Working Paper 9509

OPTIMAL FISCAL POLICY, PUBLIC CAPITAL, AND THE PRODUCTIVITY SLOWDOWN

by Steven P. Cassou and Kevin J. Lansing

Steven P. Cassou is a professor of economics at the State University of New York at Stony Brook, and Kevin J. Lansing is an economist at the Federal Reserve Bank of Cleveland. For helpful comments, the authors thank David Altig, Costas Azariadis, Terry Fitzgerald, Timothy Fuerst, Gary Hansen, Gregory Huffman, Per Krusell, Finn Kydland, B. Ravikumar, Peter Rupert, Randall Wright, and seminar participants at the Federal Reserve Banks of Cleveland and New York, and the 1995 Midwest Macroeconomics Conference.

September 1995

ABSTRACT

This paper develops a quantitative theoretical model for the optimal provision of public capital. We show that the ratio of public to private capital in the U.S. economy from 1925 to 1992 evolves in a manner that is generally consistent with an optimal transition path derived from the model. The model is also used to quantify the conditions under which an increase in the stock of public capital is desirable and to investigate the effects of hypothetical nonoptimal fiscal policies on productivity growth.

1. Introduction

In recent years, the link between public capital and private sector production has been a subject of considerable debate among policymakers and researchers. Although the idea that public capital may represent an important productive input is not new (for example, see Arrow and Kurz [1970]), work by Aschauer (1989, 1993) and Munnell (1990) stimulated renewed interest in this area because their empirical results suggested that large gains could be had by expanding public investment. These researchers also claimed that the observed decline in the rate of public capital accumulation during the 1970s and 1980s contributed significantly to the slowdown in the growth rate of U.S. labor productivity over the same period. Subsequent studies have added to the debate by attempting to confirm (or refute) the productive effects of public capital using increasingly sophisticated empirical methods.[1] Up to this point, however, little attention has been given to addressing these issues from a theoretical perspective.

In this paper, we develop a quantitative theoretical model for the optimal provision of public capital. We show that the ratio of public to private capital in the U.S. economy from 1925 to 1992 evolves in a manner that is generally consistent with an optimal transition path derived from a simple endogenous growth framework. Moreover, we are able to quantify the conditions under which an increase in the stock of public capital is justified in terms of maximizing the utility of a representative household. We find that even when the output elasticity of public capital is as high as 0.10, an increase in public capital from current levels is not called for. Finally, we show that a nonoptimal public investment policy of the type that might be interpreted as reflecting U.S. experience

[1] For instance, Aaron (1990), Tatom (1991), and Holtz-Eakin (1992) have shown that empirical methods which incorporate omitted variables, adjustments for nonstationarities, or more disaggregated data find that the output elasticity of public capital is not statistically different from zero. In contrast, Lynde and Richmond (1992), Finn (1993), and Ai and Cassou (1995) show that empirical techniques which properly handle reverse causality concerns continue to support large contributions to output from public capital.

over the last 30 years can account for only a small portion of the productivity slowdown that began in the early 1970s. In contrast, we show that the trend of increasing tax rates in the U.S. economy offers a better explanation for the productivity slowdown in the context of our model.

To perform our analysis, we embed a version of the empirical public capital model used by Aschauer (1989), Munnell (1990), and others in an equilibrium framework with an optimizing government.[2] The optimizing framework is similar to one used by Glomm and Ravikumar (1994). Our model differs from theirs in three fundamental ways. First, both private and public capital stocks are long-lived. Second, labor supply is endogenous, and third, the relevant stock of public capital for production is the per capita (or per firm) quantity. Our motivation for each of these features is as follows: By modeling capital as long-lasting, we are able to capture the lengthy transitional dynamics of an economy moving toward its balanced growth path. With endogenous labor supply, the model can be used to investigate changes in the growth rate of labor productivity arising from changes in the capital stocks. Finally, by specifying public capital as a per capita quantity, we link our model to previous empirical specifications in the literature which typically do not include any explicit congestion effects.[3] The model is used to explore the optimal transitional dynamics for an economy moving toward a balanced growth path and to quantify the effects of some hypothetical nonoptimal fiscal policies on productivity growth.[4]

[2]Early work by Kydland and Prescott (1977), Barro (1979), and Lucas and Stokey (1983) laid the groundwork for evaluating government optimization problems. Most of the recent work has been on applications to the Ramsey optimal-tax problem (e.g., Lucas [1990], Zhu [1992], Jones, Manuelli and Rossi [1993], Chari, Christiano and Kehoe [1994], and Cassou [1995]). Recently, Barro (1990) and Glomm and Ravikumar (1994) have investigated the spending side of the government budget constraint.

[3]Specifying public capital as a per capita quantity incorporates an implicit congestion effect associated with the size of the population. This differs from the explicit congestion effect in Glomm and Ravikumar (1994), where congestion is linked to the size of the private capital stock.

[4]Some recent research that also investigates transitional dynamics in neoclassical models includes King and Rebelo (1993) and Mulligan and Sala-i-Martin (1992).

The remainder of this paper is organized as follows. Section 2 presents the model. Section 3 describes the transitional dynamics of optimal fiscal policy. Section 4 describes how we obtain parameter values to carry out our quantitative exercises. Section 5 presents the quantitative exercises, and section 6 concludes.

2. The Model

The model economy consists of a private sector that operates in competitive markets and a benevolent government that solves a dynamic version of the Ramsey (1927) optimal tax problem. The private sector is typical of macroeconomic models with agents behaving optimally, taking government policy as given. In formulating its policy, the government takes into account the rational responses of the private sector. Our description of the economy proceeds in two steps and reflects this Stackelberg game hierarchy.

2.1. The Private Sector

The private sector consists of a large but fixed number of households. Each household is the owner of a single firm that produces output y_t at time t according to the technology

$$y_t = A_0 k_t^{\theta_1} (h_t l_t)^{\theta_2} k_{g,t}^{\theta_3}, \tag{1}$$

where $0 < A_0$, $0 < \theta_i$ for $i = 1,2,3$, and $\theta_1 + \theta_2 + \theta_3 = 1$.[5] With this technology, there are three factors of production: the per capita stock of private capital k_t, the per capita labor supply l_t, and the per capita stock of public capital $k_{g,t}$. The firm chooses k_t and l_t, but takes $k_{g,t}$ as exogenously supplied by the government. Defining $k_{g,t}$ as a per capita quantity ensures that there are no scale effects associated with the number of firms. Output is also affected by h_t, which is an index of knowledge outside the firm's

[5] Empirical research by Aschauer (1989), Munnell (1990), Ai and Cassou (1995), and others finds support for a technology specification with $\theta_1 + \theta_2 + \theta_3 = 1$.

control that augments the productive capacity of labor. Following Romer (1986), it is assumed that knowledge grows proportionally to, and as a by-product of, accumulated private investment and research activities, such that $h_t = \overline{k}_t$, where $\overline{k}_t$ is the average capital stock across firms. With this specification, the assumption that firms view h_t as outside their control requires that there be a sufficiently large number of firms so that no single firm has an impact on $\overline{k}_t$. Furthermore, because all firms are identical, $\overline{k}_t = k_t$ in equilibrium. Thus, the condition

$$h_t = \overline{k}_t = k_t \tag{2}$$

is imposed after firms choose their optimal labor and capital input levels.[6]

It is assumed that firms operate in competitive markets and maximize profits

$$\pi_t = y_t - w_t l_t - r_t k_t,$$

where w_t denotes the real wage and r_t denotes the real rental rate on private capital. Since $\theta_1 + \theta_2 + \theta_3 = 1$, the firm earns an economic profit equal to public capital's share of output. Our assumptions about firm ownership imply that all households receive equal amounts of total profits.[7] We assume that these profits are distributed to households as dividends and taxed as ordinary income. The market clearing prices for private capital

[6]The technology specification, the equilibrium condition $h_t = \overline{k}_t = k_t$, and the condition $\theta_1 + \theta_2 + \theta_3 = 1$, imply constant returns to scale in the two reproducible factors k_t and $k_{g,t}$. Consequently, the model exhibits endogenous growth. Kocherlakota and Yi (1995) find evidence in U.S. data supporting growth models that emphasize public capital. See Barro and Sala-i-Martin (1992) for a review of modeling structures that exhibit endogenous growth.

[7]It is possible to allow for different numbers of households and firms. However, as long as ownership of firms is uniform across households, each household will realize exactly the same level of profits as here.

and labor inputs and the resulting firm profits are given by

$$r_t = \frac{\theta_1 y_t}{k_t}, \qquad w_t = \frac{\theta_2 y_t}{l_t}$$

and

$$\pi_t = (1 - \theta_1 - \theta_2) y_t.$$

The infinitely-lived representative household chooses $\{c_t, l_t, i_t, k_{t+1} : t \geq 0\}$ to maximize

$$\sum_{t=0}^{\infty} \beta^t \log(c_t - B h_t l_t^{\gamma}) \tag{3}$$

subject to

$$\begin{aligned} c_t + i_t &= (1 - \tau_t)(w_t l_t + r_t k_t + \pi_t), \\ k_{t+1} &= A_1 k_t^{1-\delta} i_t^{\delta}, \qquad k_0 \text{ given,} \end{aligned} \tag{4}$$

where $0 < \beta < 1$, $0 \leq B$, $1 < \gamma$, $0 < A_1$, and $0 < \delta \leq 1$. In this specification, c_t denotes private consumption at time t, i_t is private investment, and τ_t is the income tax rate. The household operates in competitive markets and takes government tax policy τ_t, knowledge accumulation h_t, and dividends π_t as being determined outside of its control.

Three features of the household's problem warrant comment. First, the average capital stock affects the marginal utility of leisure via the knowledge accumulation term. This specification, which can be motivated by household production theory, ensures that the supply of labor, l_t, remains stationary along the balanced growth path.[8] Second, the parameter γ controls the elasticity of household labor supply. As γ becomes very large, the level of labor supplied approaches one, and the model reduces to one with a fixed labor supply. Third, the law of motion for private capital given by (4) implies a nonlinear relationship between current investment and next period's capital. When $\delta = 1$ and

[8] See Greenwood, Rogerson, and Wright (1994).

$A_1 = 1$, capital depreciates fully after one period, as in Glomm and Ravikumar (1994). When $0 < \delta < 1$, capital is long lasting. This nonlinear specification has been used by Hercowitz and Sampson (1991) and can be viewed as reflecting adjustment costs as in Lucas and Prescott (1971).

Using standard techniques, it can be shown that the household's decision rules are given by

$$c_t = (1 - a_0)(1 - \tau_t)y_t, \tag{5}$$

$$i_t = a_0(1 - \tau_t)y_t, \tag{6}$$

$$l_t = \left[(1 - \tau_t)\left(\frac{\theta_2 A_0}{\gamma B}\right)\left(\frac{k_{g,t}}{k_t}\right)^{\theta_3}\right]^{\frac{1}{\gamma - \theta_2}}, \tag{7}$$

where $a_0 = \frac{\beta\delta\theta_1}{1-\beta(1-\delta)}$.[9]

2.2. The Public Sector

The government chooses an optimal program of taxes and expenditures to maximize the discounted utility of the household. In addition to public investment, government expenditures include purchases of other goods and services, g_t, which do not contribute to production or household utility. We model nonproductive expenditures as a constant fraction $\phi \geq 0$ of total output, such that $g_t = \phi y_t$, but assume that the policymaker views g_t as exogenous. This specification is a simple way of ensuring that g_t continues to represent a significant fraction of output in this growing economy.[10]

To finance expenditures, the government imposes a tax on income at the rate τ_t such that

$$i_{g,t} + g_t = \tau_t y_t \tag{8}$$

[9] An appendix showing the derivation of these decision rules and other analytical results in the paper can be obtained from the authors upon request.

[10] Alternatively, we could introduce g_t as an additively separable argument in the household utility function (3). In this case, we obtain the same result–that the ratio $\frac{g_t}{y_t}$ is constant in equilibrium.

is the government budget constraint at time t, where $i_{g,t}$ represents public investment. Public investment contributes to future public capital stocks according to the following law of motion, which is analogous to (4):

$$k_{g,t+1} = A_1 k_{g,t}^{1-\delta} i_{g,t}^{\delta}, \quad k_{g,0} \text{ given.} \tag{9}$$

The government's problem can be formalized as choosing $\{\tau_t, i_{g,t}, k_{g,t+1}, c_t, l_t, i_t, k_{t+1} : t \geq 0\}$, so as to maximize (3) subject to (4), (5), (6), (7), (8), and (9). Because the model is analytically tractable, standard optimization procedures yield the following optimal policy rules:

$$i_{g,t} = a_1 y_t, \tag{10}$$

$$\tau_t = a_1 + \phi, \tag{11}$$

where $a_1 = \frac{\beta\delta\theta_3}{1-\beta(1-\delta)}$. Notice that the tax rate is constant over time and that it can be decomposed into two parts, one for public investment $i_{g,t}$ and one for nonproductive expenditures g_t.

3. Transitional Dynamics of Optimal Fiscal Policy

The model's tractable nature allows us to obtain closed-form expressions describing the optimal transition path for an economy with initial conditions that lie off the balanced growth path. To characterize the transitional dynamics, we begin by computing the optimal ratio of public to private capital, R^*, when the economy is in balanced growth. The intuition for the transitional changes is straightforward. If the current value of $R_t = \frac{k_{g,t}}{k_t}$ is less than the balanced growth ratio R^*, then optimal policy would call for an increase in R_t over time until R^* is reached. On the other hand, if R_t is greater than R^*, then a decline in R_t over time would be consistent with optimal fiscal policy.

To derive an expression for R^* as a function of the model's parameters, we combine the household and government decision rules with the laws of motion for the two capital stocks (4) and (9). Because there are two state variables, k_t and $k_{g,t}$, the decision rules must be solved jointly to obtain the equations that govern the optimal transition path leading to R^* .

Substituting the optimal decision rules (6), (7), and (11), and the production equations (1) and (2), into (4) yields

$$k_{t+1} = A_1 \left\{ A_0 a_0 \left[(1 - a_1 - \phi)^{\gamma} \left(\frac{\theta_2 A_0}{\gamma B} \right)^{\theta_2} \right]^{\frac{1}{\gamma - \theta_2}} \right\}^{\delta} k_t^{\frac{\delta\gamma(\theta_1+\theta_2)-\delta\theta_2}{\gamma-\theta_2}+1-\delta} k_{g,t}^{\frac{\delta\gamma\theta_3}{\gamma-\theta_2}}. \quad (12)$$

Equation (12) is the equilibrium law of motion governing the evolution of private capital when there is optimal behavior on the part of households, firms, and the government. Similarly, (10), (11), (7), (1), and (2) can be substituted into (9) to yield the equilibrium law of motion for public capital:

$$k_{g,t+1} = A_1 \left\{ A_0 a_1 \left[(1 - a_1 - \phi)^{\theta_2} \left(\frac{\theta_2 A_0}{\gamma B} \right)^{\theta_2} \right]^{\frac{1}{\gamma - \theta_2}} \right\}^{\delta} k_t^{\frac{\delta\gamma(\theta_1+\theta_2)-\delta\theta_2}{\gamma-\theta_2}} k_{g,t}^{\frac{\delta\gamma\theta_3}{\gamma-\theta_2}+1-\delta}. \quad (13)$$

Dividing (13) by (12) gives

$$\frac{k_{g,t+1}}{k_{t+1}} = \left[\frac{a_1}{a_0(1 - a_1 - \phi)} \right]^{\delta} \left(\frac{k_{g,t}}{k_t} \right)^{1-\delta}.$$

This equation implies that along the balanced growth path, that is, when $\frac{k_{g,t+1}}{k_{t+1}} = \frac{k_{g,t}}{k_t}$, $R^* = \frac{a_1}{a_0(1-a_1-\phi)}$, which is constant. Making use of the expressions for a_0, a_1, and (11) yields $R^* = \frac{\theta_3}{\theta_1(1-\tau)}$, where τ is the constant tax rate. By combining (12) and R^*, the following expression for the per capita growth rate can be obtained: [11]

$$\frac{k_{t+1}}{k_t} = \frac{k_{g,t+1}}{k_{g,t}} = \frac{y_{t+1}}{y_t} = A_1 a_0^{\delta} \left[A_0 (1 - a_1 - \phi)^{\theta_1 + \theta_2} \left(\frac{\theta_2}{\gamma B} \right)^{\frac{\theta_2}{\gamma}} \left(\frac{\theta_3}{\theta_1} \right)^{\theta_3} \right]^{\frac{\delta\gamma}{\gamma - \theta_2}} .$$

4. Calibration of the Model

In general, parameters are assigned values based on empirically observed features of the U.S. economy. However, for some parameters, such as the output elasticity of public capital θ_3, there is no general consensus regarding the appropriate value. Since θ_3 is important for determining R^*, we attempt to remain objective by exploring a range of values.[12] We also explore a range of values for the parameter δ, which appears in the laws of motion for the two capital stocks. In this case, the range is motivated by the lack of empirical attention given to the nonlinear specification for the relationship between current investment and next period's capital stock.

We choose baseline parameter values as follows: A discount factor of $\beta = 0.962$ implies that the real return on private assets along the balanced growth path is equal to 4 percent. Following Greenwood, Hercowitz, and Huffman (1988), we set $\gamma = 1.60$, which implies that the intertemporal elasticity of substitution in labor supply $1/(\gamma - 1)$ is equal to 1.7. Although the share of output used to compensate workers has been relatively constant over time, estimates of θ_2 are influenced by the way in which certain types of income are apportioned between labor and capital. For example, proprietor's income, indirect business taxes, and imputed services from consumer durables may affect

[11] To derive this result, we make use of the expression $\theta_1 + \theta_2 + \theta_3 = 1$. Consequently, this is a necessary condition for balanced growth in the model.

[12] The range of direct empirical estimates for θ_3 at the aggregate national level is quite large. Aschauer (1989) and Munnell (1990) estimate values of 0.39 and 0.34, respectively. Finn (1993) estimates a value of 0.16 for highway public capital. Aaron (1990) and Tatom (1991) argue that removing the effects of trends and taking account of possible missing explanatory variables, such as oil-price shocks, can yield point estimates for θ_3 that are not statistically different from zero.

estimated values of θ_2. The output elasticity of labor, $\theta_2 = 0.60$, is chosen based on empirical work by Christiano (1988), Ai and Cassou (1995), and others and is close to the value of 0.58 used by King, Plosser, and Rebelo (1988). The value $\phi = 0.17$ implies a ratio of nonproductive government spending to output of 0.17, consistent with the postwar U.S. average.

Most macroeconomic research employs a linear law of motion for capital accumulation. It is well known, however, that this specification does not yield closed-form decision rules except in the special case of 100 percent depreciation. For this reason, we employ the nonlinear form given in (4) and (9). Even in these nonlinear forms, however, δ controls the depreciation rate of existing capital.[13] Using this specification, Hercowitz and Sampson (1991) report a point estimate of $\delta = 0.34$, with a standard deviation of 0.26, using annual data on U.S. private capital from 1954 to 1987. Given the imprecise nature of the estimate, we explore a wide range of values for δ. With θ_3 set at its baseline value (described below), we find that $\delta = 0.10$ provides a reasonable fit of the U.S. time series of $R_t = \frac{k_{g,t}}{k_t}$ from 1925 to 1992. This is the period for which data on public and private capital stocks are available.[14] We also investigate values up to $\delta = 1.0$, which coincides with the value implicitly used by Glomm and Ravikumar (1994).

We examine values for θ_3 in the range $0 \leq \theta_3 \leq 0.20$. For each θ_3 in this range, we define $\theta_1 = 1 - \theta_2 - \theta_3$ to maintain the necessary condition for balanced growth in the model. Two combinations of θ_1 and θ_3 are of particular interest. The first is $\theta_1 = 0.277$ and $\theta_3 = 0.123$, which, together with $\delta = 0.10$, yield an optimal transition path that is consistent with the U.S. time series of $R_t = \frac{k_{g,t}}{k_t}$ over most of the sample

[13] In the nonlinear law of motion, δ is most properly interpreted as the elasticity of the next period capital stock with respect to current investment.

[14] The capital series are in 1987 dollars and were obtained from *Fixed Reproducible Tangible Wealth in the United States*, U.S. Department of Commerce (1993). The series for $k_{g,t}$ includes nonmilitary government-owned equipment and structures. The series for k_t includes privately owned equipment and structures. The "capital input" measure of of the net stock was used for all capital data.

period. This combination of parameters implies $R^* = 0.60$, which is slightly higher than the maximum value observed in postwar U.S. data. The second combination we examine is $\theta_1 = 0.30$ and $\theta_3 = 0.10$, which implies $R^* = 0.44$. This ratio coincides with the observed value at the end of our sample in 1992. This case is important because it shows that current levels of public capital in the U.S. economy can be consistent with optimal fiscal policy, even when θ_3 is as large as 0.10.[15]

The remaining parameters, A_0, A_1, and B, affect the scaling of the model and were calibrated using $\theta_1 = 0.271$, $\theta_3 = 0.123$, and $\delta = 0.10$. The value of $B = 3.76$ implies that household labor supply l_t is approximately equal to 0.3 along the balanced-growth path. Given a time endowment normalized to one, this means that households spend approximately one-third of their discretionary time in market work. The constants $A_0 = 4.36$ and $A_1 = 1.16$ imply that the ratio of private investment to output is 0.15 and the steady-state growth rate of labor productivity is 2.77%. This growth rate coincides with the U.S. average from 1947 to 1969. Our decision to calibrate the growth rate to this 23-year subsample of U.S. data is motivated by our interest in examining the degree to which nonoptimal fiscal policies can account for a productivity slowdown of the magnitude observed in the U.S. economy during the early 1970s.

5. Policy Evaluation

In this section, we examine how well our model can account for the evolution of the stock of public capital relative to private capital in the U.S. economy over the last 70 years. Figure 1 shows the U.S. time series of $R_t = \frac{k_{g,t}}{k_t}$ over the period 1925 to 1992. The series, which is plotted as a dashed line in the figure, grew at a rapid pace throughout the 1930s before experiencing a temporary acceleration during World War II. After the

[15] If consumer durables are included in k_t, then the ratio $R_t = \frac{k_{g,t}}{k_t}$ in 1992 is 0.37. In this case, $\theta_1 = 0.31$ and θ_3=0.09 imply $R^* = 0.37$ in the calibration.

war, the ratio declined for a few years and then settled into a long, slow growth period that peaked in the mid-1960s. Over the last 30 years, the ratio has displayed a generally declining trend.

For comparison, figure 1 also plots the optimal transition paths implied by our model for three different parameter settings. In general, the model predicts a rapid initial growth in the ratio of public to private capital, followed by a leveling off as the economy converges to the balanced growth ratio R^*. Although the U.S. data do not display this monotonicity, the model's optimal transition path with $\theta_3 = 0.123$ and $\delta = .10$ is generally consistent with the data up until about the mid-1960s, particularly if one views the war years as being influenced by a temporary shock. When $\theta_3 = 0.10$ and $\delta = .10$, the optimal transition path lies below the U.S. data for most of the sample period.

Figure 1 also shows the optimal transition path when $\theta_3 = 0.123$ and $\delta = 1.0$. Looking to the far right of the figure, we see that δ has a quantitatively small impact on the balanced growth ratio R^*.[16] Although δ has little effect on R^*, it strongly influences the length of time needed for the transition. As one would expect, higher levels of δ lead to more rapid transitions. When $\delta = 1.0$, the transition occurs in a single jump after the initial period. This illustrates a limitation of the Glomm and Ravikumar (1994) model for analyzing transitional dynamics. In the policy analysis that follows, we restrict our attention to the case of $\delta = 0.10$, since this yields a reasonable transition path in comparison to U.S. data.

5.1. Optimal Policy and the Recent Decline in Public Capital

In recent years, many policymakers and researchers have voiced concern that the decline in the ratio of public to private capital over the last 30 years is evidence that

[16]It can be shown that $\frac{\partial R^*}{\partial \delta} > 0$.

the United States has been underinvesting in public capital.[17] However, figure 1 shows that this conclusion does not necessarily follow. In particular, a declining ratio of public to private capital can be consistent with optimal fiscal policy, even when public capital contributes in a significant way to private output. When $\theta_3 = 0.10$, the optimal transition path in figure 1 lies below the U.S. time series of $R_t = \frac{k_{g,t}}{k_t}$ over the postwar period. Thus, a decline in the U.S. ratio over this period might be interpreted as bringing the economy closer to the optimal balanced growth ratio R^*.

To explore the robustness of this result, consider figure 2, which shows the effect of varying θ_3 on R^*. As public capital becomes more productive (θ_3 increases), the optimal ratio R^* along the balanced-growth path increases rapidly. Figure 2 shows that when $0 \leq \theta_3 \leq .10$, then $R^* \leq 0.44$. Note that 0.44 is the ratio observed at the end of the sample in 1992. Thus, when $0 \leq \theta_3 \leq .10$, the model implies that an increase in the ratio of public to private capital from current levels is not called for. However, if $\theta_3 > 0.10$, then figure 2 shows that $R^* > 0.44$. In this case, the model implies that the ratio of public to private capital should be increased.

It is important to note that our analysis does not resolve the debate over whether the U.S. economy is underinvested in public capital because the optimal ratio R^* depends crucially on the size of θ_3, which is the subject of much uncertainty. However, our model identifies some middle ground that neither side of the public-capital debate has formally recognized. Proponents of expanding public investment tend to make their case using empirical evidence that shows $\theta_3 > 0$. This result, together with the observation that the ratio of public to private capital has been declining over time, is often cited as evidence of nonoptimal fiscal policy. In contrast, opponents of expanding public investment tend to make their case by testing $H_0 : \theta_3 = 0$. Our analysis shows that this condition is much stronger than is needed to establish that the data do not call for an

[17] See, for example, *Economic Report of the President*, 1994, p.43.

increase in public investment. Even with θ_3 as high as 0.10, our model suggests that the decline in the U.S. ratio of public to private capital over the last 30 years is no cause for concern. This argument is made even stronger by the fact that empirical estimates of θ_3 tend to be very imprecise. For example, Finn (1993) estimates the output elasticity of public highway capital to be 0.16. However, the 95 percent confidence interval on this estimate ranges from a low of 0.001 to a high of 0.32. Thus, even for relatively large point estimates of θ_3, the data do not necessarily imply that the "true" value of θ_3 would call for an increase in public investment.

5.2. Public Capital and the Productivity Slowdown

The debate on the productive effects of public capital is often linked to discussions regarding the slowdown in the growth rate of U.S. labor productivity that began in the early 1970s. Some researchers argue that underinvestment in public capital is at least partially responsible for the slowdown.[18] In this section, we take up this issue by examining how some hypothetical nonoptimal fiscal policies can affect the growth rate of labor productivity within the context of our model.

For our first experiment, we investigate the consequences of a nonoptimal public investment policy. In the previous section, we pointed out that the observed decline in the U.S. ratio of public to private capital can be reconciled with optimal fiscal policy when $\theta_3 \leq 0.10$. However, if the optimal transition path from 1925 to 1992 is more appropriately described by the case with $\theta_3 = 0.123$ in figure 1, then the recent decline in the U.S. ratio would not be optimal. For this experiment, we adopt the latter view and set $\theta_3 = 0.123$ (and $\theta_1 = .277$), which implies $R^* = .60$. Next, as an input to the model, we construct an exogenous series for public investment, $i_{g,t}$, such that the resulting time path for $R_t = \frac{k_{g,t}}{k_t}$ coincides with the path observed in the U.S. economy

[18]See, for example, Aschauer (1993) and Munnell (1990).

from 1947 to 1992. The constructed series for R_t is shown as a crossed line in figure 3a, while the solid line shows the optimal transition path computed earlier in figure 1. Since the constructed R_t lies below the optimal transition path leading to $R^* = 0.60$, and tends to move further away over time, we interpret this experiment as capturing the type of nonoptimal public investment policy that is often cited as a possible cause of the U.S. productivity slowdown.

For this experiment, the tax rate is held constant at the optimal level implied by (11), and nonproductive government expenditures, g_t, are determined as a residual such that the government's budget constraint (8) is satisfied each period.[19] In this way, we isolate the effect of a declining public capital ratio on productivity, holding other important policy variables, such as tax rates, constant. Finally, we assume that the private sector reacts optimally to government policy, according to the decision rules (5), (6), and (7).

Figure 3b displays the results of this experiment. The crossed line shows the growth trend of labor productivity in the model, given the nonoptimal public investment policy. The solid line shows labor productivity when public investment policy is optimal, that is, when R_t follows the optimal transition path leading to R^*. The dashed line shows U.S. labor productivity from 1947 to 1992. In comparison to the optimal policy case, the nonoptimal policy produces a mild productivity slowdown beginning around 1970. Notice, however, that this slowdown is much less pronounced than the one observed for the U.S. economy. This experiment shows that a nonoptimal public investment policy of the type that might be interpreted as reflecting U.S. experience over the last 30 years can account for only a small portion of the productivity slowdown. This suggests that other forces may have contributed to the slowdown. One alternative, which can be

[19]The optimal tax rate τ^* is computed from (11) using $\theta_3 = 0.123$, $\delta = 0.10$, and $\phi = 0.17$. Nonproductive expenditures are then given by $g_t = \tau^* y_t - i_{g,t}$. Since g_t is determined as a residual for this experiment, the ratio $\frac{g_t}{y_t}$ is no longer constant.

investigated using the same methodology, represents another type of nonoptimal fiscal policy, namely, increasing tax rates.

For the second experiment, we introduce an exogenous series of tax rates, τ_t, that coincides with an average tax rate series for the U.S. economy.[20] Because this series is not constant, but displays an increasing trend over time, we interpret it as nonoptimal. To isolate the effect of this nonoptimal tax policy, we construct an exogenous series for public investment, $i_{g,t}$, such that the resulting series for $R_t = \frac{k_{g,t}}{k_t}$ generated by the model follows the optimal transition path leading to $R^* = 0.60$. As before, the private sector reacts optimally and the level of nonproductive expenditures g_t is determined as a residual such that the government budget constraint is satisfied each period.

The results of the second experiment are displayed in figures 4a and 4b. Figure 4b shows that a policy of nonoptimal tax rates can also generate a productivity slowdown. The slowdown is much more severe than in the first experiment, however. The key difference between the two exercises is that in the first experiment, the government misallocates resources between $i_{g,t}$ and g_t, while tax revenue as a fraction of total output remains constant. In the second experiment, the share of total resources claimed by the government increases over time.

Figure 4b shows that labor productivity in the model displays an abrupt change in trend around 1970 that is strikingly similar to the trend shift in U.S. labor productivity that occurred at about the same time. The cause of this trend shift in the model can be traced to the period of sharply increasing average tax rates in the late 1960s and early 1970s (see figure 4a). This experiment shows that the existence of a productivity slowdown need not imply that public investment policy is nonoptimal.

[20] We computed the average tax rate series for the U.S. economy by dividing total federal, state, and local government receipts for each year (Citibase series GGFR+GGSR+GGFSIN+GGSSIN) by GDP. This approach yields an average tax rate that is roughly consistent with the model's use of a production tax to finance all government expenditures. The resulting tax rate series displays an upward trend which is very similar to that observed for the average marginal tax rate on labor income estimated by Barro and Sahasakul (1986).

As a final experiment, we introduce both types of nonoptimal fiscal policy into the model. The results of this exercise are summarized in figures 5a and 5b. As one might expect, the productivity slowdown in the model now becomes even more severe. This occurs because an increasing fraction of total resources are now being devoted to nonproductive public expenditures g_t. Interestingly, the simulated productivity trend from the model provides a very close match to the U.S. productivity trend. Table 1 provides a quantitative comparison of the productivity effects in each of the three policy experiments.

To summarize, our experiments show that a nonoptimal public investment policy does not, by itself, provide a convincing explanation for the U.S. productivity slowdown. However, it may have been a contributing factor, together with the trend toward increasing tax rates. Finally, we note that many other explanations have been put forth to help explain the U.S. productivity slowdown. Some of the alternative hypotheses include: (1) a return to "normal" productivity growth from the unsustainably high growth rates experienced after the Great Depression and World War II; (2) changes in demographic factors that have tended to reduce the quality of the labor force; (3) a fall-off in the rate of research and development spending; (4) increased costs of complying with government regulations (such as mandated pollution control expenditures); and (5) increases in energy costs due to oil price shocks.[21]

6. Conclusion

This paper showed that optimal transitional dynamics in a simple endogenous growth model can account for much of the behavior of the stock of public capital in the U.S. economy over the last 70 years. Moreover, we showed that the observed decline in

[21] See Munnell (1990), Tatom (1991), Aschauer (1993), and the references cited therein for a more detailed discussion of these alternative hypotheses.

the U.S. ratio of public to private capital since the mid-1960s might be interpreted as a movement toward the optimal balanced growth ratio, even for output elasticities as high as 0.10. Finally, we found that a nonoptimal public investment policy of the type consistent with U.S. data does not have much impact on the growth rate of labor productivity in our model, suggesting that other explanations for the U.S. productivity slowdown should be considered.

References

Aaron, Henry J., 1990, Discussion of 'Why is infrastructure important?', in A.H. Munnell, ed., Is There a Shortfall in Public Capital Investment?, Proceedings of a conference sponsored by the Federal Reserve Bank of Boston.

Ai, Chunrong and Steven P. Cassou, 1995, A normative analysis of public capital, forthcoming in Applied Economics.

Arrow, Kenneth J. and Mordecai Kurz, 1970, Public investment, the rate of return, and optimal fiscal policy (Baltimore: Johns Hopkins Press).

Aschauer, David A., 1989, Is public expenditure productive?, Journal of Monetary Economics 23, 177-200.

———, 1993, Public capital and economic growth, Jerome Levy Economics Institute, Public Policy Brief No. 4.

Barro, Robert J., 1979, On the determination of public debt, Journal of Political Economy 87, 940-971.

———, 1990, Government spending in a simple model of endogenous growth, Journal of Political Economy 98, S103-125.

Barro, Robert J. and Chaipat Sahasakul, 1986, Average marginal tax rates from social security and the individual income tax, Journal of Business 59, 555-566.

Barro, Robert J. and Xavier Sala-i-Martin, 1992, Public finance in models of economic growth, Review of Economic Studies 89, 645-661.

Cassou, Steven P., 1995, Optimal tax rules in a dynamic stochastic economy with capital, Journal of Economic Dynamics and Control 19, 1165-1197.

Chari, V.V., Lawrence J. Christiano, and Patrick J. Kehoe, 1994, Optimal fiscal policy in a business cycle model, Journal of Political Economy 102, 617-652.

Christiano, Lawrence J., 1988, Why does inventory investment fluctuate so much?, Journal of Monetary Economics 21, 247-280.

Finn, Mary, 1993, Is all government capital productive?, Federal Reserve Bank of Richmond, Economic Quarterly 79, 53-80.

Glomm, Gerhard and B. Ravikumar, 1994, Public investment in infrastructure in a simple growth model, Journal of Economic Dynamics and Control 18, 1173-1187.

Greenwood, Jeremy, Zvi Hercowitz, and Gregory W. Huffman, 1988, Investment, capacity utilization, and the real business cycle, American Economic Review 78, 402-417.

Greenwood, Jeremy, Richard Rogerson, and Randall Wright, 1994, Household production in real business cycle theory, in T. F. Cooley, ed., Frontiers of Business Cycle Research (Princeton, NJ: Princeton University Press).

Hercowitz, Zvi, and Michael Sampson, 1991, Output growth, the real wage, and employment fluctuations, American Economic Review 81, 1215-1237.

Holtz-Eakin, Douglas, 1992, Public-sector capital and the productivity puzzle, NBER working paper 4122.

Jones, Larry E., Rodolfo Manuelli, and Peter E. Rossi, 1993, Optimal taxation in models of endogenous growth, Journal of Political Economy 101, 485-517.

Jorgenson, Dale and Martin A. Sullivan, 1981, Inflation and corporate capital recovery, in C.R. Hulten, ed., Depreciation, inflation, and the taxation of income from capital (Washington, DC: Urban Institute Press).

King, Robert G., Charles I. Plosser, and Sergio T. Rebelo, 1988, Production, growth and business cycles I. The basic neoclassical model, Journal of Monetary Economics 21, 195-232.

King, Robert G. and Sergio T. Rebelo, 1993, Transitional dynamics and economic growth in the neoclassical model, American Economic Review 83, 908-931.

Kocherlakota, Narayana R. and Kei-Mu Yi, 1995, A simple time series test of endogenous vs. exogenous growth models: An application to the United States, Review of Economics and Statistics, forthcoming.

Kydland, Finn E. and Edward C. Prescott, 1977, Rules rather than discretion: The inconsistency of optimal plans, Journal of Political Economy 85, 473-491.

Lucas, Robert E., Jr., 1990, Supply side economics: An analytical review, Oxford Economic Papers 42, 293-316.

Lucas, Robert E., Jr. and Edward C. Prescott, 1971, Investment under uncertainty, Econometrica 39, 659-681.

Lucas, Robert E., Jr. and Nancy L. Stokey, 1983, Optimal fiscal and monetary policy in an economy without capital, Journal of Monetary Economics 12, 55-94.

Lynde, Catherine and James Richmond, 1992, The role of public capital in production, Review of Economics and Statistics 74, 37-44.

Mulligan, Casey B. and Xavier Sala-i-Martin, 1992, Transitional dynamics in two-sector models of endogenous growth, NBER working paper 3986.

Munnell, Alicia H., 1990, Why has productivity growth declined?: Productivity and public investment, New England Economic Review, January/February 1990, 3-22.

Ramsey, Frank P., 1927, A contribution to the theory of taxation, Economic Journal 37, 47-61.

Romer, Paul, 1986, Increasing returns and long-run growth, Journal of Political Economy 94, 1002-1037.

Tatom, John A., 1991, Public capital and private sector performance, Federal Reserve Bank of St. Louis, Review, May/June, 3-15.

Zhu, Xiaodong, 1992, Optimal fiscal policy in a stochastic growth model, Journal of Economic Theory 58, 250-289.

Table 1: Annual Productivity Growth Rates

	1947-1969	1970-1992
U.S. Data	2.77%	0.62%
Experiment #1:		
Suboptimal Public Investment	2.66%	2.30%
Experiment #2:		
Suboptimal Tax Policy	2.68%	1.16%
Experiment #3:		
Joint Suboptimal Policy	2.57%	0.66%

Source: Authors' calculations.

FIG 1: RATIO OF PUBLIC TO PRIVATE CAPITAL

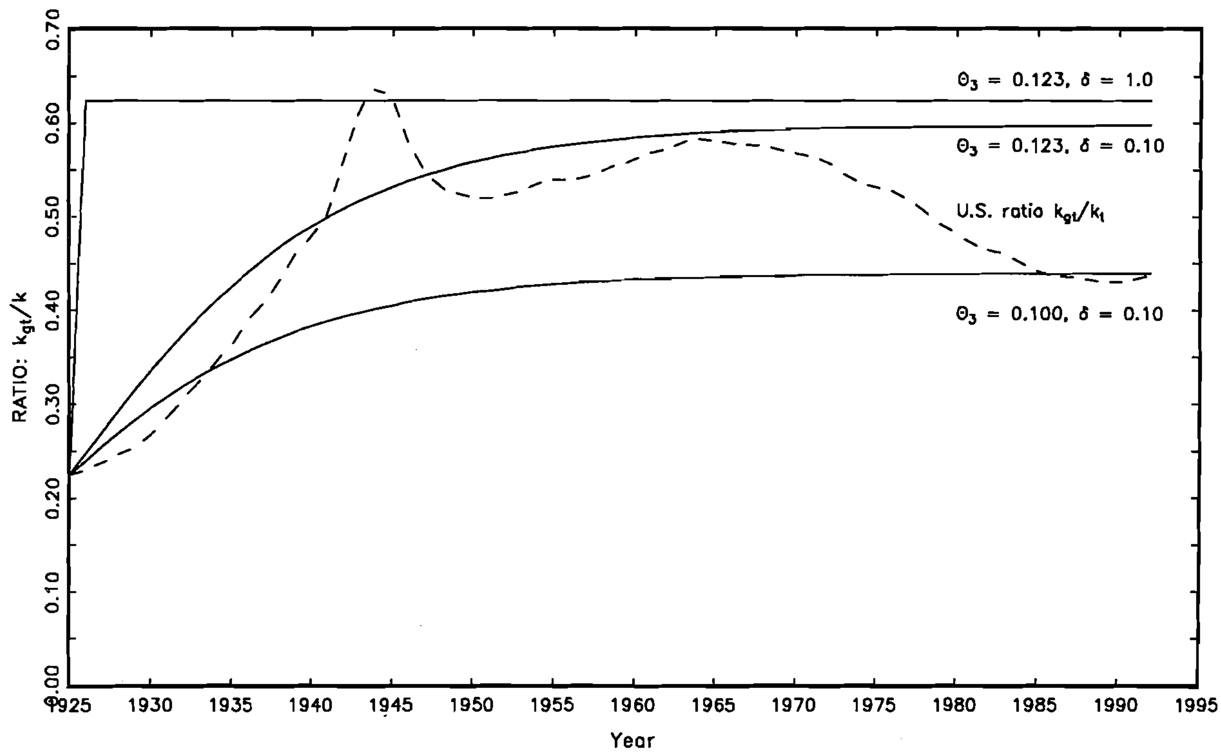
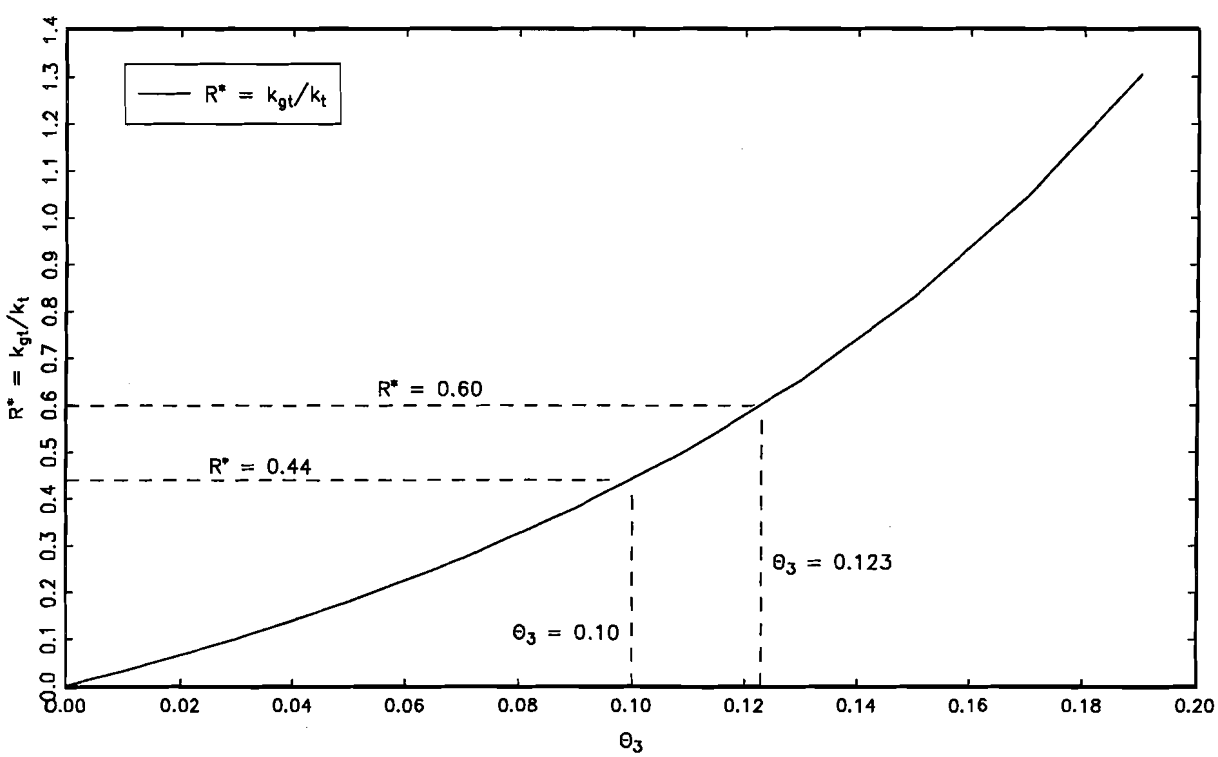


Figure 2: R* vs OUTPUT ELASTICITY OF PUBLIC CAPITAL



SOURCE: Authors' calculations.

clevelandfed.org/research/workpaper/1995/wp9509.pdf

Figure 3a: EXPERIMENT: A DECLINING PUBLIC CAPITAL RATIO

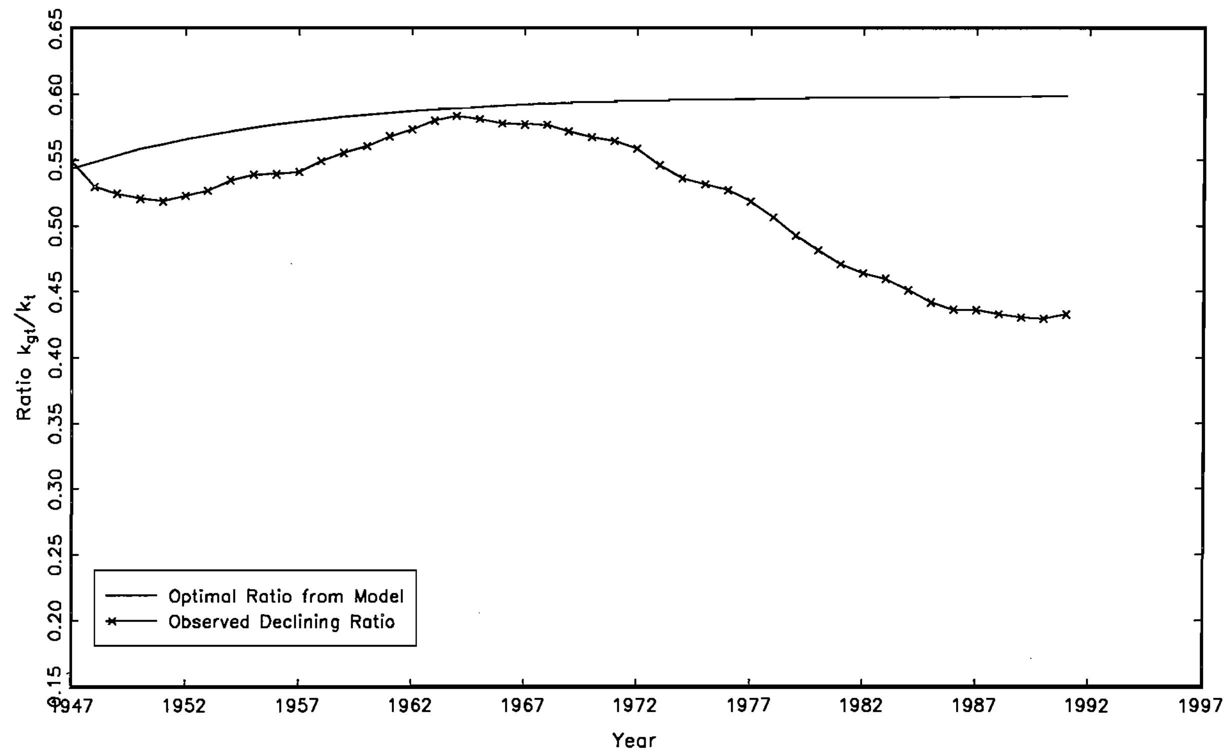
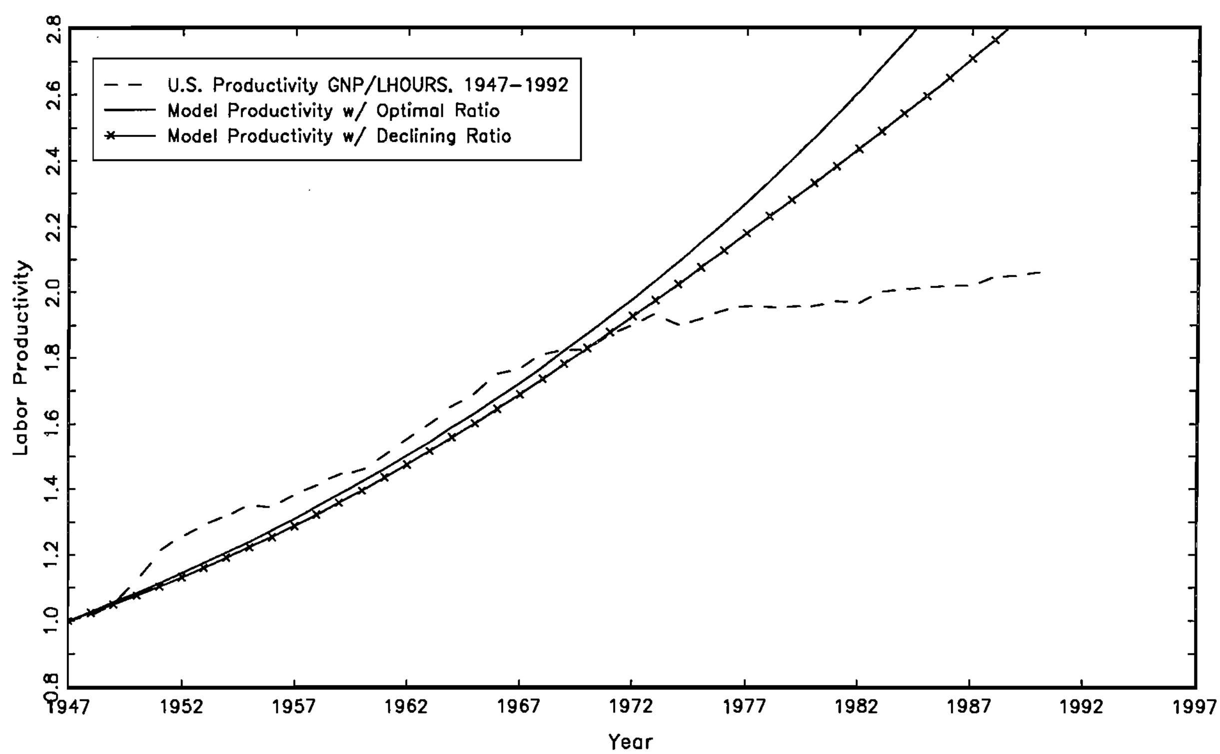


Figure 3b: EFFECT ON LABOR PRODUCTIVITY



SOURCE: Authors' calculations.

Figure 4a: EXPERIMENT: AN INCREASING TAX RATE

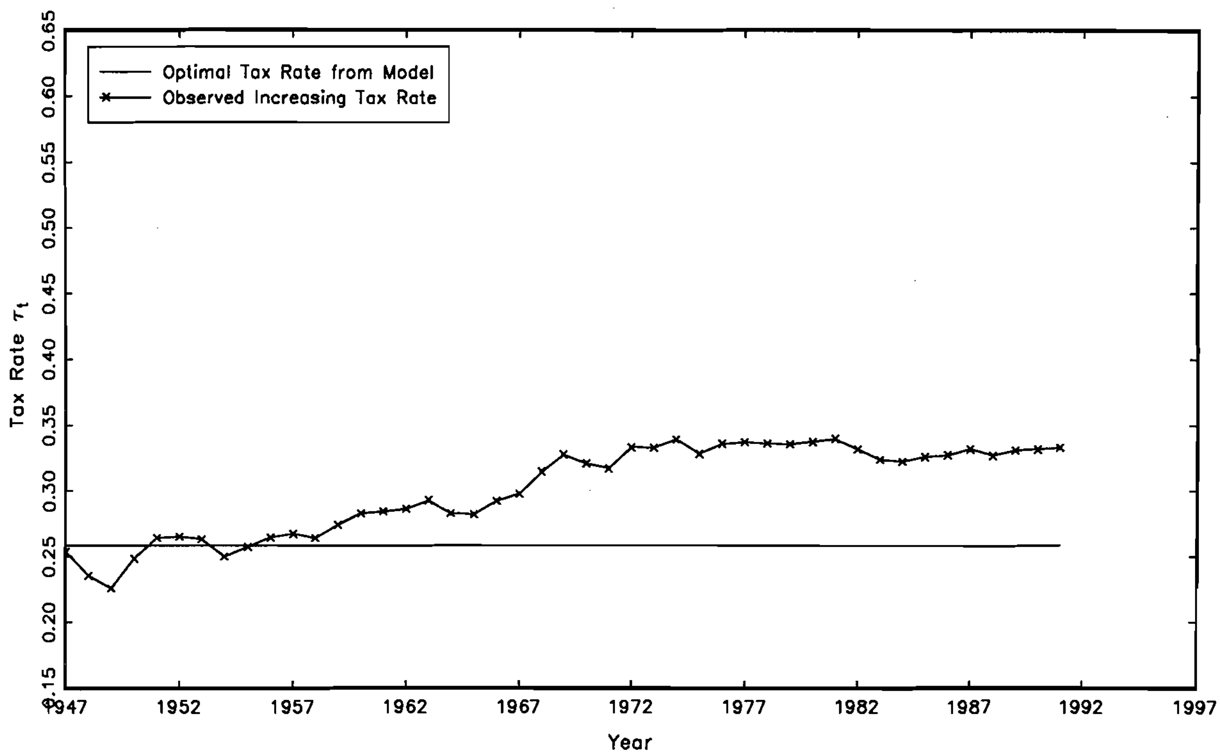
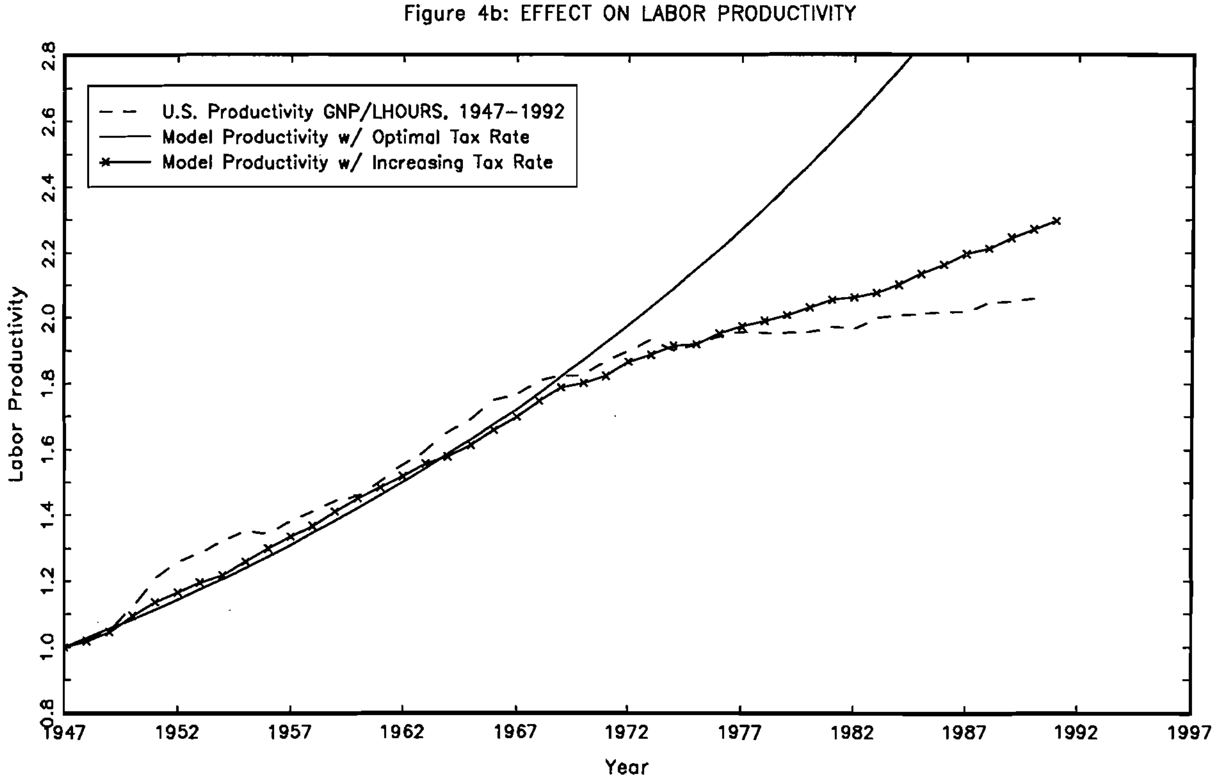


Figure 4b: EFFECT ON LABOR PRODUCTIVITY



SOURCE: Authors' calculations.

Figure 5a: EXPERIMENT: A DECLINING RATIO AND AN INCREASING TAX RATE

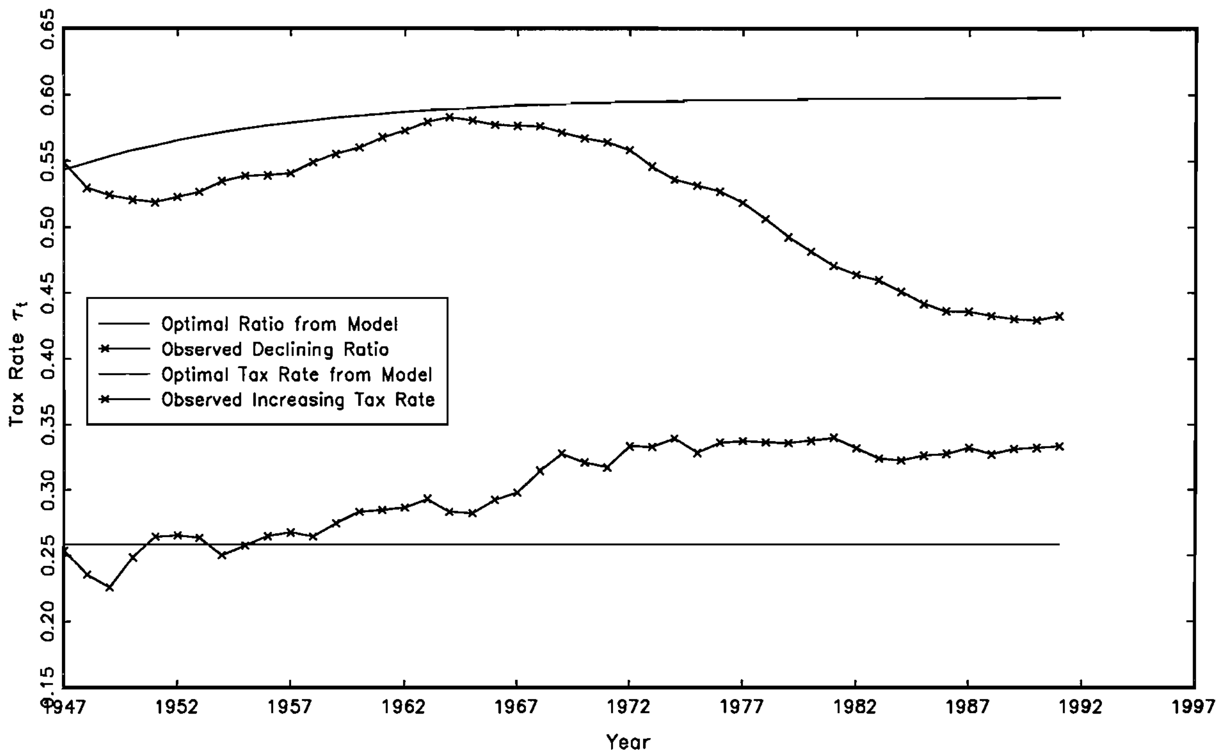
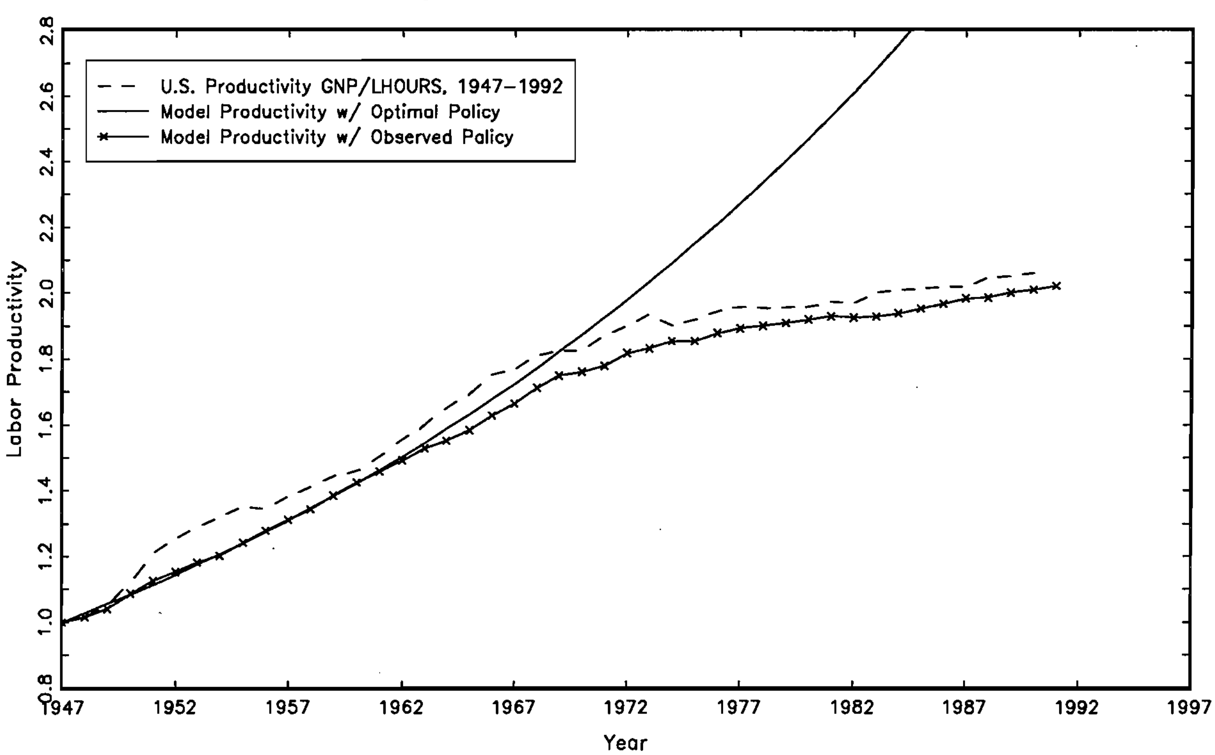


Figure 5b: EFFECT ON LABOR PRODUCTIVITY



SOURCE: Authors' calculations.

Appendix

Firm Optimization

The firm's optimization problem is straightforward. They choose l_t and k_t to maximize

$$A_0 k_t^{\theta_1}(h_t l_t)^{\theta_2} k_{g,t}^{\theta_3} \; - \; w_t l_t \; - \; r_t k_t.$$

This implies

$$r_t \; = \; \frac{\theta_1 y_t}{k_t} \quad w_t \; = \; \frac{\theta_2 y_t}{l_t}$$

and

$$\pi_t \; = \; y_t \; - \; \theta_2 y_t \; - \; \theta_1 y_t \; = \; (1-\theta_1-\theta_2) y_t.$$

Household Optimization

Write the problem as choosing $\{c_t, l_t, i_t, k_{t+1} : t \geq 0\}$ to maximize

$$\sum_{t=0}^{\infty} \beta^t \log(c_t - B h_t l_t^{\gamma}) \tag{3}$$

subject to

$$c_t \; + \; i_t \; = \; (1-\tau_t)(w_t l_t \; + \; r_t k_t \; + \; \pi_t),$$

$$k_{t+1} \; = \; A_1 i_t^{\delta} k_t^{1-\delta}. \tag{4}$$

Using the results from the firm's optimization problem and the production function, the Lagrangian for this problem can be written as

$$L(\cdot) \; = \; \sum_{t=0}^{\infty} \beta^t \left\{ \log\left(c_t \; - \; B h_t l_t^{\gamma}\right) \; + \; \lambda_t \left[(1-\tau_t)(w_t l_t + r_t k_t + \pi_t) \; - \; c_t \; - \; A_1^{\frac{-1}{\delta}} k_{t+1}^{\frac{1}{\delta}} k_t^{\frac{\delta-1}{\delta}} \right] \right\}.$$

The first-order conditions are

$$\frac{\partial L(\cdot)}{\partial c_t} = \frac{1}{c_t - Bh_t l_t^{\gamma}} - \lambda_t = 0, \tag{14}$$

$$\frac{\partial L(\cdot)}{\partial l_t} = \frac{-1}{c_t - Bh_t l_t^{\gamma}}(\gamma B h_t l_t^{\gamma-1}) + \lambda_t(1-\tau_t)w_t = 0, \tag{15}$$

$$\frac{\partial L(\cdot)}{\partial k_{t+1}} = \lambda_t\left(\frac{-1}{\delta}\right)\left(\frac{i_t}{k_{t+1}}\right) + \beta\lambda_{t+1}\left[(1-\tau_{t+1})r_{t+1} - \left(\frac{\delta-1}{\delta}\right)\left(\frac{i_{t+1}}{k_{t+1}}\right)\right] = 0, \tag{16}$$

$$\frac{\partial L(\cdot)}{\partial \lambda_t} = (1-\tau_t)(w_t l_t + r_t k_t + \pi_t) - c_t - i_t = 0. \tag{17}$$

Substituting (14) into (15) and using $r_t = \frac{\theta_1 y_t}{k_t}$ and $w_t = \frac{\theta_2 y_t}{l_t}$ yields

$$\gamma B h_t l_t^{\gamma} = (1-\tau_t)\theta_2 y_t. \tag{18}$$

Using (1) and (2) and solving (18) for l_t yields

$$l_t = \left[(1-\tau_t)\left(\frac{\theta_2 A_0}{\gamma B}\right)\left(\frac{k_{g,t}}{k_t}\right)^{\theta_3}\right]^{\frac{1}{\gamma-\theta_2}}.$$

To find the other decision rules we use the method of undetermined coefficients. We guess the functional forms

$$i_t = a_0(1-\tau_t)y_t \tag{19}$$

$$\frac{1}{\lambda_t} = b_0(1-\tau_t)y_t, \tag{20}$$

where a_0 and b_0 are constants to be determined. Substituting these into (16) and solving for a_0 gives

$$a_0 = \frac{\beta\delta\theta_1}{1-\beta(1-\delta)}.$$

We also need to verify that our guess was correct by verifying that b_0 is, in fact, a constant. To do this, we use (14) and (20) to obtain

$$c_t = b_0(1-\tau_t)y_t + Bh_t l_t^\gamma. \tag{21}$$

Substituting this expression into (17) and using (18), we can solve for b_0:

$$b_0 = (1 - a_0 - \frac{\theta_2}{\gamma}).$$

Substituting the expression for b_0 into (21) and combining with (18) gives

$$c_t = (1-a_0)(1-\tau_t)y_t.$$

We can interpret a_0 as the marginal propensity to save out of after-tax income.

Government Optimization

The government views g_t as exogenous and does not include it as part of its optimization decision. The government problem is to choose $\{\tau_t, i_{g,t}, k_{g,t+t}, c_t, l_t, i_t, k_{t+1} : t \geq 0\}$ to maximize

$$\sum_{t=0}^{\infty} \beta^t \log(c_t - Bh_t l_t^\gamma) \tag{3}$$

subject to

$$k_{t+1} = A_1 i_t^\delta k_t^{1-\delta}. \tag{4}$$

$$c_t = (1-a_0)(1-\tau_t)y_t, \tag{5}$$

$$i_t = a_0(1-\tau_t)y_t, \tag{6}$$

$$l_t = \left[(1-\tau_t)\left(\frac{\theta_2 A_0}{\gamma B}\right)\left(\frac{k_{g,t}}{k_t}\right)^{\theta_3}\right]^{\frac{1}{\gamma-\theta_2}}, \tag{7}$$

$$i_{g,t} + g_t = \tau_t y_t, \tag{8}$$

$$k_{g,t+1} = A_1 i_{g,t}^{\delta} k_{g,t}^{1-\delta}. \tag{9}$$

It will be useful to reduce the number of constraints by eliminating some of the variables. We begin by writing (6) in two forms:

$$(1-\tau_t) = \frac{i_t}{a_0 y_t}, \tag{22}$$

$$\tau_t = 1 - \frac{i_t}{a_0 y_t}. \tag{23}$$

Substitute (22) into (7) to eliminate τ_t, yielding

$$l_t = \left[\left(\frac{i_t}{a_0 y_t}\right)\left(\frac{\theta_2 A_0}{\gamma B}\right)\left(\frac{k_{g,t}}{k_t}\right)^{\theta_3}\right]^{\frac{1}{\gamma-\theta_2}}.$$

Next, substitute this expression for l_t, together with (4), into (1) to obtain

$$y_t = d_0 k_t^{e_0} k_{t+1}^{\frac{\theta_2}{\delta\gamma}} k_{g,t}^{\theta_3} \tag{24}$$

where

$$d_0 = A_0 \left(\frac{\theta_2}{a_0 \gamma B}\right)^{\frac{\theta_2}{\gamma}} A_1^{\frac{-\theta_2}{\delta\gamma}},$$

$$e_0 = (\theta_1 + \theta_2) - \frac{\theta_2}{\gamma\delta}.$$

The next step employs (8), (23), (4), (9), and (24) to obtain the following version of the government budget constraint:

$$d_0 k_t^{e_0} k_{t+1}^{\frac{\theta_2}{\delta\gamma}} k_{g,t}^{\theta_3} \; - \; \frac{A_1^{\frac{-1}{\delta}}}{a_0} k_{t+1}^{\frac{1}{\delta}} k_t^{\frac{\delta-1}{\delta}} \; - \; A_1^{\frac{-1}{\delta}} k_{g,t+1}^{\frac{1}{\delta}} k_{g,t}^{\frac{\delta-1}{\delta}} \; - \; g_t \; = \; 0,$$

where g_t is viewed as exogenous by the policymaker. An expression for the argument of the household utility function can be obtained using (5), (20), (22), and (4). The result is

$$c_t \; - \; Bh_t l_t^{\gamma} \; = \; \frac{b_0}{a_0} i_t \; = \; \frac{b_0}{a_0} A_1^{\frac{-1}{\delta}} k_{t+1}^{\frac{1}{\delta}} k_t^{\frac{\delta-1}{\delta}}. \tag{25}$$

We now can write the Lagrangian for the government problem as

$$\begin{aligned} L_g(\cdot) \; = \; & \sum_{t=0}^{\infty} \beta^t \left[\log \left(\frac{b_0}{a_0} A_1^{\frac{-1}{\delta}} k_{t+1}^{\frac{1}{\delta}} k_t^{\frac{\delta-1}{\delta}} \right) \right. \\ & \left. -\lambda_{g,t} \left(d_0 k_t^{e_0} k_{t+1}^{\frac{\theta_2}{\delta\gamma}} k_{g,t}^{\theta_3} \; - \; \frac{A_1^{\frac{-1}{\delta}}}{a_0} k_{t+1}^{\frac{1}{\delta}} k_t^{\frac{\delta-1}{\delta}} \; - \; A_1^{\frac{-1}{\delta}} k_{g,t+1}^{\frac{1}{\delta}} k_{g,t}^{\frac{\delta-1}{\delta}} \; - \; g_t \right) \right]. \end{aligned}$$

The first-order conditions are

$$\begin{aligned} \frac{\partial L_g(\cdot)}{\partial k_{t+1}} \; = \; & \frac{1}{\delta k_{t+1}} \; + \; \frac{\beta(\delta-1)}{\delta k_{t+1}} \; - \; \lambda_{g,t} \left[\left(\frac{\theta_2}{\delta\gamma} \right) \frac{y_t}{k_{t+1}} \; - \; \left(\frac{1}{a_0\delta} \right) \frac{i_t}{k_{t+1}} \right] \; - \\ & \beta\lambda_{g,t+1} \left[e_0 \frac{y_{t+1}}{k_{t+1}} \; - \; \left(\frac{\delta-1}{a_0\delta} \right) \frac{i_{t+1}}{k_{t+1}} \right] \; = \; 0, \end{aligned} \tag{26}$$

$$\frac{\partial L_g(\cdot)}{\partial k_{g,t+1}} \; = \; \lambda_{g,t} \left[\left(\frac{1}{\delta} \right) \frac{i_{g,t}}{k_{g,t+1}} \right] \; - \; \beta\lambda_{g,t+1} \left[\theta_3 \frac{y_{t+1}}{k_{g,t+1}} \; - \; \left(\frac{\delta-1}{\delta} \right) \frac{i_{g,t+1}}{k_{g,t+1}} \right] \; = \; 0, \tag{27}$$

$$\frac{\partial L_g(\cdot)}{\partial \lambda_{g,t}} \; = \; d_0 k_t^{e_0} k_{t+1}^{\frac{\theta_2}{\delta\gamma}} k_{g,t}^{\theta_3} \; - \; \frac{A_1^{\frac{-1}{\delta}}}{a_0} k_{t+1}^{\frac{1}{\delta}} k_t^{\frac{\delta-1}{\delta}} \; - \; A_1^{\frac{-1}{\delta}} k_{g,t+1}^{\frac{1}{\delta}} k_{g,t}^{\frac{\delta-1}{\delta}} \; - \; g_t \; = \; 0. \tag{28}$$

To find the decision rules, we again use the method of undetermined coefficients. We guess the functional forms

$$i_{g,t} = a_1 y_t \tag{29}$$

$$\frac{1}{\lambda_{g,t}} = b_1 y_t, \tag{30}$$

where a_1 and b_1 are constants to be determined. Substituting these expressions into (27) and solving for a_1 gives

$$a_1 = \frac{\beta\delta\theta_3}{1-\beta(1-\delta)}.$$

To find the optimal decision rule for τ_t, we substitute the expression for a_1 into (8) and make use of $g_t = \phi y_t$ to obtain

$$\tau_t = a_1 + \phi.$$

We verify that (29) and (30) are correct by showing that b_1 is in fact a constant. To do this, we substitute the optimal tax rate into (6) to obtain

$$i_t = a_0(1 - a_1 - \phi)y_t.$$

Substituting this expression, together with (29) and (30), into (26) and solving for b_1 yields

$$b_1 = \frac{\frac{\theta_2}{\gamma} + \beta e_0\delta - (1-a_1-\phi)[1-\beta(1-\delta)]}{1-\beta(1-\delta)}.$$

Since this is constant, our guess is confirmed.

Derivation of the Equilibrium Laws of Motion for Private and Public Capital

The laws of motion (12) and (13) are relatively straightforward to derive. The only tricky part is to first obtain an alternate expression for the production function.

Substituting (7) and (2) into (1) gives

$$y_t \;=\; A_0\left[(1-\tau_t)\left(\frac{\theta_2 A_0}{\gamma B}\right)\right]^{\frac{\theta_2}{\gamma-\theta_2}} k_t^{\frac{\gamma(\theta_1+\theta_2)-\theta_2}{\gamma-\theta_2}} k_{g,t}^{\frac{\gamma\theta_3}{\gamma-\theta_2}}. \tag{31}$$

Substituting (6), (11), and (31) into (4) and rearranging gives

$$k_{t+1} \;=\; A_1\left\{A_0 a_0\left[(1-a_1-\phi)^{\gamma}\left(\frac{\theta_2 A_0}{\gamma B}\right)^{\theta_2}\right]^{\frac{1}{\gamma-\theta_2}}\right\}^{\delta} k_t^{\frac{\delta\gamma(\theta_1+\theta_2)-\delta\theta_2}{\gamma-\theta_2}+(1-\delta)} k_{g,t}^{\frac{\delta\gamma\theta_3}{\gamma-\theta_2}}. \tag{12}$$

Similarly, substituting (29) and (31) into (9) and rearranging gives

$$k_{g,t+1} \;=\; A_1\left\{A_0 a_1\left[(1-a_1-\phi)^{\theta_2}\left(\frac{\theta_2 A_0}{\gamma B}\right)^{\theta_2}\right]^{\frac{1}{\gamma-\theta_2}}\right\}^{\delta} k_t^{\frac{\delta\gamma(\theta_1+\theta_2)-\delta\theta_2}{\gamma-\theta_2}} k_{g,t}^{\frac{\delta\gamma\theta_3}{\gamma-\theta_2}+(1-\delta)}. \tag{13}$$

Making use of (12) and R^* yields the following expression for the per capita growth rate:

$$\frac{k_{t+1}}{k_t} \;=\; \frac{k_{g,t+1}}{k_{g,t}} \;=\; \frac{y_{t+1}}{y_t} \;=\; A_1 a_0^{\delta}\left[A_0(1-a_1-\phi)^{\theta_1+\theta_2}\left(\frac{\theta_2}{\gamma B}\right)^{\frac{\theta_2}{\gamma}}\left(\frac{\theta_3}{\theta_1}\right)^{\theta_3}\right]^{\frac{\delta\gamma}{\gamma-\theta_2}}.$$