risk free zero interest rate from the evaluation date until the date of maturity.

A random variable X is called α (or levy-) stable if and only if X ≅ γZ+δ, γ > 0 where Z has a characteristic function:

$$E[\exp(isZ)] = \begin{cases} \exp\left(-|s|^{\alpha}\left[1 - i\beta\tan\frac{\pi\alpha}{2}\left(sign[s]\right)\right]\right) & \alpha \neq 1 \\ \exp\left(-|s|\left[1 + i\beta\frac{2}{\pi}\left(sign[s]\right)\log\left(|s|\right)\right]\right) & \alpha = 1 \end{cases} \quad (34)$$

with α ∈ (0,2],β ∈ [-1,1]. Note that for α = 2 X is distributed with a Normal distribution that has volatility 2^0.5γ and expected value δ. Thus the Normal Distribution is a special case of Z with α = 2. The parameter δ is called the location parameter and is identical with the expected value of the distribution for 1 ≤ α ≤ 2. The parameter γ is called the scale parameter and is for α = 2 identical with half the variance of the distribution. The parameter β describes the skewness of the distribution. For β = -1 the distribution is skewed completely to the left and for β = 1 skewed completely to the right. The parameter α is called the characteristic exponent and describes the tail behaviour of the distribution

Beside those models from [9], [10] [11] define a progress in financial economics they still have certain drawbacks.

a. It has been shown in other papers that there exists arbitrage in FBM See [12].

b. The findings of this paper suggests, that equity price time series are neither a stable or self similar stochastic process like levy stable distributions or FBM are.

c. The assumptions made by Mc Cullogh in [10],[11] are too restrictive to apply.

d. The estimates of the tale parameter α in this paper are clearly above 2 while for levy stable distributions holds α < 2.

Definition 5.2.1 *(ergodic / invariant measure)*

A Map M: $\mathbb{R} \to \mathbb{R}$ is called ergodic if there exists a measure (density) 0 ≤ ρ(x) ≤ 1 such that:

$$\rho(x) = \lim_{n\to\infty}\frac{1}{n}\int_{\mathbb{R}}\sum_{0}^{n}\delta\left(M^{n}\left(x_0\right) - x\right)\rho\left(x_0\right)dx_0 \quad (35)$$

δ is the Dirac delta function, M^n denotes the n^{th} iterate of the map M, the integral is over all initial conditions x_0 in $\mathbb{R}$. The measure ρ(x) is called the invariant measure of M.

Note, since we are dealing with a deterministic process ρ is not a probability measure but rather measures the frequency of appearances of a value x in $\mathbb{R}$ on average. The kernel of (2.7) $\delta(M^n(x_0)-x)$ is called the Frobenius Perron Operator.

Definition 5.2.2 *(Conditional measure)*

The conditional measure $\rho_n(x \mid x_t;\varepsilon)$ given the state x_t of a Map M and measurement error ε of initial conditions is defined by:

$$\rho_n(x|x_t;\varepsilon)=\frac{1}{\rho\left(M^n(I)\right)}\int_I \delta\left(M^n(z)-x\right)\rho(z)\,dz \tag{36}$$

Where $I \in \mathbb{R}$ is an interval such that: $I = [x_t-\varepsilon/2;\ x_t+\varepsilon/2]$, $\rho(x)$ is the invariant measure of the map and $\rho(M^n(I))$ is the "Mass" of the invariant measure on the n times iterated interval I by the Map M:

$$\rho\left(M^n(I)\right)=\int_{M^n(I)} \delta(z)\,\rho(z)\,dz \tag{37}$$

Thus the conditional measure $\rho_n(x \mid x_t;\varepsilon)$ assigns a probability mass to every point $x \in \mathbb{R}$ that it can be reached by iterating the Map M, n steps forward in time given an initial state x_t and a measurement error ε.

Since ρ does not always exist as an applicable closed form formula, consider to partition I into k subintervals of length I/k. Furthermore assign the density 1/k to each point of I. Then a discrete approximation to (36) is:

$$\bar{\rho}_n(x|x_t;\varepsilon;k)=\frac{1}{k}\sum_{s=1}^{k}\chi\left(M^n\left(x_t+\frac{sI}{k}-\frac{\varepsilon}{2}\right)-x\right) \tag{38}$$

where χ is a characteristic function defined as: $\chi(0)=1$; $\chi(x<>0)=0$ (39)

Thus Eq. (36) can be computed at least by numerical simulations when the equations of driving the diffusive process are known. The derivation of a Statistic Dynamics (SD) option pricing formula is basically straight forward:

Theorem 5.2.1 *(Statistic Dynamics option pricing formula)* Given an ergodic Map M with invariant measure $\rho(x)$, conditional measure $\rho_n(x \mid x_t;\varepsilon)$ given the state x_t of a Map M at time t and measurement error ε of initial conditions. An option with time to maturity T, and risk free interest rate r until maturity can be priced as:

$$Call\left(x_t,\varepsilon,T,r\right)=\int_0^\infty e^{-rT}Max\{S-X;0\}\rho_T\left(S|x_t;\varepsilon\right)dS \tag{40}$$

$$Put\left(x_t,\varepsilon,T,r\right)=\int_0^\infty e^{-rT}Max\{X-S;0\}\rho_T\left(S|x_t;\varepsilon\right)dS \tag{41}$$

Assuming risk neutral individuals, that can compute $\rho_n(x \mid x_t;\varepsilon)$ approximately.

To give the reader an idea on how Eq.(40) and (41) influence the option price, $\rho_n(x \mid x_t;\varepsilon)$ was computed by numerical simulations using the approximation Eq. (38). Subject of the simulation were 2000 iterates of the model. An approximation error of ε=0.01 was assumed, equivalent to that an accurate measurement only possible up to one hundredth. The interval I=1 was partitioned into one hundred equal long pieces of length 0.01 each being assigned a density of 0.01, the density was then evolved by numerical simulation until time to maturity and Theorem 5.2.1 was used to price the option.

Additional to the Statistic Dynamics option price the FBM price and the traditional Black Scholes price was computed. Two experiments were made. In the first experiment shown in Table 2 only the first 250 iterates were used in the estimation of the volatility. In the second experiment shown in Table 3 the volatility was estimated using all 2000 iterates. The Hurst Exponent was estimated always using all iterates (with value H=0.32) to give a better estimation quality. The two experiments were chosen to find out about the impact of the parameter estimation error that occurs when calibrating a homoscedastic model in a hetero- scedastic environment. In both numerical investigations the risk free interest rate r was assumed to be zero. The results of the experiments can be summarized as follows:

1) Heteroscedasticity Effect

Firstly the use of FBM and BS models for the pricing of options with the overall volatility of 9.8% is generally superior in comparison to use them with the volatility estimator of 26% of only the first 250 iterates. In particular out of the money options are extremely overpriced by the BS model with 26% volatility. The same holds for the FBM Model in a more moderate fashion. With 26% Volatility per time step a longer time to maturity has diminished influence on the BS option price. One can even see from table 2, that in the limit of infinite volatility and time to maturity the BS call price converges towards the spot and the BS put price converges towards the strike. The bias of the BS option price caused by heteroscedasticity is thus not negligible. The FMB option price has similar features, but with less strength since the presence of the Hurst Effect counter affects the heteroscedasticity bias.

2) Hurst Effect

Secondly the low Hurst Exponent of 0,32 indicates a strong anti-persistence effect that causes the log price not to diffuse unbounded, but to return to where it has come from (Recurrence). That's the reason why out of the money options are priced to high by the BS model in the Deterministic Diffusion environ- ment. The value of SD options decays rapidly to zero with decreasing moneyness. The FBM option prices also decay and match those of the SD option prices far better than the BS prices, since the FBM model captures the Hurst Effect of anti persistency.

3) Time to Maturity (Theta) Effect

Thirdly the larger the time to maturity (Theta), the larger the divergence to the pricing of the BS and FBM model of the SD pricing dependent on the heteroscedasticity bias. This goes well in line with the findings of section 4.3 that the process of Deterministic Diffusion does not posses a statistic self similarity and that the risk decreases on a long time horizon.

It can be concluded that the BS Model can lead to large price deviations from the SD price when the underlying price driving process is not the assumed process. The effect becomes more pronounced as the time evolves. The FBM Option pricing model gives a good approximation to the SD option prices, but needs a large sample of the time series to be estimated efficiently.

Table 2. Option Prices Experiment 2, Volatility = 26% of first 250 iterates was used, H=0.32 of total time series was used, Spot = 12.15

	Maturity = 30 time steps					
	Statistical Dynamics		Black Scholes		Fractional BM	
Strike	Call	Put	Call	Put	Call	Put
2	9,91	0,02	10,45	0,30	10,16	0,01
5	7,04	0,15	8,76	1,61	7,52	0,37
10	2,82	0,92	6,98	4,83	4,54	2,39
15	0,46	3,56	5,82	8,67	2,84	5,69
20	0,00	8,11	4,99	12,84	1,85	9,70
25	0,00	13,11	4,36	17,21	1,25	14,10
30	0,00	18,11	3,86	21,71	0,87	18,72
40	0,00	28,11	3,12	30,97	0,45	28,30
50	0,00	38,11	2,60	40,45	0,25	38,10
	Maturity = 100 time steps					
	Statistical Dynamics		Black Scholes		Fractional BM	
Strike	Call	PuDG	Call	Put	Call	Put
2	10,34	0,03	11,35	1,20	10,27	0,12
5	7,66	0,34	10,73	3,58	8,17	1,02
10	4,03	1,74	10,08	7,93	5,94	3,79
15	1,77	4,45	9,61	12,46	4,56	7,41
20	0,46	8,20	9,25	17,10	3,62	11,47
25	0,08	12,80	8,94	21,79	2,95	15,80
30	0,01	17,69	8,68	26,53	2,45	20,30
40	0,00	27,69	8,25	36,10	1,77	29,62
50	0,00	37,69	7,90	45,75	1,33	39,18

	Maturity = 250 time steps					
	Statistical Dynamics		Black Scholes		Fractional BM	
Strike	Call	PuDG	Call	Put	Call	Put
2	12,16	0,11	11,98	1,83	10,52	0,37
5	9,66	0,62	11,86	4,71	8,95	1,80
10	6,28	2,29	11,74	9,59	7,30	5,15
15	3,81	4,85	11,64	14,49	6,22	9,07
20	2,18	8,18	11,57	19,42	5,43	13,28
25	1,18	12,12	11,50	24,35	4,82	17,67
30	0,67	16,60	11,45	29,30	4,33	22,18
40	0,15	26,13	11,35	39,20	3,60	31,45
50	0,03	36,02	11,26	49,11	3,07	40,92

Of course in practise the approach presented here is not easily applicable because one normally does not know the true process driving the dynamics or the invariant measure of the stock price. But there should be possible methods yielding approximately comparable results. This should be part of future research.

6. Summary and conclusions

In the foregoing paper a new model named "Deterministic Diffusion" was introduced to model stock price processes. The model can be motivated by simple behavioural models of the stock market and does not need too many restrictive assumptions to be reasonable. Furthermore it helps understanding on how randomness comes about and how typical stylized facts like i.) heteroscedasticity ii.) long range dependency iii.) fat tailed frequency distributions in real world stock market data can be explained.

Table 3. Option Prices Experiment 2, Volatility = 9.8% of total 2000 iterates was used, H=0.32 of total time series was used Spot, = 11.26

	Maturity = 30 time steps					
	Statistical Dynamics		Black Scholes		Fractional BM	
Strike	Call	Put	Call	Put	Call	Put
2	9,91	0,02	10,15	0,00	10,15	0,00
5	7,04	0,15	7,24	0,09	7,15	0,00
10	2,82	0,92	3,58	1,43	2,64	0,49
15	0,46	3,56	1,68	4,53	0,54	3,39
20	0,00	8,11	0,79	8,64	0,08	7,93
25	0,00	13,11	0,39	13,24	0,01	12,86
30	0,00	18,11	0,20	18,05	0,00	17,85
40	0,00	28,11	0,06	27,91	0,00	27,85
50	0,00	38,11	0,02	37,87	0,00	37,85

	Maturity = 100 time steps					
	Statistical Dynamics		Black Scholes		Fractional BM	
Strike	Call	Put	Call	Put	Call	Put
2	10,34	0,03	10,21	0,06	10,15	0,00
5	7,66	0,34	7,87	0,72	7,17	0,02
10	4,03	1,74	5,34	3,19	3,15	1,00
15	1,77	4,45	3,81	6,66	1,15	4,00
20	0,46	8,20	2,83	10,68	0,40	8,25
25	0,08	12,80	2,17	15,02	0,14	12,99
30	0,01	17,69	1,70	19,55	0,05	17,90
40	0,00	27,69	1,10	28,95	0,01	27,86
50	0,00	37,69	0,76	38,61	0,00	37,85

	Maturity = 250 time steps					
	Statistical Dynamics		Black Scholes		Fractional BM	
Strike	Call	Put	Call	Put	Call	Put
2	12,16	0,11	10,53	0,38	10,15	0,00
5	9,66	0,62	8,99	1,84	7,27	0,12
10	6,28	2,29	7,36	5,21	3,73	1,58
15	3,81	4,85	6,29	9,14	1,85	4,70
20	2,18	8,18	5,51	13,36	0,94	8,79
25	1,18	12,12	4,91	17,76	0,49	13,34
30	0,67	16,60	4,43	22,28	0,27	18,12
40	0,15	26,13	3,70	31,55	0,09	27,94
50	0,03	36,02	3,17	41,02	0,03	37,88

Comparisons throughout the paper to real world DAX™ time series show obvious parallel features that are not neglectable and give evidence for the appropriateness of the approach. Both time series have fat tailed frequency distributions of their log returns, slowly decaying autocorrelations of their squared returns and show a large degree of heteroscedasticity.

In a Hurst analysis for the Model time series and the DAX™ time series showed strong anti-persistency with a value of $H \approx 0,3$. Log returns revert back to their mean and do not grow linearly in time since new speculators may enter the market any time or firstly invested speculators may unwind their positions from time to time to realize profits. By looking at frequency distributions on different time scales and the development of the information entropy in time both processes seem not to follow a simple self similar or stable stochastic law. An empirical estimation of Lyapunov Exponents results in clearly positive values for the DAX™ and the model time series giving strong indication for the presence of deterministic chaos. Markets do have inefficient phases where behavioural patterns of investors dominate the price evolution.

Future research should be concerned with a.) A more precise description of the Deterministic Diffusion process in terms of variables or scattering maps with empirical fitting methods to existing time series, b.) Empirical detection methods of Deterministic Diffusion and c.) Easily applicable option pricing formulas and / or methods.

Furthermore other economic time series like exchange rates and interest rates could be analyzed to see if the Deterministic Diffusion model would be reasonable for them as well. The implications for Risk Management surely should also not be out of the scope of further investigations.

7. References

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