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Maximum Likelihood in the Frequency Domain: The Importance of Time-to-Plan

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We illustrate the use of various frequency domain tools for estimating and testing dynamic, stochastic general equilibrium models. Our substantive results confirm other findings which suggest that time-to-plan in the investment technology has potentially useful role to play in business cycle models.

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1. Introduction

In recent years there has been increased interest in applying formal econometric methods to the analysis of dynamic, stochastic, general equilibrium models.[1] Researchers understand that models are abstractions and so are necessarily incorrect. As a result, there is a need for tools which are helpful for diagnosing the reasons for econometric rejections and for identifying the most fruitful avenues for further model development. This paper illustrates a set of tools which we think are useful in this sense. They are frequency domain methods which we use to diagnose Gaussian maximum likelihood estimation and testing results.[2] We apply the tools to analyze a series of real business cycle models.

We exploit the well known fact that the log, Gaussian density function has a linear decomposition in the frequency domain. This decomposition has two implications. First, the likelihood ratio statistic for testing a model can be represented as the sum of likelihood ratios in the frequency domain.[3] So, if a model is rejected because of a large likelihood ratio statistic, then it is possible to determine which frequencies of the data are responsible. Second, if parameter estimates look 'strange', then it is possible to determine which frequencies are driving the result.

We also illustrate the use of the spectrum, phase angle and coherence functions for diagnosing model estimation and testing results. Regarding the phase angle, it is well known that this object is not uniquely determined. We propose a resolution to this problem which we hope is of independent interest.

We begin with a univariate analysis which studies output growth in a version of the real business cycle model in which the only shock is a disturbance to technology. We then extend the analysis by including a second variable, business fixed investment. To avoid a statistical singularity, we must introduce a second shock into the model. For this, we include disturbances to government consumption. This aspect of our analysis illustrates how our approach can be extended to a vector of time series.

[1] Papers that use maximum likelihood methods to study general equilibrium business cycle models include Altug (1989), Bencivenga (1992), Christiano (1988), Christiano, Eichenbaum, and Marshal (1991), Hall (1996), Hansen and Sargent (1980, 1991), Ireland (1999,2000,2001), Kim (2000), Leeper and Sims (1994), McGrattan (1994), McGrattan, Rogerson, and Wright (1997).

[2] For an early paper exploiting the advantages of the frequency domain, see Engle (1974). Other papers which explore the adantages of the frequency domain - though not in the maximum likelihood context explored here - include Baxter and King (1999), Christiano and Fitzgerald (1999) and Cogley (2001a,b).

[3] A referee has pointed out to us that an analogous decomposition of the likelihood ratio statistic is also possible in the time domain. This could be constructed using the prediction error decomposition of the Gaussian density function. This would allow us to define a cumulative likelihood ratio statistic in the time domain, analogous to the cumulative likelihood ratio statistic defined in the frequency domain below. It would be of interest to explore this statistic. However, this is beyond the scope of this paper.

The key substantive finding of the paper is that the data support a 'time-to-plan' specification of the investment technology. With this specification, it takes several periods to build new capital, with a lengthy initial period being devoted to activities such as planning, which do not require an intensive application of resources. Maximum likelihood favors the time-to-plan specification because it helps the model to account for persistence in output growth and for the fact that business fixed investment lags output in the business cycle frequencies.[4] Our results complement Christiano and Todd (1996) and Bernanke, Gertler and Gilchrist (1999), which also present evidence in favor of time-to-plan.

We now consider the relationship of our paper to the existing literature. The fact that the Gaussian density function can be decomposed in the frequency domain has been exploited in several other papers. For example, Altug (1989) demonstrates its value for estimating models with measurement error. Other papers emphasize its value in the estimation of time-aggregated models.[5] Christiano and Eichenbaum (1987) and Hansen and Sargent (1993) exploit the decomposition to evaluate the consequences for maximum likelihood estimates of certain types of model specification error.[6] Finally, the value of comparing model and data spectra has also been emphasized in the recent contributions of Watson (1993), Diebold, Ohanian and Berkowitz (1998), and Berkowitz (2001).

The following section presents our econometric framework. Section 3 presents the results. Section 4 concludes.

2. Econometric Framework

This section describes the econometric framework of our analysis. First, we display the frequency domain decomposition of the Gaussian density function. Second, we derive the log-likelihood function of the unrestricted representation of the data. Third, we display the likelihood of the representation restricted by the various real business cycle models that we consider. Finally, we display the linear, frequency domain decomposition of the likelihood ratio statistic.

[4]That standard business cycle models have difficulty accounting for persistence in output growth is well known. See, for example, Christiano (1988, p. 274), Cogley and Nason (1995), and Watson (1993).

[5]See, for example, Hansen and Sargent (1980a), Christiano (1985), Christiano and Eichenbaum (1987) and Christiano, Eichenbaum and Marshall (1991).

[6]These approaches to specification error analysis are similar in spirit to the early approach taken in Sims (1972). See Berkowitz (2001) for a related discussion.

2.1. Spectral Decomposition of the Gaussian Likelihood

Suppose we have a time series of data, $y = [y_1, ..., y_T]$, where y_t is a finite-dimensional column vector with zero mean. It is well known (Harvey, 1989, p. 193) that for T large, the Gaussian likelihood for these data is well approximated by:

$$L(y) = -\frac{1}{2}\sum_{j=0}^{T-1}\left\{2\ln 2\pi + \ln\left[\det\left(F(\omega_j;\Phi)\right)\right] + tr\left(F(\omega_j;\Phi)^{-1}I(\omega_j)\right)\right\} \tag{2.1}$$

where $tr(\cdot)$ and $\det(\cdot)$ denote the trace and determinant operators, respectively. Also, $I(\omega)$ is the periodogram of the data:

$$I(\omega) = \frac{1}{2\pi T}y(\omega)y(-\omega)',\ y(\omega) = \sum_{t=1}^{T} y_t \exp(-i\omega t), \tag{2.2}$$

and

$$\omega_j = \frac{2\pi j}{T},\ j = 0, 1, ..., T-1.$$

Finally, $F(\omega;\Phi)$ is the spectral density of y at frequency ω, and Φ is a vector of unknown parameters.[7]

We find it convenient, for later purposes, to express the likelihood function as a weighted likelihood, as in Diebold, Ohanian and Berkowitz (1998):

$$L(y) = -\frac{1}{2}\sum_{j=0}^{T-1} \upsilon_j \left\{2\ln 2\pi + \ln\left[\det\left(F(\omega_j)\right)\right] + tr\left(F(\omega_j)^{-1}I(\omega_j)\right)\right\}, \tag{2.3}$$

where $\upsilon_j = 0$ or 1 for all j.

2.2. Likelihood Function for The Structural Model

The preceding discussion indicates that to estimate a model by frequency domain maximum likelihood, one needs the mapping from the model's parameters, Φ, to the spectral density

[7] Let $C(k;\Phi) = Ey_t y'_{t-k}$, for integer values of k. Then,

$$F(\omega;\Phi) = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} C(k;\Phi)e^{-i\omega k},$$

for $\omega \in (0, 2\pi)$.

matrix of the data, $F(\omega_j;\Phi)$. This subsection defines this mapping for several models. We consider the standard real business cycle model and the time-to-build model of Kydland and Prescott (1982). In each case, we consider a one-shock and a two-shock version of the model.

In the one shock model there is only a disturbance to technology. In the two shock model there is, in addition, a disturbance to government consumption. In the analysis of the one-shock model, we focus on its implications for output growth, so that

$$y_t = \ln(Y_t/Y_{t-1}), \tag{2.4}$$

where Y_t denotes gross output in period t. In the analysis of the two-shock model,

$$y_t = \begin{bmatrix} \ln(Y_t/Y_{t-1}) \\ \ln(\frac{I_t}{Y_t}) \end{bmatrix}, \tag{2.5}$$

where I_t denotes gross investment. We always consider y_t expressed as a deviation from the model's population mean.

2.2.1. The Model

Household preferences, the resource constraint and the production function are taken from Christiano and Eichenbaum (1992). Preferences and the resource constraint are:

$$E_0 \sum_{t=0}^{\infty} \beta^t \left[\ln(C_t + \xi G_t) + \psi \ln(1 - n_t)\right],\ C_t + G_t + I_t \le Y_t,$$

where C_t and G_t denote household and government consumption, respectively, and n_t denotes the fraction of available time worked. The endowment of available time is normalized to unity and $\beta = 1.03^{-0.25}$, $\psi = 3.92$. We consider two values of ξ, $\xi = 0, 1$. When $\xi = 1$, then G_t has no effect on the dynamics of output and investment.[8] We refer to this as the *one-shock*

[8] When $\xi = 1$, C_t and G_t appear symmetrically everywhere. Exploiting this, we solve the model for $\tilde{C}_t$, where $\tilde{C}_t = C_t + G_t$. There are two interpretations of this solution. One is that it is the solution to the model stated in the text, where private consumption is $C_t = \tilde{C}_t - G_t$. This is a valid interpretation as long as $C_t \ge 0$. In the stochastic process for G_t that we use below, this is true with very high probability and so we ignore the possibility, $C_t < 0$. An alternative interpretation is that ours is the solution to a model without government, where $\tilde{C}_t$ is the consumption of the household. Either way, it is clear that government shocks

model. When $\xi = 0$, then G_t does matter and so we refer to this as the *two-shock model.*

The production function is:

$$Y_t = K_t^{\theta} \left(z_t n_t\right)^{(1-\theta)}, \ 0 < \theta < 1,$$

where K_t denotes the beginning-of-period t stock of capital and z_t denotes the state of technology. The latter is assumed to evolve as follows:

$$\ln(z_t) = \ln(z_{t-1}) + \varepsilon_{zt},$$

where ε_{zt} a Normal random variable that is independently and identically distributed (i.i.d.) over time, with mean μ and variance σ_z^2. We specify the time series process for G_t as follows:

$$\ln(g_t) = (1-\rho)\ln(g) + \rho \ln(g_{t-1}) + \varepsilon_{gt}, \ g_t = \frac{G_t}{z_t},$$

where g is a constant, $-1 < \rho < 1$, and ε_{gt} is i.i.d. Normal with mean 0 and variance σ_g^2.

It remains to specify how investment contributes to the evolution of the capital stock. In the *Real Business Cycle Model* (RBC), the construction of new capital requires one period:

$$K_{t+1} - (1-\delta)K_t = I_t, \ 0 < \delta < 1.$$

We denote the parameters of the one-shock version of the model by $\Phi^r = (\sigma_z, \delta, \theta)$, where the superscript, r, stands for 'restricted'. In the two-shock version of the model, $\Phi^r = (\sigma_z, \delta, \theta, \rho, \sigma_g)$. The superscript, r, indicates that these are model parameters. This notation allows us to differentiate model parameters from those of the unrestricted reduced form, Φ^u, which are discussed below. To prevent a profusion of notation, we do not also index Φ according to the number of shocks or the type of structural model.

The *Time to Build Model* adopts Kydland and Prescott's (1982) formulation, which specifies that it takes four periods to construct new capital. Period t investment is:

$$I_t = \phi_1 x_t + \phi_2 x_{t-1} + \phi_3 x_{t-2} + \phi_4 x_{t-3},$$

have no impact on investment and output.

where $\phi_i \geq 0$ for $i = 1, 2, 3, 4$, and

$$\phi_1 + \phi_2 + \phi_3 + \phi_4 \equiv 1.$$

The investment technology specifies that if net investment in period $t+3$ is x_t, i.e.,

$$K_{t+4} - (1-\delta)K_{t+3} = x_t,$$

then, resources in the amount $\phi_1 x_t$ must be applied in period t, $\phi_2 x_t$ must be applied in period $t+1$, $\phi_3 x_t$ must be applied in period $t+2$, and finally, $\phi_4 x_t$ must be applied in period $t+3$. Once initiated, the scale of an investment project cannot be expanded or contracted. In the one-shock version of the model, $\Phi^r = (\sigma_z, \phi_1, \phi_2, \phi_3)$. In the two-shock version of the model, $\Phi^r = (\sigma_z, \phi_1, \phi_2, \phi_3, \rho, \sigma_g)$.

In each case, the solution to the model is a set of stochastic processes for y_t and the other variables, which maximize utility subject to the various constraints.

2.2.2. Reduced Form Representation and Likelihood Function

We used the undetermined coefficient method described in Christiano (2001) to develop a linear approximation to the y_t process which solves the model:

$$y_t = \alpha(L;\Phi^r)\varepsilon_t = \alpha_0(\Phi^r)\varepsilon_t + \alpha_1(\Phi^r)\varepsilon_{t-1} + \alpha_2(\Phi^r)\varepsilon_{t-2} + \ldots.. \tag{2.6}$$

In the one-shock model, y_t is defined in (2.4), and

$$\varepsilon_t = \varepsilon_{zt},\ \ V(\Phi^r) \equiv E\varepsilon_t\varepsilon_t' = \sigma_z^2.$$

The scalar polynomial, $\alpha(L;\Phi^r)$, in the lag operator, L, is the infinite moving average representation corresponding to an autoregressive, moving average process with 4 autoregressive and 8 moving average lags, i.e., an $ARMA(4,8)$.

In the two-shock model, y_t is defined in (2.5), $\alpha(L;\Phi^r)$ is 2×2 matrix polynomial in L, and

$$\varepsilon_t = \begin{pmatrix} \varepsilon_{zt} \\ \varepsilon_{gt} \end{pmatrix}, \; V(\Phi^r) = \begin{bmatrix} \sigma_z^2 & 0 \\ 0 & \sigma_g^2 \end{bmatrix}.$$

In this case, $\alpha(L;\Phi^r)$ is the infinite moving average representation corresponding to a vector ARMA model with 5 autoregressive and 8 moving average lags, i.e., a VARMA(5,8).[9]

In all cases, we restrict Φ^r so that

$$\sum_{i=0}^{\infty} \alpha_i(\Phi^r) V(\Phi^r) \alpha_i(\Phi^r)' < \infty,$$

guaranteeing that the spectral density of y_t exists. We also restrict Φ^r so that $\det[\alpha(z;\Phi^r)] = 0$ implies $|z| \geq 1$, where $|\cdot|$ denotes the absolute value operator.

The spectral density of y_t at frequency ω is

$$F^r(\omega;\Phi^r) = \frac{1}{2\pi} \alpha(e^{-i\omega};\Phi^r) V \alpha(e^{i\omega};\Phi^r)',$$

where the superscript, r, on F indicates that the form of $\alpha(L;\Phi^r)$ is restricted by the model. The frequency domain approximation to the restricted likelihood function is (2.1) with $F(\omega)$ replaced by $F^r(\omega;\Phi^r)$.

2.3. Unrestricted Reduced Form Likelihood

In order to test our model, we need to estimate an unrestricted version of (2.6):

$$y_t = \alpha(L)\varepsilon_t, \tag{2.7}$$

where

$$\alpha(L) = I + \alpha_1 L + \alpha_2 L^2 + ...$$

[9]See the appendix for details about the reduced form implications of the two structural models.

Here $I = 1$ for the one-shock model and I is the 2×2 identity matrix for the two-shock model. Also,

$$\sum_{i=0}^{\infty} \alpha_i V \alpha_i' < \infty,$$

where V is the variance, covariance matrix of ν_t. Finally, we require that $\det[\alpha(z)] = 0$ implies $|z| \geq 1$.

In the one-shock case, the polynomial in L, $\alpha(L)$, corresponds to the ratio of an 8^{th} order polynomial and a 4^{th} order polynomial, with constant terms normalized to unity. This specification nests the real business cycle model and the time to build model. It has 13 free parameters: the 12 parameters of $\alpha(L)$, and V. We denote these by the 13 dimensional vector, Φ^u.

In the two-shock model, we approximate $\alpha(L)$ with a 10 lag vector autoregression, i.e., VAR(10). The appendix explains why this is a good approximation to the VARMA(5,8) reduced form of the two-shock structural model. As in the one-shock case, we denote the parameters of the unrestricted reduced form by Φ^u. This is a 45-element vector.

Let $F^u(\omega; \Phi^u)$ denote the spectral density of y_t:

$$F^u(\omega; \Phi^u) = \frac{1}{2\pi} \alpha(e^{-i\omega}) V \alpha(e^{i\omega})'.$$

The frequency domain approximation to the unrestricted likelihood function is (2.1) with $F(\omega; \Phi)$ replaced by $F^u(\omega; \Phi^u)$.

2.4. Cumulative Likelihood Ratio

The likelihood ratio statistic is

$$\lambda = 2(L^u - L^r),$$

where L^r and L^u are the maximized values of the restricted and unrestricted log likelihoods, respectively. Under the null hypothesis that the restricted model is true, this statistic has a chi-square distribution with degrees of freedom equal to the difference between the number of parameters in the restricted and unrestricted models (Harvey, 1989, p. 235). Define

$$\lambda(\omega) = \ln \frac{\det\left[F^r(\omega; \widehat{\Phi^r})\right]}{\det\left[F^u(\omega; \widehat{\Phi^u})\right]} + tr\left[\left(F^r(\omega; \widehat{\Phi^r})^{-1} - F^u(\omega; \widehat{\Phi^u})^{-1}\right) I(\omega)\right], \qquad (2.8)$$

where a hat over a variable indicates its estimated value. Then, it is easily confirmed that,

$$\lambda = \sum_{j=0}^{T-1} \lambda(\omega_j).$$

This expression can be simplified because of the symmetry properties of $\lambda(\omega)$:[10]

$$\lambda(\omega_{\frac{T}{2}-l}) = \lambda(\omega_{\frac{T}{2}+l}),\ l = 1, 2, ..., \frac{T}{2} - 1.$$

These imply that λ can be written:

$$\lambda = \lambda(0) + 2\sum_{j=1}^{\frac{T}{2}-1} \lambda(\omega_j) + \lambda(\pi). \tag{2.9}$$

This is our linear, frequency domain decomposition of the likelihood ratio statistic.

If λ is large, then we should be able to determine which ω_j's are responsible. It is useful to define the cumulative likelihood ratio:

$$\begin{aligned} \Lambda(\omega) &= \lambda(0) + 2\sum_{\omega_j \leq \omega} \lambda(\omega_j),\ 0 < \omega < \pi \\ \Lambda(0) &= \lambda(0), \\ \Lambda(\pi) &= \lambda. \end{aligned} \tag{2.10}$$

A sharp increase in $\Lambda(\omega)$ in some region of ω's signals a frequency band where the model fits poorly. Note that although $\Lambda(\pi) \geq 0$, it is possible for $\Lambda(\omega)$ to decrease over intervals, since there is nothing preventing some $\lambda(\omega_j)$'s from being negative.

3. Results for One-Shock Models

This section presents our results for estimating and testing the one-shock versions of the RBC and time to build models. The periodogram of the data, (2.2), and the spectral density of the unrestricted reduced form are important ingredients in the analysis, and so we begin by

[10]Implicitly, we assume T is even. The adjustment when T is odd is straightforward.

presenting these. The following two subsections present the analysis of the RBC and the time to build models, respectively.

We find that when the RBC model is parameterized using standard values taken from the literature, it matches the data poorly. This is because it cannot capture the relatively high power concentrated in the lower frequencies of actual output growth. When we estimate values for the model's parameters, its fit improves significantly. However, the estimated parameter values are implausible in the light of other evidence.[11] We then turn to estimation of the time to build model. That model fails to be rejected by the data. Moreover, the estimated parameter values are consistent with other evidence.

3.1. Periodogram and Spectrum of Unrestricted Reduced Form

Figure 1 presents a smoothed version of $I(\omega)$ for $\omega \in (0,\pi)$, based on (2.2).[12] Figure 1 also displays the spectrum of our unrestricted $ARMA(4,8)$ representation of US GDP growth. An estimate of the associated 95 percent confidence interval is also reported.[13] The two estimates of the spectral density are fairly similar. This is evidence that our unrestricted time series model is a good representation of the second moment properties of the data.

Vertical bars draw attention to three frequency bands, the low frequencies (those corresponding to period 8 years to infinity), the business cycle frequencies (period 1 year to 8 years) and the high frequencies (period 2 quarters to 1 year). Note that the spectrum of output growth has relatively high power in the low end of the business cycle frequencies. In addition, it has pronounced dips in the $7-7.5$ months (near $\omega = 2.5$) range and in the

[11]Another option would be to add persistence to the growth rate of technology. This would increase the persistence of output growth in the model and improve its fit with the output data. However, as emphasized in Christiano (1988), this improvement in fit would come at the cost of counterfactual implications for the growth rate of the Solow residual.

[12]The data are seasonally adjusted, cover the period 1955Q3 to 1997Q1, are taken from the Citibase database and have mnemonic GDPQ. The sample mean of y_t is subtracted from the data, so that $I(0)$ is zero. We present the smoothed version of the periodogram because, as is well known, the unsmoothed periodogram is quite volatile. The smoothed periodogram at frequency ω_j is a centered, equally weighted average, $\sum_{i=-3}^{3} I(\omega_{j+i})/7$.

[13]This is computed using the standard 'delta function' method, which we briefly summarize here. As is well known, the maximum likelihood estimator of Φ, $\widehat{\Phi}$, has an asymptotic Normal distribution with variance covariance matrix equal to, say, W. Our estimate of this, $\widehat{W}$, is the inverse of the second derivative of the likelihood function, evaluated at $\Phi = \widehat{\Phi}$. Let $s = f(\Phi)$ denote the mapping from Φ to the vector of spectral density ordinates of interest (we consider a fine grid of frequencies on the interval, $(0,\pi)$.) Then, the point estimate of the (column) vector s is $\widehat{s} = f(\widehat{\Phi})$. An asymptotically valid estimate of the variance covariance of this estimator for s is given by $\widehat{W_s} = f'(\widehat{\Phi})\widehat{W}\left[f'(\widehat{\Phi})\right]^T$, where $f'(\widehat{\Phi})$ denotes the derivative of f and the superscript, T, denotes the matrix transposition operator. The reported confidence intervals are obtained by adding and subtracting 1.96 times the square root of the relevant diagonal element of $\widehat{W_s}$.

higher frequency component of the business cycle (near $\omega = 1.5$).[14]

3.2. Estimation and Testing of RBC Model

In our first pass at estimating and testing the RBC model, only the innovation variance of the technology shock, σ_z^2, is free. The other parameter values of the model are fixed at the estimated values reported in Christiano and Eichenbaum (1992).[15] These estimates are very similar to the ones reported in the calibration literature. We refer to the RBC model whose parameter values are assigned in this way as the 'restricted RBC'. In our second pass at estimation and testing, we also free up the values of δ and θ. The RBC model parameterized in this way is referred to as the 'unrestricted RBC'.

The spectrum of output growth implied by the restricted RBC model is displayed in Figure 2. Consistent with the findings in Watson (1993), that spectrum is essentially flat.

Figure 3 presents a formal evaluation of model fit using the cumulative likelihood ratio, (2.10). Note that λ is just under 25 (see the cumulative likelihood ratio for $\omega = \pi$). Under the null hypothesis that the restricted RBC model is true, λ is the realization of a chi-square statistic with 12 degrees of freedom. The statistic has a p-value of 1.5 percent and hence the model is rejected at the five percent significance level. To see why the model is rejected, note that the cumulative likelihood ratio displays sharp increases in the low frequency component of the business cycle, and in the frequencies corresponding to periods 7-7.5 months. Consistent with the evidence in Figure 2, the model is rejected because it fails to replicate the relatively high power at low frequencies in the actual output growth data. In time domain terms, this corresponds to the observation that actual output growth is more persistent than output growth implied by the restricted RBC model.

We now turn to the unrestricted RBC model, where we estimate θ and δ, in addition to σ_z. The estimated parameter values are $\widehat{\theta} = 0.37$ and $\widehat{\delta} = 0.73$. Although the estimated value of capital's share is reasonable, the estimated value of δ is much larger than seems plausible in light of data on investment and the stock of capital (Christiano and Eichenbaum, 1992).

To see what frequency component of the data drives this result, we recomputed $\widehat{\theta}$, $\widehat{\delta}$ several

[14] The $\omega = 1.5$ frequency corresponds roughly to a one-year seasonal, and so the dip in the spectrum here may be an effect of the seasonal adjustment procedure used by the department of commerce. (See Nerlove, Grether and Carvalo (1979) for a review of the literature on this.) We follow conventional practice in ignoring the fact that our data have been seasonally adjusted at the source. Still, the proper way to integrate seasonality into an analysis like ours remains an important outstanding subject for research. For recent work in this direction, see Hansen and Sargent (1993) and Christiano and Todd (2001). We expect that the kind of spectral techniques used in this paper will be useful in any analysis that carefully integrates seasonality.

[15] We use $\theta = 0.344$ and $\delta = 0.021$.

times using alternative weights in the weighted likelihood function, (2.3). The estimation results are displayed in Table 1 and Figure 2. These show that it is the business cycle and high frequency components of the data drive which drive δ to nearly unity. With δ near one, the model reduces to the scalar version of the model in Long and Plosser (1983), in which output growth is a first order autoregression with autoregressive parameter θ. With the spectrum of this process, proportional to $1/(1+\theta^2-2\theta\cos(\omega))$, the model is able to match the shape of the data spectrum in the business cycle and high frequencies (see 'Unrestricted RBC, Business Cycle' and 'Unrestricted RBC, High' in Figure 2). However, different values of θ work better in the two frequency ranges.

To match the low frequencies, a very different parameterization is needed, with δ nearly 0 and θ small (see 'Unrestricted RBC, Low' in Figure 2). The parameter estimates based on all frequencies are roughly an average of the results over the various frequencies.

3.3. Estimation and Testing of Time To Build Model

Next, we estimated the time-to-build model. The estimated time-to-build weights are: $\hat{\phi}_1 = 0.01$, $\hat{\phi}_2 = 0.28$, $\hat{\phi}_3 = 0.48$, and $\hat{\phi}_4 = 0.23$. Two features of these estimates are worth noting. First, the estimated value of ϕ_1 is nearly zero. This implies that in the first period of an investment project, essentially no resources are used. An interpretation is that investment projects must start with a planning period, one in which plans are drawn up, permits are secured, etc. Christiano and Todd (1996) refer to a model of this type as a 'time-to-plan' model, and argue that it is consistent with microeconomic evidence on investment projects. Second, the resource usage in later periods of investment follows a 'hump' shape.

The spectrum of output growth implied by the estimated time-to-build model is displayed in Figure 4. Note how well that spectrum conforms with the spectrum of the data. The model even matches the dip in the spectrum in the 7-7.5 month range. This is reflected in the good performance of the model's cumulative likelihood ratio (see Figure 3). The cumulative likelihood ratio rises slowly with frequency and achieves a maximum value of a little under 10. Under the null hypothesis that the model is true, this is the realization of a chi-square distribution with 9 degrees of freedom. Under these conditions, the p-value is 35 percent. As a result the model is not rejected at conventional levels.

We compare the estimated time to build model with two others: the time to build model suggested in Kydland and Prescott (1982), where $\phi_i = 0.25$, $i = 1, 2, 3, 4$; and the time to build model analyzed in Christiano and Todd (1996), where $\phi_1 \approx 0$, $\phi_i = 1/3$, $i = 2, 3, 4$.[16]

[16] What we call the Kydland-Prescott model is our Time-to-Build model, with the ϕ_i's restricted as indicated in the text. We estimate the shock variance by maximizing the likelihood function with respect to

We do not display the spectrum implied by Kydland and Prescott's model, because that essentially coincides with the spectrum of the restricted RBC model (King, 1995). As a result, Kydland and Prescott's model is rejected like the restricted RBC model.[17]

Comparing the cumulative likelihood ratio statistics for these three models allows us to understand which frequencies are responsible for the two features of the time to build weights in the estimated model that were discussed above. The time-to-plan feature helps the model primarily in the lower end of the business cycle, while the hump helps capture the dip in the spectrum of output growth in the neighborhood of frequency $\omega = 1.5$, and the rise after frequency $\omega = 2.5$. To see this, note that the Christiano-Todd and estimated models perform similarly in the lower business cycle frequencies, while the restricted RBC model (and, hence, the Kydland-Prescott model too) does relatively poorly. In the neighborhood of frequency $\omega = 1.5$ and for frequencies greater than $\omega = 2.5$, the Kydland-Prescott and Christiano-Todd models perform badly, while the estimated model does relatively well.

4. Results for Two-Shock Model

We now analyze the version of the time to build model with government consumption and technology shocks, using data on output and business investment. The first subsection below reports the spectral properties of the data. The second subsection reports the estimation and testing results for the model. We begin by briefly summarizing the key findings.

Four features of the second order properties of the data play a key role in our model analysis. First, as discussed in the previous section, the spectrum of output growth has a negative slope, suggesting persistence in those data. Second, the coherence between output and investment varies over frequencies, exhibiting high coherence in the low range of business cycle frequencies and low coherence elsewhere. Third, phase angle analysis suggests that investment lags output in the business cycle frequencies. Finally, the spectrum of the investment to output ratio exhibits a very steep, negative slope.

There are two notable features of the parameter estimates. First, as in the previous section, maximum likelihood prefers the time-to-plan specification of the investment technology.[18] In part, this is to help account for the negative slope in the spectrum of

that parameter. We treat the Christiano-Todd model in the same way.

[17] For a detailed discussion of the similarity of these models, see Christiano and Todd (1996) and Rouwenhorst (1991).

[18] The estimated values of ϕ_2, ϕ_3, ϕ_4 also display a hump-shape pattern. This is qualitatively similar to what we found in the univariate case, although the quantitative magnitude of the hump is smaller here. The statistical reason for the hump is similar to the reason for the hump in the univariate case and is not discussed here.

output growth. In addition, time-to-plan helps the model to account for the phase angle between investment and output data. Second, the parameter estimates assign a relatively large role to the government consumption shock. This is done primarily in an attempt to capture the coherence pattern between the output and investment data.

Our estimated model has weaknesses. First, the estimated variance of government consumption seems implausibly large in that it exceeds direct estimates based on government consumption data reported in Christiano and Eichenbaum (1992). Second, despite the high variance of government consumption, the likelihood ratio statistic based on output and investment data rejects the model statistically. After studying the frequency domain decomposition of our likelihood ratio statistic, we argue that the rejection reflects difficulties the model has in matching the components of the data with period 2.5 years and longer. This appears to reflect three difficulties: (i) although the model can capture the low coherence overall between investment and output, it cannot at the same time capture the relatively high coherence in the low business cycle frequencies; (ii) it has difficulty quantitatively matching the shape of the spectrum of the investment to output ratio in the low frequencies; and (iii) although it gets the sign of the phase angle between investment and output right, it misses quantitatively.

4.1. Spectral Properties of the Data

Figures 5a and 5b display the estimated spectral density for output growth and for the business investment to output ratio, respectively, implied by the unrestricted VAR(10) discussed in section 2.3.[19] The associated 95 percent confidence intervals computed using the delta function method are also displayed.[20] In addition, the figures report the smoothed periodogram estimates of the two spectra. The two spectral estimates are reasonably similar. For most frequencies, the periodogram-based estimate lies inside the confidence interval implied by the VAR(10).

The spectral densities of the two variables differ notably. The spectral power in $\ln(I_t/Y_t)$ is relatively more concentrated in the lower frequencies, and falls off sharply in the higher frequencies.

[19] We measure investment as seasonally adjusted business investment in structures and equipment, which cover the period 1955Q3 to 1997Q1, and are taken from the Citibase database. Business investment in structures has mnemonic GSVNT and business investment in equipment has mnemonic GIPNR. The investment to output ratio was measured as the ratio of the sum of these two series to nominal GDP. The latter has Citibase mnemonic GDP.

[20] See an earlier footnote for details about how we computed confidence intervals using the delta-function method.

In considering the cross-spectrum between the variables in our model, we find it convenient to work with $\ln(Y_t)$ and $\ln(I_t)$ and instead of $\ln(Y_t/Y_{t-1})$ and $\ln(I_t/Y_t)$. Focusing on levels in this way has the advantage of maximizing comparability with the business cycle literature. According to our model and to our unrestricted VAR(10), the levels data are not covariance stationary processes. Still, their matrix spectral density is well defined, as long as we do not consider frequency zero. It is obtained by a simple matrix manipulation of the spectral density of $\ln(Y_t/Y_{t-1})$ and $\ln(I_t/Y_t)$.[21]

Denote the two-dimensional spectral density of $[\ln(Y_t)\ \ln(I_t)]$ by $S(\omega)$, for $0 < \omega \leq \pi$. For fixed ω, the coherence between $\ln(Y_t)$ and $\ln(I_t)$, $R^2(\omega)$, measures the strength of the linear relationship between these variables at frequency ω:

$$R^2(\omega) = \frac{S_{12}(\omega)S_{12}(-\omega)}{S_{11}(\omega)S_{22}(\omega)}. \tag{4.1}$$

Figure 5c displays the coherence function computed based on the periodogram and on the VAR(10). In addition, that figure exhibits a 95 percent confidence interval for the coherence, based on applying the delta function method to the estimated VAR(10). Note again that the two estimates of the spectrum are quite similar. The similarity between the periodogram-based and VAR(10)-based estimates of the spectral density is consistent with the notion that the latter provides a good summary of the spectral properties of the data.

A distinctive feature of the coherence function is that it is particularly high in the business cycle frequencies. The peak is near 0.9 for cycles with period in the neighborhood of 3 years.

Figure 5d reports the phase angle between $\ln(Y_t)$ and $\ln(I_t)$. The phase angle measured in radians, $\theta_{21}^*(\omega)$, satisfies

$$S_{21}(\omega) = r(\omega)\, e^{i\theta_{21}^*(\omega)},\ \omega \in (0, 2\pi), \tag{4.2}$$

and one additional condition. In (4.2), $r(\omega)$ is the gain function:

[21]Let $F(\omega)$ denote the two-dimensional spectral density of $[\ln(Y_t/Y_{t-1})\ \ln(I_t/Y_t)]$. Then, the two-dimensional spectral density of $[\ln(Y_t)\ \ln(I_t)]$, $S(\omega)$, is given by

$$S(\omega) = H(\omega)F(\omega)H(-\omega)^T,$$

where

$$H(\omega) = \begin{bmatrix} \frac{1}{1-e^{-i\omega}} & 0 \\ -\frac{1}{1-e^{-i\omega}} & 1 \end{bmatrix}.$$

The matrix, $H(\omega)$ - and, hence, $S(\omega)$ - is well defined for $0 < \omega \leq \pi$.

$$r(\omega) = \sqrt{S_{21}(\omega)\, S_{21}(-\omega)}.$$

The phase relationship measured in units of time is given by $k_{21}^*(\omega) = -\theta_{21}^*(\omega)/\omega$. As is well known, there are many $\theta_{21}^*(\omega)$'s (and, hence $k_{21}^*(\omega)$'s) that satisfy (4.2) for each ω. We select one of these by requiring that $k_{21}^*(\omega)$ be the value of k which maximizes the covariance between the components of $\ln I_t$ and $\ln Y_{t-k}$ in a neighborhood of frequency ω, for $k = 0, \pm 1, \pm 2, \ldots$. Thus, when $k_{21}^*(\omega) > 0$, this means that investment lags output by $k_{21}^*(\omega)$ periods when only the components of the variables in a neighborhood of frequency ω are considered. Further details about the interpretation and computation of the phase angle are discussed in the appendix.

Figure 5d displays $k_{21}^*(\omega)$, $0.2 \leq \omega \leq \pi$. The grey area in the figure indicates the 95 percent confidence interval computed using the delta function method using the estimated VAR(10).[22] The figure shows that in the lower range of the business cycle frequencies, investment lags output by roughly one quarter. In a range of higher frequencies, this relationship is reversed, with investment leading output.

The results in Figure 5d, that investment leads output over nearly half the frequencies, may at first appear inconsistent with standard results in the business cycle literature. The latter suggest that investment lags output, and give no hint of any of the ambiguity in this relationship that appears to emerge from Figure 5d. Reconciling the apparent inconsistency yields insight into the gain function and into the results in the literature.

The result in the literature is based on the cross-covariance function between Hodrick-Prescott (HP) filtered output and investment data (see, e.g., Christiano and Todd (1996)).[23] This covariance function is displayed in the thick line in Figure 6a, which shows that investment lags output by 1 quarter (the other curves will be discussed later). As is well known, the HP filter is roughly a high pass filter, allowing the business cycle frequencies and higher to pass through, while zeroing out the lower frequencies (see King and Rebelo (1993)). The results in Figures 5d and 6a would seem inconsistent if in the cross-covariance function based on HP-filtered data all frequencies, business cycle and up, received equal weight. In this case,

[22]As noted in the text, $k_{21}^*(\omega)$ is the global maximum of a particular cross-covariance function with respect to lag length. In practice, this maximum could be a discontinuous function of the parameters of our underlying $VAR(10)$ time series model. In our application of the delta function method, we ignore this possibility by computing the derivative of the local maximum with respect to the VAR(10) parameters, about the point estimate, $k_{21}^*(\omega)$.

[23]For a discussion of the Hodrick-Prescott filter, see Hodrick and Prescott (1997).

the results in Figure 5d would lead one to expect that in the time domain cross-covariance between output and investment, investment *leads* output. But, not all frequencies receive equal weight in the cross-covariance. The weight assigned to frequency ω is proportional to the gain, $r(\omega)$, at that frequency (see the Appendix). Figure 6b exhibits $r(\omega)$ and shows that it takes on its largest value in the lower range of the business cycle frequencies. This pattern in the gain function, given that $k_{21}^{*}(\omega) > 0$ in the business cycle frequencies, explains why it is that the time domain covariance between investment and output exhibited in Figure 6a shows investment lagging output.

4.2. Estimation and Testing Results

Parameter estimates for the model are reported in Table 2. There are two notable features in these results. First, as in the univariate analysis, the data prefer the time-to-plan specification of investment, i.e., $\phi_1 \approx 0$. The other ϕ_i's are also similar across univariate and bivariate analyses. Second, the estimated model assigns a relatively high variance to the government consumption shock. The standard deviation of the innovation to this shock, σ_g, is nearly 3 times larger than the estimate obtained by Christiano and Eichenbaum (1992) using the government consumption data. At the same time, the estimated standard deviation of the innovation to the technology shock is one-third smaller than we obtained in the one-shock analysis.

To help understand these findings, we constructed Figures 7a-7d. Those figures report the spectral properties of the data, as implied by the estimated VAR(10), the estimated two-shock model and two perturbations on that model. The first perturbation, labeled *KP model*, sets the investment weights to Kydland and Prescott's values of $\phi_i = 0.25$ for $i = 1, ..., 4$, and leaves the other parameters unchanged from their estimated values. The second perturbation sets the parameters of the exogenous shock processes in the estimated model to the values used by Christiano and Eichenbaum (1992), and sets the remaining parameter values at their estimated values. Since the parameter values of this model are similar to those in the model of Christiano and Todd (1996), we refer to it as the *CT model* in the figure.[24] For convenience, the parameter values associated with our perturbed models are summarized in Table 2.

To understand why the estimation procedure selects the time-to-plan specification of

[24] The parameters in CT model are somewhat different from those in the model Christiano and Todd (1996) in that the CT model incorporates the hump-shape in ϕ_2, ϕ_3, ϕ_4 implied by the estimated model. CT is designed so that a comparison of the CT and the estimated models allows one to focus on the impact of the high estimated shock variances.

investment, we compare the KP and estimated two-shock model's ability to reproduce the spectral properties of the data, as captured by the VAR(10). Time-to-plan appears to play two roles in the estimated model. First, as in the univariate analysis, it helps accommodate the negative slope of the spectrum of $\log(Y_t/Y_{t-1})$ (compare KP with estimated model in Figure 7a). Second, time-to-plan helps the model to reproduce the evidence that investment lags output in the business cycle frequencies. This can be seen in the model correlation function displayed in Figure 6a and in the positive phase angle displayed in business cycle frequencies (see Figure 7d). Still, the model does not go far enough: its implied phase angle is smaller than that of the VAR(10), and Figure 6a shows that investment lags output by less in the estimated model than it does in the data.[25]

To understand why our estimation procedure assigns a relatively large variance to the government consumption shock, we compare the estimated two-shock model with the *CT model*. In the *CT model*, the government shock has low variance and so it contributes very little to the model's dynamics. If the variance of that shock were actually zero, the model would be singular, and the coherence would be unity at all frequencies. In fact, σ_g in the *CT model* is small and positive and this is why its implied coherence is close to unity. The relatively high value of σ_g in the estimated two-shock model results in a lower coherence.[26] Although the coherence function appears to have played an important role in driving the estimated values of the exogenous shock variances, Figure 7c shows that the model nevertheless has difficulty matching the steep slope of that function. In particular, the model has difficulty accommodating the high coherence in the lower end of the business cycle frequencies (see Figure 7c). These results suggest that (i) a good model must have more than one shock, so as to be able to match the low coherence function outside the business cycle and (ii) coherences in the business cycle should nevertheless be high. Our model has difficulty satisfying (i) and (ii) simultaneously.[27]

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[25]Interestingly, there is one dimension in which time-to-plan appears to *hurt* model fit. According to Figure 7b, time-to-plan reduces the ability of the model to accommodate the steep slope in the spectrum of $\log(I_t/Y_t)$. By comparison with KP, the estimated model overstates that spectrum in the higher frequencies. Evidently, this consideration does not play an important role in estimation. Presumably this is because, according to (2.1), the weight assigned in the estimation criterion to the spectrum of $\ln(I_t/Y_t)$ is proportional to the *level* of the corresponding empirical estimate. (See I in (2.1) and (2.8).) According to Figure 5b, the latter is extremely small in the higher frequencies by comparison to what it is in the lower frequencies (note that Figure 7b reports the log of the spectrum).

[26]Raising σ_g is particularly effective in reducing coherence for a second reason. In the model, government spending shocks generate a negative correlation between investment and output, while the technology shock generates a positive correlation. For a further discussion of this property of the model, see Christiano and Todd (1996).

[27]The coherence function suggests a simple measurement error model which we also explored in results not reported in the paper. We modified the CT model by treating observations on log output and log investment as the sum of the true data and orthogonal measurement error. We then estimated the variances

Finally, Figure 7b suggests that the relatively high variance in the government consumption shock in the estimated model helps that model accommodate the steep slope in the spectrum of $\log(I_t/Y_t)$. Still, the estimated model does not go far enough. It undershoots the spectrum at the low frequencies and overshoots at the high frequencies. The miss is more severe in the low frequencies and so this weighs more heavily on our estimation and testing criteria.[28]

In view of the estimated model's difficulties accommodating the phase angle and the steep slopes of the spectrum of $\log(I_t/Y_t)$ and the coherence function, it is not surprising that the model is rejected. To see this, note from Figure 8 that the likelihood ratio statistic for the estimated model is around 110 (see 'Actual Cumulative Likelihood Ratio'). Under the null hypothesis that the model is true, this is a chi-square statistic with 39 degrees of freedom. This easily exceeds conventional critical values.[29]

We can use our frequency decomposition of the likelihood ratio statistic to identify which frequencies are responsible for this rejection. Notice that the cumulative likelihood ratio rises sharply in the frequency range, $(0, 0.6)$. In a sense, the poor fit in these frequencies is the reason for the rejection. To see this, consider the Adjusted Cumulative Likelihood Ratio displayed in Figure 8. It shows what the cumulative likelihood ratio would have been if the fit in frequencies $(0, 0.6)$ had been similar to the average fit in the higher frequencies.[30] These calculations lead to the conclusion that the likelihood ratio statistic, λ, would have been 55, with a probability value of 4.6. That is, we would not have rejected the model at any significance level less than 4.6 percent.

To gain further insight into the reason the estimated model does poorly in the low fre-

of the measurement error process by maximum likelihood. As expected, the resulting model reproduces the steep slope in the estimated coherence function. However, the measurement also causes the model to overstate the high frequency component of the spectrum of $\log(I_t/Y_t)$. This model is rejected with a likelihood ratio statistic in the neighborhood of 1500.

[28]Figure 7b may appear to indicate the opposite, that the miss is larger in the higher frequencies. However, recall that Figure 7b displays the log of the spectrum of $\log(I_t/Y_t)$, not its level. As emphasized in an earlier footnote, it is the level that enters the estimation criterion.

[29]This is how we arrived at this calculation of the number of degrees of freedom. In section 2.3, we argued that the two-shock model is a restricted VARMA (5,8) model, which we approximate with a VAR (10). The unrestricted VARMA (5,8) model has 10=5×2 autoregressive and 32=8×4 moving average parameters. In addition, there are 3 parameters governing the variance-covariance of the shocks, for a total of 45 parameters. Subtract from this the 6 estimated parameters, σ_u, σ_g, ρ_g, ϕ_1, ϕ_2, ϕ_3. This brings the total number of degrees of freedom to 39.

[30]The adjusted cumulative likelihood ratio is computed as follows. We fit a regression line, $a+b\omega$, through the cumulative likelihood ratio function over the range of frequencies, $\omega \in (.6, \pi)$. The adjusted cumulative likelihood ratio is:

$$\tilde{\Lambda}(\omega) = \begin{cases} \Lambda(\omega) - a & \omega \in (.6, \pi) \\ b\omega & \omega \in (0, .6) \end{cases},$$

where $\Lambda(\omega)$ is the actual cumulative likelihood ratio, defined in (2.10).

quencies, we evaluate the model at the estimated parameter values using the cumulative likelihood ratio computed using the univariate density for $\ln(I_t/Y_t)$ alone. This is reported in Figure 9. Note that this also displays a sharp rise in the frequencies, $(0, 0.6)$. We infer from this that a part of the reason for the poor fit of the model lies in the difficulty it has in matching the steep slope in the spectrum of $\log(I_t/Y_t)$ near frequency zero. Previously, we discussed the model's difficulties in accounting for features of the cross-spectrum between investment and output at low frequencies. Presumably, these are also part of the reason for the rejection.

5. Conclusion

We applied frequency domain tools to diagnose parameter estimates and goodness of fit tests for maximum likelihood estimation of a class of real business cycle models. The principle methodological aspects of the analysis were summarized in the introduction. There are two main substantive findings. First, the results confirm other findings that suggest time-to-plan in the investment technology has a potentially useful role to play in dynamic models. Second, alternatives to government spending disturbances need to be explored in the quest for an empirically plausible business cycle model.

Although we have limited our analysis to one or two variables, we emphasize that this does not reflect an inherent limitation of the methods used. We showed how the tools apply in a bivariate setting and hopefully from this it is obvious how they can be extended to higher dimensions. To analyze a larger list of variables in the model would of course require adding more shocks. However, the literature offers plenty of candidates for these. In addition to government consumption and technology shocks, one can consider various kinds of preference shocks and also monetary shocks. In addition, there are various types of measurement error that can be incorporated into the analysis.[31] Our decision to limit the number of variables in the analysis reflected our desire to make the methodology as transparent as possible.

[31]For example, Altug (1989) assumes that the data received by the econometrician contain measurement error. Christiano (1988) and Sargent (1989) assume that the data observed by agents contain measurement error.

A. Coherence, Gain and Phase in Spectral Analysis

This appendix briefly describes the computation and interpretation of the coherence function and phase angle analyzed in the text.

We begin with the coherence. Consider the projection of y_{2t} onto y_{1t-j} for $j \in (-\infty, \infty)$:

$$y_{2t} = \sum_{j=-\infty}^{\infty} h_j y_{1t-j} + \varepsilon_t, \;\; E\varepsilon_t y_{1t-k} = 0 \text{ for all } k. \tag{A.1}$$

We assume that $y_t = (y_{1t}, y_{2t})'$ has mean zero and is covariance stationary. It is well known (see, e.g., Sargent (1987)) that $h(\omega) \equiv \sum_{j=-\infty}^{\infty} h_j e^{-i\omega j}$ satisfies

$$\begin{aligned} F_{21}(\omega) &= h(\omega)F_{11}(\omega), \\ F_{22}(\omega) &\equiv \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} C_{22}(k)e^{-i\omega k} = |h(\omega)|^2 F_{11}(\omega) + S(\omega), \end{aligned} \tag{A.2}$$

where $h(\omega) \equiv 0$ when $F_{21}(\omega) = F_{11}(\omega) = 0$. Also, $C_{ij}(k) = Ey_{it}y_{j,t-k}$ and $S(\omega)$ is the spectral density of ε_t. A measure of the information in y_{1t} about y_{2t} is given by the R^2 of the projection, (A.1):

$$R_{21}^2 = \frac{Var(\sum_{j=-\infty}^{\infty} h_j y_{1t-j})}{Var(y_{2t})} = \frac{\int_0^{2\pi} |h(\omega)|^2 F_{11}(\omega) d\omega}{\int_0^{2\pi} F_{22}(\omega) d\omega}, \tag{A.3}$$

where the subscripts on R^2 indicate which is the left and the right hand variable in the projection.

We can obtain an analogous concept of R^2 in the frequency domain. Suppose x_t is a vector stationary stochastic process with spectral density $f(\omega)$, $\omega \in (0, 2\pi)$. We define the *component of* x_t *at frequency* ω^* as the result of filtering x_t with a band-pass filter which passes power in an arbitrarily small window around frequency ω^*, and no power at other frequencies.[32] The spectral density of the component of x_t at frequency ω^* is $f^*(\omega)$, $\omega \in (0, 2\pi)$, where $f^*(\omega) = 0$ for ω outside an interval about ω^* and $f^*(\omega) = f(\omega)$ in that interval.

Let $R^2(\omega)$ be the R^2 of the regression of the frequency ω component of y_{2t} on the frequency ω component of $y_{1,t-j}$, for $j \in (-\infty, \infty)$. Then, substituting for $h(\omega)$ in (A.3) using (A.2), we obtain the coherence, (4.1). So, we see that the coherence between y_{1t} and y_{2t} is a measure of the information about y_{2t} in linear combinations of future and past y_{1t} when we consider only the frequency ω components of these variables.[33] We omit subscripts on $R^2(\omega)$ because

[32] For a discussion of the band-pass filter, see Sargent (1987, page 259).

[33] Our interpretation of the coherence function bears a similarity to the analysis in Engle (1974). He

which variable is on the left, and which on the right, does not matter when the projection underlying the R^2 occurs in the frequency domain (see (4.1)).

We now discuss the phase angle between y_{1t} and y_{2t}. Consider the object, $\theta^*_{21}(\omega)$, in (4.2). To interpret this, it is useful to express $C_{21}(k) \equiv Ey_{2t}y_{1,t-k}$ as follows:

$$
\begin{aligned}
C_{21}(k) &= \int_0^{2\pi} F_{21}(\omega)e^{i\omega k}d\omega \\
&= \int_0^{\pi} \left[F_{21}(\omega)e^{i\omega k} + F_{21}(-\omega)e^{-i\omega k}\right] d\omega \\
&= \int_0^{\pi} r(\omega) \left[e^{i\left(\theta^*_{21}(\omega)+\omega k\right)} + e^{-i\left(\theta^*_{21}(\omega)+\omega k\right)}\right] d\omega \\
&= \int_0^{\pi} r(\omega) 2\cos\left[\theta^*_{21}(\omega) + \omega k\right] d\omega \qquad \text{(A.4)} \\
&= \int_0^{\pi} C_{21}(k;\omega)d\omega,
\end{aligned}
$$

say. Here, $C_{21}(k;\omega)$ is the covariance between the component of y_{2t} at frequency ω and the component of y_{1t-k} at frequency ω. Thus, the cross-covariance function of two time series is the integral of the cross covariances between their frequency ω components, for $\omega \in (0,\pi)$. Note that the gain function, $r(\omega)$, determines how important any particular frequency is in the covariance between two variables.

It is common to characterize the lead-lag relationship between two variables by the value of k for which $C_{21}(k)$ is the largest. For example, if this happens for a value of, say, $k=-2$, then it is said that 'y_{2t} leads y_{1t} by two periods'. The analogous statements can be made in frequency domain using $C_{21}(k;\omega)$. Note that $C_{21}(k;\omega)$ is maximized for $k = -\theta^*_{21}(\omega)/\omega$. Thus, if $\theta^*_{21}(\omega) > 0$ then we say 'the frequency ω component of y_{2t} leads the frequency ω component of y_{1t} by $\theta^*_{21}(\omega)/\omega$ periods'. Similarly, if $\theta^*_{21}(\omega) < 0$, then we say that 'the frequency ω component of y_{1t} leads the frequency ω component of y_{2t} by $-\theta^*_{21}(\omega)/\omega$ periods'.

There is an ambiguity in characterizing the lead-lag relationship between variables using $C_{21}(k;\omega)$ that is not present when we do so using $C_{21}(k)$. This is because the component of a variable at frequency ω is a pure cosine wave. The length of the lead or lag between two sinusoidal functions with the same period is ill-defined. This is manifested in the observation that there are many $\theta^*_{21}(\omega)$ that solve (4.2): if $\theta^*_{21}(\omega)$ solves (4.2) then so does $\theta^*_{21}(\omega) + 2\pi l$, for $l = 0, \pm 1, \pm 2, \ldots$.

Here is an intuitive way to see this. With two series of period 11, the statement that the first leads the second by 2 periods is equivalent to the statement that the second leads the first by 8 periods. These two statements are also equivalent to the notion that the first leads the second by 12 periods, and so on.

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asks whether the linear regression relation between two variables (money growth and inflation) shifts across different frequency bands. Our analysis differs from Engle's (1974) in that we (i) focus on the R^2 of the linear relationship and (ii) examine the relationship between variables at a single frequency.

In our analysis, we adopt the following resolution to this ambiguity. Let

$$\begin{aligned} k_{21}(\omega;\Delta) &= \arg\max_{k} \int_{\omega-\Delta}^{\omega+\Delta} C_{21}(k;\omega)d\omega,\ \pi-\Delta \geq \omega \geq \Delta > 0, \\ \theta_{21}(\omega;\Delta) &= -\omega k_{21}(\omega;\Delta). \end{aligned} \tag{A.5}$$

Define

$$\begin{aligned} k_{21}^{**}(\omega) &= \lim_{\Delta\to 0} k_{21}(\omega;\Delta), \\ \theta_{21}^{**}(\omega) &= \lim_{\Delta\to 0} \theta_{21}(\omega;\Delta), \end{aligned}$$

for $\pi \geq \omega \geq 0$. Our numerical experiments suggest that $k_{21}(\omega;\Delta)$ and $\theta_{21}(\omega;\Delta)$ are single-valued functions of ω for all $\omega \in (0,\pi)$ and $\Delta > 0$, except possibly at isolated values of ω. When $\Delta = 0$, then $\theta_{21}(\omega;\Delta)$ is composed of the countable set of elements discussed above. For all but a finite set of values of ω, $\theta_{21}^{**}(\omega)$ selects one element of the set, $\theta_{21}(\omega;0)$. Similarly for $k_{21}^{**}(\omega)$. In our analysis, we identify $\theta_{21}^{*}(\omega)$ and $k_{21}^{*}(\omega)$ with $\theta_{21}^{**}(\omega)$ and $k_{21}^{**}(\omega)$, respectively. The object, $\theta_{21}^{*}(\omega)$, is our measure of the *phase angle* between the frequency ω components of y_{1t} and y_{2t}. The object, $k_{21}^{*}(\omega)$, measures this in units of time.

To gain insight into our measure of the phase angle, consider Figure 10. There, y_{1t} corresponds to $\ln(Y_t)$ and y_{2t} corresponds to $\ln(I_t)$, as implied by the estimated $VAR(10)$ for these variables, which was discussed in the text.[34] Figure 10 exhibits a subset of the elements in $\theta(\omega;0)$ for $\omega \in (0,\pi)$ (see the solid lines). The circles indicate $\theta_{21}(\omega;\Delta)$ for $\Delta = 0.3$. Note how these θ's lie close to one of the solid lines. Note too, how that phase angle function exhibits discontinuities. We found that at the points of discontinuity, there are two elements in $\theta_{21}(\omega;0.3)$. The stars indicate $\theta_{21}(\omega;\Delta)$ for $\Delta = 0.06$. Note how in each case, $\theta_{21}(\omega;\Delta)$ is now closer to one of the solid lines. This is consistent with the notion that, for each ω, $\theta_{21}(\omega;\Delta)$ converges to one of the solid lines as $\Delta \to 0$.

The dashed lines in Figure 10 indicate $+\pi$ and $-\pi$. We have included these in order to facilitate comparison to the standard method for selecting an element of $\theta_{21}(\omega;0)$, which picks $\theta \in (-\pi,\pi)$.[35] It is evident from the figure that, for our estimated $VAR(10)$, this method produces a phase angle function with a single point of discontinuity just above $\omega = 2$. In effect, the method chooses the lead-lag relationship between two variables as the smallest (in absolute value) value of k which attains the maximum for the cross-covariance function. We think ours is a more natural measure in our context, given its connection to time domain lead-lag measures used in business cycle analysis.

To obtain insight into the discontinuities in our estimate of $\theta_{21}^{*}(\omega)$ (and, hence, $k_{21}^{*}(\omega)$),

[34]Although $\log(Y_t)$ and $\log(I_t)$ are not covariance stationary, they are so after application of a band pass filter which excludes frequency zero. As a result, phase angle and coherence measures are well-defined for $\omega \neq 0$.

[35]See, e.g., Granger and Newbold (1977) or Sargent (1987).

consider Figure 11. That figure reports

$$\int_{\omega-\Delta}^{\omega+\Delta} C_{21}(k;\tilde{\omega})d\tilde{\omega},\ \Delta = 0.15,$$

for $\omega \in (1, \pi - 0.15)$ and $k \in (-15, 10)$. Note how this function oscillates with k, for each fixed ω, producing a pattern of ridges and valleys in the three dimensional surface. The dark lines indicate $k_{21}(\omega; 0.15)$. The discontinuities in $k_{21}(\omega; \Delta)$ reflect that the ridge which achieves the greatest height varies with ω, and that ridges are separated by valleys. As $\Delta \to 0$, the surface depicted in Figure 12 evolves so that for a given ω, each ridge has the same height. However, for $\Delta > 0$ we found that (except for isolated ω's) exactly one ridge achieved a global maximum for a given ω.

B. Showing that y is an ARMA(4,8) in the One-Shock Case

The policy rules that solve the time-to-build model are linear equations in the log of capital $\ln K$ and of hours-worked $\ln n$ and the technology shocks (where $\ln z_t$ equals $\ln z_{t-1} + \varepsilon_{zt}$).

$$\begin{aligned} \ln K_t &= (1 - A(1)) \ln k + A(L) \ln K_t + (1 - A(L)) \ln z_{t-4} + +B(L)(\varepsilon_{zt-4} - \mu) \\ \ln n_t &= \ln n - C(1) \ln k + C(L) \ln K_{t+4} - C(L) \ln z_t + D(L)(\varepsilon_{zt} - \mu) \end{aligned}$$

The terms $A(L)$ and $B(L)$ are polynomials of degree four and $C(L)$ and $D(L)$ are polynomials of degree three in the lag operator. Capital is a function of the past capital and the shocks to technology from four to eight periods ago. Hours worked is a function of future capital (since you have to work for the investment that you have already committed to making) and the current and lagged shocks. The lower case variables without the time subscript are the variables at steady state.

Taking the first difference of the above two equations eliminates the steady state values.

$$\begin{aligned} \triangle \ln K_t &= A(L) \triangle \ln K_t + (1 - A(L)) \triangle \ln z_{t-4} + B(L) \triangle (\varepsilon_{zt-4} - \mu) \\ &= \frac{(1 - A(L) + B(L)(1 - L)) L^4}{1 - A(L)} \varepsilon_{zt} \end{aligned}$$

$$\triangle \ln n_t = C(L) \triangle \ln K_{t+4} - C(L) \triangle \ln z_t + D(L) \triangle (\varepsilon_{zt} - \mu)$$

$$= \left(\frac{C(L)B(L)\,(1-L)}{1-A(L)} + D(L)(1-L) \right) \varepsilon_{zt}$$

The next step is to derive an equation for y. Output is produced using a Cobb-Douglas production function. Hence, the first difference of output can be written as

$$y_t = \triangle \ln Y_t = \theta \triangle \ln K_t + (1-\theta) \triangle \ln n_t + (1-\theta)\, \varepsilon_{zt}$$

Substituting in the values for $\triangle \ln K_t$ and $\triangle \ln n_t$ we have

$$\begin{aligned}(1-A(L))\, y_t \;\; = \;\; & \{\theta\,(1-A(L)+B(L)\,(1-L))\, L^4 \\ & +(1-\theta)\,[C(L)B(L)\,(1-L)+(1-A(L))\,D(L)(1-L)+1]\}\varepsilon_{zt}\end{aligned}$$

The polynomials A and C are fourth order and the polynomials D and B are third order. As the first difference operator is also present, the moving average component is an eighth order polynomial. The autoregressive term is the same order as A. The time-to-build model, therefore, can be characterized as a restricted version of an ARMA(4,8) model. The RBC model nests inside this specification.

C. Showing that Y and I/Y are a VAR(10) in the Two-Shock Case

As in the previous section, the model's solution can be expressed as two linear equations in the log of capital $\ln K$ and of hours-worked $\ln n$. The difference from the previous section is that these equations now have two shocks: technology and government consumption.

$$\ln \frac{K_{t+4}}{z_t} = \ln k + A(L) \left(\ln \frac{K_{t+4}}{z_t} - \ln k \right) + B(L)\varepsilon_t + G_K \left(\ln g_t - \ln g \right)$$

$$\ln n_t = \ln n + C(L) \left(\ln \frac{K_{t+4}}{z_t} - \ln k \right) + D(L)\varepsilon_t + G_N \left(\ln g_t - \ln g \right)$$

The equations for the shocks are:

$$\ln g_t - \ln g = \rho \left(\ln g_{t-1} - \ln g \right) + \varepsilon_{gt}$$

$$\ln z_t = \ln z_{t-1} + \varepsilon_{zt}$$

We can write the model's solution in terms of the i.i.d shocks ε_{zt} and ε_{gt} by substituting in the equations for the shocks and then simplifying

$$\ln K_{t+4} = \ln k + \left(\frac{1}{1-L} + \frac{B(L)}{(1-A(L))}\right)\varepsilon_{zt} + \frac{G_K}{(1-\rho L)(1-A(L))}\varepsilon_{gt}$$

$$\ln n_t = \ln n + \left(\frac{C(L)B(L)}{(1-A(L))} + D(L)\right)\varepsilon_{zt} + \left(\frac{C(L)G_K}{(1-\rho L)(1-A(L))} + \frac{G_N}{1-\rho L}\right)\varepsilon_{gt}$$

The moving average representation of $\triangle \ln Y_t$ is found by taking the first difference of the above two equations and substituting into the formula for the $\triangle \ln Y$. The moving average representation is

$$\triangle \ln Y = \begin{pmatrix} \varepsilon_{zt} & \varepsilon_{gt} \end{pmatrix} \begin{pmatrix} \alpha L^4 \left(1 + \frac{B(L)(1-L)}{(1-A(L))}\right) + (1-\alpha)\left(1 + C(L)\left(\frac{B(L)(1-L)}{(1-A(L))}\right) + D(L)(1-L)\right) \\ \frac{\alpha G_K (1-L) L^4}{(1-\rho L)(1-A(L))} + (1-\alpha)\left(C(L)\frac{G_K(1-L)}{(1-\rho L)(1-A(L))} + \frac{G_N(1-L)}{1-\rho L}\right) \end{pmatrix}$$

The first step in finding an equation for $\ln(I_t/Y_t)$ is to take a Taylor series expansion of $\ln(I_t/z_t)$ around the steady state values. To simplify the notation, we define the following coefficients.

$$\tilde{\phi}_1 = \phi_1 \tfrac{k}{i} \qquad \tilde{\phi}_2 = (\phi_3 - \phi_2(1-\delta))\tfrac{k}{i}\exp(-\mu)$$

$$\tilde{\phi}_3 = (\phi_4 - \phi_3(1-\delta))\exp(-2\mu)\tfrac{k}{i} \qquad \tilde{\phi}_4 = (\phi_4 - \phi_3(1-\delta))\tfrac{k}{i}\exp(-3\mu)$$

$$\tilde{\phi}_5 = \phi_4(1-\delta)\exp(-4\mu)\tfrac{k}{i}$$

The value of k/i is the steady state ratio of capital to investment. The value of μ is the steady state value of the technology shock (its mean). The Taylor series expansion results in an equation for $\ln I_t$

$$\ln I_t = \ln z_t + \ln \hat{\imath} + Q(L)\left(\ln K_{t+4} - \ln z_t - \ln \hat{k}\right) - P(L)\left(\varepsilon_{zt} - \hat{\varepsilon}_z\right) \tag{C.1}$$

where $Q(L)$ and $P(L)$ are defined as

$$Q(L) = \tilde{\phi}_1 + \tilde{\phi}_2 L + \tilde{\phi}_3 L^2 + \tilde{\phi}_4 L^3 + \tilde{\phi}_5 L^4$$

$$P(L) = \left(\tilde{\phi}_2 + \tilde{\phi}_3 + \tilde{\phi}_4 + \tilde{\phi}_5\right) + \left(\tilde{\phi}_3 + \tilde{\phi}_4 + \tilde{\phi}_5\right)L + \left(\tilde{\phi}_4 + \tilde{\phi}_5\right)L^2 + \tilde{\phi}_5 L^3$$

The next step is to subtract the equation for the $\ln Y_t$ from C.1

$$\ln I_t - \ln Y_t = \text{constant} + \left(Q(L) - \alpha L^4\right) \ln K_{t+4} + \left[\frac{(\alpha - Q(L))}{(1-L)} - P(L)\right] \varepsilon_t - (1-\alpha) \ln N_t$$

The resulting moving average representation of $\ln I_t/Y_t$ is

$$\ln \frac{I_t}{Y_t} = \begin{pmatrix} \varepsilon_{zt} & \varepsilon_{gt} \end{pmatrix} \begin{pmatrix} \alpha \left(1 + L + L^2 + L^3\right) + \frac{\left(Q(L) - \alpha L^4\right) B(L) - (1-\alpha) C(L) B(L)}{(1 - A(L))} - P(L) - (1-\alpha)\, D(L) \\ \left(Q(L) - \alpha L^4\right) \left(\frac{G_K (1-L)}{(1-\rho L)(1-A(L))}\right) - (1-\alpha) \left[C(L) \frac{G_K}{(1-\rho L)(1-A(L))} + \frac{G_N}{1-\rho L}\right] \end{pmatrix}$$

The general form of the multivariate expression used in the two-shock case is.

$$\begin{pmatrix} \Gamma_1 (L) \\ \Gamma_2 (L) \end{pmatrix} \begin{pmatrix} \ln \frac{Y_t}{Y_{t-1}} \\ \ln \frac{I_t}{Y_t} \end{pmatrix} = \begin{pmatrix} \Theta_{11}(L) & \Theta_{12}(L) \\ \Theta_{21}(L) & \Theta_{22}(L) \end{pmatrix} \begin{pmatrix} \varepsilon_{zt} \\ \varepsilon_{gt} \end{pmatrix}$$

where each polynomial $\Theta_{11}(L)$, $\Theta_{12}(L)$, $\Theta_{21}(L)$, and $\Theta_{11}(L)$ is of order 8 and Γ_1 and Γ_2 are polynomials of order 5. To approximate this VARMA(5,8), we will use a vector autoregression (VAR)

$$\begin{pmatrix} \Gamma_{11}(L) & \Gamma_{12}(L) \\ \Gamma_{21}(L) & \Gamma_{22}(L) \end{pmatrix} \begin{pmatrix} \ln \frac{Y_t}{Y_{t-1}} \\ \ln \frac{I_t}{Y_t} \end{pmatrix} = \begin{pmatrix} \varepsilon_{zt} \\ \varepsilon_{gt} \end{pmatrix}$$

The benefits of the approximation are computational. Estimating a high-order VAR is much easier than estimating a high-order VARMA process.

One must choose the number of lags for the VAR. To find a suitable lag length, we did the general-to-specific testing approach using the likelihood ratio test.

Two likelihood ratio tests are used. The first test is the standard time-domain lag-length test using the variance covariance matrix of the residuals (Hamilton, 1994). The second test is the frequency-domain test described in this paper. We tested down from a maximum number of 15 lags stopping when we reject the null that the lower-order VAR is correct at the 95% level. Data was simulated using the time-to-build model with two different sets of parameters. In one set, the investment weights are Kydland and Prescott's values of $\phi_i = 0.25$ for $i = 1, ..., 4$. In the other set, the investment weights are Christiano and Todd's values of $\phi_1 = 0.01$ and $\phi_i = 0.33$ for $i = 2, ..., 4$. For both sets, the parameters of the exogenous shock processes are the values used by Christiano and Eichenbaum (1992). Based on one hundred simulations, the average lag length chosen by the time-domain and frequency-domain tests was 11.2 and 7.8 using the Kydland-Prescott values as the data generating process and 11.03 and 9.2 using the Christiano-Todd values as the data generating process. Given these findings and how well the spectrum generated by the VAR(10) matches the smoothed periodogram estimates, the choice of ten lags seems reasonable.

References

[1] Altug, Sumru, 1989, 'Time to Build and Aggregate Fluctuations Some New Evidence', *International Economic Review,* 30(4), November, 889-920.

[2] Baxter, Marianne, and Robert King, 1999, 'Measuring Business Cycles: Approximate Band-Pass Filters for Economic Time Series,' *Review of Economics and Statistics*, 81(4), November, pages 575-93.

[3] Bencivenga, Valerie R., 1992 'An Econometric Study of Hours and Output Variation with Preference Shocks' *International Economic Review* 33 449-71

[4] Berkowitz, Jeremy, 2001, Generalized Spectral Estimation of the Consumption-Based Asset Pricing Model, forthcoming, *Journal of Econometrics*

[5] Bernanke, Ben, Mark Gertler and Simon Gilchrist, 1999, 'The Financial Accelerator in a Quantitative Business Cycle Framework,' in John Taylor and Michael Woodford, editors, *Handbook of Macroeconomics*, North-Holland.

[6] Christiano, Lawrence J., 1985, 'A Method for Estimating the Timing Interval in a Linear Econometric Model, With an Application to Taylor's Model of Staggered Contracts.' *Journal of Economic Dynamics and Control,* 9, 363-404.

[7] Christiano, Lawrence J., 1988, 'Why Does Inventory Investment Fluctuate So Much?', *Journal of Monetary Economics*, 21(2/3), March/May, 247-80.

[8] Christiano, Lawrence J., 2001, 'Solving Dynamic Equilibrium Models by a Method of Undetermined Coefficients', forthcoming. See also, *NBER Technical Working paper 225.*

[9] Christiano, Lawrence J. and Martin Eichenbaum, 1987, 'Temporal Aggregation and Structural Inference in Macroeconomics', *Carnegie Rochester Conference Series on Public Policy,* 26 (0), Spring , 63-130.

[10] Christiano, Lawrence J., Martin Eichenbaum, and David Marshall, 1991, 'The Permanent Income Hypothesis Revisited', *Econometrica*, 59(2), March, 397-423..

[11] Christiano, Lawrence J. and Martin Eichenbaum, 1992, 'Current Real-Business-Cycle Theories and Aggregate Labor-Market Fluctuations', *American Economic Review,* 82(3), June, 430-50.

[12] Christiano, Lawrence J. and Terry Fitzgerald, 1999, 'The Band Pass Filter,' National Bureau of Economic Research Working Paper number 7257.

[13] Christiano, Lawrence J. and Richard M. Todd, 1996, 'Time to Plan and Aggregate Fluctuations', *Federal Reserve Bank of Minneapolis Quarterly Review,* Winter, 14-27.

[14] Christiano, Lawrence J. and Richard M. Todd, 2001, 'The Conventional Treatment of Seasonality in Business Cycle Analysis: Does it Create Distortions?,' forthcoming, *Journal of Monetary Economics.*

[15] Cogley, Tim, and James Nason, 1995, 'Output Dynamics in Real Business Cycle Models', *American Economic Review,* 85(3), June, 492-511.

[16] Cogley, Timothy, 1998, 'Estimating and Testing Dynamic Rational Expectations Models', unpublished manuscript.

[17] Cogley, Timothy, 1998a, 'A Frequency Decomposition of Approximation Errors in Stochastic Discount Factor Models,' manuscript, University of Arizona.

[18] Diebold, Francis X., Lee E. Ohanian, and Jeremy Berkowitz, 1998, 'Dynamic Equilibrium Economies: A Framework for Comparing Models and Data', *Review of Economic Studies*, 65(3), July, 433-51.

[19] Engle, Robert F., 1974, 'Band Spectrum Regression,' *International Economic Review*, vol. 15, 1-11.

[20] Granger, Clive W. J., and Paul Newbold, 1977, *Forecasting Economic Time Series*, Academic Press, New York.

[21] Hall, George J., 1996, 'Overtime, Effort, and the Propagation of Business Cycle Shocks', *Journal of Monetary Economics*, 38 (1), August, 139-60.

[22] Hamilton, James D. 1994, *Time Series Analysis* Princeton University Press

[23] Hansen, Lars Peter, and Thomas J. Sargent, 1980, 'Formulating and Estimating Dynamic Linear Rational Expectations Models', *Journal of Economic Dynamics and Control*, 2, February, 7-46.

[24] Hansen, Lars Peter, and Thomas J. Sargent, 1980a, 'Methods for Estimating Continuous Time Rational Expectations Models from Discrete Data', Research Department Staff report 59, Federal Reserve Bank of Minneapolis.

[25] Hansen, Lars Peter, and Thomas J. Sargent, 1991, 'Exact Linear Rational Expectations Models: Specification and Estimation,' in *Underground Classics in Economics*, Boulder and Oxford: Westview Press, pages 45-76.

[26] Hansen, Lars Peter, and Thomas J. Sargent, 1993, 'Seasonality and Approximation Errors in Rational Expectations Models', *Journal of Econometrics*, 55(1-2), January-February, 21-55.

[27] Harvey, Andrew, 1989, *Forecasting Structural Time Series Models and the Kalman Filter*, Cambridge University Press.

[28] Hodrick, Robert, and Edward Prescott, 1997, 'Post-War Business Cycles: An Empirical Investigation,' *Journal of Money, Credit and Banking,* vol. 29, no. 1, February, pp. 1-16.

[29] Ireland, Peter N., 1997, 'A Small, Structural Quarterly Model for Monetary Policy Evaluation', *Carnegie Rochester Conference Series on Public Policy*, 47 (0), December, 83-108.

[30] Ireland, Peter N., 1999 'A Method for Taking Models to the Data' Federal Reserve Bank of Cleveland Working Paper 9903

[31] Ireland, Peter N., 2001 'Technology shocks and the business cycle: On empirical investigation', *Journal Of Economic Dynamics And Control* (25)5 703-19

[32] Kim, Jinill, 2000, 'Constructing and estimating a realistic optimizing model of monetary policy', *Journal Of Monetary Economics* (45)2 pp. 329-359

[33] King, Robert, 1995, 'Quantitative Theory and Econometrics', *Federal Reserve Bank of Richmond Economic Quarterly,* 81(3), 53-105.

[34] King, Robert, and Sergio Rebelo, 1993, 'Low Frequency Filtering and Real Business Cycles,' *Journal of Economic Dynamics and Control*, vol. 17, pp. 251-231.

[35] Kydland, Finn E. and Edward C. Prescott, 1982, 'Time to Build and Aggregate Fluctuations', *Econometrica,* 50, November, 1345-70.

[36] Leeper, Eric M., and Christopher A. Sims, 1994, 'Toward a Modern Macroeconomic Model Usable for Policy Analysis', Fischer, Stanley, Julio J. Rotemberg, eds. *NBER Macroeconomics Annual 1994* Cambridge and London: MIT Press, pages 81-118.

[37] Long, John, and Charles Plosser, 1983, 'Real Business Cycles', *Journal of Political Economy* 91(1), February, 39-69.

[38] McGrattan, Ellen R., 1994, 'The Macroeconomic Effects of Distortionary Taxation', *Journal of Monetary Economics*, 33, 573-601

[39] McGrattan, Ellen R., Richard Rogerson, and Randall Wright, 1997, 'An Equilibrium Model of the Business Cycle with Household Production and Fiscal Policy', *International Economic Review*, 38 (2), May, 267-90.

[40] Nerlove, Marc, David M. Grether and Jose M. Carvalho, 1979, *Analysis of Economic Time Series: A Synthesis*, Academic Press.

[41] Rouwenhorst, K. Geert, 1991, 'Time to Build and Aggregate Fluctuations A Reconsideration' *Journal of Monetary Economics,* 27, 241-54.

[42] Sargent, Thomas 1987 '*Macroeconomic Theory*' Academic Press

[43] Sargent, Thomas 1989, 'Two Models of Measurements and the Investment Accelerator,' *Journal of Political Economy*, 97(2), April, pages 251-87.

[44] Sims, Christopher, 1972, 'Approximate Prior Restrictions in Distributed Lag Estimation', *Journal of the American Statistical Association*, 67(337), 169-175.

[45] Watson, Mark, 1993, 'Measures of Fit for Calibrated Models', *Journal of Political Economy*, 101(6), 1011-1041.

D. Tables

Table 1: Weighted Likelihood Estimation Results, RBC Model						
Frequencies	θ	δ	σ_z	λ	λ_w	Observations
						Used
High	0.25	0.99	0.0126	9.6	3.7	50%
Business Cycle	0.51	0.99	0.0170	26.1	2.3	43%
Low	0.15	0	0.0100	37.3	-0.2	7%
All	0.37	0.73	0.0144	8.5	8.5	100%

- **Notes**: These are the results of estimating the unrestricted RBC model by weighted maximum likelihood (i.e., by maximizing (2.3)). Low frequencies: $v_j = 1$ only for v_j's that belong to frequencies corresponding to periods 8 years and up. Business cycle frequencies: $v_j = 1$ only for v_j's that belong to frequencies corresponding to periods 1 to 8 years. High frequencies: $v_j = 1$ only for v_j's that belong to frequencies corresponding to periods 2 quarters to 1 year; All frequencies: $v_j = 1$ for all j. Percent of observations used: fraction of $j \in \{0, 1, ..., T-1\}$ equal to unity in the weighted likelihood estimation. λ: likelihood ratio statistic based on all frequencies. λ_w: likelihood ratio statistic based only on the indicated subinterval of frequencies.

Table 2: Coefficient Values						
	Panel A: One Shock Model					
	ϕ_1	ϕ_2	ϕ_3	ρ	σ_z	σ_g
Estimated	0.0097	0.28	0.48	NA	0.014	NA
Christiano-Todd (CT model)	0.01	0.33	0.33	NA	0.018	NA
Kydland-Prescott (KP model)	0.25	0.25	0.25	NA	0.012	NA
RBC	NA	NA	NA	NA	0.011	NA
	Panel B: Two Shock Model					
	ϕ_1	ϕ_2	ϕ_3	ρ	σ_z	σ_g
Estimated	0.0079	0.30	0.41	0.94	0.012	0.063
Christiano-Todd (CT model)	0.0079	0.30	0.41	0.96	0.018	0.022
Kydland-Prescott (KP model)	0.25	0.25	0.25	0.94	0.012	0.063

Figure 1: Spectrum of $\ln(Y_t/Y_{t-1})$

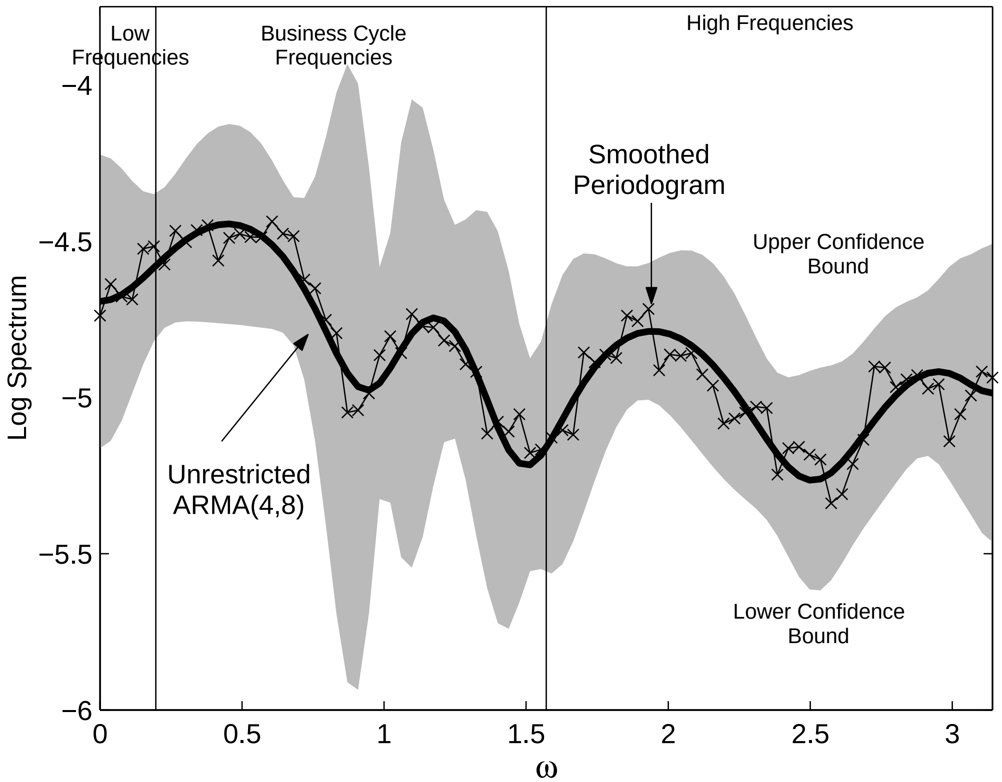
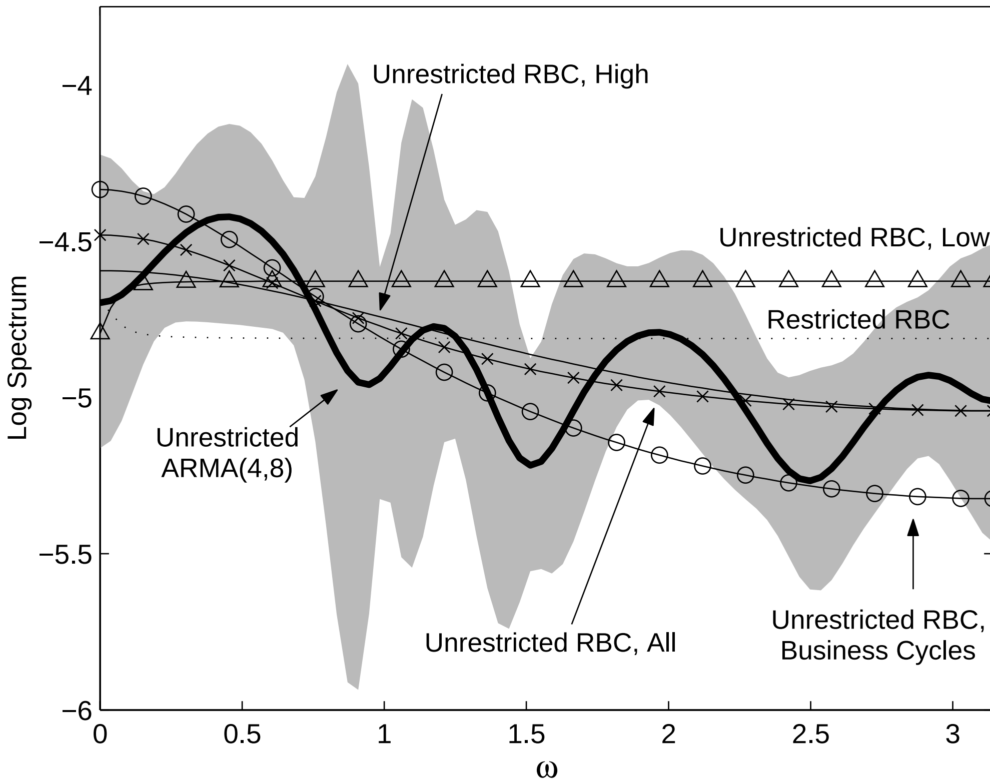


Figure 2: Diagnosing Fit of One-Shock RBC Models



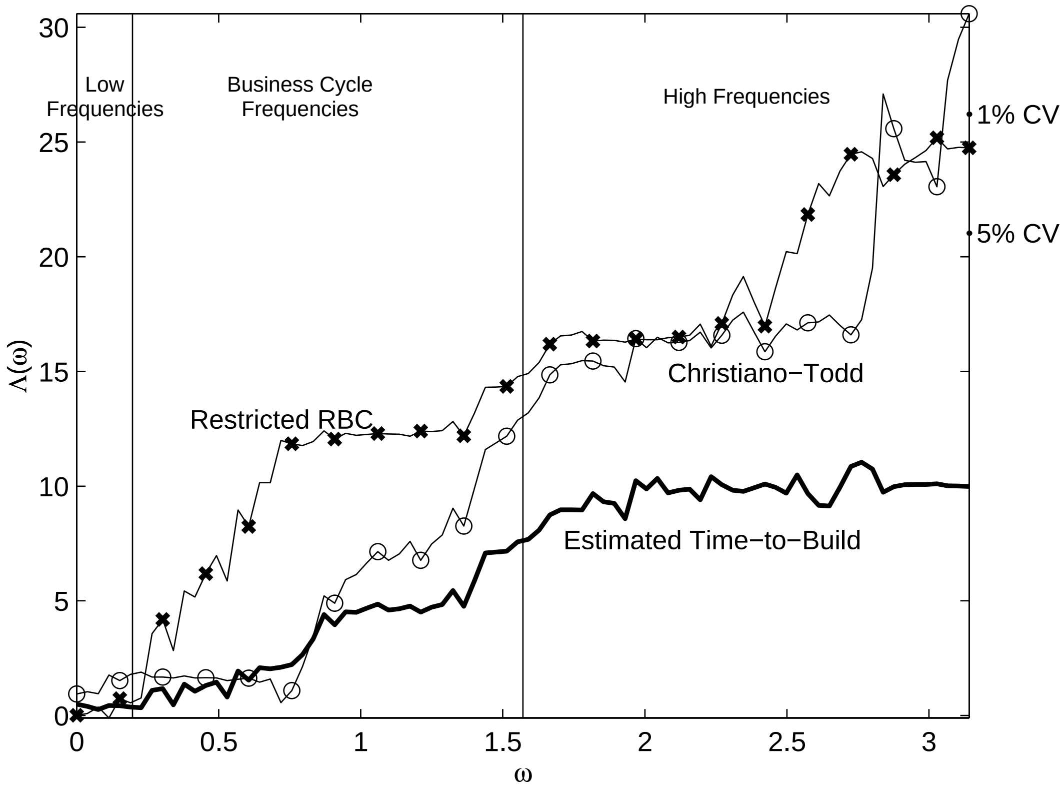
Notes: (i) Unrestricted RBC, Low ∼ results of estimation of one-shock RBC model with three free parameters estimated with $v_j \neq 0$ in (2.3) only for low frequencies. High ∼ $v_j \neq 0$ only for high frequencies; Business Cycles ∼ $v_j \neq 0$ only for business frequencies; All ∼ $v_j \neq 0$ for all frequencies.

(ii) Restricted RBC ∼ results of estimation of one-shock RBC model with only shock innovation variance free and $v_j = 1$ for all j.

(iii) Unrestricted ARMA(4,8) ∼ results for unrestricted reduced form. Reproduced from Figure 1.

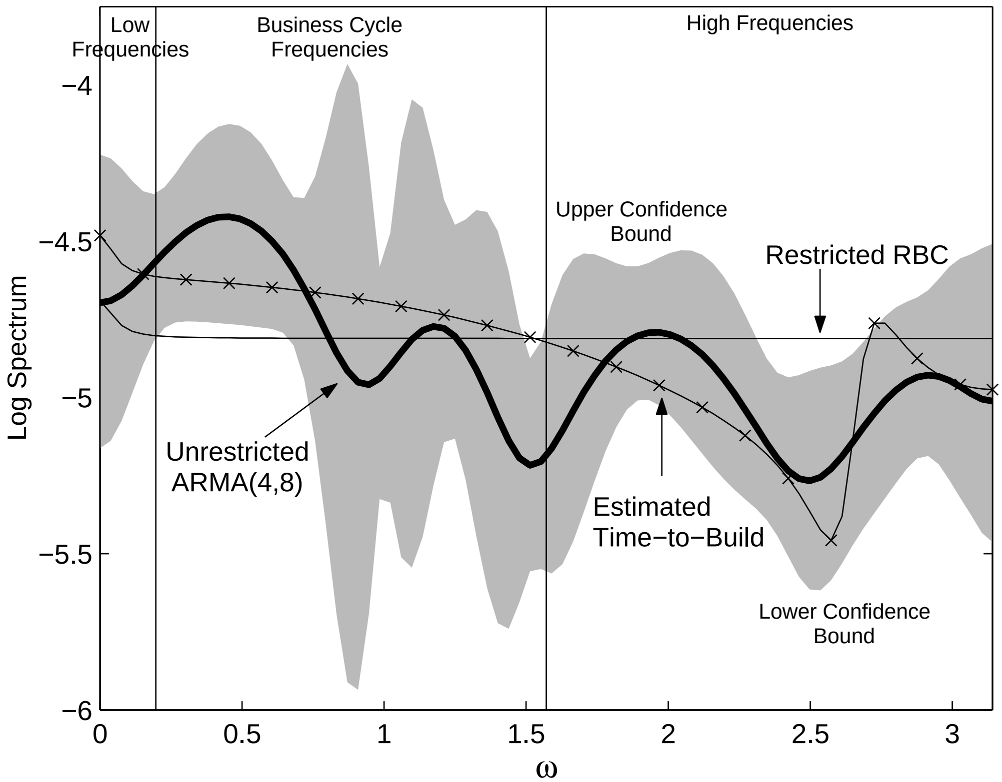
(iv) Grey area ∼ 95% confidence interval for unrestricted ARMA(4,8). Reproduced from Figure 1.

Figure 3: Cumulative Likelihood Ratio, One Shock Case



Notes: (i) Figure displays (2.10) for each of the three indicated models.
(ii) Christiano-Todd ~ time-to-build model with σ_z estimated with $v_j = 1$ in (2.3) for all frequencies and $\phi_1 = 0.01$ and $\phi_2 = \phi_3 = \phi_4 = 0.33$.
(iii) Estimated Time-to-Build ~ time-to-build model with $\{\sigma_z, \phi_1, \phi_2, \phi_3\}$ estimated with $v_j = 1$ in (2.3) for all frequencies.
(iv) Restricted RBC ~ See notes to Figure 2.
(v) CV ~ critical values for Chi-square distribution with 12 degrees of freedom.

Figure 4: Diagnosing Fit of Estimated Time-to-Build Model



Notes: (i)Unrestricted ARMA(4,8) $\sim$ results for unrestricted reduced form. Reproduced from Figure 1.
(ii) Grey area $\sim$ 95% confidence interval for unrestricted ARMA(4,8). Reproduced from Figure 1.
(iii) Restricted RBC $\sim$ Reproduced from Figure 2.
(iv) Estimated Time-to-Build $\sim$ See notes to Figure 3.

Figure 5a: Spectrum of $\ln(Y_t/Y_{t-1})$

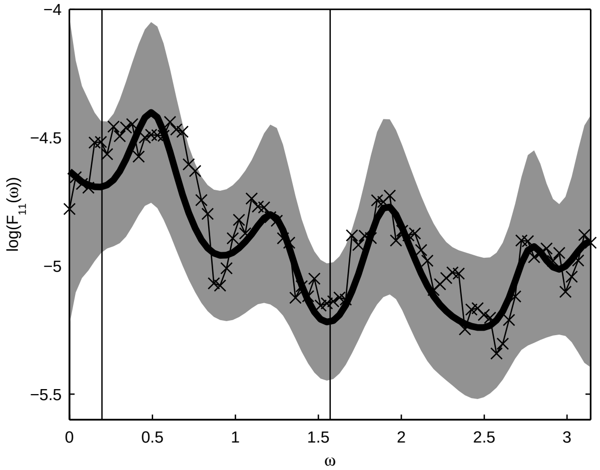


Figure 5b: Spectrum of $\ln(I_t/Y_t)$

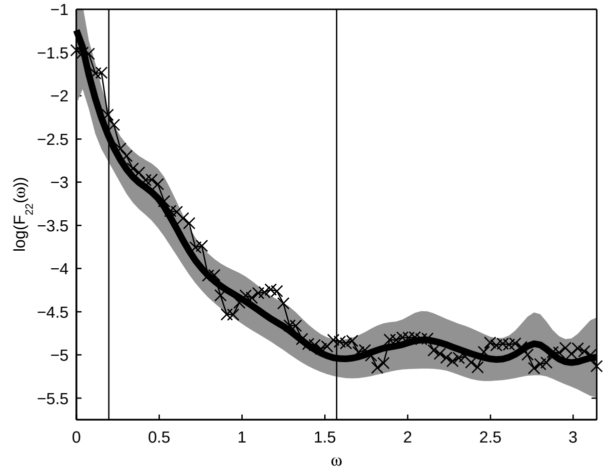


Figure 5c: Coherence For $\ln(Y_t)$ and $\ln(I_t)$

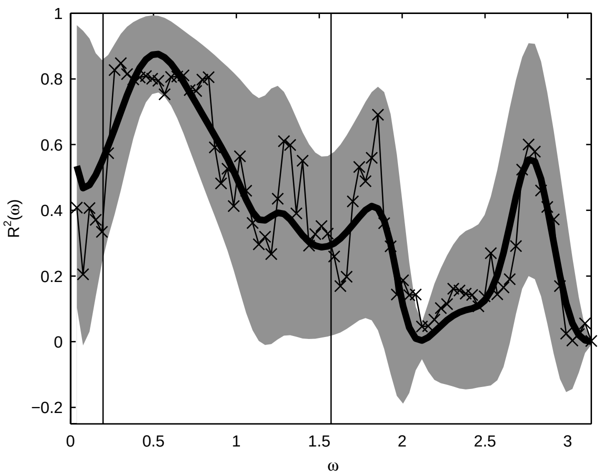
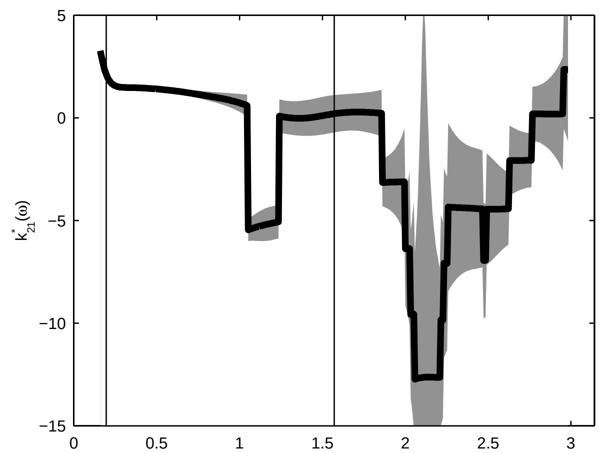


Figure 5d: The Phase Relationship in units of time For $\ln(I_t)$ and $\ln(Y_{t-k})$



Notes: (i) Unrestricted VAR(10) $\sim$ thick line. (ii) Confidence Intervals $\sim$ Grey Area. (iii) Periodogram $\sim$ x's. Periodogram from Figure 5a reproduced from Figure 1.

Figure 6a: Correlation Between HP-Filtered $\ln(I_t)$ and $\ln(Y_{t-k})$

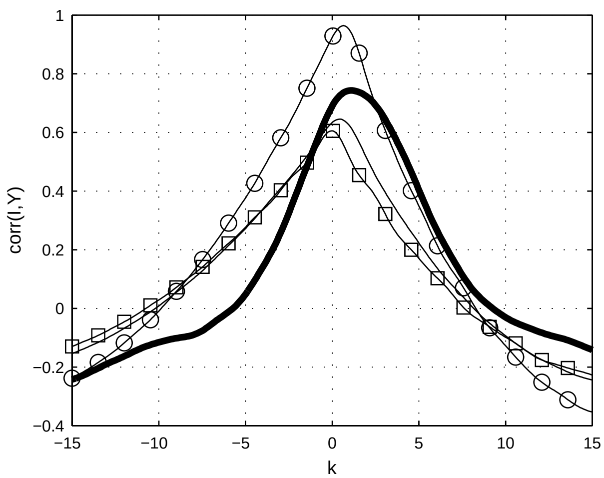
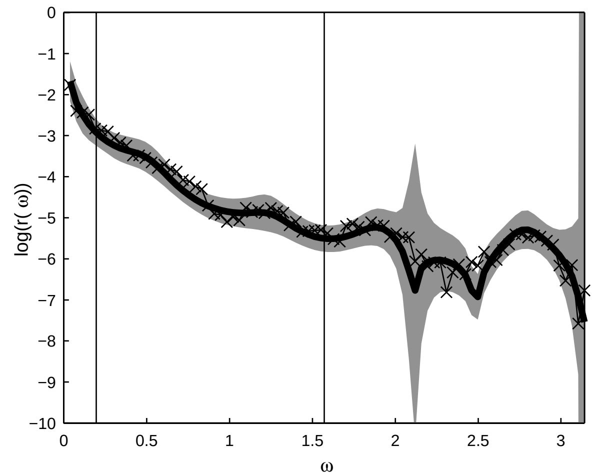


Figure 6b: Gain of Cross-Spectrum Between $\ln(I_t)$ and $\ln(Y_t)$



Notes: (i) Unrestricted VAR(10) ~ thick line. (ii) Estimated Model ~ thin line. (iii) CT ~ circles. (iv) KP ~ squares.

Figure 7a: Spectrum of $\ln(Y_t/Y_{t-1})$

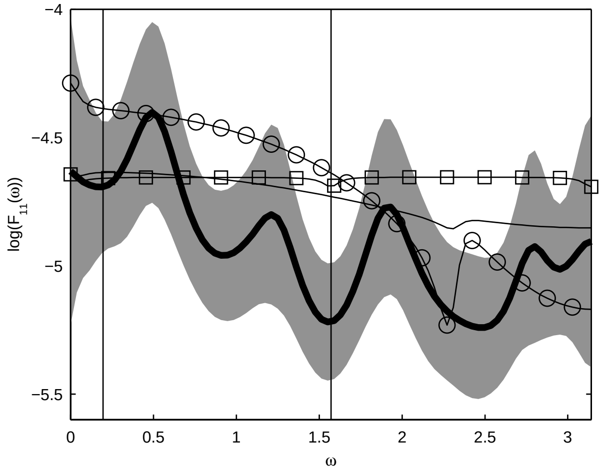


Figure 7b: Spectrum of $\ln(I_t/Y_t)$

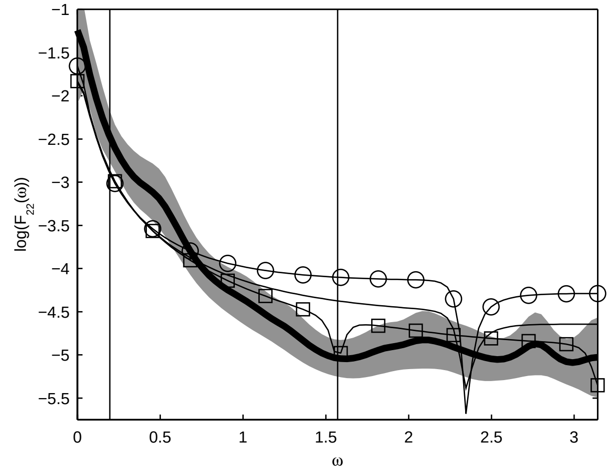


Figure 7c: Coherence For $\ln(Y_t)$ and $\ln(I_t)$

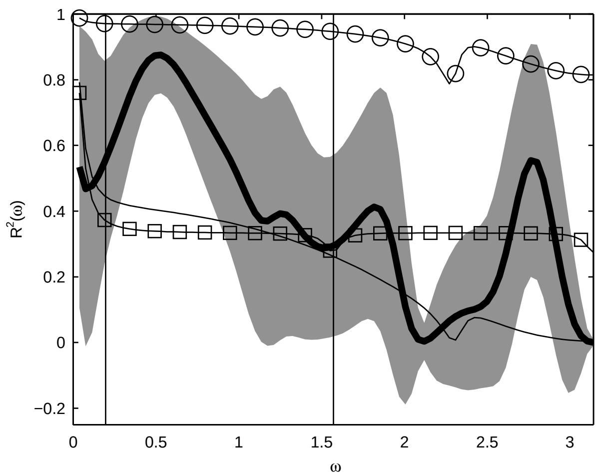
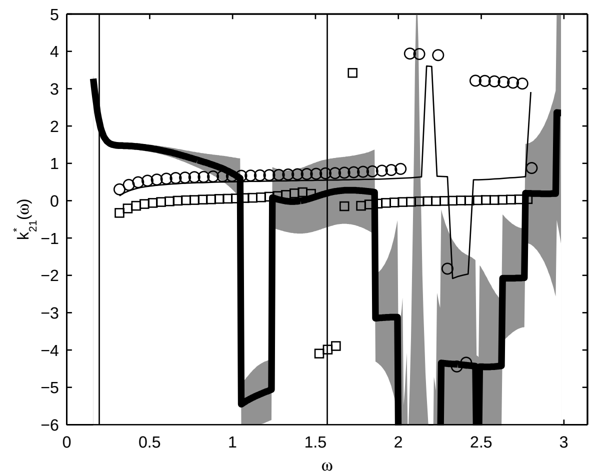
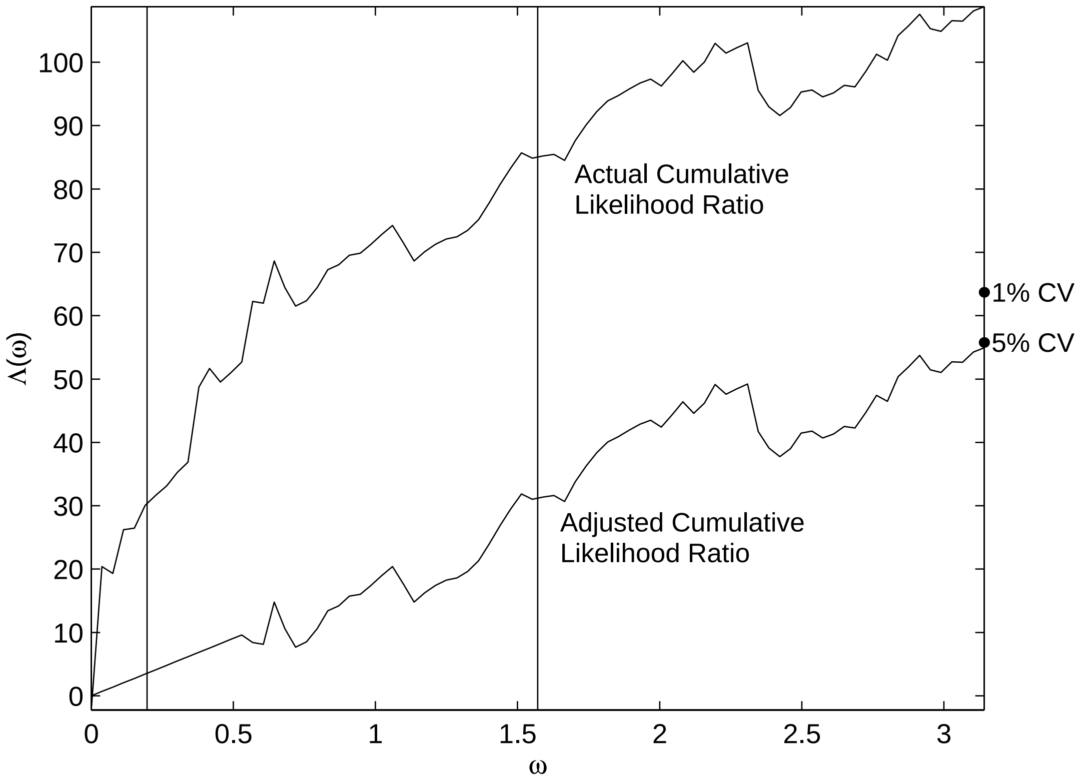


Figure 7d: The Phase Relationship in units of time For $\ln(Y_t)$ and $\ln(I_t)$



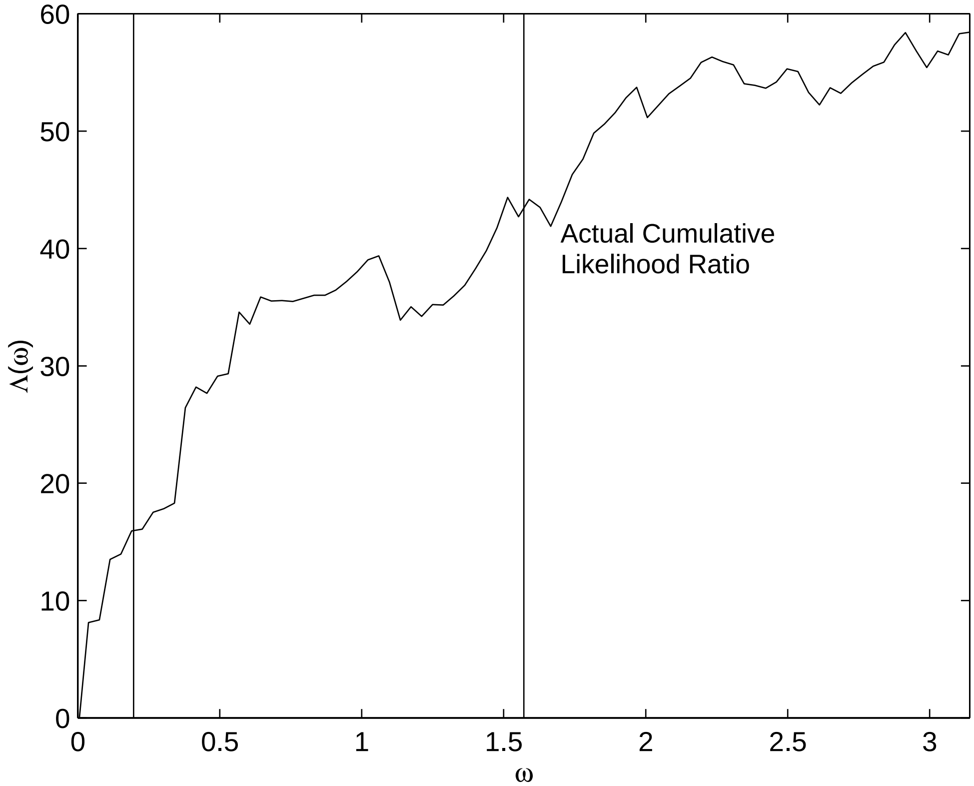
Notes: (i) Unrestricted VAR(10) $\sim$ thick line. (ii) Estimated Model $\sim$ thin line.
(iii) CT $\sim$ circles. (iv) KP $\sim$ squares.
(v) Confidence Interval and VAR(10) reproduced from Figure 5.

Figure 8: Cumulative Likelihood Ratio, Two-Shock Case



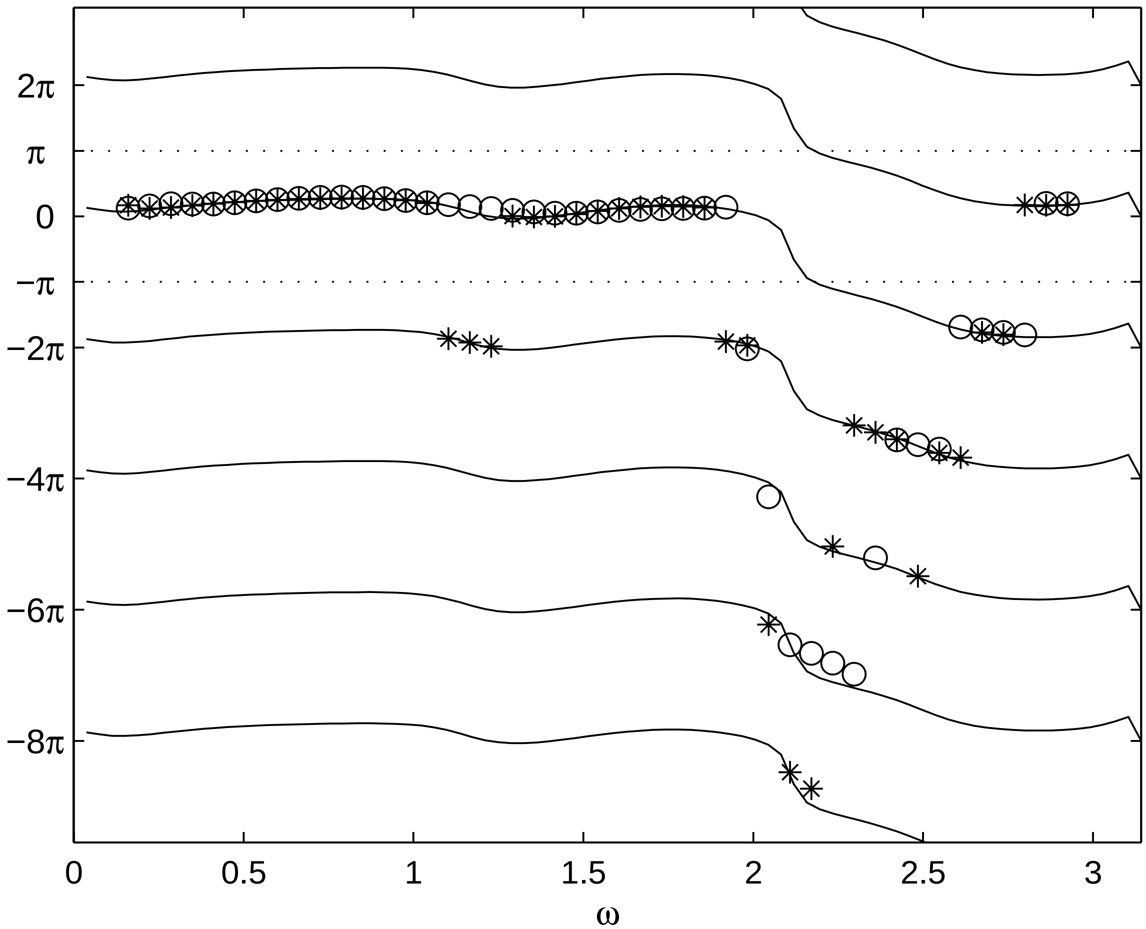
Notes (i) Actual Cumulative Likelihood Ratio $\sim$ Equation (2.10) evaluated at the indicated frequencies. (ii) Adjusted Cumulative Likelihood Ratio $\sim$ See text. (iii) CV $\sim$ critical values for Chi-square distribution with 39 degrees of freedom.

Figure 9: Cumulative Likelihood Ratio Two-Shock Case, Log(I/Y)



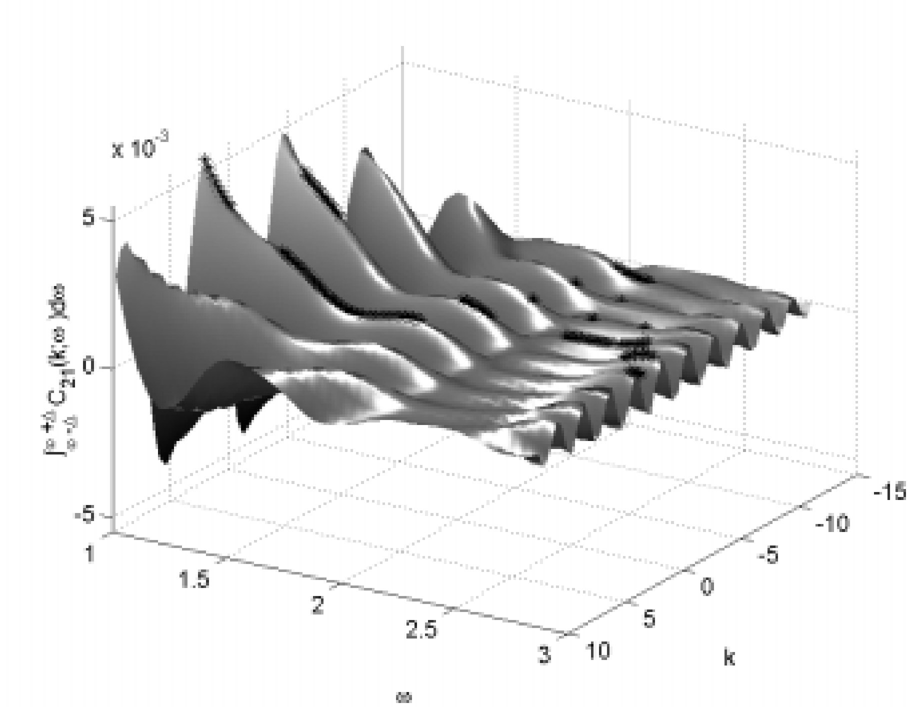
Note: Actual Cumulative Likelihood Ratio $\sim$ Equation (2.10) evaluated at the indicated frequencies using the univariate spectra for log(I/Y).

Figure 10: Various Measures of the Phase Angle,As a Function of Frequency



Notes: Solid Lines $\sim \theta_{21}(\omega;0)$. Stars $\sim \theta_{21}(\omega;0.06)$;. $\sim \theta_{21}(\omega;0.3)$.
(ii) For further discussion, see Appendix A.

Figure 11: Covariance Function for the Data



Notes: (i) $C_{21}(k;\omega) = Ey_{2,t}y_{1,t-k}$, where y_{1t} is frequency-ω component of $\ln(Y_t)$ and y_{2t} is frequency-ω component of $\ln(I_t)$. (ii) Stars $\sim$ $k_{21}(\omega;0.15)$.
(iii) For further discussion, see Appendix A.

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