Long-run Negotiations with Dynamic Accumulation*

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Abstract

Many negotiations (for instance, among political parties, partners in a business) are characterised by dynamic accumulation: current agreements affect future bargaining possibilities. We study such situations by using repeated bargaining games in which two parties can decide how much to invest and how to share the residual surplus for their own consumption. We show that there is a unique (stationary) Markov Perfect Equilibrium characterised by immediate agreement. Moreover, in equilibrium a relatively more patient party invests more than his opponent. However, being more patient can make a player worse off. In addition, we derive the conditions under which we obtain the efficient investment path. Our results are robust to different bargaining procedures, different rates of time preferences and elasticities of substitution.

JEL Classification: C61, C72, C73, C78
Key words: Bargaining, Investment, Recursive Optimisation, Markov Perfect Equilibrium.

*I wish to thank the participants of the 2007 annual conferences of the Royal Economic Society, the Scottish Economic Society, the Association of the Southern Economic Theorists and the SGPE workshop at the department of Economics of Edinburgh University for useful discussions and suggestions. ESRC financial support (grant no. RES-061-23-0084) is gratefully acknowledged. All errors remain mine.

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1 Introduction

Several bargaining situations in the most diverse contexts can be represented as bargaining games with dynamic accumulation, that is, bargaining games in which parties can invest part of the surplus and the invested surplus affects the size of future surpluses. For instance, partners in a business need to negotiate not only on how to split profits among themselves at the end of each accounting year, but also on how much profit should be re-invested for the following production period. External relationships between a business and its suppliers can also be characterised by dynamic accumulation: a supplier may agree to make ongoing investments in new production techniques to meet the specific product needs of his customer and the extent to which such investment takes place will be affected by ongoing price and quality negotiations between the two parties.

Bargaining games which allow parties to make both investment and consumption decisions for a sequential number of times are almost unexplored (for a discussion of the related literature see below). We develop a model where two risk-averse players attempt to agree on how to share a surplus between consumption and investment and on how to split the residual surplus among themselves. The level of investment affects the future capital stock and consequently, the surplus available in the following bargaining stage. The problem is complex. Not only do parties need to solve a (potentially protracted) bargaining stage, but also a dynamic accumulation problem since the agreement they reach at a specific stage affects future bargaining possibilities. To address this problem, we focus on linear strategies (we will then show that this is a weak restriction). The most important results are, first of all, that there is a unique stationary Markov Perfect Equilibrium (MPE) characterised by immediate agreement. The intuition is that, roughly speaking, a player can always invest the entire surplus to avoid a costly rejection (see section 3 for more details). Although the equilibrium is unique, there are three types of outcomes that can arise. First, when players are sufficiently impatient (or, alternatively, frictions in the bargaining process are relevant), both parties invest little on future bargaining possibilities, and extract all the residual surplus. Second, when parties are sufficiently asymmetric, such extreme consumption paths, where one party can extracts all the surplus not invested, can be obtained and the relatively impatient party makes the concessions. Third, there is an MPE in which both parties are able to consume a positive share of the current surplus even when they are responders.

An interesting result related to the third MPE (typical when parties are both fairly symmetric and patient), is that the investment plan is inefficient: as in the standard hold-up problem (described below), parties underinvest. Only at the limit, when the frictions in the bargaining stage tend to zero (or alternatively, parties are infinitely patient), does the investment path tend to the socially optimal level. Moreover, players invest more if more patient. However, patience can make a player worse off, in contrast to one of the lessons offered by classical bargaining theory.[1] In our dynamic set-up, this implies that bargaining parties can obtain a more profitable outcome if less established and more prone to dissolution.

Our results are based on a standard alternating-offer bargaining procedure. To make our analysis robust we modify the bargaining process by allowing for a random-proposer procedure. Therefore, after a rejection a player can invest again with a positive probability. We show that our results are robust. A more patient party invests more in the long-run relationship. Moreover, patience can make a player worse off. In terms of efficiency, again players that are both symmetric and infinitely patient can invest the socially optimal level of surplus.

A novel result specific to the random-proposer procedure is that when the more patient party is less likely to make a proposal, he reduces the resources invested in the long-term relationship. It is intuitive that when a party cannot control the relevant terms of the negotiations (i.e., making a proposal) he will reduce the resources devoted to the relationship. Interestingly, the opponent counteracts this trend (by increasing his investment).

Since we restrict our analysis to linear strategies, we provide a justification of this by investigating an asymptotic game in which the number of bargaining stages is finite but tends to infinity. We solve the game numerically and we show that in this asymptotic game, the strategies are linear and coincide with the ones we obtain when the game has an infinite number of rounds.

There are two main strands of literature, namely, on the *hold-up* problem and on the *tragedy of the commons,* which are related to the problem considered in this paper, however, there are fundamental differences between these models and our dynamic bargaining game with investment. In the hold-up problem, parties have the ability to make sunk investments that affect the size of a surplus, before bargaining over the division of such a surplus. Since the investor, who bears all the costs of the investment, cannot appropriate all the benefits, the resulting investment is lower than the efficient level. Typically, only one party is involved in the investment problem, moreover, the investment is once

[1] Also Sorger (2006) and Houba et al. (2000) obtained this result in their frameworks (discussed below).

and for all (see, for instance, Gibbons 1992, Muthoo 1998, Gul 2001).[2] Differently, the focus of this paper is on parties who *jointly* and *repeatedly* need to agree on how much to invest and consume.

The second strand of literature, on the tragedy of the commons, considers different parties who can extract part of a surplus for their own consumption and the remaining surplus will affect the size available in the next period (see, for instance, Levhari and Mirman, 1980, Dutta and Sandaram, 1993). The tragedy of the commons consists in the fact that parties consume more than the efficient level and therefore the surplus extinguishes quickly (over-exploitation of natural resources is a classic example). Although, the typical framework analysing the problem of the tragedy of the commons is a dynamic accumulation game, this does not include any negotiation: everyone can consume as much as he wishes, given the stock available. Bargaining has recently been introduced in these dynamic accumulation games, however, in a simplified manner (Houba et al., 2000 and Sorger, 2006). Indeed, in Houba et al. (2000) parties can potentially bargain forever (á la Rubinstein), but they need to agree once, since this agreement will be ever-lasting. Sorger (2006) is closer to our paper, since parties in each period can reach an agreement over the levels of consumption (Sorger (2006) also allows for endogenous threat points), however, the bargaining process is simplified since it is given by the solution of the Nash products. We consider different non-cooperative bargaining procedures, and characterise (analytically for some cases) the strategic behaviour that arises in equilibrium.

As far as we know the only paper with a focus on a repeated (non-cooperative) bargaining game with investment decisions in addition to the standard consumption decisions, is Muthoo (1999). However, the most important difference with our paper is that in Muthoo (1999) the focus is on steady-state stationary subgame perfect equilibria, while ours is on MPE. This implies that in the former, the investment decisions are strongly simplified since parties need to invest as much as it is necessary so as to have surpluses of the same size. Indeed, Muthoo's aim is to apply his infinitely repeated game where parties share an infinite number of cakes with the same size (Muthoo, 1995). In this sense the problem of how much parties should invest remains open.

The paper is organised as follows. In the next section we present the model. In section 3 we analyse the MPE. We show that there is a unique MPE, although, according to the value of the parameters some equilibria are at the corner (where parties consume all the surplus not invested). We investigate the solution under a random-proposer procedure in sec-

[2] An exception is in Che and Sákovics (2004) where parties keep investing until an agreement has been reached, however, once this is struck, the game ends.

tion 5. In the following section, we provide a justification for the linear strategies by analysing a game with a finite number of bargaining stages, where the number of bargaining stages increases indefinitely. Some final remarks are in section 7. Most of the proofs are in the Appendix.

2 The Model

We consider a two-player bargaining game in which bargaining and production stages alternate (and each stage can start only after the other has taken place). At the production stage, a surplus is generated according to the production function $F(k_t) = Gk_t$, where k_t is the capital stock at period t, with $t = 0, 1, ...$ and G is the constant gross rate of return. Production takes place in an interval of time τ. Once the output is generated, $F(k_t)$, the bargaining stage begins and players attempt to divide $F(k_t)$. The bargaining stage is a classic infinitely-repeated alternating-offer bargaining game (Rubinstein, 1982). A proposal by player i is a pair $({}_ix_t,{}_i I_t)$, where ${}_iI_t$ is the investment level proposed by i and ${}_ix_t$ is the share demanded by i over the remaining surplus. The subscript t indicates the dependence of the proposal $({}_ix_t,{}_i I_t)$ on capital at time t, denoted by k_t, that is the state variable in the model. If there is an acceptance, the bargaining stage ends and the proposer's current per-period utility is $u_i({}_ix_t,{}_i I_t)$ with

$$u_i({}_ix_t,{}_i I_t) = \begin{cases} \frac{{}_ic_t^{1-\eta}}{1-\eta} \text{ for } \eta \neq 1 \\ \ln({}_ic_t) \text{ for } \eta = 1 \end{cases} \tag{1}$$

where[3] ${}_ic_t = {}_ix_t(F(k_t) - {}_iI_t)$ is the level of consumption. The output available at the next bargaining stage (at $t+1$) is $F(k_{t+1})$, where k_{t+1} is the capital stock in the next period and it is given by the investment level ${}_iI_t$ and the capital remaining after depreciation, $k_{t+1} = {}_iI_t + (1-\lambda)k_t$, where λ is the depreciation rate ($0 < \lambda \leq 1$). Regardless of whether the proposal at t has been successful the responder at t is the next proposer. If there is a rejection, after an interval of time Δ, the rejecting player can make a counter-offer. We assume that the capital stock remains unchanged.[4] In perpetual disagreement, parties consume ${}_ic_t = 0$. For the cases of $\eta \geq 1$, we need to impose that players do not receive any utility during a temporary disagreement but only once an agreement

[3]Sorger (2006) uses the same per-period utility forms while Muthoo (1995, 1999) can assume linear per-period utilities, given that the investment problem is strongly simplified.

[4]Alternatively, production takes place again and therefore the capital stock depreciates. In this case, the qualitatity results we show in the paper are unaffected, since the equilibrium is characterised by no delays even when the capital stock does not shrink after a rejection.

has been reached or, alternatively, when the (perpetual) disagreement outcome arises.[5] Player i's time preference is represented by his discount rate h_i (with $i = 1, 2$). Since intervals of time have different lengths, we introduce in our model two distinct discount factors[6]: the between-cake discount factor $\alpha_i = exp(-h_i \tau)$ which takes into account that production takes time and the between-cake discount factor $\delta_i = exp(-h_i \Delta)$ that takes into account that there is an interval of time between a rejection and a new proposal. In the first period, at $t = 0$, a bargaining stage starts and the surplus available is 1, by assumption. The time line in a specific example of this game is represented in figure 1 below.

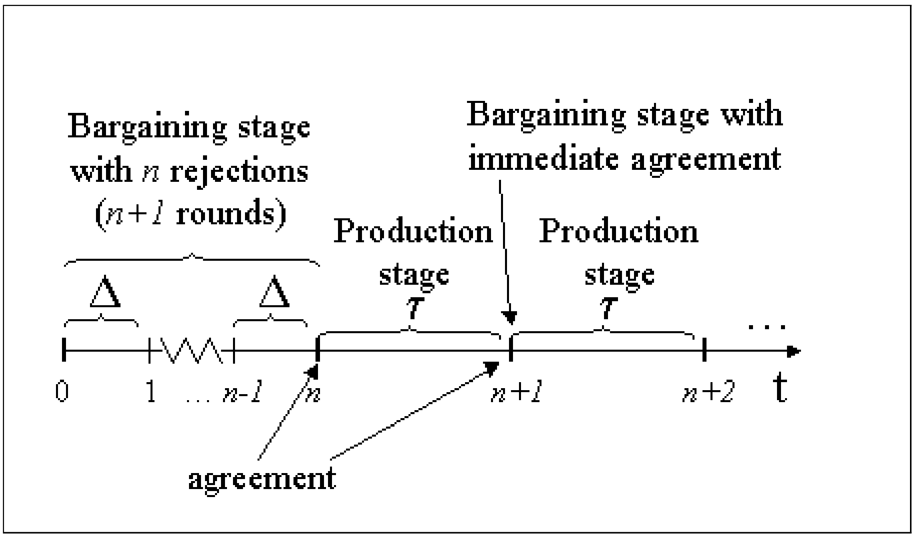

Figure 1. Time line for a game where the first two bargaining stages are characterised by n rejections (with $n \geq 0$) and immediate agreement respectively.

The focus is on (stationary) MPE, where the Markov strategies specify players' actions for each time period t as a function of the state of the system at the beginning of that period, k_t. Moreover, the aim of our analysis is to define time-invariant linear rules which describe the investment and consumption path as a linear function of the state k_t. In other words, the rules are identified by the share of the surplus invested, φ_i, and the share of the residual surplus that is consumed by proposer i (responder j), x_i ($1 - x_i$, respectively).

Player i is willing to make an acceptable offer to player j if this is

[5] The case of $\eta > 1$ is empirically relevant according to recent macroeconomic studies (see, for instance, Leith and Malley, 2005 where η is estimated to be 2).

[6] Muthoo (1995) considers the possibility of an interval of time between different bargaining stages. In Flamini (2007) we study the effects of this complex discounting structure within the agenda formation problem.

profitable, that is, $V_i(k_t) \geq \delta_i W_i(k_t)$ where $V_i(k_t)$ is the optimal expected utility to player i as a proposer, while $W_i(k_t)$ is the optimal expected utility to player i as a responder. Then, the problem for player i can be written in the following recursive form:

$$V_i(k_t) = \max_{\substack{x_i \in [0,1] \\ \varphi_i \in [-(1-\lambda), G]}} \frac{[x_i(G-\varphi_i)k_t]^{1-\eta}}{1-\eta} + \alpha_i W_i(k_{t+1}) \tag{2}$$

$$\text{s.t. } \frac{[(1-x_i)(G-\varphi_i)k_t]^{1-\eta}}{1-\eta} + \alpha_j V_i(k_{t+1}) \geq \delta_j V_j(k_t) \tag{3}$$

if the offer is accepted, otherwise

$$V_i(k_t) = \delta_i W_i(k_t) \text{ and } W_j(k_t) = \delta_j V_j(k_t) \tag{4}$$

$$\text{with } k_{t+1} = \begin{cases} (1-\lambda+\varphi_i)k_t \text{ if there is an acceptance} \\ k_t \text{ otherwise} \end{cases} \tag{5}$$

with $i, j = 1, 2$ and $i \neq j$. $V_i(k_t)$ is also called the *value function* of the Bellman equation (2). The equation of motion, (5), specifies whether and how the capital stock is modified once a proposal is either accepted or rejected.

Problem (2)-(5) is a recursive constrained problem with a complex structure since not only does (2) have a recursive form, but the constraint (3) embodies another recursive problem (via the value function $V_j(k_t)$) Although, generally such problems cannot be solved (see Stokey and Lucas, 1989, Ljungqvist and Sargent, 2000), we can characterise the properties of the equilibrium outcome and we can also obtain an analytical solution under certain conditions. The crucial assumption is that the Markov strategies are linear, that is, the surplus invested and consumed by each player is a linear function of the capital stock. As explained in the introduction, this assumption is not simply for tractability. In section 6, we provide a justification for this choice, in particular, we focus on a game with a finite number of bargaining stages (which asymptotically tends to infinity) and we show that in such a game the equilibrium strategies are linear.

3 Linear MPE Strategies

In this section, we focus on the CES utility. Since in the model the capital stock is unchanged (or non-increasing) after a rejection, in a stationary Markov equilibrium delays cannot be sustained. The intuition is that when the parameter η is smaller than 1, a player can always invest an appropriate amount of surplus so that a rejection is unprofitable to the responder. For the case of η larger than 1 instead, the stationarity of the MPE strategies allows us to exclude temporary delays (note that temporary delays are not costly in our model for $\eta > 1$, see note 3). This is formally proved in the following lemma.

Lemma 1. *Delays are not sustainable in a stationary linear MPE.*
Proof. In Appendix. ■

When the equilibrium strategies have a linear form, the value function is linear as well. Therefore, we write the value function as a linear function of the state variable with coefficients which are left undetermined, we then derive the values of the coefficients that make the guess correct. Since an MPE is characterised by no delays (Lemma 1), then players' optimisation problem can be written as follows:

$$\phi_i \frac{k_t^{1-\eta}}{1-\eta} = \max_{\substack{x_i \in [0,1] \\ \varphi_i \in [-(1-\lambda), G]}} \frac{(x_i(G-\varphi_i)k_t)^{1-\eta}}{1-\eta} + \alpha_i \mu_i \frac{k_{t+1}^{1-\eta}}{1-\eta} \tag{6}$$

$$\frac{[(1-x_i)(G-\varphi_i)k_t)]^{1-\eta}}{1-\eta} + \alpha_j \phi_j \frac{k_{t+1}^{1-\eta}}{1-\eta} \geq \delta_j \phi_j \frac{k_t^{1-\eta}}{1-\eta} \tag{7}$$

$$\mu_i \frac{k_t^{1-\eta}}{1-\eta} = \frac{[(1-x_j)(G-\varphi_j)k_t)]^{1-\eta}}{1-\eta} + \alpha_i \phi_i \frac{k_{t+1}^{1-\eta}}{1-\eta} \tag{8}$$

$$k_{t+1} = (1-\lambda+\varphi_i)k_t \tag{9}$$

with $i, j = 1, 2$ and $i \neq j$. Let $l = G + 1 - \lambda$ and $b_i = [\alpha_i^\eta (\alpha_j l)^{1-\eta}]^{\frac{2}{2\eta-1}}$ with $b_i \in (0,1)$ and $\eta \neq 1/2$. The following proposition defines the first type of MPE, which we call ultimatum-like, since a player can extract all the surplus not invested as he does typically under an ultimatum procedure.

Proposition 1 *If $\eta \in (0,1)$ but $\eta \neq 1/2$ and*

$$\frac{\delta_j}{\left(\alpha_j^{1-2\eta+2\eta^2} \alpha_i^{2\eta(1-\eta)}\right)^{\frac{1}{2\eta-1}}} \leq l^{\frac{1-\eta}{2\eta-1}} < \left(\frac{1}{\alpha_i^\eta \alpha_j^{1-\eta}}\right)^{\frac{1}{2\eta-1}} \tag{10}$$

with

$$\delta_j < \left(\frac{\alpha_j}{\alpha_i}\right)^\eta \tag{11}$$

there is a unique MPE in which a proposer consumes all the surplus not invested ($x_i = 1$) and invests a share given by

$$\varphi_i = (lb_i - (1-\lambda)) = \left(l(\alpha_i^\eta \alpha_j^{1-\eta})^2\right)^{\frac{1}{2\eta-1}} - (1-\lambda) \tag{12}$$

for $i, j = 1, 2$ with $i \neq j$.

Proof. in Appendix. ■

In an ultimatum-like MPE a proposer can consume all the surplus not invested as long as he makes an appropriate proposal over the investment path (so as to obtain an acceptance). An important feature of this equilibrium is that patience can make a player worse off.[7] Indeed, if $\eta < 1/2$, the share invested by a player decreases when he (or his opponent) becomes more patient. Moreover, a proposer's expected payoff is also lowered by a higher discount factor. Instead, for η in $(1/2, 1)$, the investment increases with patience. When a player or his opponent becomes more patient, his investment increases. As a result, the expected payoffs of both a proposer and a responder are higher.

It is known that in long-run relationships, when players share an infinite number of cakes of the same size (see Muthoo, 1995), players can have extreme forms of bargaining power where proposers consume all the residual surplus. This is the case when simply $\Delta \geq \tau$ (see Muthoo 1995, p. 594). In our framework where parties can make decisions over the investment level, an ultimatum-like MPE can be sustained when[8]

$$\delta_j \leq \alpha_j \left(lb_i\right)^{1-\eta}$$

$i, j = 1, 2$ with $i \neq j$. Therefore, the ultimatum-like MPE can hold not only when the production stage is relatively quick in comparison to the length of a round ($\Delta \geq \tau$) but also in the more interesting case in which this is not true ($\alpha_i \leq \delta_i$, for any i) as long as the investment is sufficiently large, that is, $lb_i \geq (\delta_j/\alpha_j)^{1/(1-\eta)}$. However, at the limit of the interval Δ that tends to 0, condition (11) cannot hold for both $i, j = 1, 2$ with $i \neq j$.

Assuming that the between-cake discount factor is lower than the within-cake discount factor ($\Delta < \tau$), the equilibrium defined in proposition 1 exists only for a given range of values of l (given by (10)). To give an idea of how relevant this range can be, let's consider the case of symmetric players. Then Proposition 1 holds if

$$\delta \leq \left(\alpha l^{1-\eta}\right)^{\frac{1}{2\eta-1}} < 1$$

For instance, if $\eta = 0.3$, $\delta_i = 0.9$, $\alpha_i = 0.8$, condition (10) implies that $l = G+1-\lambda$ is in $(1.38, 1.46]$. When the frictions in the bargaining stage tend to disappear (δ increases), the relevant interval for the parameters gets smaller and smaller. And, as already mentioned, at the limit for Δ that tends to 0, proposition 1 does not hold.

[7] This is in accordance with the results obtained by Houba et al. (2000) and Sorger (2006).

[8] Indeed, this condition, together with $b_i \in (0, 1)$, is equivalent to (10) and (11).

The next two propositions define the MPE when proposition 1 does not hold. In particular, in the next proposition we show that strongly asymmetric demands can be sustained in equilibrium (only one player is able to extract all the surplus not invested), while in proposition 3, both players leave a positive share of consumption to the responder.

Proposition 2 *For $\eta < 1$, assume that the conditions (10) and (11) do not hold, instead, if there is a solution (ψ_i, ψ_j, m_j) to the following system*

$$\psi_i = 1 + \frac{\alpha_i \delta_i l^{1/\eta} m_j^{(1-\eta)/\eta}}{\delta_i \psi_j^{1-\eta} - \alpha_i l^{1-\eta}(\psi_j - 1 - m_j^{1/\eta})^{1-\eta}}$$

$$\psi_j = \Big(\frac{(\alpha_j l^{1-\eta})^2 (\psi_i - 1)^{1-\eta}}{(\psi_i \psi_j)^{1-\eta} - (\alpha_i l^{1-\eta})^2 (\psi_i - 1)^{1-\eta} (\psi_j - 1 - m_j^{1/\eta})^{1-\eta}} + \\ + \frac{\alpha_i m_j l^{1-\eta}}{\psi_i^{1-\eta} - \alpha_i \delta_i l^{1-\eta}(\psi_i - 1)^{1-\eta}} \Big)^{1/\eta} + 1 + m_j^{1/\eta} \tag{13}$$

$$\frac{m_j^{(1-\eta)/\eta}}{\delta_i \psi_j^{1-\eta} - \alpha_i l^{1-\eta}(\psi_j - 1 - m_j^{1/\eta})^{1-\eta}} = \frac{1}{\psi_i^{1-\eta} - \alpha_i \delta_i l^{1-\eta}(\psi_i - 1)^{1-\eta}}$$

with $\psi_i, \psi_j, m_j > 0$ and s.t.

$$\frac{\delta_j}{\alpha_j l^{1-\eta}} \leq \left(1 - \frac{1}{\psi_i}\right)^{1-\eta} < \frac{1}{l^{1-\eta}\alpha_i \delta_i} \tag{14}$$

$$0 < \left(1 - \frac{1 + m_j^{1/\eta}}{\psi_j}\right) < \frac{\delta_i}{\alpha_i l^{1-\eta}}$$

then the MPE proposals are the following:

$$x_i = 1 \text{ and } \varphi_i = G - \frac{l}{\psi_i} \tag{15}$$

$$x_j = \frac{1}{1 + m_j^{1/\eta}} \tag{16}$$

$$\varphi_j = G - l\frac{1 + m_j^{1/\eta}}{\psi_j} \tag{17}$$

with $i, j = 1, 2$ and $i \neq j$.

Proof. in Appendix. ∎

In general, we cannot solve analytically system 13, however, we can solve it numerically and we found that the solution is unique when it exists. Moreover, the MPE in which player i is able to behave as in an ultimatum can exist for

$$\frac{\delta_j}{\alpha_j} \leq \frac{1}{\alpha_i \delta_i}$$

see (14). This implies that such MPE are more common when player i is sufficiently impatient. Indeed, if both α_i and δ_i tend to 1, the MPE can still exist but only for $\delta_j \leq \alpha_j$. For instance, for $\eta = 2/3$, $\alpha_i = 0.8$, $\delta_i = 0.9$, $l = 1.8$, there is an MPE for $\alpha_j = 0.4 = \delta_j$ (where $m_j = 0.292$, $\psi_i = 3.675$; $\psi_j = 2.252$). However, if the within-cake discount factor δ_j increases (say to 0.45), then such an MPE does not exist. Generally, a change in the within-cake discount factor δ_j may affect the inequality (14), however, if this still holds, a change in δ_j will not affect players' strategies (the system 13 does not depend on δ_j, the within-cake discount factor of the weaker player). Indeed, player i will still be able to obtain a share x_i equal to 1, while player $j's$ proposal is such that player i is indifferent between accepting and rejecting it.

The most aggressive player (i) not only consumes all the residual surplus (after investment), but can also obtain a larger payoff both as a proposer (using the previous example $\phi_i = 3.716 > \phi_j = 1.114$) and as a responder ($\mu_i = 3.344 > \mu_j = 0.488$). Some numerical comparative statics show that in general in the ultimatum-like MPE, patience makes players better off (for any $\eta < 1$). In particular, if α_i increases (say, using the previous example, to 0.85), then all payoffs increases ($\phi_i = 5.225$, $\phi_j = 1.138$, $\mu_i = 4.703$ and $\mu_j = 0.512$), mainly due to an increase in player 1's investment (ψ_i increases to 4.997). Instead, an increase in the within-cake discount factor δ_i increases player i's power only (in the sense that his payoffs increase) and makes player j worse off (e.g., if $\delta_i = 0.95$, then ϕ_i and μ_i increase to 4.978 and 4.729, respectively while ϕ_j and μ_j decrease to 0.966 and 0.435, respectively). Finally, an increase in α_j increases player j's power only but leaves player i's payoffs (almost) unaffected (e.g., if $\alpha_j = 0.45$, then ϕ_j and μ_j increase to 1.178 and 0.580, respectively while ϕ_j and μ_j are unchanged). Indeed, player j is able to consume more for a given state k_t (x_j increases since the multiplier decreases to 0.274) by investing a higher share.

Next, we assume that the propositions above do not hold, the following proposition defines the MPE when the shares demanded are interior. Let

$$M_i = \left\{ (m_i, \psi_i) | m_i, \psi_i > 0, 0 < l^{1-\eta} \left(1 - \frac{(1 + m_i^{1/\eta})}{\psi_i} \right)^{1-\eta} < \min \left(\frac{\delta_j}{\alpha_j}, \frac{1}{\alpha_i} \right) \right\}$$

for $i = 1, 2$.

Proposition 3 *Assume that proposition 1 and 2 do not hold, instead, if there is a solution $(\psi_i, m_i) \in M_i$ to the following system:*

$$g_i = l^{1-\eta}\left(\frac{\alpha_i\delta_i m_j^{\frac{1-\eta}{\eta}}}{\psi_j^{1-\eta}\delta_i - \alpha_i l^{1-\eta} g_j^{\frac{1-\eta}{\eta}}} + \frac{\alpha_j m_i}{\psi_j^{1-\eta} - \alpha_i\delta_i l^{1-\eta} g_j^{\frac{1-\eta}{\eta}}}\right) \tag{18}$$

$$\frac{m_j^{\frac{1-\eta}{\eta}}}{\psi_j^{1-\eta}\delta_i - \alpha_i l^{1-\eta} g_j^{\frac{1-\eta}{\eta}}} = \frac{1}{\psi_i^{1-\eta} - \alpha_i\delta_i l^{1-\eta} g_i^{\frac{1-\eta}{\eta}}} \tag{19}$$

where $g_i = \left(\psi_i - (1 + m_i^{1/\eta})\right)^\eta$, *then the MPE demands* (x_i, φ_i) *are as follows:*

$$x_i = \frac{1}{1 + m_i^{1/\eta}} \tag{20}$$

$$\varphi_i = G - \frac{l(1 + m_i^{1/\eta})}{\psi_i} \tag{21}$$

while the value of the (undetermined) coefficients are defined below:

$$\phi_i = \frac{l^{1-\eta}}{\psi_i^{1-\eta} - \alpha_i\delta_i l^{1-\eta} g_i^{\frac{1-\eta}{\eta}}} \text{ and } \mu_i = \frac{l^{1-\eta}\delta_i m_j^{\frac{1-\eta}{\eta}}}{\psi_j^{1-\eta}\delta_i - \alpha_i l^{1-\eta} g_j^{\frac{1-\eta}{\eta}}} \tag{22}$$

with $i, j = 1, 2$ *with* $i \neq j$.

Proof. in Appendix. ■

The properties of the equilibrium can be highlighted in the following remarks and corollaries.[9]

Remark 1. *The most patient party invests more.*

Let's assume that player 2 is more patient than 1 (for instance, $\alpha_1 = 0.8$, $\alpha_2 = 0.85$, $\delta_1 = 0.9$ and $\delta_2 = 0.95$ with $\eta = 1/4$ and $l = 1$), then player 2 consumes less than player 1 ($m_2 = 0.789 < 1.095 = m_1$, see (20)) but invests more (the gross investment is $r_i = 1 + \lambda + \varphi_i$ and $r_1 = 0.4766 < r_2 = 0.4787$) and his discounted payoffs are larger than his opponent (μ_2 =0.851 > μ_1 = 0.483, ϕ_1 =0.537< ϕ_2 = 0.896).

Remark 2. *Patience can make players worse off.*

We first consider the effects of a change in players' between-cake discount factors and show the remark above then we study the effects of

[9] We obtain unique numerical solutions to the system (18) and (19). when it exists.

a change in the within-cake discount factors and show that the effect on payoff is positive (patience makes a player better off). Let's assume that player 2 is the most patient party, for instance, $\alpha_1 = 0.7$, $\alpha_2 = 0.8$, $\delta_1 = 0.8$ and $\delta_2 = 0.9$, with $l = 1.1$ and $\eta = 2$. Since $\eta > 1$, then the optimal expected utility of a proposer (responder) is a decreasing function of the guessed parameters ϕ (μ). The impatient party obtains less than his rival (in this example $\phi_1 = 131.9 > 116.8 = \phi_2$ and $\mu_1 = 105.5 > 105.1 = \mu_2$). However, if player 1 becomes more patient, in particular, α_1 increases to 0.72 (note that he is still more impatient than 2), then player 1 is worse off (ϕ_1 increases to 176.8 and μ_1 increases to 141.5) while player 2 is better off (ϕ_2 decreases to 112.3 and μ_2 to 101.1).

If the most patient party (player 2 in this example) becomes even more patient, then not only he is worse off (his payoffs decrease) but his expected discounted payoffs become lower than his rival's. For instance, if player 2's between-cake discount factor increases to $\alpha_2 = 0.82$ (fixing $\alpha_1 = 0.7$, $\delta_1 = 0.8$ and $\delta_2 = 0.9$), then ϕ_1 decreases to 124.37 and μ_1 to 99.5, while ϕ_2 increases to 152.1 and μ_2 increases to 136.9 (therefore, $\phi_1 < \phi_2$ and $\mu_1 < \mu_2$). This is due to the fact that while both players increase their investment shares, player 2's consumption decreases while player 1's increases (for a given k_t). Therefore, patience can make a player worse off.

The effect of an increase in the within-cake discount factor on players' payoffs is more straightforward. Indeed, if the within-cake discount factor δ_i increases, player i is better off while player j is worse off. For instance, using the example above ($\alpha_1 = 0.7$, $\alpha_2 = 0.8$, $\delta_1 = 0.8$ and $\delta_2 = 0.9$, with $l = 1.1$ and $\eta = 2$) if δ_2 increases to 0.92, then player 2's payoffs decrease ($\mu_2 = 84.5$ and $\phi_2 = 91.8$) while player 1's payoffs increase ($\mu_1 = 120.2$ and $\phi_1 = 150.3$) and the most patient party obtains the largest payoffs.

Corollary 1. *For $\eta = 1/2$, $h_i = h$ (i.e., $\delta_i = \delta$, $\alpha_i = \alpha$ for $i = 1, 2$), if $\alpha^2 l < 1$, there is a unique symmetric equilibrium in which the each player proposes the following division:*

$$x = \frac{1}{1+m^2} \tag{23}$$

$$\varphi = G - \frac{l(1+m^2)}{\psi} \tag{24}$$

where the auxiliary variable ψ and the multiplier m are the following:

$$\psi = \frac{(1+m^2)(1-\delta m)^2 \alpha^2 l}{\alpha^2 l (1-\delta m)^2 - (\delta - m)^2} \tag{25}$$

$$m = \frac{-(1-\delta^2)(1+\alpha^2 l) + \Delta^{\frac{1}{2}}}{2\delta(1-\alpha^2 l)} \tag{26}$$

with

$$\Delta = (1+\delta^4)(1+\alpha^2 l)^2 + 2\delta^2(1-\alpha^2 l)^2 - 8\alpha^2\delta^2 l$$

Proof. When $\eta = 1/2$ and $h_i = h$, it is straightforward to solve system (18)-(19). ■

Corollary 2. *For $h_1 = h_2$, at the limit for Δ that tends to 0, the stationary MPE is socially optimal:*

$$\lim_{\Delta\to 0} x = \frac{1}{2} \tag{27}$$

$$\lim_{\Delta\to 0} \varphi = (\alpha l)^{1/\eta} - (1-\lambda) \tag{28}$$

Proof. At the limit for Δ that tends to 0, (19) implies that the multiplier is 1, therefore the share consumed x is 1/2. Moreover, from (18) we obtain that ψ is $2(1-\alpha l^{1-\eta})^{-1/\eta}$. The latter, together with $m = 1$ in (21), implies (28). It can be shown that when players are symmetric, a social planner, who maximises the sum of players' discounted payoffs, would set $x = 1/2$ and φ as in (28). ■

That is, players with the same rate of time preference consume half of the residual surplus and invest a non-negative amount of surplus if sufficiently patient (i.e., $\alpha \geq (1-\lambda)^{1/2}/l$) otherwise players disinvest as a social planner would efficiently choose to do.[10]

Remark 3. *Bargaining can lead to underinvestment.*

Using (25) and (26), we can show that the investment strategy in (24) is smaller than the investment in (28) when players have some degree of impatience (i.e., $\delta < 1$, with $\eta = 1/2$, and $\alpha^2 l < 1$). Only infinitely patient players behave efficiently.

4 Logarithmic utility

A general result obtained in the previous section is that players with a CES utility agree on how to share a surplus immediately. Moreover, when the frictions in bargaining stage get smaller (i.e., $\Delta \to 0$), symmetric players invest efficiently and split the residual surplus equally. In this section, we investigate linear MPE without delays for players with logarithmic utilities, and show that such equilibria are not as pervasive as for the case of the general CES per-period utilities. These can exist

[10]This result is in accordance with Lockwood and Thomas (2002), which shows that the level of cooperation among players tends to the efficient level in the limit as players become patient, although their framework is quite different from ours: players are symmetric, cannot bargain and cannot reverse their actions (while in our model, parties are allowed to underinvest, $\varphi < 0$, for instance).

only under two regimes: either at the steady-state or at the limit for the interval Δ that tends to zero. Moreover, as for the CES utility, we obtain that symmetric players invest efficiently in frictionless negotiations.

Suppose that there is a linear MPE with no delays, where player i demands a share equal to x_i^* and invests a share φ_i^*. Let $r_i = 1-\lambda+\varphi_i$. Then the sum of the expected payoff of a proposer in equilibrium is

$$V_i(k_t) = \sum_{s=0}^{\infty} \alpha_i^{2s} \ln x_i^*(G-\varphi_i^*)k_t(r_ir_j)^s + \alpha_i \sum_{s=0}^{\infty} \alpha_i^{2s} \ln(1-x_j^*)(G-\varphi_j^*)k_t r_i^{s+1} r_j^s k_t$$

that is

$$V_i(k_t) = \frac{\ln\left(x_i^*(G-\varphi_i^*)k_t\right) + \alpha_i \ln\left((1-x_j^*)(G-\varphi_j^*)k_t r_i\right)}{1-\alpha_i^2} + \frac{\alpha_i^2 \ln(r_ir_j)}{(1-\alpha_i^2)(1-\alpha_i)}$$

Similarly the optimal expected payoff of a responder in this candidate equilibrium is

$$W_i(k_t) = \frac{\ln\left((1-x_j^*)(G-\varphi_j^*)k_t\right) + \alpha_i \ln\left(x_i^*(G-\varphi_i^*)k_t r_j\right)}{1-\alpha_i^2} + \frac{\alpha_i^2 \ln(r_ir_j)}{(1-\alpha_i^2)(1-\alpha_i)}$$

If such a pair (x_i^*, φ_i^*) exists then it maximise the following Lagrangian:

$$L_i(k_t) = \max_{\substack{x_i\in[0,1] \\ \varphi_i\in[-(1-\lambda),G]}} V_i(k_t) - m_i(\delta_j V_j - W_j)$$

where $m_i \geq 0$ is the Kuhn-Tucker multiplier. The first order conditions of $L_i(k_t)$ with respect to x_i and φ_i are as follows:

$$x_i^* = \frac{1}{1+m_i} \tag{29}$$

$$\varphi_i^* = G - l\frac{a(m_i)}{a(m_i)+b(m_i)} \tag{30}$$

where $a(m_i) = \frac{1}{1-\alpha_i^2} + \frac{m_i}{1-\alpha_j^2}$ while $b(m_i) = \frac{\alpha_i}{(1-\alpha_i^2)(1-\alpha_i)} + \frac{\alpha_j m_i}{(1-\alpha_j^2)(1-\alpha_j)}$, with $i,j = 1,2$ and $i \neq j$. Note that in the case of symmetry $(h_i = h)$ the share invested becomes

$$\varphi^* = l\alpha - (1-\lambda) \tag{31}$$

The investment plan (31) is consistent with the investment in the ultimatum-like MPE, (12), for $\eta = 1$ and the socially optimal level (see (28), again for $\eta = 1$).

The first order condition of $L_i(k_t)$ with respect to m_i, is the constraint $W_j(k_t) = \delta_j V_j(k_t)$, or using (29) and (30):

$$(1-\alpha_i\delta_i)\left[\ln\left(\frac{m_j}{1+m_j}\frac{la(m_j)}{a(m_j)+b(m_j)}\right)+\frac{\alpha_i}{1-\alpha_i}\ln\left(\frac{lb(m_j)}{a(m_j)+b(m_j)}\right)\right]+$$

$$-(\delta_i-\alpha_i)\left[\ln\left(\frac{1}{1+m_i}\frac{la(m_i)}{a(m_i)+b(m_i)}\right)+\frac{\alpha_i}{1-\alpha_i}\ln\left(\frac{lb(m_i)}{a(m_i)+b(m_i)}\right)\right]+$$

$$+(1-\delta_i)(1+\alpha_i)\ln(k_t)=0 \tag{32}$$

for $i, j = 1, 2$ and $i \neq j$. This is a system in two unknown variables m_i with $i = 1, 2$. In general, the constraint is binding, otherwise by the complementary slackness conditions, m_i must be zero (and $x_i = 1$), however an offer where a proposer does not leave any current positive consumption to a responder is always rejected (since the right-hand side of (32) is negative or $W_j(k_t) < \delta_j V_j(k_t)$). This implies that generally the solution to (32) is a function of the state k_t. If the multiplier m_i is time-dependent, then the share consumed x_i^* and invested φ_i^* are are also time-dependent in contradiction with our assumption (the guessed parameters would also be wrong since are based on the time-invariant linear rules of the consumption and investment paths). However, under certain conditions we can obtain an analytical solution to the problem as shown below.

Proposition 4 *There is a steady-state MPE for $h_i = h$, $\Delta = \tau$ and $l\alpha = 1$, where players invest only to maintain the same capital stock k_0 ($\varphi_i = \lambda$) and demand to consume a share*

$$x_i^* = 1 - \frac{1}{(1-\alpha)lk_0} = 1 - \frac{1}{(l-1)k_0} \tag{33}$$

with $i = 1, 2$. Alternatively, at the limit for $\Delta \to 0$, symmetric players (i.e., $h_i = h$) invest the efficient level $\varphi^* = l\alpha - (1-\lambda)$ and consume a share equal to

$$x^* = \frac{1}{2} \tag{34}$$

with $i = 1, 2$.

Proof. in Appendix. ■

For the first case considered in this proposition, we obtain an MPE where players invest more and decrease the share x_i^* if they become more patient (α increases), while for the second case, we obtain that bargaining can be efficient in a frictionless bargaining game (as in section 3).

5 Random Proposer Procedure

To establish the robustness of our result we relax the assumption of the alternating-offer bargaining procedure and assume a random-proposer procedure. That is, both after a rejection and after an acceptance (and production), there is a positive (constant) probability that a player becomes a proposer. Let p_i be the probability that player i becomes a proposer, with $i = 1, 2$ and $p_1 = 1 - p_2$. We will now show that most of our qualitative results are robust to this change. First of all, we can establish the following result.

Remark 4. *Delays are not sustainable in a stationary linear MPE.*

The proof follow the same argument as the proof of Lemma 1 and it is therefore omitted. The intuition is that since for the alternating-offer bargaining procedure the stationary MPE is characterised by immediate agreement, a fortiori, in an game where there is a positive probability that a player will become again the proposer in the next period, a stationary MPE must be characterised by immediate agreement.

Given the linearity of the MPE strategies, we can assume that the value function has the same form as the per-period utility function. Then, a proposer's recursive problem is as follow:

$$\frac{\phi_i k_t^{1-\eta}}{1-\eta} = \max_{\substack{x_i \in [0,1] \\ \varphi_i \in [-(1-\lambda), G]}} \frac{(x_i(G-\varphi_i)k_t)^{1-\eta}}{1-\eta} + \alpha_i \beta_i \frac{k_{t+1}^{1-\eta}}{1-\eta} \tag{35}$$

$$\beta_i = p_i \phi_i + (1-p_i)\mu_i \tag{36}$$

$$\mu_j \geq \delta_j \beta_j \tag{37}$$

$$k_{t+1} = k_t(1-\lambda+\varphi_i) \tag{38}$$

As for the case of the alternating-offer bargaining procedure, ultimatum-like MPE can be sustainable when parties are sufficiently impatient and/or asymmetric (see proposition 1 and 2). However, when players are sufficiently patient and symmetric, they do have some bargaining power in equilibrium, in the following proposition, we focus on the case where the indifference conditions are binding. Let $c_i = 1 - (1 - p_i)\delta_i$ and

$$M_i = \left\{ (m_i, \psi_i) | m_i, \psi_i > 0, 0 < l^{1-\eta} \left(1 - \frac{(1 + (c_j m_i)^{1/\eta})}{\psi_i} \right)^{1-\eta} < \min\left(\frac{\delta_j}{\alpha_j}, \frac{1}{\alpha_i} \right) \right\}$$

for $i = 1, 2$.

Proposition 5 *There is a linear MPE characterised by the following proposal by player i :*

$$x_i = \frac{1}{1 + (c_j m_i)^{\frac{1}{\eta}}} \tag{39}$$

$$\varphi_i = G - \frac{l(1 + (c_j m_i)^{1/\eta})}{\psi_i} \tag{40}$$

where $(\psi_i, m_i) \in M_i$ is the solution to the following system[11]*:*

$$\psi_i = (\alpha_i \beta_i + \alpha_j \beta_j c_j m_i)^{1/\eta} + 1 + (c_j m_i)^{1/\eta} \tag{41}$$

$$\frac{(c_i m_j)^{\frac{1-\eta}{\eta}}}{\psi_j^{1-\eta} \delta_i - \alpha_i l^{1-\eta} g_j} = \frac{p_i}{c_i \psi_i^{1-\eta}} \left(1 + \frac{\alpha_i l^{1-\eta} g_i (c_i m_j)^{\frac{1-\eta}{\eta}}}{\psi_j^{1-\eta} \delta_i - \alpha_i l^{1-\eta} g_j} \right) \tag{42}$$

with

$$\mu_i = \frac{l^{1-\eta} \delta_i (c_i m_j)^{\frac{1-\eta}{\eta}}}{\psi_j^{1-\eta} \delta_i - \alpha_i l^{1-\eta} g_j}, \beta_i = \frac{p_i}{c_i} \phi_i \tag{43}$$

$$\phi_i = \frac{l^{1-\eta}}{\psi_i^{1-\eta}} \left(1 + \frac{\alpha_i l^{1-\eta} g_i (c_i m_j)^{\frac{1-\eta}{\eta}}}{\psi_j^{1-\eta} \delta_i - \alpha_i l^{1-\eta} g_j} \right)$$

with $g_j = (\psi_j - 1 - (c_i m_j)^{\frac{1}{\eta}})^{1-\eta}$, $i, j = 1, 2$ and $i \neq j$.

Proof. We omit the proof[12] since the argument is as in the proof of proposition 3. ■

The proposition shows that the change in the bargaining procedure from alternating-offer to random-proposer does not change the nature of the equilibrium (the structure of the MPE proposal is similar). In the following, we first highlight some of the similarities with the alternating-offer procedure and then the properties which are instead typical only under the random-proposer procedure. As in the previous section we can show that, a more patient player invests more than his rival (e.g., let $\eta = 3/2$, α_1=0.8, δ_1=0.95, α_2=0.8, δ_2=0.99, p_1=1/2, l=1.1, then $r_1 = 1 + \lambda + \varphi_1$ =0.922, while r_2 =0.926). Moreover, as shown in the following corollary, we can re-establish the efficiency of bargaining.

Corollary 3. *For $h_1 = h_2$, $p_i = 1/2$, $\eta = 1/2$, the MPE strategies are characterised by the following unique solution:*

$$\psi = \frac{(1 + m^2)(1 - \delta m)^2 \alpha^2 l}{\alpha^2 l (1 - \delta m)^2 - (\delta - m)^2} \tag{44}$$

[11] A pair (m_i, ψ_i) in M_i characterises a real and positive MPE proposal. Moreover the tranversality condition is satisfied. There is a weaker condition for the tranversality condition, that is, $\alpha_i(p_i r_i^{1-\eta} + p_j r_j^{1-\eta}) < 1$, where r_i is the gross investment $(1 + \lambda + \varphi_i)$, with $i, j = 1, 2$ and $i \neq j$.

[12] This is available upon request.

$$m = 2\frac{-2(1-\delta) + \Delta^{\frac{1}{2}}}{(2-\delta)(\delta - \alpha^2 l - \delta(1-\delta))} \tag{45}$$

with

$$\Delta = 4[(1-\delta^3) - 2\delta(1-\delta)] + (\delta^2 + \alpha^2 l)^2 - 4\alpha^2 \delta l$$

Therefore, at the limit for Δ that tends to 0, the MPE strategies are efficient, that is:

$$\lim_{\Delta \to 0} x = \frac{1}{2}$$
$$\lim_{\Delta \to 0} \varphi = (\alpha l)^2 - (1-\lambda)$$

By comparing the multiplier m in the corollary above with the corresponding solution under the alternating-offer procedure (see (26)), we can show that a player is able to extract a larger share under the random-proposer procedure (cm is smaller than the multiplier in (26)). It is intuitive that if there is a probability that a proposer will be again in power and propose again, then he will be able to extract a larger surplus. Finally, an additional feature of the random-proposer procedure is highlighted in the following remark.

Remark 5. *If player i is more impatient than player j, then he invests less (for a given k_t) but if the probability that player i proposes increases then player j's level of investment decreases while player i's increases.*

We first show this result numerically. Let's assume that player 2 is more patient than player 1, for instance, $\alpha_1 = 0.7$, $\alpha_2 = 0.8$, $\delta_1 = 0.8$ and $\delta_2 = 0.9$, with $l = 1$ and $\eta = 1/2$. If the probability of proposing increases from $p_1 = 1/2$ to $2/3$ then the patient party decreases his gross investment from a share of $r_2 = 0.630$ to 0.572, while the impatient party increases his investment from $r_1 = 0.545$ to 0.550. The intuition is that a more patient player always prefers to invest more than his opponent. However, if his opponent is more likely to make an offer, then it is more likely that the total share of the surplus consumed increases (since the less impatient player will reduce the share invested as soon as he can make an offer), and the high investment made by a patient player is partially lost. For this reason, when the probability of becoming a proposer increases for the impatient player, the patient player reduces his investment plan. On the other hand, the impatient player has to increase the investment level to compensate for the fact that the investment made by his opponent is not only lower but also less likely. Additionally, the incentive to make an acceptable offer gives the impatient player an additional reason to invest more. Therefore, the incentive to increase current consumption, typical of an impatient player, is mitigated by his higher probability to propose in such a dynamic set-up.

6 An Asymptotic Case

Often the linearity of the strategies is assumed for tractability (see Houba et al., 2000 and Sorger, 2006). In this section, we provide an additional reason to limit the analysis to linear strategies. We show that in an asymptotic game in which the number of bargaining stages n is finite, but tends to infinity, the equilibrium strategies are linear. To simplify the analysis we assume that $\eta < 1$, the depreciation rate is at its maximum, $\lambda = 1$, and the gross rate G is 1. Moreover, we assume that players have the same rate of time preference and we focus on symmetric equilibria.

If n is finite the game can be solved backwards. Assume first that $n = 2$. At the last stage, players should invest nothing (or disinvest if the depreciation rate is less than 1) and divide the surplus as in the classic Rubinsteinian game (Rubinstein, 1982), with CES utilities. Therefore at the beginning of the game the problem of a proposer, say i, is given by the Lagrangian function L_i :

$$\begin{aligned} L_i = \max_{x_i, I_i} & \frac{[x_i(k_t - I_i)]^{1-\eta}}{1-\eta} + \frac{\alpha}{1-\eta}\left(\frac{\delta^{1/\eta}}{1+\delta^{1/\eta}} I_i\right)^{1-\eta} + \\ & -\lambda_i \left(\frac{[(1-x_i)(k_t - I_i)]^{1-\eta}}{1-\eta} + \frac{\alpha}{1-\eta}\left(\frac{I_i}{1+\delta^{1/\eta}}\right)^{1-\eta}\right) + \\ & +\lambda_i \delta \left(\frac{[x_j(k_t - I_j)]^{1-\eta}}{1-\eta} + \frac{\alpha}{1-\eta}\left(\frac{I_j}{1+\delta^{1/\eta}}\right)^{1-\eta}\right) \end{aligned} \tag{46}$$

where λ_i is the (non-positive) multiplier. The first order conditions imply that

$$x_i = \frac{1}{1 + (-\lambda_i)^{\frac{1}{\eta}}} \tag{47}$$

$$I_i = \varphi_i k_t \text{ where } \varphi_i = \frac{1}{1 + \frac{\left(1+(-\lambda_i)^{\frac{1}{\eta}}\right)(1+\delta^{1/\eta})^{\frac{1-\eta}{\eta}}}{\alpha^{\frac{1}{\eta}}(\delta^{(1-\eta)/\eta} - \lambda_i)^{\frac{1}{\eta}}}} \tag{48}$$

If the acceptance condition is not binding the multiplier λ_i is zero and a player consumes the entire surplus not invested and invests a share equal to

$$\varphi_i = \frac{1}{1 + \frac{1}{\alpha^{\frac{1}{\eta}}}\left(\frac{1+\delta^{1/\eta}}{\delta^{1/\eta}}\right)^{\frac{1-\eta}{\eta}}} \tag{49}$$

which is increasing in players' discount factors. Since the focus is on symmetric solutions, let $\lambda_i = \lambda$ and $\varphi_i = \varphi$. Using the first order conditions, the immediate agreement conditions,

$$\left(\frac{[(1-x_i)(k_t-I_i)]^{1-\eta}}{1-\eta}+\frac{\alpha}{1-\eta}\left(\frac{I_i}{1+\delta^{1/\eta}}\right)^{1-\eta}\right)=\delta\frac{[x_j(k_t-I_j)]^{1-\eta}}{1-\eta}+$$
$$+\delta\frac{\alpha}{1-\eta}\left(\frac{I_j}{1+\delta^{1/\eta}}\right)^{1-\eta}$$

can be written as

$$(1+\delta^{1/\eta})^{\frac{1-\eta}{\eta}}\left((-\lambda)^{\frac{1-\eta}{\eta}}-\delta\right)+\alpha^{\frac{1}{\eta}}(1-\delta)(\delta^{(1-\eta)/\eta}-\lambda)^{\frac{1-\eta}{\eta}}=0 \quad (50)$$

The inequality $(-\lambda)^{\frac{1-\eta}{\eta}}\geq\delta$ cannot be satisfied, otherwise the right-hand side of (50) is always positive and therefore by the principle of complementary slackness, the multiplier λ must be zero and clearly this is in contradiction with $(-\lambda)^{\frac{1-\eta}{\eta}}\geq\delta>0$. For $(-\lambda)^{\frac{1-\eta}{\eta}}<\delta$, then (50) becomes

$$(1+\delta^{1/\eta})^{\frac{1-\eta}{\eta}}\left(\delta-(-\lambda)^{\frac{1-\eta}{\eta}}\right)=\alpha^{\frac{1}{\eta}}(1-\delta)(\delta^{(1-\eta)/\eta}-\lambda)^{\frac{1-\eta}{\eta}} \quad (51)$$

Remark 6. The investment and consumption levels in equilibrium are a linear function of the capital stock.

Proof. The first order conditions, (47) and (48), and the indifference condition, (51), imply that the multiplier, the consumption and investment shares are independent of the capital stock. By iteration, this can be proved also for $n>2$ (see (58) below). ■

In the next proposition we derive an algorithm to solve the iteration process when the number of bargaining stages tends to infinity. To obtain a simple analytical solution for the multiplier, we will focus on the case in which $\eta=1/2$.

Proposition 6 *When the number of bargaining rounds is* n *(with* $n\geq 3$*), the equilibrium proposal is given by the following shares:*

$$x_{ns}=\frac{1}{1+(-\lambda_{ns})^{\frac{1}{\eta}}} \quad (52)$$

$$\varphi_{ns}=\frac{a_{ns}}{a_{ns}+b_{ns}} \quad (53)$$

where

$$a_{ns}=\left(\alpha(c_{(n-1)s}-d_{(n-1)s}\lambda_{ns})\right)^{\frac{1}{\eta}} \quad (54)$$

$$b_{ns}=1+(-\lambda_{ns})^{\frac{1}{\eta}} \quad (55)$$

$$c_{ns}=((1-x_{ns})(1-\varphi_{ns}))^{1-\eta}+\alpha\varphi_{ns}^{1-\eta}d_{(n-1)s} \quad (56)$$

$$d_{ns}=(x_{ns}(1-\varphi_{ns}))^{1-\eta}+\alpha\varphi_{ns}^{1-\eta}c_{(n-1)s} \quad (57)$$

and the multiplier λ_{ns} is the solution of the following equation

$$\frac{(-\lambda_{ns})^{\frac{1-\eta}{\eta}} - \delta}{b_{ns}^{1-\eta}}(1-\varphi_{ns})^{1-\eta} + \alpha\varphi_{ns}^{1-\eta}(d_{(n-1)s} - c_{(n-1)s}) = 0 \tag{58}$$

with

$$c_{2s} = ((1-x_{2s})(1-\varphi_{2s}))^{1-\eta} + \alpha\left(\frac{1}{1+\delta^{1/\eta}}\varphi_{2s}\right)^{1-\eta} \tag{59}$$

$$d_{2s} = (x_{2s}(1-\varphi_{2s}))^{1-\eta} + \alpha\left(\frac{\delta^{1/\eta}}{1+\delta^{1/\eta}}\varphi_{2s}\right)^{1-\eta} \tag{60}$$

Moreover, for $\eta = 1/2$, at the limit of Δ that tends to zero, players invest and consume efficiently.

Proof. in Appendix. ■

In the finite-horizon game, we did not impose that the share invested and consumed are time-invariant, however, when we consider the limit for n that tends to infinity we obtain a unique solution x and φ. Moreover, such solution coincide with the MPE shares obtained in the previous section.

7 Final Remarks

We showed that in a dynamic set-up, under an alternating bargaining procedure there is a unique stationary linear MPE. Under certain conditions, proposers have all the bargaining power and consume all the surplus not invested. Interestingly, this type of MPE can also be sustained in a game with little frictions, as long as there is at least a party who is sufficiently impatient. Often in non-cooperative games of bargaining to simplify parties are assumed to be symmetric, however asymmetries among players are pervasive in many negotiations and our analysis also demonstrates that asymmetries matter.

We also established that bargaining can lead to underinvestment. Only symmetric players who are infinitely patient bargain efficiently (both under random-proposer and alternating-offer bargaining procedures). However, if parties could commit to share all the surplus not invested equally, then, even if impatient, players with similar rates of time preference can behave efficiently (regardless of the bargaining procedure adopted). Suppose for instance that before entering a business two partners could sign a contract that specifies that each will obtain half of the profits not re-invested. Then, since the players have the same rate of time preference and consume the same share of the residual surplus, it is intuitive that even if they can make a proposal with different probabilities, their investment plan is unaffected by these probabilities. An implication of this result for policy makers is therefore to guarantee an equal division of mutual gains to encourage efficient investment paths for impatient players.

APPENDIX

Proof of Lemma 1[13]

The proof first focuses on the case in which $\eta < 1$ and then on the case of $\eta > 1$.

Consider any subgame where player i proposes first. Let k_t be the state variable and $V_i(k_t)$ ($W_j(k_t)$), the optimal expected discounted utility to player i (j), with $i, j = 1, 2$ and $i \neq j$.

a) The case of $\eta < 1$. The optimal expected utilities $V_i(k_t)$ ($W_i(k_t)$) are lower bounded (for instance, in the case of disagreement) and can be upper bounded:

$$V_i(k_t), W_i(k_t) \in \left[0, \frac{k_t^{1-\eta}}{1-\eta} \frac{l^{1-\eta}}{\left(1-\left(\alpha_i l^{1-\eta}\right)^{1/\eta}\right)^{\eta}}\right]$$

if $\alpha_i l^{1-\eta} < 1$. The upper bound has been derived by assuming that the investment and consumption paths are to maximise player i' payoff as in a standard growth model with no bargaining. Using the value function iteration method it can be shown that the per-period consumption for player i is given by

$$l\left(1-\left(\alpha_i l^{1-\eta}\right)^{1/\eta}\right) k_t$$

and all the surplus not consumed by player i is invested. The condition $\alpha_i l^{1-\eta} < 1$ must hold to satisfy the transversality condition.

Let's now focus on the case in which the strategies are linear. Then if player j accepts a proposal (x_i, φ_i) he gets

$$\frac{[(1-x_i)(G-\varphi_i)k_t)]^{1-\eta}}{1-\eta} + \alpha_j V_j(k_{t+1})$$

while if he rejects it he obtains $\delta_j V_j(k_t)$. Therefore, the proposal is accepted if and only if

$$\frac{[(1-x_i)(G-\varphi_i)k_t]^{1-\eta}}{1-\eta} \geq \delta_j V_j(k_t) - \alpha_j V_j(k_{t+1}) \tag{61}$$

[13] The proof generalises the arguments in Muthoo (1995) to the case of dynamic accumulation and CES per-period utility.

If the RHS of (61) is non-positive for $i, j = 1, 2$ and $i \neq j$, player i can consume all the surplus not invested ($x_i = 1$) without facing a rejection. Instead, if the RHS of (61) is positive, there are two cases. A proposer could prefer either to make an acceptable offer or to make an unacceptable one. In the former, a proposer must ask for a share x_i strictly smaller than 1, although he would be better off in asking for a share as large as possible so as to obtain an acceptance. Then, the two cases can be represented as follows. Either the proposal (x_i, φ_i) is such that the following holds

$$[(1-x_i)(G-\varphi_i)]^{1-\eta}\frac{k_t^{1-\eta}}{1-\eta} = \delta_j V_j(k_t) - \alpha_j V_j(k_{t+1}) \tag{62}$$

and this is the case when

$$[x_i(G-\varphi_i)]^{1-\eta}\frac{k_t^{1-\eta}}{1-\eta} + \alpha_i W_i(k_{t+1}) \geq \delta_i W_i(k_t)$$

or the proposal is such that

$$[(1-x_i)(G-\varphi_i)]^{1-\eta}\frac{k_t^{1-\eta}}{1-\eta} < \delta_j V_j(k_t) - \alpha_j V_j(k_{t+1}) \tag{63}$$

and this is the case when

$$[x_i(G-\varphi_i)]^{1-\eta}\frac{k_t^{1-\eta}}{1-\eta} + \alpha_i W_i(k_{t+1}) < \delta_i W_i(k_t) \tag{64}$$

We now show that (62) must hold. Suppose, by contradiction, that it does not, then there are 2 cases. First, assume that (63) holds for both players. Then, there is no acceptable offer when the state is k_t (therefore, the state k_{t+1} is never reached) and $V_j(k_t) = V_j(k_{t+1}) = W_i(k_t) = W_i(k_{t+1}) = 0$ for any $i, j = 1, 2$, which lead to a contradiction. Assume that (63) holds only for one player, without loss of generality, say 1. Then, player 1 makes an unacceptable offer, that is, (64) holds for $i = 1$ and $j = 2$ while player 2 makes an acceptable offer, that is (62) holds for $i = 2$ and $j = 1$. As a result, $V_1(k_t) = \delta_1 W_1(k_t)$, $W_2(k_t) = \delta_2 V_2(k_t)$,

$$\begin{aligned} V_2(k_t) &= (x_2(G-\varphi_2))^{1-\eta}\frac{k_t^{1-\eta}}{1-\eta} + \alpha_2 W_2(k_{t+1}) \\ W_1(k_t) &= [(1-x_2)(G-\varphi_2)]^{1-\eta}\frac{k_t^{1-\eta}}{1-\eta} + \alpha_1 V_1(k_{t+1}) \end{aligned}$$

The latter is also equivalent to $W_1(k_t) = \delta_1 V_1(k_t)$. Then it must be $W_1(k_t) = 0 = V_1(k_t)$ and therefore there is a contradiction. In conclusion, for any state k_t, a proposal (x_i, φ_i) is part of an MPE if immediately accepted.

b) The case of $\eta > 1$. In this case the optimal expected values $V_i(k_t)$ and $W_i(k_t)$ assume negative values and are not lower bound. Indeed, in case of disagreement $V_i(k_t) = W_i(k_t) = -\infty$ for any $i = 1, 2$. When the state variable is k_t, player j accepts a proposal (x_i, φ_i) if and only if

$$-\frac{1}{(\eta-1)[(1-x_i)(G-\varphi_i)k_t]^{\eta-1}} \geq \delta_j V_j(k_t) - \alpha_j V_j(k_{t+1}) \tag{65}$$

When the RHS of this inequality is negative ($\delta_j V_j(k_t) < \alpha_j V_j(k_{t+1})$), there are two cases: either player i prefers to make an acceptable offer or not. This is as for the case of $\eta < 1$ (and a positive RHS) and since the proof follows a similar argument we omit it (the only differences are that $W_i(k_t)$ and $V_i(k_t)$ are negative numbers and in case of disagreement are equal to $-\infty$ rather than 0).

When the RHS of (65) is non-negative, there is no offer (x_i, φ_i) which player j can accept. Again there are two cases: either this holds for both players or only for one player. In the former, no offer is ever accepted, and therefore $V_j(k_t) = W_j(k_t) = -\infty$ but if player j rejects an offer, then it must be that $W_j(k_t) < \delta_j V_j(k_t)$ and therefore a contradiction. In the latter, without loss of generality, assume that for the state k_t, player 2 could make an acceptable offer (i.e., RHS of (65) is negative for $j = 1$), while player 1's offer is rejected (i.e., RHS of (65) is non-negative for $j = 2$). In this scenario, $V_1(k_t) = \delta_1 W_1(k_t)$, $W_2(k_t) = \delta_2 V_2(k_t)$, $W_1(k_t) = \delta_1 V_1(k_t)$ and

$$V_2(k_t) = -\frac{1}{\eta-1}\frac{1}{[x_2(G-\varphi_2)k_t]^{\eta-1}} + \alpha_2 W_2(k_{t+1})$$

This implies that $V_1(k_t) = W_1(k_t) = -\infty$. Then it must be that player 1 accepts $x_2 = 1$ when the state variable is k_t (note that $\delta_1 V_1(k_t) < \alpha_1 V_1(k_{t+1})$). However, the condition $\delta_1 V_1(k_t) = -\infty < \alpha_1 V_1(k_{t+1})$ implies that a demand of $x_2 = 1$ could not be accepted for k_{t+j} for any $j = 1, 2...$(because otherwise $V_1(k_{t+1}) = -\infty$, leading to a contradiction). The stationarity of the MPE requires that player 2 still demand $x_2 = 1$ at k_{t+j} but this must be rejected. Using the arguments above we can show that there is a contradiction. Indeed, if player 1's proposal is still unacceptable then any proposal is always rejected and therefore we have a contradiction. If player 1's proposal becomes acceptable, we can repeat the same arguments as for the case in which only a player can obtain an acceptable offer, with player i replaced by j (with $i, j = 1, 2$ and $i \neq j$). Then, x_1 must be equal to 1 but this proposal cannot be accepted for more than one bargaining stage, therefore a contradiction. To conclude also for the case of $\eta > 1$, for any state k_t, a proposal (x_i, φ_i) is part of an MPE if immediately accepted.

Proof of Proposition 1

We show that each proposer can consume the entire residual surplus and his investment level will make such offer acceptable to the responder (indeed, the responder is strictly better off in accepting rather than rejecting the offer). The first order condition of the Bellman equation (6) with respect of φ_i implies that

$$\varphi_i = \frac{(\alpha_i \mu_i)^{1/\eta} G - (1-\lambda)}{(\alpha_i \mu_i)^{1/\eta} + 1} \tag{66}$$

while the first order condition with respect to x_i implies that the share demanded should be as high as possible. We now input the first order condition in the Bellman equation (6) and after simplifying we obtain that $\phi_i = \psi_i^{\eta} l^{1-\eta}$ with $\psi_i = 1 + (\alpha_i \mu_i)^{1/\eta}$. The rate of investment (66) can now be written as $\varphi_i = G - \frac{l}{\psi_i}$. Consequently, the unknown coefficient related to the responder's payoff is

$$\mu_j \equiv \alpha_j \phi_j (1 - \lambda + \varphi_i)^{1-\eta} = \alpha_j \phi_j l^{1-\eta} \left(1 - \frac{1}{\psi_i}\right)^{1-\eta}$$

since $\eta < 1$. This and the definition of ψ_i, that is $(\psi_i - 1)^{\eta} = \alpha_i \mu_i$, implies the following system:

$$\alpha_j^2 l^{2(1-\eta)} \left(\frac{\psi_i - 1}{\psi_i}\right)^{1-\eta} = \left(\frac{\psi_j - 1}{\psi_j}\right)^{\eta}$$

For[14] $\eta \neq 1/2$, there is a unique solution given by

$$\psi_i = \frac{1}{1 - b_i}$$

with $b_i \in (0,1)$. This defines an acceptable offer if the responder is better off in accepting rather than rejecting the offer:

$$\alpha_j \phi_j l^{1-\eta} \left(1 - \frac{1}{\psi_i}\right)^{1-\eta} \geq \delta_j \phi_j$$

that is, $\alpha_j (l b_i)^{1-\eta} \geq \delta_j$ for $i, j = 1, 2$ with $i \neq j$. The latter, together with $b_i \in (0,1)$, implies the conditions set in (10) and (11).

[14] For $\eta = 1/2$ the solution is given by $\psi_i = 1$ for any i. Therefore the ultimatum-like MPE cannot be sustained since no responder would accept a proposal where capital is fully disinvested and no surplus is offered (i.e., $\varphi_i = -(1-\lambda)$, $x_i = 1$)

Proof of Proposition 2

The proof consists in constructing a set of strategies in which player $i's$ demand is $x_i = 1$ while player $j's$ is $x_j < 1$. Then we show under what conditions these strategies can be sustained as an MPE.

The Bellman equations are given by (6) with $x_i = 1$ for player i and with $x_j < 1$ for player j. Therefore, the constraint of the acceptance condition (7) is not binding for player j (so that $x_i = 1$), while it must be binding for player i. Therefore, player j's Lagrangian is as follows:

$$L_j(k_t) = \max_{\substack{x_j \in [0,1] \\ \varphi_j \in [-(1-\lambda), G]}} V_j(k_t) - m_j(\delta_i V_i - W_i) \tag{67}$$

where m_j is the (non-negative) Kuhn-Tucker multiplier. We focus on $m_j > 0$, otherwise proposition 1 must hold. The first order conditions of the lagrangian $L_j(k_t)$ with respect to x_j and φ_j can be written as (16) and (17) respectively, with

$$\psi_j = (\alpha_j \mu_j + m_j \alpha_i \phi_i)^{1/\eta} + 1 + m_j^{1/\eta} \tag{68}$$

Similarly, the first order conditions for the Bellman (6), in which the constraint is not binding can be written as (16) and (17) where the multiplier is 0, that is (15), where

$$\psi_i = 1 + (\alpha_i \mu_i)^{1/\eta} \tag{69}$$

We input the first order conditions in the Lagrangian $L_j(k_t)$ and the Bellman equation $V_i(k_t)$ and after some manipulations we obtain

$$\phi_i = \left(\frac{l}{\psi_i}\right)^{1-\eta} \left(1 + \alpha_i \mu_i \left(\psi_i - 1\right)^{1-\eta}\right)$$

$$\mu_i = \left(\frac{l}{\psi_j}\right)^{1-\eta} \left(m_j^{(1-\eta)/\eta} + \alpha_i \phi_i \left(\psi_j - 1 - m_j^{1/\eta}\right)^{1-\eta}\right)$$

$$\phi_j = \left(\frac{l}{\psi_j}\right)^{1-\eta} \left(1 + \alpha_j \mu_j \left(\psi_j - 1 - m_j^{1/\eta}\right)^{1-\eta}\right)$$

$$\mu_j = \left(\frac{l}{\psi_i}\right)^{1-\eta} \left(\alpha_j \phi_j \left(\psi_i - 1\right)^{1-\eta}\right)$$

Using the first order condition of the Lagrangian $L_j(k_t)$ with respect to m_j (i.e., $\mu_i = \delta_i\phi_i$), the system can be re-written as a function of the auxiliary variables ψ_1 and ψ_2 :

$$\phi_i = \frac{l^{1-\eta}}{\psi_i^{1-\eta} - \alpha_i\delta_i l^{1-\eta}\left(\psi_i - 1\right)^{1-\eta}}$$

$$\mu_i = \frac{m_j^{(1-\eta)/\eta}\delta_i l^{1-\eta}}{\psi_j^{1-\eta}\delta_i - \alpha_i l^{1-\eta}\left(\psi_j - 1 - m_j^{1/\eta}\right)^{1-\eta}}$$

$$\phi_j = \frac{l^{1-\eta}\psi_i^{1-\eta}}{(\psi_i\psi_j)^{1-\eta} - (\alpha_j l^{1-\eta})^2\left(\psi_j - 1 - m_j^{1/\eta}\right)^{1-\eta}(\psi_i - 1)^{1-\eta}}$$

$$\mu_j = \frac{\alpha_j l^{2(1-\eta)}\left(\psi_i - 1\right)^{1-\eta}}{(\psi_i\psi_j)^{1-\eta} - (\alpha_j l^{1-\eta})^2\left(\psi_j - 1 - m_j^{1/\eta}\right)^{1-\eta}(\psi_i - 1)^{1-\eta}}$$

It must be that $\mu_j \geq \delta_j\phi_j$ and $\mu_z, \phi_z \geq 0$ with $z = 1, 2$. Such inequalities are equivalent to the constraints in the proposition. Moreover, given the definition of the auxiliary variables (68) and (69) and the indifference condition $\mu_i = \delta_i\phi_i$, we can derive the system in the proposition. A solution to the system, (ψ_i, ψ_j, m_j), satisfying the constraints above, uniquely defines the MPE strategies and payoffs. The conditions (10) and (11) cannot hold. Suppose by contradiction that (10) and (11) held, then proposer j would not optimise by leaving a positive consumption to player i. He could have consumed all the surplus not invested without facing any rejection.

Proof of Proposition 3

The Lagrangian for the constraint problem in (6) is as follows:

$$L_i(k_t) = \max_{\substack{x_i \in [0,1] \\ \varphi_i \in [-(1-\lambda), G]}} \quad V_i(k_t) - m_i(\delta_j V_j - W_j) \tag{70}$$

where m_i is the (non-negative) Kuhn-Tucker multiplier, with $i, j = 1, 2$ and $i \neq j$. The first order condition of (70) with respect of φ_i is as follows:

$$\varphi_i = \frac{(\alpha_i\mu_i + m_i\alpha_j\phi_j)^{1/\eta}G - (1 + m_i^{1/\eta})(1-\lambda)}{(\alpha_i\mu_i + m_i\alpha_j\phi_j)^{1/\eta} + 1 + m_i^{1/\eta}} \tag{71}$$

while the first order condition with respect to x_i is (20). By the complementary slackness condition, if the constraint $W_j \geq \delta_j V_j$ is not binding, the multiplier m_i is zero and the share φ_i in (71), coincides with (66).

The first order condition with respect of the multiplier implies $W_j = \delta_j V_j$. Using (71), the Bellman equation can be re-written as

$$\phi_i = l^{1-\eta} \frac{1 + \alpha_i \mu_i g_i{}^{(1-\eta)/\eta}}{\psi_i^{1-\eta}} \tag{72}$$

where $g_i = \alpha_i \mu_i + m_i \alpha_j \phi_j$ and $\psi_i = g_i^{1/\eta} + 1 + m_i^{1/\eta}$ as in (68) while the coefficient related to the responder is

$$\mu_j = l^{1-\eta} \frac{m_i^{\frac{1-\eta}{\eta}} + \alpha_j \phi_j g_i^{(1-\eta)/\eta}}{\psi_i^{1-\eta}}$$

Using the constraint $\mu_j = \delta_j \phi_j$, the undetermined coefficients μ_j,ϕ_j can be written as in (22). This implies that, the indifference condition $\mu_j = \delta_j \phi_j$ can be written as in (19), while from the definition for the auxiliary variable, that is,

$$\psi_i = (\alpha_i \mu_i + m_i \alpha_j \phi_j)^{1/\eta} + 1 + m_i^{1/\eta}$$

we obtain (18). Therefore, a solution (ψ_i, m_i) to system (19) and (18) defines the undetermined coefficients, μ_i,ϕ_i in (22), the share consumed (20) and invested (21). Moreover, the solution is (ψ_i, m_i) must belong to M_i to obtain a positive equilibrium (in which the undetermined coefficients are real and positive), additionally, the transversality condition is satisfied. Finally, it must be that proposition 1 and 2 do not hold, otherwise, at least a multiplier m_i with $i = 1$,2 must be zero.

Proof of Proposition 4
First, when $l\alpha_i = 1$, then $k_{t+1} = k_t$, and the system (32) is only affected by the initial capital stock k_0, which is a constant. Assuming that $h_i = h$, $\Delta = \tau$ and $l\alpha = 1$, then (32) becomes:

$$(1 - \alpha^2) \left(\ln(k_0) + \ln \left(\frac{ml}{1+m} (1-\alpha) \right) + \frac{\alpha}{1-\alpha} \ln (l\alpha) \right) = 0 \tag{73}$$

This implies that there is a symmetric solution for

$$m = (k_0 l(1-\alpha) - 1)^{-1}$$

Therefore, using (29), we obtain (33).

For the second part of the proposition, we consider the limit for Δ that tends to 0. Then, the indifference conditions in (32) become

$$(1 - \alpha) \ln(m) = 0$$

or $m = 1$, from which we obtain (34).

Proof of Proposition 6
To solve (51), let $\eta = 1/2$. Then, the multiplier is:

$$\lambda = -\frac{\delta[(1+\delta^2) - \alpha^2(1-\delta)]}{(1+\delta^2) + \alpha^2(1-\delta)}$$

Moreover, at the limit for Δ that tends to 0, the multiplier λ tends to -1 and therefore parties share equally the surplus not invested and invest a share equal to

$$\varphi = \frac{\alpha^2}{1+\alpha^2}$$

We add a subscript $2s$ to φ and x to indicate the solution related to the game for $n = 2$. Let j be the first mover at $t = 0$ when $n = 3$. Then the problem is

$$\max_{x_j, I_j} \frac{[x_j(Gk_0 - I_j)]^{1-\eta}}{1-\eta} + \frac{\alpha}{1-\eta} I_j^{1-\eta} c_{2s}$$

where c_{2s} is defined in (59). The immediate agreement condition implies that the expression below must be zero:

$$\frac{[(1-x_j)(k_0 - I_j)]^{1-\eta}}{1-\eta} + \frac{\alpha}{1-\eta} I_j^{1-\eta} d_{2s} - \delta\left(\frac{[x_j(k_0 - I_i)]^{1-\eta}}{1-\eta} + \frac{\alpha}{1-\eta} I_j^{1-\eta} c_{2s}\right)$$

where d_{2s} is defined in (60). Then the first order condition implies that x_j and I_j have the same structure as in (47) and (48) where the multiplier now is referred to as λ_{3s} and φ_{3s}, a_{3s} and b_{3s} are as in (53)-(57) with $n = 3$.

For $\eta = 1/2$ and at the limit for δ that tends to 1 it can be shown that $-\lambda_{3s}$ tends to 1 and therefore x_{3s} tends to 1/2 while the share invested tend to

$$\imath\varphi_{3s} = \frac{\alpha^2}{1+2\alpha^2}$$

Similarly, for a game with n bargaining stages (with $n \geq 3$) x_j and I_j have the same structure as in (47) and (48), that is,

$$x_{ns} = \frac{1}{1 + (-\lambda_{ns})^{\frac{1}{\eta}}}$$
$$\varphi_{ns} = \frac{a_{ns}}{a_{ns} + b_{ns}}$$

with a_{ns} and b_{3s} as in (54)-(57). Then, the indifference condition can be written as in (58). Condition (58) is an equation in one unknown, the multiplier λ_{ns}. A solution to (58) defines the solution to (x_{ns}, φ_{ns}). At this stage we can solve the problem numerically. We obtain that for $\eta = 1/2$ with $\delta = 0.9$, $\alpha = 0.8$ the share invested is 0.62 and a proposer obtains a share equal to 0.63.[15] When patience increases, for instance, $\delta = 0.99$ and $\alpha = 0.9$, the share invested increases to 0.81, and a proposer obtains a share equal to 0.52. Finally for $\delta = 0.9999$ and $\alpha = 0.9$, the share consumed decreases to 0.5 and the share invested increases to 0.81. Since the social planner would set $x = 1/2$ and $\varphi = (\alpha l)^2 - (1 - \lambda) = 0.81$, we can conclude that at the limit for $\delta \to 1$, parties invest and consume efficiently.

References

[1] Che, Y.-K. and J., Sákovics (2004) A Dynamic Theory of Holdup. Econometrica, 1063-103

[2] Dutta, P.K. and R.K. Sandaram (1993): The Tragedy of the Commons? Economic Theory 3, 413-26.

[3] Flamini, F. (2007): First Things First? The Agenda Selection Problem in Multi-issue Committees. Journal of Economic Behavior and Organization 63 (1), 138-57.

[4] Gibbons, R. (1992): Game Theory for Applied Economists, Princeton University Press, Princeton, New Jersey.

[5] Gul, F. (2001): Unobservable Investment and the Hold-up Problem. Econometrica 69 (2), 343-76.

[6] Leith, C. and J. Malley (2005): Estimated General Equilibrium Models for the Evaluation of Monetary Policy in the US and Europe, European Economic Review 49 (8), 2137-59.

[7] Levhari, D. and L. Mirman (1980): The Great Fish War: An Example Using a Dynamic Cournot-Nash Solution, Bell Journal of Economics 11, 322-34.

[8] Lockwook, B. and J. Thomas (2002): Gradualism and Irreversability 69, 339-56

[9] Ljungqvist, L. and T. Sargent (2000) Recursive Macroeconomic Theory. MIT Press, Cambridge, Massachusetts.

[10] Muthoo, A. (1995): Bargaining in a Long Run Relationship with Endogenous Termination, Journal of Economic Theory 66, 590-98.

[11] Muthoo, A. (1998): Sunk Costs and the Inefficiency of Relationship-Specific Investment, Economica, 97-106.

[12] Muthoo, A. (1999): Bargaining Theory with Applications. Cambridge University Press, Cambridge, England.

[15] We present only the most interesting case in which parties are fairly patient (α is large), however, the result can be shown also for smaller α.

[13] Rubinstein, A. (1982): Perfect Equilibrium in a Bargaining Game. Econometrica 50, 97-109.
[14] Sorger, G. (2006): Recursive Nash Bargaining over a Productive Asset. Journal of Economic Dynamic and Control 30, 2637-59.
[15] Stokey, N. and R. Lucas (1989): Recursive Methods in Economic Dynamics. Harvard University, Cambridge, Massachusetts.