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BANK DIVERSIFICATION: LAWS AND FALLACIES OF LARGE NUMBERS

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ABSTRACT

The conventional wisdom on bank diversification confuses risk with failure. This paper clarifies that distinction and shows how increasing bank size may increase bank risk even though it lessens the probability of failure and lowers the expected loss. The key result is an application of Samuelson's "fallacy of large numbers."

Introduction

Conventional wisdom states that large banks are safer than small banks because they can diversify more. This conventional wisdom, however, confuses risk with probability of failure. While the law of large numbers does imply that a large bank is less likely to fail than a small bank, equating this tendency to less risk falls into what Samuelson termed the *fallacy* of large numbers. A $10 billion bank may be less likely to fail than a $10 million bank, but it may also saddle the investor with a $10 billion loss.

In this paper, I hope to clarify what this distinction means for banks. Banks diversify by growing-- by adding risks-- something distinctly different from the subdivision of risk behind standard portfolio theory. A simple mean-variance example will make the point that a bank's owner need not value diversification. After that, I take a regulator's perspective and consider how a bank guarantee fund, such as the FDIC, views bank growth and diversification. After a short review of why diversification by adding risks decreases the probability of bank failure, I look at how such diversification alters the expected value of FDIC payments, and then diversification's impact on the FDIC's expected utility, using recent results from the theory of *standard* risk aversion.

To concentrate on the cleanest example, this paper stays with the case of independent and identically distributed risks. This admittedly ignores the alleged ability of large banks to diversify regionally[1] or the possibly adverse incentives of deposit insurance (Boyd and Runkle [1993], Todd and Thomson [1991]).

[1] Compare Haubrich (1990) with Kryzanowski and Roberts (1993). Even small banks may diversify, however, by selling loans or participating in mortgage pools or other forms of securitization.

1. A Simple Example

Probably the easiest way to understand the effects of diversification by adding risks is to consider a bank financed exclusively by an owner/investor who cares only about means and variances. With no debt, failure disappears as an issue, and instead the question becomes the utility-maximizing portfolio for the bank's owner.

The owner and sole equity holder has, conveniently for us, sunk his entire wealth W into the bank. He faces the problem of dividing his portfolio between holding S safe government bonds with a certain return of zero and investing in some number K of risky, independent bank loans with returns r_i normally distributed as $N(\overline{\mu},\overline{\sigma})$. If each loan costs a dollar, the investor's budget constraint is $W=S+K$. These bank loans are indivisible--the bank cannot diversify by spreading one dollar across many loans. Then the return on the portfolio is

$$\tilde{R}_p = \frac{\sum_{i=1}^{K} r_i}{W}.$$

Since $\sum_{i=1}^{K} r_i$ is distributed $N(K\overline{\mu},\sqrt{K}\overline{\sigma})$, standard techniques (Fama and Miller [1972], chapter 6, section III) imply that

$$(1)\quad E(\tilde{R}_p) = \frac{\overline{\mu}}{\overline{\sigma}}\sqrt{K}\sigma(\tilde{R}_p).$$

In mean-standard deviation space, equation (1) defines a portfolio opportunity set illustrated by figure 1 (for $W=5$). The opportunity set is disjointed, since the decision to add another loan is discrete. Depending on the shape of the indifference curves, the bank owner may put none, all, or some of his wealth into bank loans.

Figure 1 shows a typical case with an interior solution. This illustrates quite clearly that the bank does not always wish to diversify. Stated another way, the portfolio return is distributed $N(\frac{K}{W}\overline{\mu}, \frac{\sqrt{K}}{W}\overline{\sigma})$, so that as the bank invests in more loans, the standard deviation as well as the expected return increases. Which one matters more depends on preferences.

An all-equity bank offers a nice illustration, but does not provide a very representative case. Even a stylized bank should have deposits.

2. Does the FDIC Want Banks to Diversify?

Allowing banks to take in deposits means allowing banks to fail. The return on assets may not cover the payments promised to the depositors. In the U.S., this liability devolves upon the Federal Deposit Insurance Corporation. This provides a natural focal point for our discussion. Actual banks raise money in many different ways, using several types of preferred stock, subordinated bonds, and commercial paper. What happens in bankruptcy is at best complicated and at worst unknown, as the courts must determine the validity of claims as diverse as offsetting deposits and the source-of-strength doctrine. A detailed consideration of how each class of investors views diversification, then, is beyond the scope of this paper. Instead, to make what is admittedly a simple point, I concentrate on the FDIC, which ultimately bears the liability for bank failures.

The FDIC steps in if the realization of bank assets Y is too low to repay the face value of the debt F, that is, if $Y<F$. This is a fairly general formulation in that the assets producing Y may be funded by means other than deposits, but it is not completely general because it ignores the possibility that the FDIC may have priority over some

investors. For the rest of the paper, however, I will restrict myself to purely deposit-financed banks.

What is the face value of the debt, F? With no capital, if the bank funds n projects each requiring funds f, the face value is the sum of the deposits, $F = n\,f$. The payout of bank assets is likewise the sum over the different projects,

$$Y_n = \sum_{i=1}^{n} x_i \ ,$$

where n indexes the number of projects in which the bank has invested.

A. The Probability of Bank Failure

How likely is it that this bank will fail? The answer is $\Pr(Y_n < n \cdot f)$ or

(2) $\Pr(\sum_i x_i < n \cdot f)$.

Assume the x_i's are independent and identically distributed (i.i.d.) with mean $E(x_i)=\mu$ and further assume that $f<\mu$, so that the face value of the debt is smaller than the expected payout of the assets.

We can rewrite expression (2) as

(3) $\Pr(\frac{Y_n}{n} < f)$

because the set $\{y{:}\, y < n \cdot f\}$ is the same as the set $\{y{:}\, \frac{y}{n} < f\}$.

The weak law of large numbers (Shirayev [1984], theorem 2, p. 323) tells us that provided $E|x_i|<\infty$ and $Ex_i = \mu$, then for all $\varepsilon>0$,

$$\Pr\{|\frac{Y_n}{n} - \mu| \geq \varepsilon\} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

In particular, since f<μ, $\Pr(\frac{Y_n}{n} < f) < \Pr(|\frac{Y_n}{n} - \mu| \geq \mu - f)$. That is, we can represent the values of $\frac{Y_n}{n}$ below *f* as values more than $\mu - f$ away from the mean μ. Thus, as Diamond (1984) explicitly states, the weak law of large numbers implies that diversification by adding risks reduces the probability of bank failure.

B. The Expected Value of the FDIC's Liabilities

As Samuelson points out, a rational utility maximizer maximizes expected utility, not the probability of success. The probability of each outcome must be weighted by the utility of that outcome. As mentioned before, a $10 billion bank that does fail may cost the FDIC more to resolve than a $10 million bank.

In the simplest case of risk neutrality, expected utility corresponds to expected value. The first question, then, equivalent to assuming risk neutrality on the part of the agency, concerns the expected value of the FDIC's payout. Though the calculation is not particularly difficult, I have not seen it before in the literature. The expected payout value becomes a question of finding the expected value of a particular function. The FDIC must pay

$$(4) \qquad \begin{cases} 0 & if \quad Y_n \geq F. \\ F - Y_n & if \, Y_n < F. \end{cases}$$

Figure 2 plots the function along with a typical density function.

It is worth noting that the expected value of (4) is *not* a conditional expectation. If the set A = $\{Y_n: Y_n < F\}$, then the expected value of (4) is P(*A*) E($Y_n|A$) rather than E($Y_n|A$). A simple example will make this clear. Take a four-point distribution, P(1)=P(2)=P(3)=P(4)=$\frac{1}{4}$. Then E(*X*)= $\frac{1}{4}(1+2+3+4) = \frac{5}{2}$. Now define the function g(*x*) as $g(x)=\{0$, if $x \geq 2.5$ and x, if $x < 2.5\}$. Then $E[g(x)]=\frac{1}{4}(1+2)=\frac{3}{4}$, while

$E[x|\ x \leq 2.5]=\frac{3}{2}$.

The question before us is what happens to the expected value of the FDIC's payments as the bank diversifies. Recall that the FDIC pays off if $\sum_{i=1}^{n} x_i < n \cdot f$ or, equivalently, $\frac{\sum_{i=1}^{n} x_i}{n} < f$. By the strong law of large numbers, the mean of the partial sums $\frac{\sum_{i=1}^{n} x_i}{n}$ converges to a mass point on E(x), and intuition suggests that the expected value of anything below the mean (and *a fortiori* anything below f) will have very little importance, that is, an expected value approaching zero.

To establish this rigorously and to understand what diversification does to the expected value of the FDIC's payments requires a more formal approach. Let each random variable be defined on the probability space (Ω, F, P) and identify Ω with R, the real numbers, without loss of generality. The random variables are then functions on this space, $X_i(\omega)$, and define $Z_n(\omega)$ as

$Z_n(\omega)=\sum_{i=1}^{n}\frac{X_i(\omega)}{n}$. Next, define the function $g(\omega)$ as

$$g(\omega)=\begin{cases} f - X(\omega) & \text{if} \quad X(\omega) \leq f \\ 0 & \text{if} \quad X(\omega) > f \end{cases}.$$

Note that we can think of the expectation $E(X(\omega))$ as a random variable, and so $g(E[X(\omega)])=g(\mu)=0$ since $f<\mu$. Further define $g_n(\omega)$ as $g(Z_n(\omega))$.

The value of diversification can then be expressed as saying that as n approaches infinity, the expected value of $g(Z_n)$ approaches zero, or

$$(5)\quad \lim_{n\to\infty} \int g_n(\omega) = g(\mu) = 0 .$$

To prove (5), we use the Lebesgue dominated convergence theorem (Royden [1968], p. 229), which says that if $h(\omega) \geq 0$ is integrable, if $|g_n(\omega)| \leq h(\omega)$, and if $g_n(\omega) \xrightarrow{a.s.} g(\omega)$, then $\lim_{n\to\infty} \int g_n(\omega) = g(\omega)$.

The theorem first requires that we prove $g_n(\omega) \xrightarrow{a.s.} g(\mu)$. To do this we use the strong law of large numbers. The strong law of large numbers for i.i.d random variables (see Breiman [1992], p. 52, theorem 3.30) says that for i.i.d. $X_1, X_2, X_3 \ldots$, if $E|X_1| < \infty$ then $\frac{\sum X_i}{n} \xrightarrow{a.s.} E(X_1)$, where $\xrightarrow{a.s.}$ denotes almost sure convergence, that is, convergence on all but a set of measure (probability) zero.

Hence, given an ω, except for a set of measure 0, we have that for any $\varepsilon > 0$, there exists an N such that if $n > N$, $|Z_n(\omega) - \mu| < \varepsilon$. Choose $\varepsilon < \mu - f$, which implies that if $|Z_n(\omega) - \mu| < \varepsilon$, then $Z_n(\omega) > \mu - \varepsilon > f$. This, with the definition of g, in turn implies that $g_n(\omega) = 0$. For this ω, then, $g_n(\omega) = g(\mu) = 0$, and *a fortiori* $|g_n(\omega) - g(\mu)| < \varepsilon$. Since $g_n(\omega) \to g(\mu)$ for each ω where $Z_n \to \mu$, the almost sure convergence of the strong law implies the almost sure convergence $g_n(\omega) \xrightarrow{a.s.} g(\mu)$.

All that remains to be shown is existence of the integrable bound $h(\omega)$. For this, use $|X_1(\omega)+\mu-f|$, which bounds g_n and is integrable because $E|X_1|<\infty$ is a hypothesis of the strong law. Hence, the Lebesgue dominated convergence theorem applies.

As a bank makes more loans, the expected value of FDIC payouts tends toward zero. Diversification works.

3. A Risk-Averse FDIC

Strictly speaking, what Samuelson terms the *fallacy* of large numbers enters only with risk aversion. Applying this to an agency such as the FDIC, rather than to an individual, requires some justification. The FDIC obtains its funds by taxing people, either indirectly through its assessment on banks, or directly by congressional appropriation. Risk aversion by the FDIC may then reflect risk aversion on the part of those taxed, or nonlinearities associated with distortionary taxation. Alternatively, the risk aversion may result from the incentives, constraints, and information facing the organization. (Of course, as Kane [1989] points out, these may at times promote risk-seeking behavior, as in the FSLIC case.)

A. Conditions for the Fallacy

Samuelson (1963) shows that if a consumer rejects a bet at *every* wealth level, then he will always reject any independent sequence of those bets. Under the Samuelson condition, if the FDIC found one bank loan too risky, it would find a portfolio of any number too risky.

Samuelson posits a rather stringent condition. It rules out, for example, constant relative risk averse (CRRA) utility, because CRRA exhibits decreasing absolute risk aversion (DARA), and so some unacceptable gambles would become acceptable at higher wealth levels. Pratt and Zeckhauser (1987) improve considerably on the condition with their notion of *proper* risk aversion. The conditions for proper risk aversion answer the question, "An individual finds each of two independent monetary lotteries undesirable. If he is required to take one, should he not continue to find the other undesirable?" (Pratt and Zeckhauser, p.143). Proper risk aversion shares one defect with Samuelson's condition, however. It is difficult to characterize and difficult to determine whether a particular utility function satisfies the condition.

A slightly stronger condition proposed by Kimball (1993) has a simple characterization. Kimball's *standard* risk aversion implies proper risk aversion. It thus applies a slightly stronger condition than is strictly necessary for the fallacy. If a utility function displays standard risk aversion, then an investor disliking a bet will also dislike a collection of such bets.

Kimball (1993) shows that necessary and sufficient conditions for standard risk aversion are (monotone) DARA and (monotone) decreasing absolute prudence. If the utility function in question has a fourth derivative, then these conditions become

$$(6)\ \frac{d}{dW}\left(-\frac{u'}{u''}\right) < 0 \text{ or } u^{(3)} > \frac{(u'')^2}{u'} > 0$$

$$(7)\ \frac{d}{dW}\left(-\frac{u^{(2)}}{u^{(3)}}\right) < 0 \text{ or } u^{(4)} < \frac{(u^{(3)})^2}{u^{(2)}} < 0\,.$$

A key point here is that the individual finds each independent risk undesirable. (Kimball has a slightly weaker, more technical condition that he calls loss aggravation.) This certainly applies to the problem as we have defined it, because the payoff to the FDIC is nonpositive--at best it pays nothing. This is not the only way to structure the problem, however, because the FDIC collects deposit insurance premiums from banks. A major strand in banking research has been to ascertain whether the insurance premiums are fairly priced, that is, whether they represent a tax or a subsidy on the bank (Pennacchi [1987], Thomson [1987]). The empirical results are mixed, varying by time period and by bank, and in any case assume risk neutrality so do not directly answer the question most relevant here. It makes sense, then, to think about both possibilities--the case where the FDIC finds insuring a single loan undesirable and the case where it finds insuring a single loan desirable.

In the first case, where the FDIC dislikes insuring an individual loan, expressions (6) and (7) provide sufficient conditions for the agency to dislike insuring any portfolio of such loans. That is, diversification by adding risks does not work; adding risks makes the insurance agency (guarantee corporation) worse off.

In the second case, where the FDIC likes insuring an individual loan, equations (6) and (7) do not help. Their derivation presupposes that the agency dislikes the risk it bears. For *favorable* bets, Diamond (1984) builds on Kihlstrom, Romer, and Williams (1981) to develop sufficient conditions for when the fallacy of large numbers is not a fallacy.

Diamond poses the problem in terms of risk premiums and notes that adding risks provides true diversification if it reduces the risk premium. That is, diversification works if the incremental risk premium for adding the second risky loan to the portfolio is lower than for adding the first, identical risky loan. Kihlstrom, Romer, and Williams (1991) show how to handle risk aversion with two sources of uncertainty by defining a new utility function, given initial wealth W_0 and initial risky bet $\tilde{x}_1$, as

(8) $v(x_2) = Eu(W_0 + \tilde{x}_1 + x_2)$.

Now $v(x_2)$ defined in equation (8) can be treated as a utility function, so Diamond's question comes down to whether $u(\bullet)$ is more risk averse than $v(\bullet)$. If it is, then the risk premium for bearing the second risk is lower than for the first, and the fallacy of large numbers is not a fallacy.

Diamond derives two sufficient conditions for $u(\bullet)$ to be more risk averse than $v(\bullet)$. Using Jensen's inequality, he shows that

(9) $u^{(3)} > 0$

(10) $u^{(4)} > 0$

are sufficient conditions when the risk has zero expected value. When the risk is freely chosen, he must append decreasing absolute risk aversion, equation (6). The reason for this is that a freely chosen gamble increases mean wealth, which requires us to augment the sufficient conditions.

Notice that inequalities (7) and (10) cannot both hold: (7) demands a negative fourth derivative, and (10) demands a positive fourth derivative. The inequalities apply in different situations, however. Inequality (7) concerns unfavorable bets and describes when bearing one such risk makes the agent less willing to bear another. Inequality (10) concerns favorable bets and describes when bearing such a risk makes the agent even more willing to bear another. The conditions really answer two quite different questions. Since each inequality provides a sufficient but not necessary condition, any contradiction between the answers is more apparent than real.

An important caveat is that this analysis is consciously partial equilibrium, concentrating on the risk of a single bank. If the bank grows by absorbing smaller banks, the total number of loans insured by the system does not change. In a bank with many loans, the profitable loans may offset unprofitable loans and lessen the guarantor's liability. Since the FDIC does not share in the positive profits, it cannot undertake a similar offset if the loans are in different banks. This is not the only scenario, however. The bank may grow at the expense of nonbank intermediaries or by making loans that would not be made without the guarantee. Either case results in new liabilities for the deposit guarantor.

B. An Exponential Example

A simple example can serve to illustrate some of the subtleties involved. To illustrate what can happen, I use an exponential utility function and an exponential distribution. The exponential distribution keeps the algebra simple because sums of exponentials are gamma distributions. Exponential utility exhibits constant, rather than decreasing, absolute risk aversion. It does not satisfy the sufficiency conditions of Kimball ([6] and [7])or of Diamond ([9] and [10]). Therefore, adding risks can sometimes help and sometimes hurt the investor.

Whether diversification helps or hurts depends on the risk premium. If the risk premium decreases as the investor adds i.i.d. risks, diversification helps. If the risk premium increases, diversification hurts. The simplicity of the example allows us to calculate the risk premium explicitly.

Recall from equation (4) that for one loan, the FDIC pays nothing if the loan's payoff exceeds its face value, and otherwise pays the difference. Denoting this function by $g(x)$ (as in section II), the risk premium is defined as the π_1 that satisfies

(11) $u(W_0 + E(g(\tilde{x})) + \pi_1) = Eu(W_0 + g(\tilde{x}))$.

With x following the simplest exponential distribution, e^{-x}, the expected value in (11) becomes

$Eg(\tilde{x}) = (f-1) + e^{-f}$.

Using exponential utility of the form e^{-aW} allows us to solve for π_1 :

$$(12)\ \pi_1 = -\frac{1}{a}\log[e^{-f} - \frac{1}{1+a}e^{af}(e^{-(1+a)f} - 1) + (f-1) + e^{-f}].$$

For two loans, $g(x)$ is zero if x exceeds $2f$ and $2f-x$ otherwise. The random variable x, as the sum of two independent exponentials, has a gamma distribution,

$$x \sim \frac{xe^{-x}}{\Gamma(2)} = \frac{x}{2}e^{-x}.$$

Then the expected value becomes $Eg(\tilde{x}) = (f-1) + (f+1)e^{-2f}$. Solving for the risk premium implicitly defined by $u(W_0 + E(g(\tilde{x})) + \pi_2) = Eu(W_0 + g(\tilde{x}))$ yields

(13)

$$\pi_2 = -\frac{1}{a}\log\{\frac{1}{2}(1+2f)e^{-2f} + \frac{1}{2}\frac{1}{(1+a)^2}e^{2af}[1-(1+2f(1+a))e^{-(1+a)f}](f-1) + (1+f)e^{-2f}\}.$$

To complete the example, set f, the face value of the debt, to 1, and risk aversion to 1 and 2, and evaluate (12) and (13).

risk aversion, a	face value	π_1	π_2
1	1	-0.0659	-0.0640
2	1	-0.1302	-1.1742

This example illustrates two points. First, diversification can work. For low risk aversion, the required risk premium for two loans is lower than for only one. Conversely, for higher risk aversion, adding risks does not help: The risk premium for two risks exceeds twice the risk premium for one risk. Both points emphasize the sufficiency of expressions (6), (7), (9), and (10), because the example satisfies neither set and still illustrates both gains and losses from diversification.

4. Conclusion

Discussions of banking have been obscured by a false analogy with portfolio theory. A bank diversifies differently than does a mutual fund. The bank adds risks, rather than subdivides risks. Using the weak law of large numbers to establish that diversified banks have a lower expected failure rate neglects the deeper question of whether this represents a decrease in economic risk. Clearly posing that question is the main point of this paper.

Just because a bank is less likely to fail does not mean the bank is less risky. If the insurer, or owner, is risk neutral, a more complicated argument shows that the bank is less risky in the sense of expected value. With risk aversion, however, the question become ambiguous. As a practical matter, sufficient conditions exist, and the combination of exponential utility with exponential distributions provides a tractable framework for further exploration.

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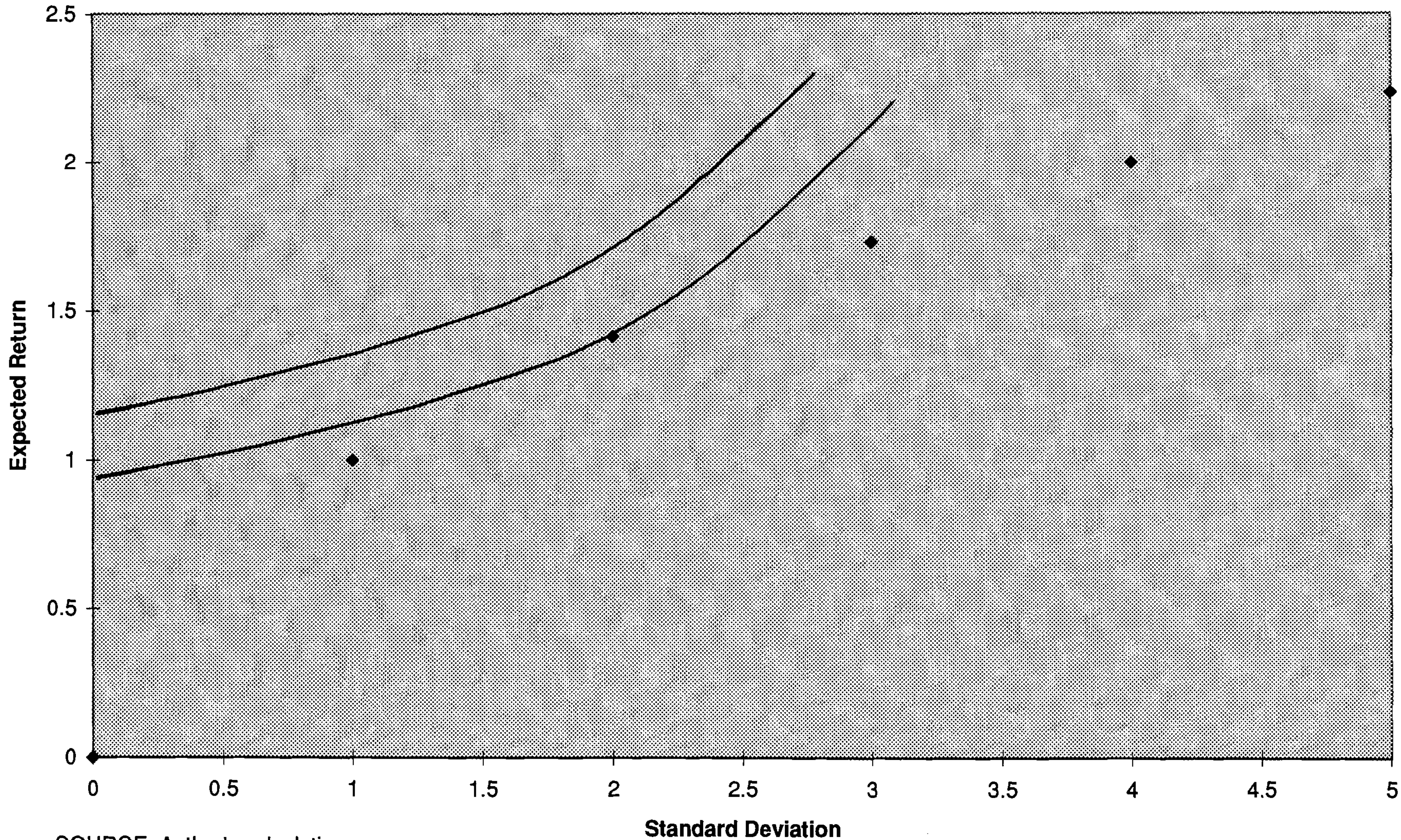
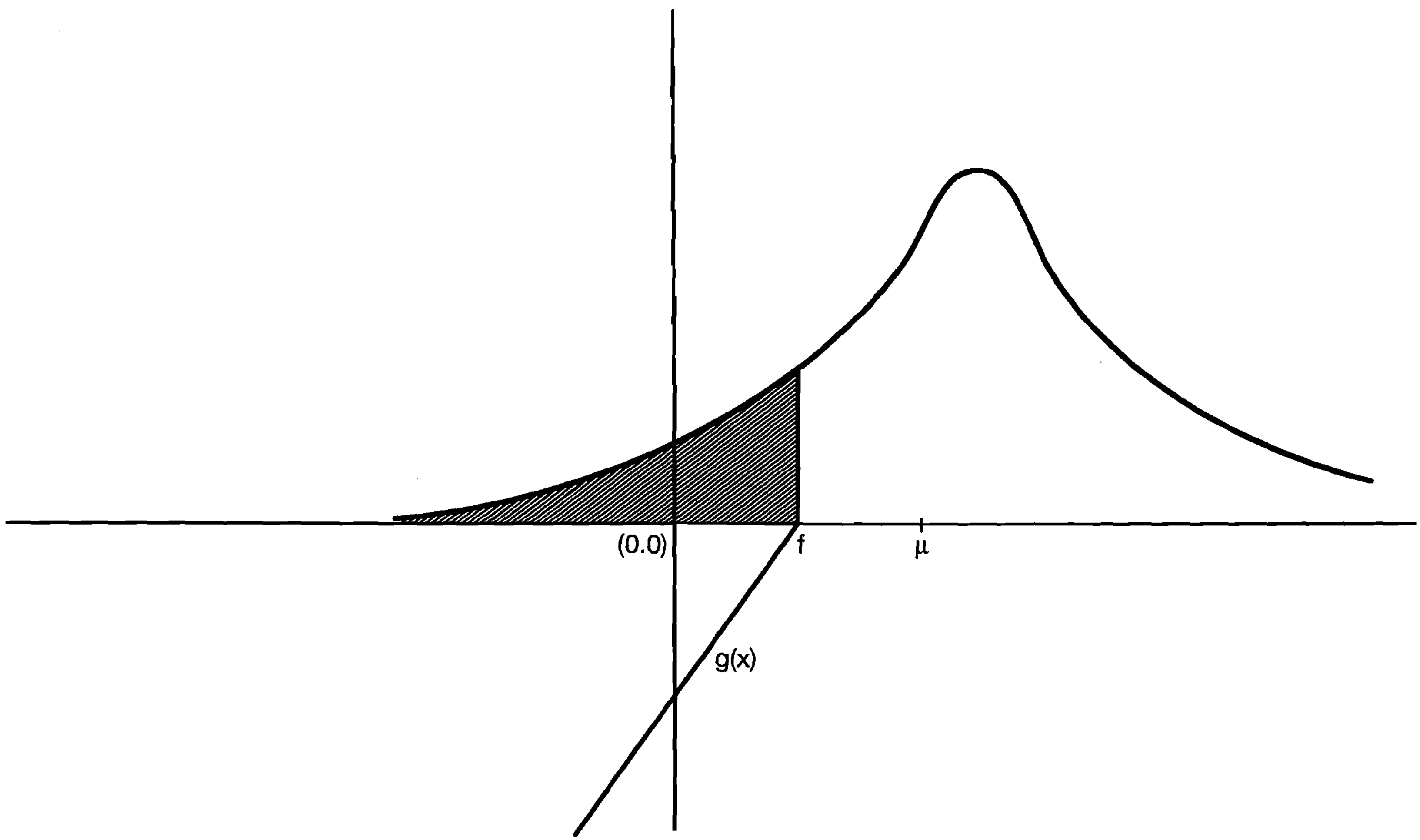
Figure 1: Bank Opportunity Set



SOURCE: Author's calculations.

Figure 2: FDIC Payout—Function and Probability Distribution



SOURCE: Author's calculations.