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ON MARKOVIAN REPRESENTATIONS OF THE TERM STRUCTURE

by Peter Ritchken and
L. Sankarasubramanian

Peter Ritchken is a professor at the Weatherhead School of Management, Case Western Reserve University, and L. Sankarasubramanian is an assistant professor in the School of Business Administration, University of Southern California. For helpful comments, the authors would like to thank seminar participants at the Federal Reserve Bank of Cleveland, the University of Southern California, the University of British Columbia, and Queens University.

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Abstract

The linkages between term structures separated by finite time periods can be complex. Indeed, in general, the dynamics of the term structure could depend on the *entire set* of information revealed since the earlier date. This path dependence, which causes difficulties in pricing interest rate claims, is usually *eliminated* by making specific assumptions on investment behavior or on the evolution of interest rates. In contrast, this article identifies the class of volatility structures that permits the path dependence to be *captured* by a single sufficient statistic. An equilibrium framework is provided where beliefs and technologies are restricted so that the resulting term structures have volatilities that belong to the restricted class. The models themselves can be characterized by a parsimonious set of parameters and can be initialized to an observed term structure without the introduction of ad-hoc time-varying parameters. Furthermore, since the dynamics of the resulting term structures are two-state Markovian, simple pricing mechanisms can be developed for interest rate claims.

I. Introduction

In an economy with continuous trading, the linkages between term structures of interest rates separated by a finite time period can be quite complex, even when uncertainty is generated by a single source. In particular, even though all bonds are *instantaneously* perfectly correlated, over finite time periods this perfect relationship may break down. Indeed, to characterize the term structure at any date *relative* to the term structure at an earlier date, *all* the information revealed in the interim period may be required. This feature appears, for example, in the general Heath, Jarrow and Morton (hereafter HJM) [1992] paradigm of the term structure. In their approach, the evolution of the term structure is influenced by the entire history of the process, and this history generally cannot be summarized by a finite number of state variables. This path dependence is endemic to all models of the term structure and can only be eliminated if specific structure is imposed, either directly or indirectly, on forward rate volatilities.

One way to eliminate path dependence is to impose sufficient structure on the economy to ensure that, at every date, the level and shape of the term structure can be described by a few state variables. For example, in the single-factor equilibrium model of Cox, Ingersoll and Ross (hereafter CIR) [1985], the entire term structure at each date can be described by the spot interest rate.[1] Such models have the property that whenever the interest rate returns to a previous level, so do all other forward rates. Hence, unless time-varying parameters are used, these models impose restrictions on the shapes that term structures can take.

A second approach that eliminates path dependence involves restricting volatilities of all spot and forward rates to being deterministic. Under this structure, the path dependence issues disappear, and all forward interest rates are perfectly correlated over finite time horizons. Consequently, knowing any one rate is sufficient to characterize the rest of the term structure. Examples of such approaches are provided by HJM [1992] and Jamshidian [1989], who assume dynamically complete markets, and by Turnbull and Milne [1991], who establish general equilibrium pricing models for interest rate claims. In both approaches, the initial term structure is supplied exogenously. The fact that the term structure can be initialized, coupled with the simplicity of the resulting expressions, has led to the popularity of these models, especially for valuing claims that have prices quoted relative to an observed term structure.[2,3] Unfortunately, recent empirical evidence does not support the deterministic volatility structure for interest rates. For example, Chan, Karolyi, Longstaff and

[1] Additional examples of such models include Brennan and Schwartz [1977], CIR [1980], Cox and Ross [1976], Dothan [1978] and Vasicek [1977].

[2] Such pricing problems have become increasingly more important, primarily due to the rapid growth of the over-the-counter market, where the majority of prices are quoted relative to the term structure. Indeed, in 1991, over-the-counter trading comprised a larger than $4 trillion market of notional principal. This exceeded the $2.2 trillion interest rate futures market and the $77 billion stock index futures market.

[3] Examples of such approaches include Amin and Jarrow [1991], Musiela, Turnbull and Wakeman [1992] and Ritchken and Sankarasubramanian [1992].

Sanders [1992] find that spot rate volatility appears to be highly sensitive to the level of the spot rate itself, and they conclude that models with deterministic volatilities are misspecified.

This article considers an alternative approach to dealing with path dependence. In particular, rather than structure the economy so that the path dependence is completely eliminated, we identify situations where it can be captured by a single sufficient statistic, common across all bonds. To do this, we first derive the conditions that must prevail if path dependence is to be captured by a single statistic. We then demonstrate that in all these cases, the intertemporal linkages between term structures can be completely characterized in terms of two state variables. That is, given any initial term structure, all future term structures are determined by the values of two state variables. The first of these is shown to be the spot rate; the second is a path dependent statistic that accumulates information along the path in such a way that no information is lost.[4] This simplification of the intertemporal relationships between term structures is attained not by constraining the volatility structure for the spot interest rate, but rather by restricting the linkage between forward rate volatilities and spot rate volatilities.

Our framework has several useful properties. First, models for pricing interest rate claims can be developed in which the volatility structures for spot and forward rates are not deterministic, yet the term structure can be initialized to any exogenous specification. In this respect, these models generalize the *Markovian* models presented in HJM [1992] and in Turnbull and Milne [1991]. Second, unlike existing single-factor models in which interest rates have nondeterministic volatility, our models can be initialized to a term structure without requiring the ad-hoc introduction of time dependent parameters.[5] Finally, since the structure for the spot rate volatility is unrestricted, our framework permits the term structure to be modelled by a large class of processes. Since a general equilibrium approach is used, the resulting models suffer no internal inconsistencies and admit no dynamic arbitrage opportunities.

The paper proceeds as follows. In the next section we develop a simple economy with a single representative investor and a single production technology. Within this framework, we then establish the intertemporal linkages between term structures and identify the path dependence issues that arise. Section III provides the main theorem, which identifies conditions that must be

[4] Since path dependence is not eliminated in our approach, bond prices are not perfectly related over finite time horizons, even though they are perfectly instantaneously correlated.

[5] To initialize almost all existing single-state models of the term structure to an arbitrary initial term structure usually requires the introduction of time-varying parameters in the evolution of the spot interest rate. These are then estimated by inverting the term structure. Unfortunately, these inversions are nontrivial, computationally difficult, and, if the term structure is observed with measurement error, the estimates may be unreliable. Indeed, using time-varying parameters has come under some criticism. Dybvig [1989] discusses the ad-hoc nature of using time-varying parameters to initialize the term structure. For further details on inversion procedures, see CIR [1985], Hull and White [1990] and Jarrow [1988].

In contrast, our approach deals only with the intertemporal linkages between term structures, and not with their levels or shapes. As a result, the parameters of our model are specified independent of the term structure.

satisfied if the path dependence is to be captured by a single sufficient statistic, regardless of the volatility structure of the spot rate. The consequences of this result are then fully explored. In particular, we make explicit the intertemporal linkages of the term structure and its evolution. Section IV focuses on an economy in which technology innovations follow a square root process, similar to that considered by CIR [1985]. We demonstrate that in this economy, path dependence can be captured by a single statistic, provided restrictions are imposed on the manner by which investors revise their beliefs about future levels of technology, in response to current technological innovations. Section V summarizes the paper.

II. The Intertemporal Linkages of Bond Prices

Assume all physical investment is performed by a single stochastic constant-returns-to-scale technology that produces a good that is either consumed or invested in production. The return on physical investment is governed by

$$\frac{dQ(t)}{Q(t)} = \alpha\, X(t,t)\, dt + \sigma_1 \sqrt{X(t,t)}\, dz_1(t) \tag{1}$$

Here, $X(t,t)$ is the level of technology at date t, with dynamics given by

$$dX(t,t) = \mu_X(t,t)\, dt + \sigma_X(t,t)\, dz_2(t) \tag{2}$$

where $dz_1(t)$ and $dz_2(t)$ are standard Wiener increments with $\mathbf{E}[dz_1(t)\, dz_2(t)] = \theta dt$. We assume that the structure for $\sigma_X(t,t)$ is given. For example, $\sigma_X(t,t)$ could be proportional to the square root of $X(t,t)$, in which case its structure would be identical to that of CIR [1985].

The drift term, $\mu_X(t,t)$, is not directly specified. Rather, we assume that at date t, investor beliefs about the level of technology for each future date, τ, are provided. In particular, let $\mathcal{I}(t)$ denote the belief set at date t. Specifically,

$$\mathcal{I}(t) = \{X(t,\tau) | \tau \geq t\} \tag{3}$$

where

$$X(t,\tau) = \mathbf{E}_t[X(\tau,\tau)] \tag{4}$$

Here, $X(t,\tau)$ represents the date t assessment of the level of technology for date τ.[6]

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[6] Note that if the drift term is specified as in the CIR model by

$$\mu_X(t,t) = \beta[\mu - X(t,t)]\,,$$

then it can be shown that

$$X(t,\tau) = \mu - [\mu - X(t,t)]e^{-\beta(\tau-t)} \tag{F6.1}$$

Rather than specify the drift term directly, the CIR model could have been obtained by specifying this belief structure. The drift term would then be recovered using equation (6).

From equations (2) and (4), it follows that

$$\frac{\partial X(t,\tau)}{\partial \tau} = \mathbf{E}_t[\mu_X(\tau,\tau)] \tag{5a}$$

$$\left.\frac{\partial X(t,\tau)}{\partial \tau}\right|_{\tau=t} = \mu_X(t,t) \tag{5b}$$

That is, if a structure for the belief set is given, then the drift term for the technology process could be recovered. The volatility term, $\sigma_X(t,t)$, captures the sensitivity of technological innovations to the Brownian disturbances, $dz_2(t)$. Changes in this technology level cause investors to alter their beliefs about its future level. Let $\sigma_X(t,\tau)$ measure the sensitivity of beliefs for date τ to the Brownian disturbance. In particular, we have

$$dX(t,\tau) \;=\; \sigma_X(t,\tau)\,dz_2(t) \quad \text{for} \quad \tau > t \tag{6}$$

If the structure for $X(t,\tau)$ were given (or equivalently, if the drift term in equation (2) were provided) then the exact relationship between $\sigma_X(t,\tau)$ and $\sigma_X(t,t)$ could be recovered.[7] Let $\omega(t,\tau)$ be defined as

$$\omega(t,\tau) \;=\; \frac{\sigma_X(t,\tau)}{\sigma_X(t,t)} \tag{7}$$

$\omega(t,\tau)$ is referred to as the *belief revision scheme*, since it identifies the manner by which beliefs about future levels of technology, $X(t,\tau)$, are revised in response to a change in the current level, $X(t,t)$.

We assume that the economy is composed of identical individuals, each of whom seeks to maximize an objective function of the form

$$\mathbf{E}_t\left\{\int_t^\infty e^{-\rho(x-t)}\,\ln[C(x)]\,dx\right\} \tag{8}$$

Here, $C(x)$ represents time x consumption and ρ is the discount rate reflecting time preferences. Markets are assumed to be perfectly competitive and frictionless. Furthermore, traders can borrow or lend at the riskless rate, as well as trade in other types of contingent claims.

An exact specification of the belief set, $\mathcal{I}(t)$, is essential to characterize the term structure. However, our objective is not to characterize the term structure itself, but rather to develop the intertemporal relationships that exist between term structures separated by finite time horizons. Hence, at this juncture we leave the exact structure for $\mathcal{I}(t)$ unspecified.

[7] In our case, an exact structure for $X(t,\tau)$ is not provided. In the CIR model, however, from footnote 6 we can apply Ito's lemma to equation (F6.1) to identify the volatility structure. In particular, we obtain

$$dX(t,\tau) \;=\; e^{-\beta(\tau-t)}\,\sigma_X(t,t)\,dz_2(t)$$

Hence, in the CIR model, the Brownian disturbance affects future beliefs in an exponentially dampened fashion.

In this economy, the investor's problem is completely determined by the belief set, $\mathcal{I}(t)$, and the investor's wealth, $W(t)$. Following Merton (1973), the investor's derived utility of wealth function is partially separable in wealth and in the elements of the belief set. Setting up the Bellman optimization problem, establishing the first-order conditions, and imposing the market clearing condition that all wealth is invested in physical production then leads to

$$C(t) = \rho W(t) \tag{9a}$$

$$r(t) = a X(t,t) \tag{9b}$$

where $a = \alpha - \sigma_1^2$ and $r(t)$ denotes the spot interest rate at date t. We also assume that $\alpha > \sigma_1^2$ to ensure that $r(t)$ and $X(t,t)$ are positively related. From this it follows that the dynamics of the spot interest rate can be obtained as

$$dr(t) = \mu_r(t)\,dt + \sigma_r(t)\,dz_2(t) \tag{10a}$$

where

$$\mu_r(t) = a\,\mu_X(t,t) \tag{10b}$$

and

$$\sigma_r(t) = a\,\sigma_X(t,t) \tag{10c}$$

Further, from equation (5) we obtain

$$\mathbf{E}_t[\mu_r(\tau)] = a\left[\frac{\partial X(t,\tau)}{\partial \tau}\right]$$

Hence, the belief set, $\mathcal{I}(t)$, permits unbiased expectations of the drift terms of future interest rates to be computed.

Let $P(t,\tau)$ be the date t price of a default-free pure discount bond that matures at date τ. In addition, denote by $f(t,\tau)$ the forward rate at date t for the time increment $[\tau, \tau + d\tau]$, with $f(t,t) \equiv r(t)$. By definition, bond prices are related to forward rates as

$$P(t,\tau) = e^{-\int_t^\tau f(t,s)ds} \tag{11}$$

Since forward rates and bond prices do not depend on the level of wealth in this economy, their dynamics can be represented by

$$df(t,\tau) = \mu_f(t,\tau)\,dt + \sigma_f(t,\tau)\,dz_2(t)\,, \quad \text{for } \tau > t\,, \tag{12a}$$

and

$$\frac{dP(t,\tau)}{P(t,\tau)} = \mu_p(t,\tau)\,dt + \sigma_p(t,\tau)\,dz_2(t)\,, \quad \text{for } \tau > t \tag{12b}$$

where $\mu_f(t,\tau)$, $\mu_p(t,\tau)$, $\sigma_f(t,\tau)$ and $\sigma_p(t,\tau)$ are the instantaneous drift and volatility terms, which in general could depend on the level of all the state variables in the belief set, $\mathcal{I}(t)$.

Including bonds in the representative investor's portfolio choice problem leads to the following additional restriction:

$$\frac{\mu_p(t,T) - r(t)}{\sigma_p(t,T)} = \frac{\theta\sigma_1\sqrt{r(t)}}{\sqrt{a}} \equiv \lambda(t) \tag{13}$$

Using equations (13) and (11), it is then readily established that[8]

$$\mu_f(t,T) = \sigma_f(t,T)[\lambda(t) - \sigma_p(t,T)] \tag{14a}$$

and

$$\sigma_f(t,T) = -\frac{\partial}{\partial T}\sigma_p(t,T) \tag{14b}$$

or equivalently,

$$\sigma_p(t,T) = -\int_t^T \sigma_f(t,u)\,du \tag{14c}$$

Substituting equation (14a) into (12b) and computing the dynamics of the spot rate using the relationship

$$dr(t) = \frac{\partial}{\partial u}f(u,t)\bigg|_{u\to t} dt + \frac{\partial}{\partial u}f(t,u)\bigg|_{u\to t} dt$$

then leads to

$$dr(t) = \mu_r(t) + \sigma_r(t)dz_2(t) \tag{15a}$$

where

$$\mu_r(t) = \lambda(t)\sigma_r(t) + \frac{\partial}{\partial u}f(t,u)\bigg|_{u\to t} \tag{15b}$$

Equations (15a-b) highlight the relationships that exist between the drift and volatility terms, the market price of risk, and the shape of the term structure. In the CIR model, the drift and volatility terms are given, as is the market price of risk. From equation (15b), this information uniquely

[8] These results were also obtained by HJM [1992] using arbitrage arguments. To see these results, note from equation (11) that $f(t,T) = -\partial \ln[P(t,T)]/\partial T$, and that using Ito's lemma yields

$$df(t,T) = -\frac{\partial}{\partial T}[\mu_p(t,T) - \frac{1}{2}\sigma_p^2(t,T)]dt - \frac{\partial}{\partial T}[\sigma_p(t,T)]dz_2(t)$$

Hence,

$$\sigma_f(t,T) = -\frac{\partial}{\partial T}\sigma_p(t,T)$$

or equivalently,

$$\sigma_p(t,T) = -\int_t^T \sigma_f(t,u)\,du$$

and

$$\mu_f(t,T) = -\frac{\partial}{\partial T}[\mu_p(t,T) - \frac{1}{2}\sigma_p^2(t,T)] = -\frac{\partial}{\partial T}[\mu_p(t,T)] - \sigma_f(t,T)\sigma_p(t,T)$$

From equation (13) we then obtain

$$\frac{\partial}{\partial T}[\mu_p(t,T)] = \lambda(t)\frac{\partial}{\partial T}[\sigma_p(t,T)] = -\lambda(t)\,\sigma_f(t,T)$$

Equation (14a) then follows.

defines the shape of the term structure. In contrast, in what follows, the drift term is defined by the volatility structure, the market price of risk and the shape of the term structure through equation (15*b*).

Finally, using equations (14*a*), (12*a*-*b*) and (11) leads to

$$f(t,T) = f(0,T) + \int_0^t \sigma_f(u,T)\,[\lambda(u) - \sigma_p(u,T)]\,du + \int_0^t \sigma_f(u,T)\,dz_2(u) \tag{16a}$$

and

$$P(t,T) = \left[\frac{P(0,T)}{P(0,t)}\right] e^{\int_t^T \int_0^t \sigma_f(u,s)[\sigma_p(u,s)-\lambda(u)]\,du\,ds - \int_t^T \left[\int_0^t \sigma_f(u,s)\,dz_2(u)\right] ds} \tag{16b}$$

Equations (16*a*-*b*) establish the intertemporal linkages between term structures. In particular, the term structure at date t depends on the setting at date 0, together with all the Brownian disturbances over the period $[0,t]$. Note that knowledge of the term structure at date 0, together with a single point on the term structure at date t is not generally sufficient to uniquely characterize the term structure at date t. Indeed, to establish the term structure at date t, all the path information over the period $[0,t]$ may be required. This path dependent feature also appears in the general HJM [1992] framework. In their approach, the evolution of the term structure is influenced by the entire history of the process, and this history cannot, in general, be summarized into a finite number of state variables. As a result, for quite general volatility structures, the lattice methods described by HJM [1990] grow exponentially. Due to this computational complexity, empirical tests have been lacking for all but the simplest of the HJM option models. Equations (16*a*-*b*) also highlight the fact that path dependence is endemic to all models of the term structure and can only be eliminated if additional assumptions are made on the volatilities of all forward rates.

As mentioned earlier, path dependence in the term structure can be eliminated if sufficient structure is imposed to ensure that the evolution and setting of all bond prices can be described, at all dates, by a finite number of points on the term structure. For example, in the equilibrium single-factor model of CIR [1985], as well as in the arbitrage models of Vasicek [1977], Dothan [1978] and others, a single state variable is either explicitly or implicitly assumed to determine the level and shape of the term structure at all times. This variable is usually chosen, without any loss of generality, to be the spot interest rate. This assumption imposes a deterministic relationship between all bond returns *over finite intervals of time.* Empirical research, however, has not provided strong support for any of them.

The second approach commonly used to eliminate path dependence involves restricting the volatilities of all spot and forward rate volatilities to being deterministic. It can be readily verified from equation (16*a*) that under a deterministic volatility structure for spot and forward interest rates, the forward rate, $f(t,T)$, can be expressed as a simple linear function of the spot rate, $r(t)$, with deterministic coefficients. This implies that changes in forward rates are perfectly correlated over finite time intervals. While this simplification allows the term structure to be initialized to an exogenous structure, without requiring time-varying parameters, the volatility structure required to accomplish this has not been supported by recent empirical analyses.

In the next section, we consider an alternative approach to deal with path dependence. The idea is not to eliminate it outright, but rather to capture it by a single sufficient statistic, common across bonds of all maturities. In the resulting framework, the term structure is fully determined by two state variables. Unlike other single-factor Markovian models, our models do not have deterministic relationships between the returns on any two bonds over finite time intervals.

III. Path Dependent Models of the Term Structure

In this section, we first identify the relationship that must prevail between the volatilities of spot and forward rates if path dependence in the intertemporal linkages between term structures is to be captured by a single statistic, without imposing restrictions on either the spot rate volatility or the shape of the term structure itself. We then demonstrate that whenever this relationship prevails, a *two-state* Markovian representation of the term structure will result.

Theorem 1

To capture the path dependence illustrated in equations (16a-b) by a single statistic, without imposing any additional assumptions on the volatility of spot interest rates or on the shape of the term structure, it is necessary that the volatilities of all forward rates be related to each other as

$$\sigma_f(t,T) = \sigma_r(t)\, e^{-\int_t^T \kappa(x)dx} \qquad \textit{for all } T \geq t \tag{17}$$

where $\sigma_r(t) = \sigma_f(t,t)$ *is the volatility of the spot interest rate, and* $\kappa(x)$ *is some deterministic function.*

Proof See Appendix 1.

Theorem 1 implies that if the volatility structure of forward rates does not satisfy equation (17), then, without making more assumptions about either the spot rate volatility or the shape of the term structure, it is not possible for a single sufficient statistic to capture the path dependence.[9] The volatility of the spot rate, $\sigma_r(t)$, is itself unrestricted and could depend on the entire term structure

[9] Of course, forward rate volatilities that do not satisfy equation (17) may, in conjunction with specific structures for the volatility of the spot rate, lead to one- or two-state Markovian term structure models. However, the restriction imposed by equation (17) is the only one that works with *all* spot rate volatility specifications. This result hence permits the development of models where the structure for the spot rate volatility is itself treated as a "free parameter." For example, the spot rate volatility could be described by $\sigma_r(t) = \sigma r(t)^\eta$, where both σ and η are empirically estimable parameters.

at date t. From equation (10c), this implies that in order to eliminate the path dependence in bond prices, no assumptions are needed on the volatility of technological innovations, $\sigma_X(t,t)$, in the economy.

The exponential term in equation (17) identifies the mechanism by which uncertainty at one end of the term structure is transmitted to the rest of the term structure. This transmission scheme is completely described by the deterministic function $\kappa(\cdot)$. This function may be specified exogenously and need not be restricted to attain a Markovian representation of the term structure.[10] As we demonstrate in the next section, $\kappa(\cdot)$ is fully determined by the manner in which investors revise their beliefs about future levels of technology, $X(t,T)$, in response to information that alters the current level, $X(t,t)$. Specifically, $\kappa(\cdot)$ is determined by the belief revision scheme, $\omega(\cdot,\cdot)$. In contrast to the volatility of $X(t,t)$, which is left unrestricted, bond prices will be path dependent unless the belief revision scheme is curtailed.

The class of term structure models that result from equation (17) have the property that forward rates are linear in the state variables. The exact form of the bond prices, which is provided below, establishes that the restriction in equation (17) is also sufficient to obtain a two-state Markovian representation of the term structure.

Theorem 2

Under the restriction imposed by equation (17), the price of a bond at any future date t can be represented in terms of its forward price at date 0, the short interest rate at date t, and the path of interest rates as

$$P(t,T) \;=\; \left(\frac{P(0,T)}{P(0,t)}\right) e^{-\beta(t,T)\left(r(t)-f(0,t)\right)-\frac{1}{2}\beta^2(t,T)\,\phi(0,t)} \tag{18a}$$

where $\beta(t,T)$ is given by

$$\beta(t,T) \;=\; -\frac{\sigma_P(t,T)}{\sigma_r(t)} \;=\; \frac{1}{\kappa(t)}-\frac{1}{\kappa(T)}e^{-\int_t^T \kappa(x)dx} \tag{18b}$$

and $\phi(0,t)$, the state variable that captures path dependence, is given by

$$\phi(0,t) \;=\; \int_0^t \sigma_f^2(u,t)\,du \tag{18c}$$

Proof See Appendix 2.

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[10] As we show later in this section, $\kappa(\cdot)$ measures the degree of mean reversion in spot interest rates. In many models of the term structure, this is assumed to be constant. Other models, however, achieve term structure matching by selecting the mean reversion function appropriately. This is in contrast to our approach, where the choice of $\kappa(\cdot)$ does not alter the term structure at date 0. Hence, $\kappa(\cdot)$ could be assumed constant.

As the bond pricing equation above illustrates, the two state variables, $r(t)$ and $\phi(0,t)$, capture all relevant information required to update the entire term structure relative to an earlier term structure. The first state variable, $r(t)$, is the stochastic spot interest rate that appears in most other single-factor models. The second state variable, $\phi(0,t)$, is a statistic that captures information revealed over the time interval $[0,t]$. Even though the latter is uncertain viewed from date 0, it is locally deterministic and its evolution does not contain a stochastic component. To see this, substitute from equation (17) into equation (18c) and use Ito's rule to obtain

$$d\phi(0,t) = \Big(\sigma_r^2(t) - 2\kappa(t)\phi(0,t)\Big)dt$$

Hence, at date t, once the levels of the state variables, $r(t)$ and $\phi(0,t)$, are observed, the value of $\phi(0,t+dt)$ can be predicted with certainty. All the uncertainty in the term structure over the time increment $[t,t+dt]$ is hence captured by the change in the spot rate, $dr(t)$. To gain a better understanding of the uncertainty revealed by $\phi(0,t)$, consider the term structure at date 0. At that initial date, investors reflect their beliefs about the future spot interest rate, $r(t)$, and its volatility, in the term structure and, in particular, in the forward rate, $f(0,t)$. The evolution of this forward rate ultimately culminates, at date t, in the spot rate, $r(t)$. Unless the volatility, $\sigma_f(u,t)$, is deterministic, investors cannot predict with perfect certainty the accumulated volatility of this rate. It is exactly this uncertainty that is revealed by $\phi(0,t)$ and that is reflected in the term structure at date t.

In practice, it is extremely difficult to compute the value of $\phi(0,t)$ directly. Fortunately, any two points on the term structure at date t can be used to reconstruct the entire yield curve. To see this, set $T = s$, take logarithms on both sides of equation (18a) and differentiate with respect to s to obtain

$$\begin{aligned}\phi(0,t)\,\beta(t,s) &= f(0,t) - r(t) + [f(t,s) - f(0,s)]\,\frac{\sigma(r,t)}{\sigma_f(t,s)} \\ &= f(0,t) - r(t) + [f(t,s) - f(0,s)]\,e^{\int_t^s \kappa(x)dx}\end{aligned}$$

Hence, the state variable, $\phi(0,t)$, can be replaced in the bond pricing equation by any other forward rate, $f(t,s)$.[11] Further, regardless of the volatility structure of spot interest rates, a linear relationship exists between all forward and spot interest rates.

Since forward rates are two-state Markovian, we can recover the drift of the spot rate, $\mu(r,t)$. Its form is summarized below.

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[11] If the second state variable, $\phi(0,t)$, is constant, then bond prices collapse to a single-state Markov representation. From equation (18c), this occurs if the volatility of the spot rate, $\sigma_r(u)$, is deterministic for all u.

Theorem 3

Under the restriction imposed by equation (17), the evolution of the two state variables, $r(t)$ and $\phi(0,t)$, can be expressed as

$$dr(t) = \Big(\kappa(t)[f(0,t) - r(t)] + \phi(0,t) + \lambda(t)\,\sigma_r(t) + \frac{d}{dt}f(0,t)\Big)dt + \sigma_r(t)dz_2(t) \qquad (19a)$$

$$d\phi(0,t)) = \Big(\sigma_r^2(t) - 2\kappa(t)\phi(0,t)\Big)dt \qquad (19b)$$

where $r(0)$ is initialized to the observed spot rate.

Proof See Appendix 3.

The drift term in the evolution of the spot interest rate, $r(t)$, can be decomposed into four terms. The first, $\kappa(t)[f(0,t) - r(t)]$, captures the effect of the mean reversion that arises whenever spot rates deviate from their forward rates. The parameter $\kappa(t)$, which represents the degree of mean reversion, could be assumed constant.[12] The second term, $\phi(0,t)$, incorporates information revealed since date 0. In a certain world, this adjustment would be zero, independent of the path of interest rates over the time period $[0,t]$. The third term, $\lambda(t)\sigma_r(t)$, is the market risk premium for interest rate risk. The final term is the time-varying slope of the initial forward rate curve, representing the anticipated "creep" in interest rates that appears implicitly in all models. For example, consider a certain economy in which future spot rates equal their forward rates. Here, $r(t) = f(0,t)$ for all t, and the evolution of the spot rate is given by

$$dr(t) = \left[\frac{d}{dt}f(0,t)\right] dt \qquad (20)$$

Hence, even with no uncertainty, this term must be present in one form or the other to preclude arbitrage among bonds of differing maturities. The introduction of uncertainty, by itself, does not eliminate this term.

Observe from equation (18a) that bond prices at date t are expressed *relative to* some initial set of bond prices at date 0. Hence, the evolution of bond prices may be readily initialized to their observed values, irrespective of the parameters describing the volatility structure of forward rates. As such, these models do not require time dependent parameters to initialize the term structure to its observed value. This is advantageous, since it avoids the term structure inversions associated with many single-state models. Further, unlike the general HJM models, path dependence is no longer an issue. As a result, the enormous computational difficulties faced in implementing their models are overcome. In particular, since the forward rate volatility restriction allows the Markovian property to be recovered, efficient numerical procedures such as finite difference methods, lattice

[12] In the next section, we explicitly describe an economy in which $\kappa(x)$ is constant.

approximations, Gaussian quadrature, and Monte Carlo simulation methods can be invoked to price complex interest rate claims.

Since the models developed here are consistent with the HJM [1992] pricing paradigm, the valuation of interest rate claims involves taking expectations under the equivalent martingale measure obtained by setting $\lambda = 0$ in equations (19*a*-*b*). The joint distribution of the terminal spot rate, $r(t)$, and the path statistic, $\phi(0,t)$, is, however, nonstandard for arbitrary specifications of the spot rate volatility. For specific structures, such as the square root volatility considered by CIR [1985], however, it is possible to establish all joint moments of the terminal distribution, from which pricing relationships can be derived using Edgeworth expansions, as in Jarrow and Rudd [1982].

IV. Economies with Square Root Technology Innovations

From equation (9*b*), we see that the evolution of the spot interest rate, $r(t)$, is similar to that of the level of technology, $X(t,t)$. As demonstrated below, the restriction imposed by equation (17) in turn constrains the manner by which beliefs about future levels of technology are revised in response to the Brownian disturbance, $dz_2(t)$. In particular, equation (17) imposes restrictions on the volatilities of these beliefs, $\sigma_X(t,T)$. To keep the analysis focused, this section restricts attention to economies in which the evolution of $X(t,t)$ is given by

$$dX(t,t) \ = \ \mu_X(t,t)\,dt + \sigma_2\sqrt{X(t,t)}\,dz_2(t) \tag{21}$$

This implies that $\sigma_r(t) = \sigma\sqrt{r(t)}$, where $\sigma = \sigma_2\sqrt{a}$. Further, we assume that forward rate volatilities are as specified by equation (17). In other words,

$$\sigma_f(t,T) \ = \sigma\sqrt{r(t)}e^{-\int_t^T \kappa(x)dx} \tag{22}$$

From this, it follows that the volatilities of all discount bonds can be computed as

$$\sigma_p(t,T) \ = \ -\,\beta(t,T)\,\sigma\sqrt{r(t)} \tag{23}$$

where

$$\beta(t,T) \ = \frac{1}{\kappa(t)} - \frac{1}{\kappa(T)}e^{-\int_t^T \kappa(x)dx} \tag{24}$$

To establish the relationship between forward rates and the state variables, take $T = t$ in equation (16*a*) and compute expectations. This leads to

$$E_0[r(t)] = f(0,t) + E_0\left[\int_0^t \sigma_f(u,t)[\lambda(u) - \sigma_p(u,t)]du\right]$$

Substituting equations (13), (22) and (23) into the above expression and using equations (4) and

(9*b*) yields

$$f(0,t) = a\Big[X(0,t) - \int_0^t \gamma(u,t)X(0,u)du\Big] \tag{25a}$$

where
$$\gamma(u,t) = [\lambda^* + \sigma^2\beta(u,t)]e^{-\int_u^t \kappa(x)dx} \tag{25b}$$

and
$$\lambda^* = \theta\sigma_1\sigma_2 \tag{25c}$$

More generally, over the period $[t,\tau]$ we have

$$f(t,\tau) = a\Big[X(t,\tau) - \int_t^\tau \gamma(u,\tau)\,X(t,u)\,du\Big] \tag{25d}$$

Equation (25*d*) establishes the link between forward rates and the belief set $\mathcal{I}(t)$ at date t. Using Ito's lemma then results in

$$\sigma_f(t,\tau) = a\Big[\sigma_X(t,\tau) - \int_t^\tau \gamma(u,\tau)\,\sigma_X(t,u)\,du\Big] \tag{26}$$

Substituting for $\sigma_f(t,\tau)$ from equation (22), recognizing that $\sigma_r(t) = a\sigma_X(t,t)$, and rearranging leads to

$$e^{-\int_t^\tau \kappa(x)dx} = \omega(t,\tau) - \int_t^\tau \gamma(u,\tau)\,\omega(t,u)\,du \tag{27}$$

where $\omega(t,\tau) = \sigma_X(t,\tau)/\sigma_X(t,t)$ is the belief revision scheme for the level of technology that reflects the manner by which beliefs about future levels of technology, $X(t,\tau)$, are revised, in response to a change in the current level, $X(t,t)$.

Equation (27) shows that given any belief revision scheme $\omega(t,\tau)$, we can estimate the functional representation for $\kappa(\tau)$. However, for some representations of the belief revision scheme, $\omega(t,\tau)$, the resulting expression for the mean reversion function, $\kappa(\tau)$, will not be independent of time, t. In such cases, from Theorem 1, additional restrictions must be explicitly imposed on the belief set, $\mathcal{I}(t)$, in order to eliminate the resulting path dependence. For example, assume that the volatility structure of forward rates is exactly that of CIR [1985]. Then using equation (22) leads to a functional form for $\kappa(\tau)$ that depends on time t. This implies that the volatility structure of forward rates, as postulated by CIR, is not by itself sufficient to allow path dependence to be characterized by a single statistic. Indeed, in order to establish the CIR model, an explicit structure is required for the drift term, $\mu_X(t,t)$, that allows the belief set, $\mathcal{I}(t)$, to be fully characterized (see footnote 6). As a result, the term structures are themselves described by a single statistic and all path dependence linking term structures over discrete time periods is eliminated. If, on the other hand, the belief revision scheme leads to an expression for $\kappa(\tau)$ that is independent of time t, then a two-state Markovian representation exists for the term structure.

In contrast to the CIR model, our approach does not impose any functional relationship between the state variables in the belief set, $\mathcal{I}(t)$. However, as evidenced by equation (27), requiring the mean reversion function to be deterministic and independent of time also constrains the belief revision scheme. To make these constraints more explicit, differentiate equation (27) with respect to T and eliminate the integral term in equation (27) to obtain

$$0 = \frac{\partial^2}{\partial T^2}\omega(t,T) + b(T)\frac{\partial}{\partial T}\omega(t,T) + c(T) \tag{28a}$$

where

$$b(T) = 3\kappa(T) - \lambda^* - \frac{\frac{\partial}{\partial T}g(T)}{g(T)} \tag{28b}$$

$$c(T) = \frac{\partial}{\partial T}\kappa(T) - \sigma^2 g(T) + \left[\frac{\frac{\partial}{\partial T}g(T)}{g(T)} - 2\kappa(T)\right][\lambda^* - \kappa(T)] \tag{28c}$$

$$g(T) = 1 + \frac{\frac{\partial}{\partial T}\kappa(T)}{[\kappa(T)]^2} \tag{28d}$$

subject to the boundary conditions

$$\omega(t,t) = 1 \tag{28e}$$

and

$$\left.\frac{\partial}{\partial T}\omega(t,T)\right|_{T=t} = \lambda^* - \kappa(t) \tag{28f}$$

where the last boundary condition is obtained by differentiating equation (27) and setting $T = t$. Equation (28a) is a second-order linear homogeneous differential equation in $\omega(t,T)$. Hence, as long as $\kappa(T)$ is a twice differentiable function, a unique solution for $\omega(t,T)$ exists. Further, since the coefficients $b(T)$ and $c(T)$ are deterministic, so is the solution for $\omega(t,T)$. This leads to the following result, which is stated without proof.

Lemma

If interest rates follow an Ito process with square root volatility, $\sigma_r(t) = \sigma\sqrt{r(t)}$, and if the mean reversion function, $\kappa(T)$, is twice differentiable, then a unique, deterministic, twice differentiable belief revision scheme $\omega(t,T)$ exists.

A special case of the above occurs when the mean reversion parameter is constant. In particular, let $\kappa(T) = \kappa$. In that case,

$$\sigma_f(t,T) = \sigma\sqrt{r(t)}\,e^{-\kappa(T-t)} \tag{29}$$

and

$$\beta(t,T) = -\frac{\sigma_p(t,T)}{\sigma\sqrt{r(t)}} = \frac{1}{\kappa}\left(1 - e^{-\kappa(T-t)}\right) \tag{30}$$

The above structure for forward rate volatility is similar to that considered by Vasicek [1977], with the notable exception that, unlike in his model, spot rates do not have constant volatility. From the bond pricing equation (18a), we also note that our model is completely described by the observed term structure at date 0 and the unobservable parameters σ and κ.[13] Hence, it lends itself readily to empirical investigation.[14]

The belief revision scheme for this structure is readily obtained by solving equations (28a-f) as

$$\omega(t,T) = \alpha^* e^{-P_1(T-t)} + (1-\alpha^*)e^{-P_2(T-t)} \tag{31a}$$

where

$$P_1 = \frac{3\kappa - \lambda^*}{2} + \frac{1}{2}\sqrt{(\kappa+\lambda^*)^2 - 4\sigma^2}$$

$$P_2 = \frac{3\kappa - \lambda^*}{2} - \frac{1}{2}\sqrt{(\kappa+\lambda^*)^2 - 4\sigma^2}$$

$$\alpha^* = \frac{1}{2} - \frac{1}{2}\frac{\kappa+\lambda^*}{\sqrt{(\kappa+\lambda^*)^2 - 4\sigma^2}}$$

and where we have assumed that $\kappa + \lambda^* > 2\sigma$. If $\kappa + \lambda^* < 2\sigma$, then

$$\omega(t,T) = e^{-\zeta_1(T-t)}\cos[\zeta_2(T-t)] + \left(\frac{\lambda^* + \kappa}{2\zeta_2}\right)\sin[\zeta_2(T-t)] \tag{31b}$$

where

$$\zeta_1 = \frac{3\kappa - \lambda^*}{2}$$

$$\zeta_2 = \frac{\sqrt{4\sigma^2 - (\kappa+\lambda^*)^2}}{2}$$

In particular, equation (31a) shows that the desired volatility structure for forward rates results from a belief scheme that is a weighted sum of exponentially dampened functions.

Note that the belief scheme in equations (31a-b) differs from the one implicit in the CIR model (see footnotes 6 and 7). In our approach, if innovations are modelled by equation (21), and if the belief revision scheme is given as in equations (31a) or (31b), then the resulting volatility structure of forward rates satisfies Theorem 1. However, the volatility and belief schemes are themselves not sufficient to fully characterize the drift term. Indeed, in our approach a specification of the term structure is required to uniquely determine the drift term.

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[13] The market price for the interest rate risk parameter, λ^*, is not relevant for pricing interest rate claims, since the martingale measure relevant for pricing described by HJM [1991] can be obtained by setting $\lambda^* = 0$ in the drift terms of the evolution of the state variables, $r(t)$ and $\phi(0,t)$. Furthermore, unlike the CIR [1985] model, this parameter does not appear explicitly in the volatility of all forward rates. Hence, unlike in their model, λ^* does not appear *explicitly* in any of our pricing relationships. However, λ^* does appear implicitly in all the model prices through the initial term structure, which we assume is observable.

[14] For empirical purposes, a more general model in which $\sigma_r(t) = \sigma[r(t)]^\eta$ can also be cast without any additional complexities.

V. Conclusion

In an economy with continuous trading, the linkages between term structures separated by a finite time period can be quite complex. Indeed, characterizing the term structure in accordance with its value at a previous date, together with a finite number of state variables, is not possible unless assumptions are placed, directly or indirectly, on the volatilities of spot and forward rates. In most cases, these assumptions completely eliminate the path dependence. In this article, we investigated alternative representations for the term structure in which the path dependence is not eliminated, but rather is captured by a single sufficient statistic. This statistic, together with the spot rate, was shown to fully characterize the term structure and its dynamics. The ability to capture the path dependence by a single statistic stemmed, in part from the manner in which investors revise their beliefs about future levels of technology.

Since our analysis is concerned only with the intertemporal links between term structures at different points in time, an initial term structure must be provided. Any observed term structure is a viable candidate. Further, this initialization is achieved without restricting the volatility of spot rates or introducing ad-hoc time-varying parameters that may be difficult to estimate. Finally, unlike the general HJM models, these benefits are obtained without sacrificing the advantages of the finite state space methodology. In particular, even with extremely general structures for the spot rate volatility, efficient computational procedures can be developed to price interest rate claims. In addition, the parameters for the volatility structure can be readily estimated.

This article restricted attention to economies in which the evolution of all bonds was driven by a single source of uncertainty. Many term structure models have been postulated where uncertainty is driven by two or more sources of uncertainty. For example, in Brennan and Schwartz [1979], the evolution is described by the short and long interest rates. More recently, Longstaff and Schwartz [1992] developed a model in which the volatility of spot rates itself is stochastic. In these models, bonds are not instantaneously perfectly correlated, but the instantaneous canonical correlation between every pair of bonds is perfect. In other words, the evolution of the term structure can be derived from the evolution of any pair of bonds. Over finite intervals, however, the path dependence issues that we outlined in this paper may persist, so it may not be possible to reconstruct the entire term structure given just two points. For example, in a two-factor economy where the uncertainty in the term structure is linked to a short and a long rate, the path taken by *both* factors generating uncertainty may influence the intertemporal relationship between term structures. Once again, to obtain a Markovian path-independent representation requires restrictions on either the volatility of both sources of uncertainty or the shape of the term structure. Alternatively, necessary and sufficient conditions on volatility structures can be identified that permit path dependence to be captured by two sufficient statistics. These statistics, together with the short and long rate, would then be sufficient to describe the term structure at any point in time, relative to an earlier term structure. This simplification leads to models in which the entire term structure can be described by any four distinct points. The development of these multi-factor economies and the handling of the resulting complexities are the subjects of future research.

Finally, this article provides exciting directions for future research. First, empirical tests of the two-state single-factor model need to be performed. Since more information on the term structure is incorporated into the dynamics of the spot interest rate than in the single-state models, the two-state models are likely to perform better. Second, specific two-state, single-factor option pricing models need to be investigated, as does the impact of alternative volatility structures for the spot rate process. Since the two-state option models do not require time-varying parameters, they may have advantages over existing single-state option models.

Appendix 1

Proof of Theorem 1

Equation (16*b*) can be rewritten as

$$f(t,T) - f(0,T) - \int_0^t \sigma_f(u,T)[\lambda(u) - \sigma_p(u,T)]du = \int_0^t \sigma_f(u,T)\,dz_2(u)$$
$$= \Lambda(0,t,T) \qquad (A1.1)$$

$\Lambda(0,t,T)$ reflects the path dependence as the "weighted sum" of all Brownian disturbances realized from time 0 to time t. This path dependence can be captured by a single statistic provided that a common "weighting scheme" exists for all forward rates. However, if a unique weighting scheme is to exist for all forward rates, it must be the case that the weighting function is independent of T. This in turn implies that

$$\frac{\sigma_f(u,T)}{\int_0^t \sigma_f(x,T)\,dx} = \frac{\sigma_f(u,t)}{\int_0^t \sigma_f(x,t)\,dx} \quad \forall u \in [0,t]$$

Equivalently,
$$\sigma_f(u,T) = \sigma_f(u,t)\left[\frac{\int_0^t \sigma_f(x,T)\,dx}{\int_0^t \sigma_f(x,t)\,dx}\right] = \sigma_f(u,t)\,k(t,T)$$

$\sigma_f(u,t)$ is possibly stochastic, with values depending on the belief set at date u. Substituting $t = u$ leads to

$$\sigma_f(u,T) = \sigma_r(u)\,k(u,T) \qquad (A1.2)$$

Also, $\sigma_f(u,t) = \sigma_r(u)k(u,t)$. Hence,

$$k(u,T) = k(u,t)\,k(t,T) \quad \forall t \in [u,T] \qquad (A1.3)$$

with
$$k(t,t) = 1 \quad \forall t \in [u,T]$$

Setting $u = 0$ in equation (A1.3) leads to

$$k(t,T) = \frac{k(0,T)}{k(0,t)} \qquad \forall t \in [0,T]$$

This implies that the function $k(t,T)$ is known at date 0 and is hence deterministic. Finally, note that the class of functions that satisfy equation (A1.3) can be written in the form

$$k(u,T) = e^{-\int_u^T \kappa(x)\,dx} \qquad (A1.4)$$

Since $k(u,T)$ is deterministic, so is $\kappa(x)$. Equations (A1.4) and (A1.2) together complete the proof.

□

Appendix 2

Proof of Theorem 2

Note from equation (16b) that

$$R(0,t;\tau) \equiv f(t,\tau) - f(0,\tau) = \int_0^t \sigma_f(u,\tau)\Big[\lambda(u) - \sigma_p(u,\tau)\Big]du + \int_0^t \sigma_f(u,\tau)dz_2(u) \qquad (A2.1)$$

Here, $R(0,t;\tau)$ is defined as the difference between the forward rate at date t and that at the original date. Using the restriction imposed by equation (17) on equation (A2.1) we obtain

$$\frac{R(0,t;\tau)}{e^{-\int_t^\tau \kappa(x)\,dx}} + \int_0^t \sigma_f(u,t)\sigma_p(u,\tau)du = \int_0^t \sigma_f(u,t)\lambda(u)du + \int_0^t \sigma_f(u,t)dz_2(u) \qquad (A2.2)$$

Since the right-hand side of equation (A2.2) is independent of τ, take $\tau = t$ to obtain

$$\frac{R(0,t;\tau)}{e^{-\int_t^\tau \kappa(x)\,dx}} + \int_0^t \sigma_f(u,t)\sigma_p(u,\tau)du = R(0,t;t) + \int_0^t \sigma_f(u,t)\sigma_p(u,t)du$$

which simplifies to

$$\begin{aligned}
R(0,t;\tau) &= e^{-\int_t^\tau \kappa(x)\,dx}\left\{R(0,t;t) + \int_0^t \sigma_f(u,t)\Big[\sigma_p(u,t) - \sigma_p(u,\tau)\Big]du\right\} \\
&= e^{-\int_t^\tau \kappa(x)\,dx}\left\{R(0,t;t) + \int_0^t \sigma_f(u,t)\int_t^\tau \sigma_f(u,x)dx\,du\right\} \\
&= \frac{\sigma_f(t,\tau)}{\sigma_r(t)}\left\{R(0,t;t) + \int_0^t \sigma_f^2(u,t)du\int_t^\tau e^{-\int_t^x \kappa(x)\,dx}dx\right\} \\
&= \frac{\sigma_f(t,\tau)}{\sigma_r(t)}\left\{R(0,t;t) + \int_0^t \sigma_f^2(u,t)du\int_t^\tau \frac{\sigma_f(t,x)}{\sigma_r(t)}dx\right\} \\
&= \sigma_f(t,\tau)\frac{R(0,t;t)}{\sigma_r(t)} - \frac{\sigma_f(t,\tau)}{\sigma_r^2(t)}\sigma_p(t,\tau)\phi(0,t)
\end{aligned}$$

where $$\phi(0,t) = \int_0^t \sigma_f^2(u,t)du$$

Equivalently,

$$f(t,\tau) = f(0,\tau) + \frac{\sigma_f(t,\tau)\Big(r(t) - f(0,t)\Big)}{\sigma_r(t)} - \frac{\sigma_f(t,\tau)}{\sigma^2(r,t)}\sigma_p(t,\tau)\phi(0,t) \qquad (A2.3)$$

Notice that the forward rate at time t is completely determined by the spot rate, $r(t)$, and the path statistic, $\phi(0,t)$. Also, observe that the state variable, $\phi(0,t)$, which captures the information relating to the path of interest rates, is independent of the forward rate maturity. Further, since the two state variables, $r(t)$ and $\phi(0,t)$, permit us to compute all forward rates at time t, they also allow us to price all discount bonds. Hence, the entire term structure at time t is fully determined by these two state variables, which means that all bond prices can be expressed in terms of them. To see this, note that equation (A2.3) can be written as

$$f(t,T) - f(0,T) = -\frac{\partial}{\partial T}\left[\sigma_p(t,T)\frac{R(0,t;t)}{\sigma_r(t)} - \frac{\phi(0,t)}{2\sigma_r^2(t)}\sigma_p^2(t,T)\right]$$

Further, from equation (1) we have

$$P(t,T) = e^{-\int_t^T f(0,s)ds + \int_t^T \frac{\partial}{\partial x}\left[\sigma_p(t,x)\frac{R(0,t;t)}{\sigma_r(t)} - \frac{\phi(0,t)}{2\sigma_r^2(t)}\sigma_p^2(t,x)\right]dx}$$

which upon simplification yields

$$P(t,T) = \left(\frac{P(0,T)}{P(0,t)}\right) e^{-\frac{1}{2}\beta^2(t,T)\,\phi(0,t) + \beta(t,T)\,[f(0,t) - r(t)]}$$

where

$$\beta(t,T) = -\frac{\sigma_p(t,T)}{\sigma(r,t)} = \frac{1}{\kappa(T)}\left[1 - \frac{\kappa(t)}{\kappa(T)}e^{-\int_t^T \kappa(x)dx}\right]$$

$$\phi(0,t) = \int_0^t \sigma_f^2(u,t)\,du = \int_0^t \sigma_r^2(u)\,e^{-\int_u^t \kappa(x)dx}du \qquad (A2.4)$$

This completes the proof. □

Appendix 3

Proof of Theorem 3

The dynamics of the spot rate, $r(t)$, involve the simultaneous movement in both arguments of the forward rate. Hence, its evolution is given by

$$\begin{aligned} dr(t) &= \frac{\partial}{\partial u} f(u,t)\Big|_{u\to t} dt + \frac{\partial}{\partial u} f(t,u)\Big|_{u\to t} dt \\ &= df(t,t) + \frac{\partial}{\partial u} f(t,u)\Big|_{u\to t} dt \end{aligned}$$

Computing these terms using equations (14a), (A2.3) and (A2.4) and rearranging yields

$$dr(t) = \Big(\kappa(t)[f(0,t) - r(t)] + \phi(0,t) + \lambda(t)\,\sigma_r(t) + \frac{d}{dt} f(0,t)\Big) dt + \sigma_r(t) dz_2(t) \quad (A3.1)$$

$$d\phi(0,t) = \Big(\sigma_r^2(t) - 2\kappa(t)\phi(0,t)\Big) dt \quad (A3.2)$$

This completes the proof. □

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