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Dynamics, Cycles and Sunspot Equilibria in "Genuinely Dynamic, Fundamentally Disaggregative" Models of Money

by Ricardo Lagos and Randall Wright

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This paper pursues a line of Cass and Shell, who advocate monetary models that are "genuinely dynamic and fundamentally disaggregative" and incorporate "diversity among households and variety among commodities." Recent search-theoretic models fit this description. We show that, like overlapping generations models, search models generate interesting dynamic equilibria, including cycles, chaos, and sunspot equilibria. This helps us understand how alternative models are related, and lends support to the notion that endogenous dynamics and uncertainty matter, perhaps especially in monetary economies. We also suggest such equilibria in search models may be more empirically relevant than in some other models.

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1 Introduction

Cass and Shell (1980) advocate macro and monetary models with an explicit microeconomic structure. The framework they had in mind was the overlapping generations model of fiat money [Samuelson [1958]; Shell [1971]; Wallace [1980]). They argue "this basic structure has two general features which we believe are indispensable to the development of macroeconomics as an intellectually convincing discipline... *First it is genuinely dynamic... Second it is fundamentally disaggregative*." They also say that they "firmly believe that a satisfactory general theory must, at a minimum, encompass some diversity among households as well as some variety among commodities," although for technical reasons these features "will typically impose a significant constraint on our ability to derive substantial propositions concerning qualitative effects."

Since they wrote these words a new monetary model has been developed that is also genuinely dynamic and fundamentally disaggregative and takes seriously diversity among households and variety among commodities – the search model. While ostensibly quite different, on some dimensions search models deliver very similar predictions to overlapping generations models. We show here that, like overlapping generations models, search models generate interesting dynamic equilibria including cycles, chaos, and sunspot equilibria. Knowing this helps us understand how alternative monetary theories are connected, and lends further support to the notion that extrinsic dynamics and uncertainty matter, perhaps especially in monetary economies (Shell [1977]; Cass and Shell [1983]).

There are several search-based monetary models in the literature. Many assume severe restrictions on how much money agents can carry, typically

$m \in \{0,1\}$ (Kiyotaki and Wright [1991]; Trejos and Wright [1995]; Shi [1995]). This is convenient for the same reason two-period-lived agents are convenient in the overlapping generations model: at any point in time we can partition the set of agents into a subset with money who want to spend it and a subset without money who want to get it. Yet $m \in \{0,1\}$ is not an appealing assumption. Recent search models allow agents to carry any $m \in \mathcal{R}_+$. Some versions of this model are notoriously difficult to analyze (Molico [1999]). Here we focus on the much more tractable version in Lagos and Wright (2002) – hereafter referred to as LW.

Section 2 reviews the basic LW model. Section 3 goes beyond previous analyses by considering dynamic equilibria, including cycles and chaos. Section 4 considers sunspots. Section 5 concludes.[1]

2 The Basic Model

Time is discrete. The set of agents $\mathcal{A}$ has measure 1. Agents live forever and have discount factor $\beta \in (0,1)$. There is a set of nonstorable goods $\mathcal{G}$. Our version of Cass and Shell's (1980) "diversity among households [and] variety among commodities" is as follows. Each $i \in \mathcal{A}$ consumes a subset of goods $\mathcal{G}^i \subset \mathcal{G}$, and produces one good g^i where $g^i \notin \mathcal{G}^i$. Also, given any two agents i and j drawn at random from $\mathcal{A}$, the probability of a single coincidence is $prob(g^i \in \mathcal{G}^j) = \sigma \in (0, \frac{1}{2})$ and the probability of a double coincidence is $prob(g^i \in \mathcal{G}^j \wedge g^j \in \mathcal{G}^i) = 0$. In a single-coincidence meeting, if $g^i \in \mathcal{G}^j$ we

[1] A textbook treatment of cycles and sunspots in overlapping generations (and some other) models is Azariadis (1993). Nonmonetary search models can also display interesting dynamics, but only if the meeting technology has increasing returns (Diamond and Fudenberg [1989]), which is *not* the case here. Dynamics have been studied in monetary search models by Wright (1994), Shi (1995), Coles and Wright (1998), Li and Wright (1998), Ennis (2001,2002), and Lomeli and Temzelides (2002); these papers all assume $m \in \{0,1\}$.

call j the *buyer* and i the *seller*.[2]

Let $u(q)$ be i's utility from consuming any good in $\mathcal{G}^i$ and $c(q) = q$ the disutility of production. The assumption that c is linear is what makes things tractable. Assume u is $\mathcal{C}^n$ with $u' > 0$ and $u'' < 0$. Also, $u(0) = 0$ and $u(\bar{q}) = \bar{q}$ for some $\bar{q} > 0$. Let q^* denote the *efficient* quantity, which solves $u'(q^*) = 1$. We sometimes need $u''' \leq (u'')^2/u'$, which holds if u' is log-concave. In addition to goods, there is another object called *money* that is storable, divisible, and can be held in any quantity $m \geq 0$. For now, it has no intrinsic value – it is fiat money.

Each period has two sub-periods, day and night. During the day there is a decentralized market with anonymous bilateral matching and bargaining; at night there is a centralized frictionless market. Agents do not discount between subperiods here but this is easily generalized (see Rocheteau and Wright [2002]). All trade in the decentralized market must be quid pro quo, as there can be no credit between anonymous agents (Kocherlakota [1998]; Wallace [2001]). We could allow intertemporal trade in the centralized market, but in equilibrium it will not happen since we cannot find one agent who wants to save and another to borrow at the same interest rate.

Let F_t be the distribution of money holdings in the decentralized market at t, where $M_t = \int m dF_t(m) = M$ for all t.[3] Let $V_t(m)$ be the value function of an agent with m dollars in the decentralized market and $W_t(m)$ the value function in the centralized market. Let $q_t\,(m, \tilde{m})$ and $d_t\,(m, \tilde{m})$ be the quantity of goods and dollars traded in a single-coincidence meeting between a buyer with m and a seller with $\tilde{m}$ dollars. Let α be the probability of meeting

[2]In LW, double coincidence meetings can occur with positive probability and the results are basically unchanged. Also, in LW there are some general goods that everyone consumes, but this is not important for the model.

[3]We keep M_t constant here so we can focus on dynamics due exclusively to beliefs.

anyone in the decentralized market and $\lambda = \alpha\sigma$. Bellman's equation is[4]

$$\begin{aligned} V_t(m) &= \lambda \int \{u[q_t(m,\tilde{m})] + W_t[m - d_t(m,\tilde{m})]\} \, dF_t(\tilde{m}) \\ &\quad + \lambda \int \{-q_t(\tilde{m},m) + W_t[m + d_t(\tilde{m},m)]\} \, dF_t(\tilde{m}) \qquad (1) \\ &\quad + (1-2\lambda) W_t(m). \end{aligned}$$

We normalize the price of goods in the centralized market to 1 and let ϕ_t be the price of money. Then the problem is

$$\begin{aligned} W_t(m_t) &= \max_{x_t, y_t, m_{t+1}} \{u(x_t) - y_t + \beta V_{t+1}(m_{t+1})\} \qquad (2) \\ \text{s.t. } x_t &= y_t + \phi_t(m - m_{t+1}), \end{aligned}$$

where x_t is total consumption of goods in $\mathcal{G}^i$, y_t is production, and m_{t+1} is money left over after trading. We impose nonnegativity on m_{t+1} and x_t but *not* y_t. Our approach is to allow $y_t < 0$ when solving (2); then, after finding equilibrium, one can introduce conditions to insure $y_t \geq 0$ (see LW).

If we insert y_t from the budget equation,

$$W_t(m) = \phi_t m_t + \max_{x_t, m_{t+1}} \{u(x_t) - x_t - \phi_t m_{t+1} + \beta V_{t+1}(m_{t+1})\}. \qquad (3)$$

From (3), $x_t = q^*$ where $u'(q^*) = 1$. Also, $W_t(m)$ is linear:

Lemma 1 *$W_t(m) = W_t(0) + \phi_t m$.*

Given this, we can simplify (1) to

$$\begin{aligned} V_t(m) &= \lambda \int \{u[q_t(m,\tilde{m})] - \phi_t d_t(m,\tilde{m})\} \, dF_t(\tilde{m}) \qquad (4) \\ &\quad + \lambda \int \{-q_t(\tilde{m},m) + \phi_t d_t(\tilde{m},m)\} \, dF_t(\tilde{m}) + W_t(m). \end{aligned}$$

[4]The first term is the expected payoff from buying $q_t(m,\tilde{m})$ and going to the centralized market that night with $m - d_t(m,\tilde{m})$ dollars. The second term is the expected payoff from selling $q_t(\tilde{m},m)$ and going to the centralized market with $m + d_t(\tilde{m},m)$ dollars. Notice the roles of m and $\tilde{m}$ are reversed in these two terms. The final term is the payoff from going to the centralized market without trading in the decentralized market.

We now turn to the decentralized terms of trade. Consider a single-coincidence meeting when the buyer has m and the seller $\tilde{m}$. We use the generalized Nash solution:

$$\max_{q,d} \ [u(q) + W_t(m-d) - W_t(m)]^{\theta} [-q + W_t(\tilde{m}+d) - W_t(\tilde{m})]^{1-\theta}$$

s.t. $d \leq m$. By Lemma 1, this reduces to:

$$\max_{q,d} \ [u(q) - \phi_t d]^{\theta} [-q + \phi_t d]^{1-\theta} \tag{5}$$

s.t. $d \leq m$. The solution (q, d) to (5) does not depend on $\tilde{m}$, and depends on m iff the constraint $d \leq m$ binds. Thus, we write $q_t(m, \tilde{m}) = q_t(m)$ and $d_t(m, \tilde{m}) = d_t(m)$ in what follows.

Lemma 2 *The bargaining solution is*

$$q_t(m) = \begin{cases} \widehat{q}_t(m) & \text{if } m < m_t^* \\ q^* & \text{if } m \geq m_t^* \end{cases} \qquad d_t(m) = \begin{cases} m & \text{if } m < m_t^* \\ m^* & \text{if } m \geq m_t^* \end{cases} \tag{6}$$

where $m_t^* = [\theta q^* + (1-\theta)u(q^*)]/\phi_t$ *and* $\widehat{q}_t(m)$ *solves*

$$\phi_t m = \frac{\theta q u'(q) + (1-\theta)u(q)}{\theta u'(q) + 1 - \theta}. \tag{7}$$

Proof: One can easily check that the proposed solution satisfies the first order conditions from (5). ■

Lemma 3 *For all* t*, for all* $m < m_t^*$*, we have* $q_t'(m) > 0$ *and* $q_t(m) < q^*$*.*

Proof: The implicit function theorem implies $\widehat{q}_t(m)$ is C^{n-1} and

$$\begin{aligned} \widehat{q}_t' &= \frac{\phi_t(\theta u' + 1 - \theta)}{u' - \theta(\phi_t m - q)u''} \\ &= \frac{\phi_t(\theta u' + 1 - \theta)^2}{u'(\theta u' + 1 - \theta) - \theta(1-\theta)(u-q)u''} > 0 \end{aligned}$$

for all $m < m_t^*$, where the second equality follows from (7). Since $\lim_{m \to m_t^*} \widehat{q}_t(m) = q^*$, we conclude that $q_t(m) = \widehat{q}_t(m) < q^*$ for all $m < m_t^*$. ■

Inserting (6) and (3) into (4), we have

$$V_t(m) = v_t(m) + u(q^*) - q^* + \phi_t m + \max_{m_{t+1}} \{-\phi_t m_{t+1} + \beta V_{t+1}(m_{t+1})\} \quad (8)$$

where

$$v_t(m) = \lambda\{u[q_t(m)] - \phi_t d_t(m)\} + \lambda \int [\phi_t d_t(\tilde{m}) - q_t(\tilde{m})] dF_t(\tilde{m}). \quad (9)$$

In LW we show there exists a unique $V_t(m)$ in a certain class of functions satisfying (8).[5] To avoid these details in this paper, we can work with the sequence problem,

$$V_0(m_0) = \max_{\substack{\{m_{t+1}\} \\ m_0 \text{ given}}} \sum_{t=0}^{\infty} \beta^t [v_t(m_t) + u(q^*) - q^* + \phi_t(m_t - m_{t+1})] \quad (10)$$

where the agent takes as given the entire path for (ϕ_t, F_t).

Since $q_t(m)$ and $d_t(m)$ are C^{n-1} for all $m \neq m_t^*$, so is $v_t(m)$. We have the first order conditions

$$-\phi_t + \beta v'_{t+1}(m_{t+1}) + \beta \phi_{t+1} \leq 0, \ = 0 \text{ if } m_{t+1} > 0 \ \forall t. \quad (11)$$

In a monetary equilibrium at least one agent must choose $m_{t+1} > 0$, and for him (11) holds with equality. Indeed, we claim that all agents choose the same $m_{t+1} = M > 0$ in any equilibrium. We show this in several steps.

Lemma 4 *In any equilibrium, $\beta\phi_{t+1} \leq \phi_t$ for all t.*

[5] This is a nonstandard dynamic programming problem because it is nonstationary and because $V_t(m)$ is unbounded due to the term $\phi_t m$. The problem can be rendered stationary by including ϕ_t as an aggregate state variable as in Duffie et al. (1994). The fact that V_t is unbounded can be finessed by working in the space of functions $\bar{V}(m) = \bar{v}(m) + \phi m$, where $\bar{v}$ is continuous and bounded. See LW.

Proof: From (9) we have

$$v'_{t+1}(m) = \begin{cases} \lambda u'\left[\widehat{q}_{t+1}(m)\right]\widehat{q}'_{t+1}(m) - \lambda\phi_{t+1} & \text{if } m < m^*_{t+1} \\ 0 & \text{if } m \geq m^*_{t+1} \end{cases} \tag{12}$$

where $\widehat{q}'_{t+1}$ is given above. If $\beta\phi_{t+1} > \phi_t$ then the left hand side of (11) is strictly positive for $m_{t+1} > m^*_{t+1}$ and so the problem has no solution. ■

Lemma 5 *In any equilibrium, $d_t(m_t) = m_t < m^*_t$ and $q_t(m_t) = \widehat{q}_t(m_t) < q^*$ for all $t \geq 1$.*

Proof: One can easily check $\lim_{m_{t+1}\to m^*_{t+1}} v'_{t+1}(m_{t+1}) < 0$. This combined with Lemma 4 implies that the left hand side of (11) is strictly negative for all $m_{t+1} \in (m^*_{t+1} - \varepsilon, m^*_{t+1})$ for some $\varepsilon > 0$, and weakly negative for all $m_{t+1} > m^*_{t+1}$. Hence, we have $m_{t+1} < m^*_{t+1}$. The rest follows from Lemma 2. ■

We now know that agents choose $m_{t+1} < m^*_{t+1}$. If we could show $v'' < 0$ for $m_{t+1} < m^*_{t+1}$ then we could conclude they all choose the same m_{t+1}, simply because the marginal cost of acquiring money is constant at ϕ_t given our linear production cost. While v'' cannot be signed in general (since it depends on u''') we do have the following result.

Lemma 6 *Assume that $\theta \approx 1$ or that u' is log-concave. Then in any equilibrium, for all $t \geq 1$, we have $v''(m_{t+1}) < 0$ for all $m_{t+1} < m^*_{t+1}$, and F_t is degenerate at M.*

Proof: Differentiation implies v'' is the same sign as $\Gamma + (1-\theta)\left[u'u''' - (u'')^2\right]$, where Γ is a function of parameters that is strictly negative but otherwise need not concern us. Hence, $v'' < 0$ for all m in the relevant range if either we assume $u'u''' \leq (u'')^2$ for all q, which follows from log-concavity, or we assume $\theta \approx 1$. Given $v'' < 0$, there is a unique solution to (11) at equality, and so all agents choose the same m_{t+1}. That is, F is degenerate. ■

The next step is to reduce the model to 1 equation in 1 unknown. To begin, insert v'_{t+1} from (12) into (11) to get

$$\phi_t = \beta\phi_{t+1}\left[1 - \lambda + \lambda e(q_{t+1})\right], \tag{13}$$

where $e(q_{t+1}) = u'[q_{t+1}(m_{t+1})]\widehat{q}'_{t+1}(m_{t+1})/\phi_{t+1}$. Inserting $\widehat{q}'_{t+1}$, we have

$$e(q) = \frac{u'(q)[\theta u'(q) + 1 - \theta]^2}{u'(q)[\theta u'(q) + 1 - \theta] - \theta(1-\theta)\left[u(q) - q\right]u''(q)}. \tag{14}$$

Notice $e(q_{t+1})$ does not depend on t or ϕ_{t+1} directly, but only on q_{t+1}.

Now (7) with $m = M$ implies $\phi_{t+1} = f(q_{t+1})/M$, where

$$f(q) = \frac{\theta q u'(q) + (1-\theta)u(q)}{\theta u'(q) + 1 - \theta}. \tag{15}$$

Inserting (15) into (13), we get a simple difference equation:

$$f(q_t) = \beta f(q_{t+1})[1 - \lambda + \lambda e(q_{t+1})]. \tag{16}$$

A *monetary equilibrium* is a path $\{q_t\}$ satisfying (16) that remains in $(0, q^*)$ for all t. A steady state q^s is a solution to

$$1 = \beta\left(1 - \lambda\right) + \beta\lambda e(q^s). \tag{17}$$

Proposition 1 *Assume either that $\theta \approx 1$ or that u' is log concave. Then there can be at most one monetary steady state q^s. A steady state q^s exists iff $e(0) > (1 - \beta + \lambda\beta)/\lambda\beta$. When it exists, q^s is increasing in θ, λ and β. Also, $q^s \to q^*$ as $\beta \to 1$ iff $\theta = 1$.*

Proof: Let $T(q)$ denote the right hand side of (17). Calculation shows the stated conditions imply $e' < 0$ and hence $T' < 0$, so there cannot be more than one solution to $T(q) = 1$. Notice

$$T(q^*) = \frac{\beta\lambda}{1 - \theta(1-\theta)[u(q^*) - q^*]u''(q^*)} + \beta(1-\lambda).$$

Hence, $T(q^*) \leq \beta < 1$, and there is a solution $q^s \in (0, q^*)$ to $T(q) = 1$ iff $T(0) > 1$ iff $e(0) > \frac{1-\beta+\lambda\beta}{\lambda\beta}$. Also, $T(q^*) = \beta$ if $\theta = 1$ and $T(q^*) < \beta$ if $\theta < 1$. Hence, $q^s \to q^*$ as $\beta \to 1$ iff $\theta = 1$. The rest is routine. ■

3 Dynamics

We now consider equilibria where q changes over time. To begin, denote the right hand side of (16) by $R(q_{t+1})$. Notice $R(q_{t+1}) \geq 0$, $f(0) = 0$, and $\lim_{q\to\infty} f(q) = \infty$ (at least if we assume u' is bounded away from 0). Also notice

$$f'(q) = \frac{u'(q)[\theta u'(q) + 1 - \theta] - \theta(1-\theta)[u(q) - q]u''(q)}{[\theta u'(q) + 1 - \theta]^2} > 0.$$

Hence, $\forall q_{t+1} \geq 0$ there exists a unique $q_t = g(q_{t+1})$ such that $f(q_t) = R(q_{t+1})$. Thus g is single-valued, although $h = g^{-1}$ is generally a correspondence.

Clearly $q = 0$ is a (nonmonetary) steady state.[6] Under the conditions in Proposition 1 there is a unique monetary steady state $q^s = g(q^s) > 0$. Hence g intersects the 45^o line exactly twice, at 0 and at q^s.

Lemma 7 *Assume a unique monetary steady state q^s exists; then $g'(q^s) < 1$ and $g'(0) > 1$.*

Proof: Differentiation implies

$$g'(q_{t+1}) = \frac{\beta}{f'(q_t)} \left\{ f'(q_{t+1})[1 - \lambda + \lambda e(q_{t+1})] + \lambda e'(q_{t+1}) f(q_{t+1}) \right\}.$$

[6] Notice $e(q) \leq \theta u'(q) + 1 - \theta$. Hence, $f(q)e(q) \leq \theta q u'(q) + (1-\theta)u(q)$. Therefore the right hand side of (16) is 0 at $q = 0$.

At $q_t = q_{t+1} = q^s$ we have

$$g'(q^s) = \beta(1-\lambda) + \beta\lambda e(q^s) + \frac{\beta\lambda e'(q^s) f(q^s)}{f'(q^s)}.$$

Inserting (17), we have

$$g'(q^s) = 1 + \frac{\beta\lambda e'(q^s) f(q^s)}{f'(q^s)}. \tag{18}$$

As in Proposition 1, $T(q^*) < 1$ where $T(q) = \beta\lambda e(q) + \beta(1-\lambda)$. Hence, if there exists a unique solution to $T(q^s) = 1$ we must have $T'(q^s) < 0$. This means $e'(q^s) < 0$, so (18) implies $g'(q^s) < 1$ and $g'(0) > 1$. ■

FIGURE 1

Figure 1 shows $q_t = g(q_{t+1})$ and $q_{t+1} = h(q_t)$ in the (q_t, q_{t+1}) plane when h is single-valued. Since $g'(0) > 1$ we have $h'(0) < 1$, which means there exists a continuum of dynamic equilibria: for all $q_0 \in (0, \bar{q}_0)$, where $\bar{q}_0 \geq q^s$, there is a path from q_0 staying in $(0, q^*)$.[7] We summarize this observation as follows.

Proposition 2 *If a steady state $q^s > 0$ exists, there is a $\bar{q}_0 \geq q^s$ such that $\forall q_0 \in (0, \bar{q}_0)$ there is an equilibrium starting at q_0 where $q_t \to 0$.*

FIGURE 2

In Figure 2, h is a correspondence: given q_t there can be multiple values of q_{t+1} consistent with equilibrium. The Figure is drawn with $g'(q^s) < -1$, which implies there is a 2-period cycle. We state this formally, but as the result is standard we only sketch the proof (see Azariadis [1993]).

[7] In the case shown in the figure we have $\bar{q}_0 = q^s$ because h is single valued; in general, when h is not single valued we can have $\bar{q}_0 > q^s$ (see Figure 2 below). Also, if there are multiple monetary steady states, the result holds if we interpret q^s as the lowest one.

Proposition 3 *Assume a monetary steady state q^s exists. If $g'(q^s) < -1$ then $\exists(q^l, q^h)$, with $q^l < q^s < q^h$, such that $q^i = g^2(q^i)$, $i = l, h$, and if $g'(q^s)$ is close to -1 then $q^i \in (0, q^*)$.*

Proof: If $g'(q^s) < -1$ then in the (q_t, q_{t+1}) plane g is steeper than h where they intersect on the 45^o line. Given $g(0) = h(0) = 0$, the curves must intersect off the 45^o line at some point (q^l, q^h). This means $g^2(q^l) = q^l$ and $g^2(q^h) = q^h$. Hence the system has a 2-cycle. Technically, as g' passes through -1 the system undergoes a flip bifurcation that gives rise to the cycle, which guarantees q^l and q^h will be close to q^s if g' is close to -1. ■

In (18) we found $g'(q^s) = 1 + \beta\lambda f(q^s)e'(q^s)/f'(q^s)$. For instance, if $\theta = 1$ then $g'(q^s) = 1 + \beta\lambda q^s u''(q^s)$, which suggests that sufficient curvature in u is required for $g'(q^s) < -1$. To consider an explicit example, suppose

$$u(q) = \frac{(b+q)^{1-\eta} - b^{1-\eta}}{1-\eta},$$

where $b \in (0, 1)$ and $\eta > 0$. With $\theta = 1$, we can solve explicitly for

$$q^s = \left(\frac{\lambda\beta}{1-\beta+\lambda\beta}\right)^{1/\eta} - b.$$

For $b \approx 0$ we have $g'(q^s) \approx 1 - \eta(1 - \beta + \beta\lambda)$, and so $g'(q) < -1$ when $\eta > \eta_0 = 2/\left(1 - \beta + \beta\lambda\right)$. For η close to η_0 the cycle stays in $(0, q^*)$. More generally, given $b > 0$ there is a critical η_b such that as we increase η beyond η_b the system bifurcates and a 2-cycle emerges. As η increases further cycles of other periodicity emerge; e.g. with $\alpha = 1$, $\sigma = 0.5$, $b = 0.01$ and $\beta = 1/1.1$, cycles of period 3 emerge at $\eta_b \approx 3.9$, as can be verified numerically by looking for fixed points of g^3. Once we have cycles of period 3 we have cycles of all periods, including chaos (Li and Yorke [1975]).

Similar results hold in other models, including overlapping generations models, although there seems to be a view that "unrealistic" parameter values are required in that context. In our model we need $q_{t+1} = h(q_t)$ to be backward bending, just like in an overlapping generations model, but in that model the condition means the *saving function* is non-monotone in the rate of return, while here it means the *demand for liquidity* is non-monotone in inflation, which seems to be a different requirement.[8]

We now introduce a flow return γ per nominal unit of money. This can have interesting effects on the equilibrium set here, as it can in overlapping generations models (Farmer and Woodford [1997]), and will also be useful in the next section. One can interpret γ as interest on currency if $\gamma > 0$ and as a storage cost or tax if $\gamma < 0$. Interpreting it as interest on currency will allow us below to relate to Friedman's (1969) policy discussion. Note that γ is a *real* return per unit of currency: the 'government' pays γm goods to an agent holding m dollars (it does not matter which good since every $g \in \mathcal{G}$ has the same price). We balance the budget with lump sum taxes.

To derive the equilibrium conditions with $\gamma \neq 0$, simply add the term γm_t minus the lump sum tax to the right hand side of the budget constraint in (2), and notice that now $W_t(m) = W_t(0) + (\phi_t + \gamma)m$. The bargaining solution is as in (6) except we replace ϕ_t with $\phi_t + \gamma$ in the definitions of m_t^* and $\widehat{q}_t(m)$. The steps leading to (16) now yield

$$f(q_t) = \beta f(q_{t+1})[1 - \lambda + \lambda e(q_{t+1})] + \gamma M, \tag{19}$$

where e and f are as before, although now $\phi_t = f(q_t)/M - \gamma$. Hence, $q_t = g_\gamma(q_{t+1})$ where g_γ is defined by (19). Figure 3 shows the dynamical system for three cases: $\gamma < 0$, $\gamma = 0$, and $\gamma > 0$. The $\gamma = 0$ case was analyzed above, and we now consider the cases $\gamma < 0$ and $\gamma > 0$ in turn.

[8] The essential point is that here money is a genuine medium of exchange, not merely a store of value, and so the empirical implications have different interpretations. Moreover, our model can be easily and naturally calibrated to an arbitrarily period length – a year, a day, or whatever – which also changes the empirical implications.

FIGURE 3

If $\gamma < 0$ then g_γ is shifted to the left of g_0. There is some $\underline{\gamma} < 0$ such that if $\gamma < \underline{\gamma}$ there is no monetary steady state, or indeed any monetary equilibria. If $\gamma \in (\underline{\gamma}, 0)$, there are generically an even number of monetary steady states, all below the q^s that prevailed with $\gamma = 0$. The lowest monetary steady state $\underline{q}$ inherits the stability properties of the nonmonetary steady state from the case $\gamma = 0$, so it is locally stable. Now $q = 0$ is still a steady state (where agents dispose of money, given $\gamma < 0$) but it is unstable; indeed, for any $q_0 \in (0, \underline{q})$ we have $q_t \to \underline{q}$. Although $\underline{q}$ is locally stable, q_t may not converge to it if $q_0 > \underline{q}$; it is possible that q_t will converge to another steady state $\overline{q}$ or cycle around $\overline{q}$ (this cannot happen in Figure 3, because h is single valued, but it can happen in general).

If $\gamma > 0$ then g is shifted to the right. Then there is a unique monetary steady state, and it is above the q^s that prevailed with $\gamma = 0$. One can generalize Lemma 3 to show that q_t can never exceed q^*: as we will argue shortly, equilibria only exist for γ below some threshold $\hat{\gamma}$, and given $\theta = 1$ and $\gamma = \hat{\gamma}$, we have $q = q^*$ is a steady state, while otherwise $q_t < q^*$ $\forall t$. First, notice that $\gamma > 0$ implies $q = 0$ is *not* an equilibrium. Thus, $\gamma > 0$ eliminates the nonmonetary steady state, and any path converging to $q = 0$. Also, suppose a monetary steady state does not exist with $\gamma = 0$; i.e., suppose $q_t = g_0(q_{t+1})$ in Figure 3 does not intersect the 45^o line when $\gamma = 0$. When $\gamma > 0$, it is obvious from (19) that g_γ must intersect the 45^o line at some $q > 0$, so there necessarily is a monetary steady state.[9]

To complete the analysis of $\gamma > 0$ we characterize the maximum feasible

[9]Intuitively, even if you believe money has no exchange value, you would still be willing to give some positive q for it so as to get the interest γ. Similar results hold in models where $m \in \{0, 1\}$ (Li and Wright [1998]).

interest payment $\hat{\gamma}$. First, with $\gamma \neq 0$ (17) becomes

$$e(q) = 1 + \frac{1-\beta}{\lambda\beta} - \frac{M}{\lambda\beta f(q)}\gamma. \tag{20}$$

Second, we can generalize Lemma 5 to show $\beta(\phi_{t+1} + \gamma) \leq \phi_t$. In steady state this reduces to $\gamma \leq \frac{1-\beta}{\beta}\phi$ or, since $f(q) = (\phi + \gamma) M$,

$$\gamma \leq \frac{1-\beta}{M} f(q) = \gamma(q). \tag{21}$$

Any equilibrium has to satisfy this condition.

Suppose we set γ to the maximum feasible γ, satisfying (21) with equality. Then (20) becomes

$$e(q) = 1 + \frac{1-\beta}{\lambda\beta} - \frac{M}{\lambda\beta f(q)}\gamma(q) = 1.$$

When $\theta = 1$, $e(q) = 1$ holds at $q^s = q^*$ and $\gamma = \gamma(q^*) = (1-\beta)q^*/M = \gamma^*$; for any $\theta < 1$, however, $e(q) = 1$ holds at $q^s < q^*$ and $\gamma = \gamma(q^s) < \gamma^*$. That is, given $\theta = 1$ the maximum interest we can pay is γ^*, and it implies $q^s = q^*$ $\forall t$ is an equilibrium (indeed, the unique equilibrium); given $\theta < 1$, however, the maximum interest we can pay is $\gamma < \gamma^*$, and it implies the steady state is $q^s < q^*$.

We interpret $\gamma = \gamma^*$ as the Friedman Rule for interest on currency. One can derive similar results if, rather than paying interest, we contract the money supply so ϕ_t increases with t: the maximum deflation rate consistent with monetary equilibrium ($\phi_{t+1}/\phi_t = 1/\beta$) is the optimal policy, and it achieves the efficient outcome iff $\theta = 1$. Yet there is one significant difference: with deflation and fiat money ($\gamma = 0$) the nonmonetary equilibrium always exists; with interest payment γ^*, given $\theta = 1$ the efficient outcome is the *unique* equilibrium.[10] We summarize the key results as follows:

[10] We already argued that $\gamma = \gamma^*$ implies the steady state is q^* and that for any $\gamma > 0$

Proposition 4 *Suppose $\gamma > 0$. Then there is some maximum feasible $\hat{\gamma}$. If $\theta = 1$, $\hat{\gamma} = \gamma^*$ and it implies the unique equilibrium is $q_t = q^*$ $\forall t$. If $\theta < 1$, $\hat{\gamma} < \gamma^*$ and $q_t < q^*$ $\forall t$. In any case, $\gamma > 0$ implies there is no equilibrium with $q_t = 0$ or $q_t \to 0$. Now suppose $\gamma < 0$. Then there is some $\underline{\gamma} < 0$ such that $\gamma \in (\underline{\gamma}, 0)$ implies multiple equilibria, all of which have $q_t < q^*$ $\forall t$, while $\gamma < \underline{\gamma}$ implies there are no monetary equilibria.*

4 Sunspots

Let s_t be a stochastic process with no effect on fundamentals but potentially with an effect on behavior due to expectations. Assume s_t is known at t before the decentralized market opens, but s_{t+1} is random with distribution function $G(s_{t+1}|s_t)$. In general, all variables now need to be indexed by s as well as t. In a stationary equilibrium we need to index things by s but not by t. Much of this section will focus on stationary equilibria, but we will also consider briefly nonstationary sunspot equilibria.

We begin with an example where $\gamma = 0$. The example constructs a nonstationary sunspot equilibrium along the lines of Cass and Shell (1983) by randomizing across deterministic equilibria. Assume the graph of $q_{t+1} = h(q_t)$ is a correspondence, so that given some q_0 there are two distinct values, q_1^H and q_1^L, both in $(0, q^*)$ and both satisfying $q_1 = h(q_0)$. This means that

$$f(q_0) = \beta f(q_1) \left[1 - \lambda + \lambda e(q_1)\right] \tag{22}$$

holds at both $q_1 = q^H$ and $q_1 = q^L$.

there are no equilibrium paths leading to $q = 0$. If $h' > 0$ at q^* then there clearly can be no nonstationary equilibria. If $h' < 0$ at q^* then paths starting near q^* must oscillate around q^*, which violates the result that $q_t \leq q^*$. Hence $q_t = q^*$ for all t is the unique equilibrium.

Now suppose we set $q_1 = q^H$ with probability π and $q_1 = q^L$ with probability $1-\pi$. To be more precise, we set $prob(s = s_1) = \pi$ and $prob(s = s_2) = 1-\pi$, and the equilibrium is as follows: $s = s_1$ implies $q_1 = q^H$ while $s = s_2$ implies $q_1 = q^L$, and for $t > 1$ we follow a deterministic equilibrium path from q_1. Proceeding as above, the generalized equilibrium condition is

$$f(q_0) = \beta\pi f(q^H)\left[1-\lambda+\lambda e(q^H)\right] + \beta(1-\pi)f(q^L)\left[1-\lambda+\lambda e(q^L)\right]. \quad (23)$$

Clearly, if the equilibrium condition for the deterministic economy holds at both q^H and q^L then the equilibrium condition for the economy with sunspots also holds.[11]

We now consider stationary sunspot equilibria, where at every date $q = q(s)$. We omit t subscripts and write, e.g., $q_t = q$ and $q_{t+1} = q_{+1}$. The generalization of (3) is

$$W(m,s) = u(q^*) - q^* + \phi(s)m + \gamma m + \max_{m_{+1}}\left\{-\phi(s)m_{+1} + \beta E V_{+1}\left(m_{+1}, s_{+1}\right)\right\}$$

where E is the expectation with respect to $G(s_{+1}|s)$. As in Lemma 1, $W(m,s)$ is linear. The bargaining solution is as in (6), except $m^*(s)$ and $\widehat{q}(m,s)$ are indexed by s, and (8) becomes

$$\begin{aligned} V(m,s) &= v(m,s) + u(q^*) - q^* + \phi(s)m + \gamma m \\ &\quad + \max_{m_{+1}}\left\{-\phi(s)m_{+1} + \beta E V_{+1}\left(m_{+1}, s_{+1}\right)\right\}, \end{aligned}$$

[11] Following Peck (1988) one can also construct a sunspot equilibrium which is not a randomization over deterministic equilibria, even if h is a function. Consider (22), and suppose that $R' > 0$ where $R(q_1)$ denotes the right hand side (it is not hard to make assumptions to guarantee this). Then let $q^H = q_1 + \varepsilon$ and $q^L = q_1 - \varepsilon$. Then clearly $R\left(q^L\right) < f\left(q_0\right) < R\left(q^H\right)$. Therefore, we can choose the probability π so that the equilibrium condition for the sunspot economy (23) will hold.

where $v(m, s)$ is the obvious generalization of (9).

The relevant first order conditions are

$$-\phi(s) + \beta E\left[v'_{+1}(m_{+1}, s_{+1}) + \phi(s_{+1}) + \gamma\right] = 0 \ \forall s, \tag{24}$$

where

$$v'(m, s) = \begin{cases} \lambda u'\left[\widehat{q}(m, s)\right]\widehat{q}'(m, s) - \lambda[\phi(s) + \gamma] & \text{if } m < m^*(s) \\ 0 & \text{if } m \geq m^*(s) \end{cases}$$

and $\widehat{q}'$ is as above. Notice that $m_{+1} \geq m^*(s_{+1})$ with probability 1 implies $v'_{+1}(m_{+1}, s_{+1}) = 0$ with probability 1, and thus (24) implies $E\left[\phi(s_{+1}) + \gamma\right] = \beta^{-1}\phi(s) > \phi(s)$ with probability 1. At least with $\gamma \leq 0$, this is cannot be – we cannot expect $\phi(s)$ to rise in every state in a stationary equilibrium – and so we have $m_{+1} < m^*(s_{+1})$ with positive probability. However, it is not necessarily true that $m_{+1} < m^*(s_{+1})$ with probability 1.

The conditions in Lemma 6 now imply $v'' \leq 0 \ \forall s$ with strict inequality if $m_{+1} < m^*(s_{+1})$. Hence, there is a unique solution to (24), and F is again degenerate at $m_{+1} = M$. Let $B = \{s : M < m^*(s)\}$ be the set of realizations for s such that $q < q^*$. Inserting v' into (24), we have

$$\begin{aligned} \phi(s) &= \beta \int_B [\phi(s_{+1}) + \gamma]\left\{1 - \lambda + \lambda e\left[Q(s_{+1})\right]\right\} dG(s_{+1}|s) \\ &\quad + \beta \int_{B^c} [\phi(s_{+1}) + \gamma] dG(s_{+1}|s), \end{aligned}$$

where $Q(s)$ is the equilibrium value of q in state s and $e(q)$ is defined above.

Inserting $\phi(s) = f[Q(s)]/M - \gamma$, where f is defined above, we have a functional equation in $Q(s)$:

$$\begin{aligned} f[Q(s)] &= \beta \int_B f\left[Q(s_{+1})\right]\left\{1 - \lambda + \lambda e\left[Q(s_{+1})\right]\right\} dG(s_{+1}|s) \\ &\quad + \beta \int_{B^c} f[Q(s_{+1})] dG(s_{+1}|s) + \gamma M. \end{aligned}$$

It is always possible for agents to ignore sunspots; e.g. if $\gamma = 0$ one solution is $Q(s) = q^s \ \forall s$ where q^s is the steady state in the model without sunspots, and another is $Q(s) = 0 \ \forall s$. Or, if $\gamma < 0$ but $|\gamma|$ is not too big, then there are two monetary steady states without sunspots, called $\underline{q}$ and $\overline{q}$ in the previous section. Here we are interested in a *proper* sunspot equilibrium, or a non-constant solution $Q(s)$.

At this point, mainly to reduce notation, we set $\theta = 1$ so that $e(q) = u'(q)$ and $f(q) = q$. Now consider the case $s \in \{s_1, s_2\}$, with $prob(s_{+1} = s_2 | s = s_1) = H_1$ and $prob(s_{+1} = s_1 | s = s_2) = H_2$ (H_j is the probability of leaving state s_j). Let $q_j = Q(s_j)$. One possibility is that B contains both states – say, $q_1 < q_2 < q^*$ – in which case we have

$$\begin{aligned} q_1 &= \beta(1 - H_1) q_1 \left[1 - \lambda + \lambda u'(q_1)\right] + \beta H_1 q_2 \left[1 - \lambda + \lambda u'(q_2)\right] + \gamma M \\ q_2 &= \beta H_2 q_1 \left[1 - \lambda + \lambda u'(q_1)\right] + \beta(1 - H_2) q_2 \left[1 - \lambda + \lambda u'(q_2)\right] + \gamma M. \end{aligned}$$

We seek a solution to this system with $q_1 < q_2 < q^*$. For this, we mimic what one does in overlapping generations models.

One way to proceed is to follow Guesnerie (1986) or Azariadis and Guesnerie (1986) by considering the limiting case where $H_1 = H_2 = 1$. In this case the system becomes

$$\begin{aligned} q_1 &= \beta q_2 \left[1 - \lambda + \lambda u'(q_2)\right] + \gamma M = g_\gamma(q_2) \\ q_2 &= \beta q_1 \left[1 - \lambda + \lambda u'(q_1)\right] + \gamma M = g_\gamma(q_1) \end{aligned}$$

where $q_t = g_\gamma(q_{t+1})$ is precisely the dynamical system in the previous section. That is, with $H_1 = H_2 = 1$ we have reduced our search for sunspot equilibria to a search for 2-cycles. We know a 2-cycle exists if $g' < -1$ by Proposition 3 and the explicit example. Since the system of equations in question is continuous in (H_1, H_2) there also exist stationary sunspot equilibria with $q_1 < q_2 < q^*$ for some $H_j < 1$.

Another way to proceed is to set $\gamma < 0$ and then try to construct sunspot equilibria from the two monetary steady states, similar to Cass and Shell (1983). To begin, set $H_1 = H_2 = 0$, so that

$$\begin{aligned} q_1 &= \beta q_1 \left[1 - \lambda + \lambda u'(q_1)\right] + \gamma M \\ q_2 &= \beta q_2 \left[1 - \lambda + \lambda u'(q_2)\right] + \gamma M. \end{aligned}$$

These two equations are identical, but can have two different solutions: the two monetary steady states in Figure 3. Thus, we can set $q_1 = \underline{q} \in (0, q^*)$ and $q_2 = \overline{q} \in (\underline{q}, q^*)$. By continuity, there exists a solution $q_1 < q_2 < q^*$ for some $H_j > 0$.

We summarize the key results of this section as follows:

Proposition 5 *When h is a correspondence, sunspot equilibria exist where given q_0 we randomize over q_1. When there is a 2-cycle, stationary sunspot equilibria exist when H_j is near 1. When there exist multiple monetary steady states, stationary sunspot equilibria exist when H_j is near 0.*

5 Conclusion

This paper has pursued a line, due to Cass and Shell (1980), that says it is desirable to work with monetary models that are "genuinely dynamic and fundamentally disaggregative" and also allow "diversity among households and variety among commodities." Recent search-theoretic models fit this description well. We showed here that these models, like some other models, generate interesting dynamics. This helps us to understand how alternative monetary theories are related, and gives further support to the notion that extrinsic dynamics and uncertainty matter. One possible advantage of our framework is that the parameter values needed for fluctuating equilibria may not be as empirically implausible, and certainly the implied period length seems more empirically relevant, that in some other models.

References

[1] Azariadis, C. *Intertemporal Macroeconomics*. Oxford: Blackwell Publishers, 1993.

[2] Azariadis, C., and R. Guesnerie. "Sunspots and Cycles." *Review of Economic Studies* 53 (1986), 725-737.

[3] Cass, D., and K. Shell. "In Defence of a Basic Approach." In *Models of Monetary Economies*, ed. by J. Kareken and N. Wallace, Federal Reserve Bank of Minneapolis, 1980.

[4] Cass, D., and K. Shell. "Do Sunspots Matter?" *Journal of Political Economy* 91 (1983): 193-227.

[5] Coles, M., and R. Wright. "A Dynamic Equilibrium Model of Search, Bargaining and Money." *Journal of Economic Theory* 78 (1998): 32-54.

[6] Diamond, P. and D. Fudenberg. "Rational Expectations Business Cycles in Search Equilibrium." *Journal of Political Economy* 89 (1987): 606–619.

[7] Duffie, D., J. Geanakoplos, A. Mas-Colell and A. McLennan. "Stationary Markov Equilibria." *Econometrica* 62 (1994): 745-782.

[8] Ennis, H. "Search, Monetary Equilibria and Sunspots." *Macroeconomic Dynamics* 5, 2001.

[9] Ennis, H. "Macroeconomic Fluctuations and Bargaining." Unpublished manuscript, 2002.

[10] Farmer, R. and M. Woodford. "Self-fulfilling Prophecies and the Business Cycle." Macroeconomic Dynamics 1 (1997): 74-769.

[11] Friedman, M. *The Optimal Quantity of Money and Other Essays.* Chicago: Adeline, 1969.

[12] Guesnerie, R. "Stationary Sunspot Equilibria in an N-Commodity World." *Journal of Economic Theory* 40 (1986): 103-128.

[13] Kiyotaki, N. and R. Wright. "A Contribution to the Pure Theory of Money." *Journal of Economic Theory* 53 (1991): 215-235.

[14] Kocherlakota, N. "Money is Memory." *Journal of Economic Theory* 81 (1998): 232-251.

[15] Lagos, R. and R. Wright. "A Unified Framework for Monetary Theory and Policy Analysis." Mimeo, 2002.

[16] Li, Y. and R. Wright. "Government Transaction Policy, Media of Exchange, and Prices." *Journal of Economic Theory* 81 (1998): 290-313.

[17] Li, T. and J. Yorke. "Period Three Implies Chaos." *American Mathematical Monthly* 82 (1975): 985-992.

[18] Lomeli, H., and T. Temzelides. "Discrete Time Dynamics in a Random Matching Monetary Model." *Economic Theory* 20 (2002): 259-269

[19] Molico, M. "The Distribution of Money and Prices in Search Equilibrium." Mimeo, 1999.

[20] Peck, J. "On the Existence of Sunspot Equilibria in an Overlapping Generations Model." *Journal of Economic Theory* 44 (1988): 19-42.

[21] Rocheteau, G. and R. Wright. "Money in Search Equilibrium, Competitive Equilibrium, and Competitive Search Equilibrium." Mimeo, 2001.

[22] Samuelson, P. "An Exact Consumption-Loan Model of Interest with or without the Social Contrivance of Money." *Journal of Political Economy* 66 (1998): 467-481.

[23] Shell, K. "Notes on the Economics of Infinity." *Journal of Political Economy* 79 (1971): 1002-1011.

[24] Shell, K. "Monnaie et Allocation Intertemporelle." Mimeo, 1977.

[25] Shi, S. "Money and Prices: A Model of Search and Bargaining." *Journal of Economic Theory* 67 (1995): 467-496.

[26] Trejos, A., and R. Wright. "Search, Bargaining, Money and Prices." *Journal of Political Economy* 103 (1995): 118-141.

[27] Wallace, N. "The Overlapping Generations Model of Fiat Money." In *Models of Monetary Economies*, ed. by J. Kareken and N. Wallace, Federal Reserve Bank of Minneapolis, 1980.

[28] Wallace, N. "Whither Monetary Economics?" *International Economic Review* 42 (2001): 47-870.

[29] Wright, R. "A Note on Sunspot Equilibria in Search Models of Fiat Money." *Journal of Economic Theory* 64 (1994): 234-41.

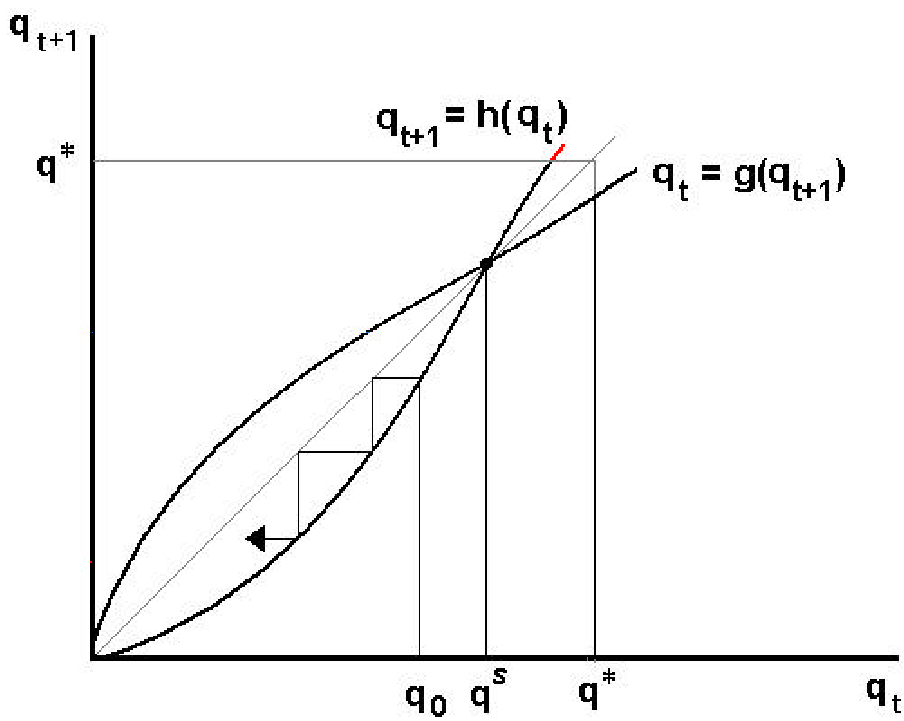

Figure 1: Case where h is a function

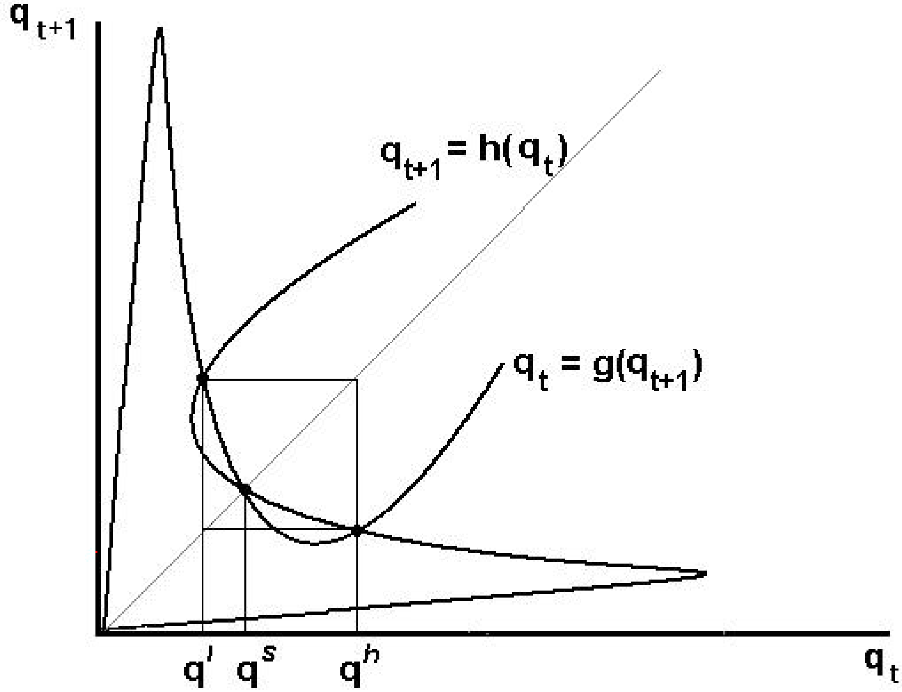

Figure 2: Case where h is a correspondence

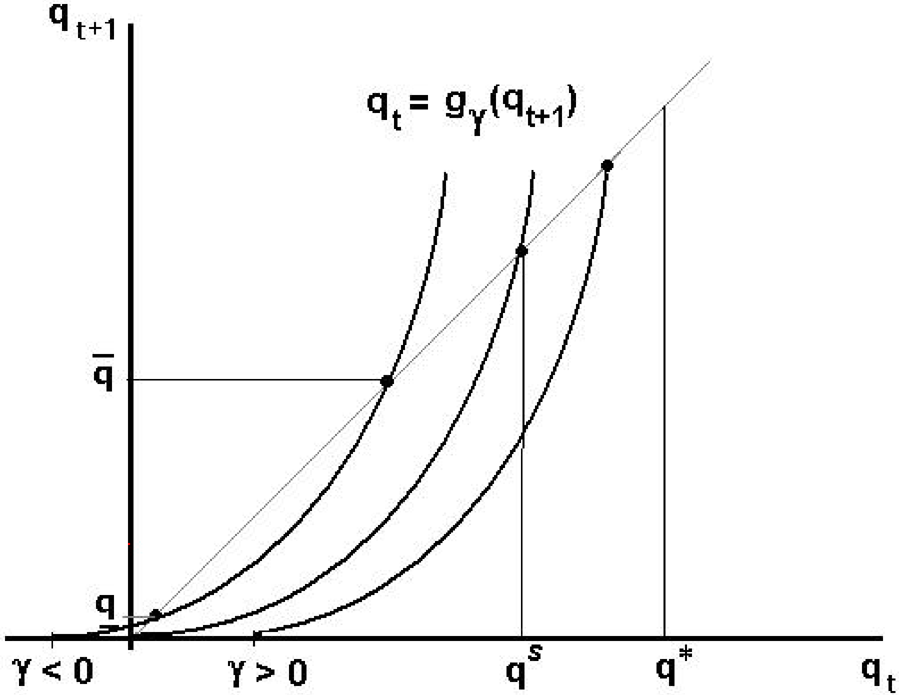

Figure 3: Equilibria for different γ

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