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This paper analyzes the restrictions necessary to ensure that the interest rate policy rule used by the central bank does not introduce local real indeterminacy into the economy. It conducts the analysis in a Calvo-style sticky price model. A key innovation is to add investment spending to the analysis. In this environment, local real indeterminacy is much more likely. In particular, all forward-looking interest rate rules are subject to real indeterminacy.

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I. Introduction.

The celebrated Taylor (1993) rule posits that the central bank uses a fairly simple rule when conducting monetary policy. This rule is a reaction function linking movements in the nominal interest rate to movements in inflation and possibly other endogenous variables. A rule is called active if the elasticity with respect to inflation (denoted by τ) is greater than one; the rule is called passive if τ is less than one. Recently there has been a considerable amount of interest in ensuring that such rules do no harm. The problem is that by following a rule in which the central bank responds to endogenous variables, the central bank may introduce real indeterminacy and sunspot equilibria into an otherwise determinate economy. These sunspot fluctuations are welfare-reducing and can potentially be quite large.

A standard result is that to avoid real indeterminacy the central bank should respond aggressively ($\tau > 1$) to either expected inflation (see Bernanke and Woodford (1997) and Clarida, Gali, Gertler (2000)) or current inflation (see Kerr and King (1996)). We will refer to the former rule as "forward-looking" and the latter as "current-looking". These analyses are all reduced-form sticky price models, where the underlying structural model is a labor-only economy and money is introduced via a money-in-the-utility function (MIUF) model with a zero cross-partial between consumption and real balances (ie., $U_{cm} =0$).[1]

Our analysis differs from those of Bernanke and Woodford (1997), Clarida, Gali, Gertler (2000)), and Kerr and King (1996) on one important dimension, the addition of

[1] Benhabib, Schmitt-Grohe and Uribe (2001a) analyze Taylor rules in a continuous time MIUF environment with an arbitrary cross-partial U_{cm} and demonstrate that the conditions for determinacy

capital and investment spending. This added margin makes determinacy much harder to achieve. In contrast to these papers we demonstrate that essentially all forward-looking rules are subject to local indeterminacy, and that a sufficient condition for local determinacy is for the monetary authority to react aggressively to *current* movements in inflation. These findings are of more than academic interest since several central banks currently use inflation forecasts as an important part of their decision-making on policy issues.

A recent paper by Dupor (2001) analyzes a similar sticky price environment with investment but comes to substantially different policy prescriptions than those presented here. He demonstrates that a passive rule ($\tau < 1$) is necessary and sufficient for local equilibrium determinacy. The essential difference between the two papers is that Dupor utilizes a continuous-time model. A key difference between a discrete-time and continuous-time model is the no-arbitrage relationship between bonds and capital. In discrete time the *future* marginal productivity of capital equals the real interest rate; in continuous time *today's* marginal productivity of capital equals the real interest rate. This contemporaneous condition provides an extra restriction in continuous time. In a model in which the central bank conducts policy with an interest rate instrument, this extra restriction alters the determinacy conditions across the two models.

This paper extends Dupor's (2001) analysis in three ways. First, as noted, we examine a discrete time model and demonstrate that this timing assumption has an important implication on equilibrium determinacy. Second, we consider a more general utility specification in that we make no assumption on the sign of U_{cm}. Finally, the

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depend on the sign of U_{cm}. Their analysis abstracts from investment spending.

discrete-time model in the current paper allows us to examine both current- and forward-looking Taylor rules.

The outline of the paper is as follows. The next section develops the basic model. Section III provides the determinacy analysis. Section IV compares these results to the continuous time analysis of Dupor (2001). Finally section V concludes. An appendix proves the main propositions.

II. A Sticky Price Model

The economy consists of numerous households and firms each of which we will discuss in turn. We are concerned with issues of local determinacy. Hence, without loss of generality we limit the discussion to a deterministic model. As is well known, if the deterministic dynamics are not unique, then it is possible to construct sunspot equilibria in the model economy. Below we will use the terms "real indeterminacy," "local indeterminacy," and "sunspot equilibria" interchangeably.[2]

Households are identical and infinitely-lived with preferences over consumption, real money balances and leisure given by

$$\sum_{t=0}^{\infty} \beta^t U(c_t, M_{t+1}/P_t, 1-L_t),$$

where β is the personal discount rate, c_t is consumption, and $1-L_t$ is leisure. We utilize a MIUF environment because of its generality (see Feenstra (1986)). In contrast to Carlstrom and Fuerst (2001a), we adopt the convention of end-of-period money balances in the utility function. We do this to be consistent with Dupor's continuous time

[2] Benhabib et al. (2001b) conduct a global analysis of equilibrium determinacy in a continuous time model

analysis.[3] For the purposes of this paper this timing issue is of limited importance. The utility function is given by:

$$U(c,m,1\text{-}L) \equiv V(c,m) - L.$$

Most studies investigating the conditions for determinacy have assumed a zero cross-partial between consumption and real balances (ie., $V_{cm} = 0$). In contrast we make no assumption on the nature of V(c,m). The conditions for determinacy are completely independent of V. Of course this generality comes at a price. Most notably we assume an infinite labor supply elasticity. Carlstrom and Fuerst (2000, 2001b) provide a complementary determinacy analysis in a flexible price environment in which the determinacy conditions are independent of labor supply elasticity. We thus restrict our analysis to an infinite labor supply elasticity.

The household begins the period with M_t cash balances and B_{t-1} holdings of nominal bonds. Before proceeding to the goods market, the household visits the financial market where it carries out bond trading and receives a cash transfer of $M_t^s(G_t - 1)$ from the monetary authority where M_t^s denotes the per capita money supply and G_t is the gross money growth rate. After engaging in goods trading, the household ends the period with cash balances given by the intertemporal budget constraint:

$$M_{t+1} = M_t + M_t^s(G_t - 1) + B_{t-1}R_{t-1} - B_t + P_t\{w_t L_t + [r_t + (1-\delta)]K_t\} - P_t c_t - P_t K_{t+1} + \Pi_t \ .$$

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without capital. Local indeterminacy is a sufficient condition for global indeterminacy. Similarly, local determinacy is a necessary condition for global determinacy.

[3] Carlstrom and Fuerst (2001a) refer to this as "cash-when-I'm-done timing" (CWID) as it is assumed that the money balances that aid in transactions are the money balances that the household has after *leaving* the store. The natural alternative is cash-in-advance (CIA) timing. That is, the money balances that aid in transactions are the money balances that the household has upon entering goods market trading so that $A_t = M_t + M_t^s(G_t - 1) + B_{t-1}R_{t-1} - B_t$. As noted by Benhabib et al. (2001a), the discrete time analog to a continuous time MIUF model is CWID timing.

K_t denoes the households accumulated capital stock that earns rental rate r_t and depreciates at rate δ. The real wage is given by w_t while Π_t denotes the profit flow from firms. The first order conditions to the household's problem include the following:

$$\frac{U_L(t)}{U_c(t)} = w_t \tag{1}$$

$$U_c(t) = \beta\{U_c(t+1)[r_{t+1} + (1-\delta)]\} \tag{2}$$

$$\frac{U_c(t)}{P_t} = \beta R_t \left\{\frac{U_c(t+1)}{P_{t+1}}\right\} \tag{3}$$

$$\frac{U_m(t)}{U_c(t)} = \frac{R_t - 1}{R_t} \tag{4}$$

Equation (1) is the familiar labor supply equation, while (2) is the asset accumulation margin. Equation (3) is the Fisherian interest rate determination in which the nominal rate varies with expected inflation and the real rate of interest on bonds. Equation (4) is the model's money demand function.

As for firm behavior, we follow Yun (1996) and utilize a model of imperfect competition in the intermediate goods market. Final goods production in this economy is carried out in a perfectly competitive industry that utilizes intermediate goods in production. The CES production function is given by

$$Y_t = \{\int_0^1 [y_t(i)^{(\eta-1)/\eta}]di\}^{\eta/(\eta-1)}$$

where Y_t denotes the final good, and $y_t(i)$ denotes the continuum of intermediate goods, each indexed by $i \in [0,1]$. The implied demand for the intermediate good is thus given by

$$y_t(i) = Y_t\left[\frac{P_t(i)}{P_t}\right]^{-\eta}$$

where $P_t(i)$ is the dollar price of good i, and P_t is the final goods price. Perfect competition in the final goods market implies that the final goods price is given by

$$P_t = \{\int_0^1 [P_t\ (i)^{(1-\eta)}]di\}^{1/(1-\eta)}. \tag{5}$$

Intermediate goods firm i is a monopolist producer of intermediate good i. Each intermediate firm rents capital and hires labor from households utilizing a CRS Cobb-Douglass production function denoted by $f(K,L) \equiv K^{\alpha}L^{1-\alpha}$. Imperfect competition implies that factor payments are distorted. With z_t as marginal cost, we then have $r_t = z_t f_K(K_t,L_t)$ and $w_t = z_t f_L(K_t,L_t)$. Since factor markets are competitive, the intermediate goods firms take z_t as given.

As for intermediate goods pricing, we follow Yun (1996) and utilize the assumption of staggered pricing in Calvo (1983). Each period fraction $(1-\nu)$ of firms get to set a new price, while the remaining fraction ν must charge the previous period's price times steady-state inflation (denoted by π). This probability of a price change is constant across time and is independent of how long it has been since any one firm has last adjusted its price. Suppose that firm i wins the Calvo lottery and can set a new price in time t. It's optimization problem is given by:

$$\max_{P_t(i)} \left\{ \sum_{j=0}^{\infty} (\nu\beta)^j \frac{\Lambda_{t+j}}{\Lambda_t} \left[\left(\frac{P_t(i)}{P_t} \right)^{-\eta} Y_t \left(\frac{P_t(i)}{P_t} - z_t \right) \right] \right\}$$

where $\Lambda_{t+j} \equiv U_c(t+j)/P_{t+j}$ denotes the marginal utility of a dollar. The optimization condition is given by

$$P_t(i) = \frac{\eta \sum_{j=0}^{\infty} (\nu\beta)^j \Lambda_{t+j} P_{t+j}^{\eta} Y_{t+j} z_{t+j}}{(\eta-1)\sum_{j=0}^{\infty} (\nu\beta\pi)^j \Lambda_{t+j} P_{t+j}^{\eta-1} Y_{t+j}} \tag{6}$$

If $\nu = 0$ so that all prices are flexible each period, $z_t = (\eta-1)/\eta < 1$. This latter term $\bar{z} \equiv (\eta-1)/\eta$ is a measure of the steady-state distortion arising from monopolistic competition. In the case of sticky prices ($\nu > 0$), z_t will not typically equal z and will reflect the time varying monopoly distortion.

A recursive competitive equilibrium is given by stationary decision rules that satisfy (3), (4), (5), (6), and the following:

$$\frac{U_L(t)}{U_c(t)} = z_t f_L(t) \tag{7}$$

$$U_c(t) = \beta\{U_c(t+1)[z_{t+1} f_K(K_{t+1}, L_{t+1}) + (1-\delta)]\} \tag{8}$$

$$c_t + K_{t+1} - (1-\delta)K_t = f(K_t, L_t) \equiv Y_t \tag{9}$$

Note that equations (7)-(9) are essentially the real business cycle (RBC) conditions distorted by marginal cost and the effect of real money balances on the marginal utility of consumption. This latter distortion is proxied by the nominal rate of interest. If these distortions were held fixed, we would have the RBC model, and would thus be assured of a unique equilibrium. Indeterminacy arises because of endogenous fluctuations in the nominal interest rate and marginal cost.

To close the model we need to specify the central bank reaction function. In what follows we assume a reaction function where the current nominal interest rate is a function of inflation.[4] We will consider two variations of this simple rule:

$$R_t = R_{ss}\left(\frac{\pi_{t+i}}{\pi_{ss}}\right)^{\tau}, \; where \;\; \tau \geq 0, \; R_{ss} = \frac{\pi_{ss}}{\beta},$$

where i = 1 is a forward-looking rule, and i = 0 is a current-looking rule.

Under any such interest rate policy the money supply responds endogenously to be consistent with the interest rate rule. It is this endogeneity of the money supply that leads to the possibility of indeterminacy. By real indeterminacy, we mean a situation in which the behavior of one or more real variables is not pinned down by the model. This possibility is of great importance as it immediately implies the existence of sunspot equilibria which in the present environment are necessarily welfare reducing.

III. Equilibrium Determinacy.

We will now discuss each Taylor timing convention in turn. Because we are interested in highlighting the effects of capital on the determinacy conditions, in each subsection we will first present results for a labor-only economy and then note how the determinacy range is affected by the inclusion of the investment margin.

Forward-Looking Taylor Rules:

Consider a labor-only economy in which production is linear in labor ($\alpha = 0$). Combining (3), (7) and the forward-looking policy rule gives

[4] Including output in the Taylor rule would have only minor effects on the local determinacy conditions.

$$\tilde{z}_{t+1} = \tilde{z}_t + \tau\tilde{\pi}_{t+1} - \tilde{\pi}_{t+1}. \tag{10}$$

The tildes denote log deviations from steady-state values, except for the inflation rate in which case it is simply the deviation from the steady-state value. As for marginal cost, Yun (1996) demonstrates that (5)-(6) can be combined to yield the following log-linearized "Phillips curve"

$$\tilde{\pi}_t = \lambda\tilde{z}_t + \beta\tilde{\pi}_{t+1}, \tag{11}$$

where $\pi_t = (P_t/P_{t-1}) - 1$, denotes the inflation rate, and $\lambda = (1-\nu)(1-\nu\beta)/\nu$.

Equations (10)-(11) represent a system in z_t and π_t. This is essentially the model analyzed by Clarida, Gali, and Gertler (2000). Their analysis suggested that to avoid indeterminacy the central bank should react aggressively (but not too aggressively) to future inflation. In particular, the necessary and sufficient condition for determinacy is

$$1 < \tau < \frac{2(\beta+1)+\lambda}{\lambda}.$$

For plausible parameter values (eg., β is close to one, λ and α each about 1/3) indicate that the determinacy range is quite large with an upper bound of about 13. Hence, an aggressive (but not too aggressive) forward-looking rule is determinate in this environment.

Adding capital basically eliminates all these determinate equilibria. One important reason is because in the model with capital there is always a zero eigenvalue. To see this substitute (8) and (9) into (7) to obtain

$$\frac{R_t}{\pi_{t+1}} = z_{t+1} f_k(t+1) + (1-\delta).$$

We thus abstract from it for simplicity.

This is an arbitrage relationship linking the real return on bonds to the real return on capital accumulation. Note that both elements are forward-looking. In particular, for a forward-looking policy rule this arbitrage relationship does not depend on time t variables. This immediately suggests a zero eigenvalue.[5]

Since capital is the only state variable in the system a zero eigenvalue implies that if there is determinacy capital must immediately jump to the steady state. This suggests that even if there is determinacy the welfare properties of the rule would be disastrous. Conversely, if there are sunspots, one can always construct them in a model in which capital is set to steady-state for all periods. Hence, the conditions for determinacy in a model with capital are at least as tight as in the model with labor. But in fact, the conditions are sufficiently tighter: ***(NEED MORE DISCUSSION HERE.)***

Proposition 1: Suppose that monetary policy is given by a forward-looking Taylor rule. In the Calvo sticky price model with investment a necessary condition for determinacy is that τ be in one of the following two regions:

$$\frac{(1+3\beta)a_2}{a_1} < \lambda(\tau-1) < \min\left\{\frac{2(1+\beta)a_2}{a_1+\alpha}, \frac{a_2}{\alpha}\right\},$$

$$0 < \lambda(\tau-1) < \frac{a_2(1-\beta)}{\alpha},$$

where $a_1 = 1-\beta(1-\delta)(1-\alpha)$ *and* $a_2 = 1-\beta(1-\delta)$.

[5] The appendix formally shows the presence of a zero eigenvalue. The labor equation has time t elements but can be substituted out given the assumption of linear leisure. However, a zero eigenvalue arises even in models without an infinite labor supply elasticity—separable preferences are a sufficient condition.

These regions for determinacy are remarkably narrow. Suppose that $\alpha = \lambda = 1/3$, $\delta = .02$, and $\beta = .99$, so that $a_1 \approx .35$ and $a_2 \approx .03$. In this case, the first region is an empty set, and the second region is $1 < \tau < 1.0027$. In comparison to the labor-only economy, the presence of capital makes determinacy essentially impossible.

Current-Looking Taylor Rules:

As before, consider the labor-only economy. The system is given by (11) and the counterpart to (10)

$$\tilde{z}_{t+1} - \tilde{z}_t = \tau\tilde{\pi}_t - \tilde{\pi}_{t+1}.$$

It is straightforward to show that we have real determinacy if and only if $\tau > 1$. To understand why, assume to the contrary that there is indeterminacy when $\tau > 1$. Consider a sunspot increase in π_{t+1}. Indeterminacy implies that $|z_{t+1}| < |z_t|$ and $|\pi_{t+1}| < |\pi_t|$. If $\tau > 1$ then marginal cost (z_t) and inflation (π_t) must be inversely related. But the Phillips curve implies that they move together which gives us our contradiction. A similar argument can be used to show that when $\tau < 1$ indeterminacy is possible. Remarkably this conclusion is not affected by the addition of investment to the model:

Proposition 2: Suppose that monetary policy is given by a current-looking Taylor rule. A necessary condition for determinacy is that $\tau > 1$. Furthermore if $\lambda a_1 > (2\beta - 1)a_2$ where $a_1 = 1 - \beta(1-\delta)(1-\alpha)$ and $a_2 = 1 - \beta(1-\delta)$ then in the Calvo sticky price model with investment $\tau > 1$ is both a necessary and sufficient condition for determinacy.

Note: Given the above calibration $\alpha = 1/3$, $\delta = .02$, and $\beta = .99$, so that $a_1 \approx .35$ and $a_2 \approx .03$, there can only be indeterminacy for some values of $\tau > 1$ if prices are extremely sticky, $\lambda < 0.083$. Given that $\lambda = (1-\nu)(1-\nu\beta)/\nu$ this implies that less than 25% of firms adjust their prices every quarter (where ν denotes the probability that a firm can adjust prices in the current period). The above, however, is only a sufficient condition for determinacy. Even if $\lambda < 0.05$ the range for indeterminacy is very small $1.1 < \tau < 1.69$.

IV. A Comparison with a Continuous Time Analysis.

In a recent paper Dupor (2000) conducts a similar determinacy analysis in a continuous time MIUF model. The environments appear to be the same: prices are sticky as in Calvo (1983), output is produced using both labor and capital, and preferences are linear over labor. However, Dupor reaches a quite different outcome: he reports that $\tau < 1$ is necessary and sufficient for local determinacy.

In this section we will explore the reasons for the different results. Dupor assumes a utility function of the form $U(c,m,1-L) \equiv \ln(c) + V(m) - L$. For simplicity we maintain this restriction. Following Proposition 1, however, we need not make any assumption regarding $V(c,m)$. We will first present a discrete time analysis of the model using this functional form, and then turn to Dupor's continuous time version.

To compare the models we must specify whether in discrete time we have a current- or a forward-looking Taylor rule. In continuous time limit, there is no distinction between a current-looking and forward-looking rule. We initially assume that the discrete time model is given by a forward-looking rule since the comparison between discrete and continuous time is especially straight forward and dramatic.

Using the Fisher equation (3) the capital accumulation equation is

$$\frac{R_t}{\pi_{t+1}} = \beta\{z_{t+1} f_K(K_{t+1}, L_{t+1}) + (1-\delta)\} \tag{12}$$

Equation (12) is key in what follows. As noted above, this is a no-arbitrage relationship: the real return on bonds must be equal to the real return on capital accumulation. With a forward-looking rule this expression is entirely in terms of time t+1 variables, and is the cause of the zero eigenvalue above.

Note that since cash balances are separable, the money demand curve (4) is irrelevant for determinacy issues. Money is the residual that can be backed out at the end. Let x ≡ L/K. The log-linearized equilibrium conditions are given by

$$\tilde{z}_t = \alpha\tilde{x}_t + \tilde{c}_t \tag{13}$$

$$(\tau-1)\tilde{\pi}_{t+1} = a_2\tilde{z}_{t+1} + a_2(1-\alpha)\tilde{x}_{t+1} \tag{14}$$

$$\tilde{c}_{t+1} - \tilde{c}_t = (\tau-1)\tilde{\pi}_{t+1} \tag{15}$$

$$\tilde{K}_{t+1} = \left(1 + \frac{c_{ss}}{K_{ss}}\right)\tilde{K}_t + \left(\frac{(1-\alpha)Y_{ss}}{K_{ss}}\right)\tilde{x}_t - \left(\frac{c_{ss}}{K_{ss}}\right)\tilde{c}_t \tag{16}$$

$$\tilde{\pi}_t = \lambda\tilde{z}_t + \beta\tilde{\pi}_{t+1}, \tag{17}$$

where $a_2 \equiv 1-\beta(1-\delta)$, β ≡ 1/(1+ρ), and c_{ss} denotes steady-state consumption, etc.

The continuous time counterpart to this discrete-time system is given by

$$\tilde{z}_t = \alpha\tilde{x}_t + \tilde{c}_t \tag{18}$$

$$(\tau-1)\tilde{\pi}_t = a_2\tilde{z}_t + a_2(1-\alpha)\tilde{x}_t \tag{19}$$

$$\dot{\tilde{c}}_t = (\tau-1)\tilde{\pi}_t \tag{20}$$

$$\dot{\tilde{K}}_t = \left(\frac{c_{ss}}{K_{ss}}\right)\tilde{K}_t + \left(\frac{(1-\alpha)Y_{ss}}{K_{ss}}\right)\tilde{x}_t - \left(\frac{c_{ss}}{K_{ss}}\right)\tilde{c}_t \tag{21}$$

$$\dot{\tilde{\pi}}_t = \rho\tilde{\pi}_t - \lambda\tilde{z}_t \tag{22}$$

The fundamental difference in this system is that the no-arbitrage capital accumulation (19) equation is solely in time-t variables. We can use (18) and (19) to eliminate z_t and x_t from the system. Doing so yields the following:

$$\begin{pmatrix}\dot{\tilde{c}}_t \\ \dot{\tilde{\pi}}_t \\ \dot{\tilde{K}}_t\end{pmatrix} = \begin{pmatrix} 0 & \tau-1 & 0 \\ -\lambda(1-\alpha) & \rho-\lambda\alpha(\tau-1)/a_2 & 0 \\ -\left(\frac{(1-\alpha)Y_{ss}+c_{ss}}{K_{ss}}\right) & \frac{(1-\alpha)Y_{ss}(\tau-1)}{a_2K_{ss}} & \frac{c_{ss}}{K_{ss}} \end{pmatrix}\begin{pmatrix}\tilde{c}_t \\ \tilde{\pi}_t \\ \tilde{K}_t\end{pmatrix}$$

For determinacy, we need exactly one negative eigenvalue. By inspection, we have one eigenvalue equal to $c_{ss}/K_{ss} > 0$. The remaining two are the solution to the following quadratic equation:

$$h(q) \equiv a_2q^2 + [(\lambda\alpha(\tau-1)) - \rho a_2]q + \lambda(1-\alpha)a_2(\tau-1).$$

In continuous time determinacy requires that one root be negative and one be positive. The sign of h(0) is given by the sign of (τ-1). Since h $\to \infty$ as either q $\to -\infty$ or q $\to \infty$, $\tau < 1$ is both a necessary and sufficient for determinacy. (This is analogous to Dupor's Theorem 1.) If $\tau > 1$ the system is either overdetermined or underdetermined depending on the sign of $h'(0) = [(\lambda\alpha(\tau-1)/a_2) - \rho]$. If this term is positive, the remaining two roots are negative and the system is underdetermined. If this term is negative, the remaining two roots are positive and the system is overdetermined. In the former case, there are a continuum of equilibria, while in the latter there are no stationary equilibria. This corresponds to Dupor's Theorem 2.

How does the analysis differ in discrete time? The key difference is that the no-arbitrage equation (14) is entirely in time t+1 variables so that it does not provide any restriction on time-t behavior. This comparison can be made exact in the following way. Scroll the expressions (13), (15), (16) and (17) forward one period. Scrolling everything but the capital accumulation forward implies that the system is given by

$$\widetilde{z}_{t+1} = \alpha\widetilde{x}_{t+1} + \widetilde{c}_{t+1} \tag{23}$$

$$(\tau - 1)\widetilde{\pi}_{t+1} = a_2\widetilde{z}_{t+1} + a_2(1-\alpha)\widetilde{x}_{t+1} \tag{24}$$

$$\widetilde{c}_{t+2} - \widetilde{c}_{t+1} = (\tau - 1)\widetilde{\pi}_{t+2} \tag{25}$$

$$\widetilde{K}_{t+2} = \left(1 + \frac{c_{ss}}{K_{ss}}\right)\widetilde{K}_{t+1} + \left(\frac{(1-\alpha)Y_{ss}}{K_{ss}}\right)\widetilde{x}_{t+1} - \left(\frac{c_{ss}}{K_{ss}}\right)\widetilde{c}_{t+1} \tag{26}$$

$$\widetilde{\pi}_{t+1} = \lambda\widetilde{z}_{t+1} + \beta\widetilde{\pi}_{t+2}\,, \tag{27}$$

plus the following time t restrictions

$$\widetilde{z}_t = \alpha\widetilde{x}_t + \widetilde{c}_t \tag{28}$$

$$\widetilde{c}_{t+1} - \widetilde{c}_t = (\tau - 1)\widetilde{\pi}_{t+1} \tag{29}$$

$$\widetilde{K}_{t+1} = \left(1 + \frac{c_{ss}}{K_{ss}}\right)\widetilde{K}_t + \left(\frac{(1-\alpha)Y_{ss}}{K_{ss}}\right)\widetilde{x}_t - \left(\frac{c_{ss}}{K_{ss}}\right)\widetilde{c}_t \tag{30}$$

$$\widetilde{\pi}_t = \lambda\widetilde{z}_t + \beta\widetilde{\pi}_{t+1}\,. \tag{31}$$

Using (23) and (24) to eliminate z_{t+1} and x_{t+1} from the system, we can write the dynamic part of the system in matrix form as

$$\begin{pmatrix} 1 & 1-\tau & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \widetilde{c}_{t+2} \\ \widetilde{\pi}_{t+2} \\ \widetilde{K}_{t+2} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -\lambda(1-\alpha)/\beta & \left[a_2 - \lambda\alpha(\tau-1)\right]/(\beta a_2) & 0 \\ -\left(\dfrac{(1-\alpha)Y_{ss} + c_{ss}}{K_{ss}}\right) & \dfrac{(1-\alpha)Y_{ss}(\tau-1)}{a_2 K_{ss}} & 1 + \dfrac{c_{ss}}{K_{ss}} \end{pmatrix}\begin{pmatrix} \widetilde{c}_{t+1} \\ \widetilde{\pi}_{t+1} \\ \widetilde{K}_{t+1} \end{pmatrix}$$

Or

$$A\psi_{t+2} = B\psi_{t+1}$$

Notice the similarity in the discrete time versus continuous time matrices. This system has three eigenvalues. The time t restrictions (27)-(31) provide three restrictions on the equilibria. But we have seven unknowns. Hence, for determinacy, we need all three of the eigenvalues to lie outside the unit circle. By inspection, one of the eigenvalues is given by $1+c_{ss}/K_{ss}$. The remaining two are the solution to the following quadratic equation:

$$g(q) \equiv \beta a_2 q^2 + \{\lambda(\tau-1)[\alpha+(1-\alpha)a_2] - (1+\beta)a_2\}q + a_2 - \lambda\alpha(\tau-1)\,.$$

Notice the similarity between this quadratic and the h(q) function that arises in the continuous time model. In continuous time we need only one explosive root; in discrete time we need two explosive roots. The quadratic g(q) is identical to the quadratic equation that arises in the proof of Proposition 1 (see the Appendix). Hence, there is indeterminacy for all values of τ.

We have now duplicated the results of Dupor and demonstrated how the local indeterminacy that arises with the discrete time model becomes local determinacy (with $\tau < 1$) in the case of continuous time. The matrix for the continuous time case is the continuous time analogue to the discrete time matrix. However, for determinacy we need both roots to be explosive with discrete time and only one explosive root for continuos time. The key difference is that in continuous time there is an additional restriction: at time t the marginal productivity of capital equals the real interest rate while in discrete time model this relationship is in terms of the future realizations of returns. Therefore the key difference is in the no-arbitrage relationship between bonds and capital.

What about the comparison between continuous time and a discrete time model with a current-looking rule? As noted before with continuous time there is no distinction between current and forward-looking rules. The difference between the discrete-time model with a current-looking rule and a continuous time model is just as striking. A necessary (and sufficient for plausible parameter values) condition for determinacy with a current-looking rule is $\tau > 1$, while with continuous time $\tau < 1$ is necessary and sufficient. Once again the extra restriction present with continuos time plays an important role in the difference between the two models.

V. Conclusion.

The central issue of this paper is to identify the restrictions on the Taylor interest rate rule needed to ensure real determinacy. A classic result in the literature is that an aggressive response to future or forecasted inflation is sufficient for determinacy (eg., Bernanke and Woodford (1997) and Clarida, Gali, Gertler (2000)). These earlier analyses ignore the central role of investment spending. The role of investment across the business cycle has a long tradition in monetary economics so that ignoring it seems like a bad idea. We have demonstrated above that in the case of forward-looking policy, inclusion of the investment choice dramatically shrinks the region of determinacy. The end result is that for anything but the most extreme parameter values this model with Calvo stickiness implies that monetary policy must respond aggressively to current inflation to generate determinacy. In short, there is a clear danger to any policy that is forward-looking.

Appendix

Proposition 1: Suppose that monetary policy is given by a forward-looking Taylor rule. Then in the Calvo sticky price model with investment a necessary condition for determinacy is that τ be in one of the following two regions:

$$\frac{(1+3\beta)a_2}{a_1} < \lambda(\tau-1) < \min\left\{\frac{2(1+\beta)a_2}{a_1+\alpha}, \frac{a_2}{\alpha}\right\},$$

$$0 < \lambda(\tau-1) < \frac{a_2(1-\beta)}{\alpha},$$

where $a_1 = 1-\beta(1-\delta)(1-\alpha)$ *and* $a_2 = 1-\beta(1-\delta)$.

Proof:

Given the assumption that utility is linear in leisure the first order conditions (7) and (8) can be written as

$$U_c(t) = \frac{x_t^{\alpha}}{z_t(1-\alpha)}, \textit{where } x_t = \frac{L_t}{K_t} \tag{A1}$$

$$U_c(t) = \beta\left(U_c(t+1)\left\{\alpha z_{t+1}x_{t+1}^{1-\alpha} + (1-\delta)\right\}\right) \tag{A2}$$

Substituting (A1) and equation (A1) scrolled forward one period into (A2) yields

$$\frac{x_t^{\alpha}}{z_t} = \beta\left(\frac{x_{t+1}^{\alpha}}{z_{t+1}}\left\{\alpha z_{t+1}x_{t+1}^{1-\alpha} + (1-\delta)\right\}\right). \tag{A3}$$

We can express this as

$$x_{t+1} = F(x_t, z_{t+1}, z_t). \tag{A4}$$

The resource constraint (9) provides another equation:

$$K_{t+1} = K_t x_t^{1-\alpha} + (1-\delta)K_t - c_t. \tag{A5}$$

The money demand curve (4) implies that real balances depend only on c_t and R_t. Using (A1) we then have that c_t depends only on R_t, z_t and x_t. Using the policy rule we can then write (A5) as

$$K_{t+1} = G(x_t,\pi_{t+1},K_t,z_t). \qquad \text{(A6)}$$

Using (A1) and the policy rule, we can express the Fisher equation (3) as

$$z_{t+1} = H(\pi_{t+1},x_t,x_{t+1},z_t). \qquad \text{(A7)}$$

In summary, we have three equations (A4), (A6), and (A7). The Phillips curve (11) yields $\pi_{t+1} = P(\pi_t, z_t)$. The linearized Euler equations are given by

$$x_{t+1} = F(x_t,z_t,z_{t+1})$$

$$K_{t+1} = G(x_t,\pi_{t+1},K_t,z_t)$$

$$z_{t+1} = H(x_t,x_{t+1},\pi_{t+1},z_t)$$

$$\pi_{t+1} = P(\pi_t,z_t).$$

Let w_t denote the vector $[x_t, z_t, \pi_t, K_t]$ so that the linearized system can be expressed as

$$Aw_{t+1} = Bw_t$$

where A and B are 4x4 matrices with elements given by the derivatives of F, G, H and P. After inverting A, we are left with the matrix $A^{-1}B$ which has four eigenvalues. Since there is only one state variable in this system (K_t), we need three explosive eigenvalues for determinacy. Once again one eigenvalue is zero, while another is given by

$$1+\frac{c_{ss}}{K_{ss}}>1$$

where ss denotes steady-state levels. Hence, a necessary and sufficient condition for determinacy is that the remaining two roots be outside the unit circle. The relevant quadratic equation is given by

$$J_2 q^2 + J_1 q + J_0$$

where

$$J_2 = \beta a_2$$

$$J_1 = \lambda(\tau - 1)a_1 - (1+\beta)a_2$$

$$J_0 = a_2 - \alpha\lambda(\tau - 1)$$

$$a_1 = 1 - \beta(1-\delta)(1-\alpha) \; and \; a_2 = 1 - \beta(1-\delta).$$

The above implies

$$J(1) = \lambda(\tau - 1)(a_1 - \alpha).$$

If $\tau < 1$ then $J(1) < 0$ and $J(0) > 0$ which means one root is in (0,1). Hence, a necessary condition for determinacy is $\tau > 1$. Under this restriction, $J(1) > 0$ so that additional necessary conditions are $J(0) > 0$ and $J(-1) > 0$. These put an upper bound on τ:

$$\lambda(\tau - 1) < \min\left\{\frac{2(1+\beta)a_2}{a_1 + \alpha}, \frac{a_2}{\alpha}\right\} \qquad \text{(A12)}$$

To make further progress and tighten the bound more closely, we must consider the two cases where the solutions to J are real or complex.

<u>Suppose first that the two roots of J are real</u>:

We first note that J is quadratic and convex. In this case (assuming the roots are real, A12, and $\tau > 1$) we have the following two potential regions of determinacy:

$$\frac{-J_1}{2J_2} < -1 \quad \text{and} \quad \frac{-J_1}{2J_2} > 1\text{, or} \qquad \text{(A13)}$$

$$\frac{(1+3\beta)a_2}{a_1} < \lambda(\tau-1) \quad \text{and} \quad 0 < \lambda(\tau-1) < \frac{a_2(1-\beta)}{a_1} \qquad \text{(A14)}$$

Combined with (A12) and $\tau > 1$ these two regions of determinacy (assuming the roots are real) are

$$0 < \lambda(\tau-1) < \frac{a_2(1-\beta)}{a_1} \quad \text{and} \qquad \text{(A13)}$$

$$\frac{(1+3\beta)a_2}{a_1} < \lambda(\tau-1) < \min\left\{\frac{2(1+\beta)a_2}{a_1+\alpha}, \frac{a_2}{\alpha}\right\}. \qquad \text{(A14)}$$

<u>Suppose instead that the roots are complex:</u>

The norm of these roots is given by

$$\frac{J_0}{J_2} = \frac{a_2 - \alpha\lambda(\tau-1)}{\beta a_2}.$$

Since $J_0 > 0$, this yields the following necessary and sufficient condition for determinacy (if the roots are complex):

$$0 < \lambda(\tau-1) < \frac{a_2(1-\beta)}{\alpha}. \qquad \text{(A15)}$$

Combining the real and complex regions and noting that $\alpha < a_1$ a necessary condition for determinacy is that τ be in one of the following two regions:

$$\frac{(1+3\beta)a_2}{a_1} < \lambda(\tau-1) < \min\left\{\frac{2(1+\beta)a_2}{a_1+\alpha}, \frac{a_2}{\alpha}\right\},$$

$$0 < \lambda(\tau-1) < \frac{a_2(1-\beta)}{\alpha}.$$

QED

Proposition 2: Suppose that monetary policy is given by a current-looking Taylor rule. A necessary condition for determinacy is that $\tau > 1$. Furthermore if $\lambda a_1 > (2\beta - 1)a_2$ where $a_1 = 1 - \beta(1-\delta)(1-\alpha)$ and $a_2 = 1 - \beta(1-\delta)$ then in the Calvo sticky price model with investment $\tau > 1$ is both a necessary and sufficient condition for determinacy.

Proof: The linearized Euler equations are given by

$$x_{t+1} = F(x_t,z_t,z_{t+1})$$

$$K_{t+1} = G(x_t,\pi_t,K_t,z_t)$$

$$z_{t+1} = H(x_t,x_{t+1},\pi_{t+1},\pi_t,z_t)$$

$$\pi_{t+1} = P(\pi_t,z_t).$$

Let w_t denote the vector $[x_t, z_t, \pi_t, K_t]$ so that the linearized system can be expressed as

$$Aw_{t+1} = Bw_t$$

where A and B are 4x4 matrices with elements given by the derivatives of F, G, H and P. After inverting A, we are left with the matrix $A^{-1}B$ which has four eigenvalues. Since there is one state variable in this system (K_t), we need three explosive eigenvalues for determinacy. Once again one is given by

$$1 + \frac{c_{ss}}{K_{ss}} > 1$$

where ss denotes steady-state levels. Hence, a necessary and sufficient condition for determinacy is that two of the remaining three roots be outside the unit circle. Hence, there must be one real root within the unit circle. The relevant cubic equation is given by

$$J_3 q^3 + J_2 q^2 + J_1 q + J_0$$

where

$$J_3 = \beta a_2$$

$$J_2 = -\lambda a_1 - (1+\beta)a_2$$

$$J_1 = a_2 + \alpha\lambda + \lambda \tau a_1$$

$$J_0 = -\alpha \lambda \tau$$

$$a_1 = 1 - \beta(1-\delta)(1-\alpha) \ and \ a_2 = 1 - \beta(1-\delta).$$

The above implies J(0) < 0 and

$$J(1) = \lambda(\tau - 1)(a_1 - \alpha) .$$

For determinacy we need exactly one real root in (0,1). Hence, $\tau > 1$ is necessary for determinacy. The assumed condition in the proposition implies that J is concave at 1. Hence, the remaining two roots, either real or complex, must be outside the unit circle. QED

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