# A model-free approach to delta hedging

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**Abstract**

Delta hedging, which plays a crucial rôle in modern financial engineering, is a tracking control design for a "risk-free" management. We utilize the existence of trends for financial time series (Fliess M., Join C.: *A mathematical proof of the existence of trends in financial time series*, Proc. Int. Conf. Systems Theory: Modelling, Analysis and Control, Fes, 2009. Online: `http://hal.inria.fr/inria-00352834/en/`) in order to propose a model-free setting for delta hedging. It avoids most of the shortcomings encountered with the now classic Black-Scholes-Merton setting. Several convincing computer simulations are presented. Some of them are dealing with abrupt changes, *i.e.*, jumps.

*Keywords*—Delta hedging, trends, quick fluctuations, abrupt changes, jumps, tracking control, model-free control.

# 1 Introduction

*Delta hedging*, which plays an important rôle in financial engineering (see, *e.g.*, [24] and the references therein), is a tracking control design for a "risk-free" management. It is the key ingredient of the famous Black-Scholes-Merton (BSM) partial differential equation ([3, 22]), which yields option pricing formulas. Although the BSM equation is nowadays utilized and taught all over the world (see, *e.g.*, [18]), the severe assumptions, which are at its bottom, brought about a number of devastating criticisms (see, *e.g.*, [6, 16, 17, 20, 25, 26] and the references therein), which attack the very basis of modern financial mathematics and therefore of delta hedging.

We introduce here a new dynamic hedging, which is influenced by recent works on *model-free* control ([8, 10]), and bypass the shortcomings due to the BSM viewpoint:

- In order to avoid the study of the precise probabilistic nature of the fluctuations (see the comments in [9, 11]), we replace the various time series of prices by their *trends* [9], like we already did for redefining the classic beta coefficient [12].
- The control variable satisfies an elementary algebraic equation of degree 1, which results at once from the *dynamic replication* and which, contrarily to the BSM equation, does not need cumbersome final conditions.
- No complex calibrations of various coefficients are required.

**Remark 1.1** *Connections between mathematical finance and various aspects of control theory has already been exploited by several authors (see, e.g., [2, 23] and the references therein). Those approaches are however quite far from what we are doing.*

Our paper is organized as follows. The theoretical background is explained in Section 2. Section 3 displays several convincing numerical simulations which

- describe the behavior of $\Delta$ in "normal" situations,
- suggest new control strategies when abrupt changes, *i.e.*, jumps, occur, and are forecasted via techniques from [13] and [11, 12].

Some future developments are listed in Section 4.

# 2 The fundamental equations

## 2.1 Trends and quick fluctuations in financial time series

See [9], and [11, 12], for the definition and the existence of *trends* and *quick fluctuations*, which follow from the Cartier-Perrin theorem [4].[1] Calculations of the trends and of its derivatives are deduced from the denoising results in [14, 21] (see also [15]), which generalize the familiar *moving average* techniques in technical analysis (see, *e.g.*, [1, 19]).

[1] The connections with *technical analysis* (see, *e.g.*, [1, 19]) are obvious (see [9] for details).

## 2.2 Dynamic hedging

### 2.2.1 The first equation

Let $\Pi$ be the value of an elementary portfolio of one long option position $V$ and one short position in quantity $\Delta$ of some underlying $S$:

$$\Pi = V - \Delta S \tag{1}$$

Note that $\Delta$ is the control variable: the underlying asset is sold or bought. The portfolio is *riskless* if its value obeys the equation

$$d\Pi = r(t)\Pi dt$$

where $r(t)$ is the risk-free rate interest of the equivalent amount of cash. It yields

$$\Pi(t) = \Pi(0)\exp\int_0^t r(\tau)d\tau \tag{2}$$

Replace Equation (1) by

$$\Pi_{\text{trend}} = V_{\text{trend}} - \Delta S_{\text{trend}} \tag{3}$$

and Equation (2) by

$$\Pi_{\text{trend}} = \Pi_{\text{trend}}(0)\exp\int_0^t r(\tau)d\tau \tag{4}$$

Combining Equations (3) and (4) leads to the tracking control strategy

$$\boxed{\Delta = \frac{V_{\text{trend}} - \Pi_{\text{trend}}(0)e^{\int_0^t r(\tau)d\tau}}{S_{\text{trend}}}} \tag{5}$$

We might again call *delta hedging* this strategy, although it is of course an approximate dynamic hedging via the utilization of trends.

### 2.2.2 Initialization

In order to implement correctly Equation (5), the initial values $\Delta(0)$ and $\Pi_{\text{trend}}(0)$ of $\Delta$ and $\Pi_{\text{trend}}$ have to be known. This is achieved by equating the logarithmic derivatives at $t = 0$ of the right handsides of Equations (3) and (4). It yields

$$\boxed{\Delta(0) = \frac{\dot{V}_{\text{trend}}(0) - r(0)V_{\text{trend}}(0)}{\dot{S}_{\text{trend}}(0) - r(0)S_{\text{trend}}(0)}} \tag{6}$$

and

$$\boxed{\Pi_{\text{trend}}(0) = V_{\text{trend}}(0) - \Delta(0)S_{\text{trend}}(0)} \tag{7}$$

**Remark 2.1** *Let us emphasize once more that the derivation of Equations* (5), (6) *and* (7) *does not necessitate any precise mathematical description of the stochastic process $S$ and of the volatility. The numerical analysis of those equations is moreover straightforward.*

## 2.3 A variant

When taking into account variants like the *cost of carry* for commodities options (see, *e.g.*, [27]), replace Equation (3) by

$$d\Pi_{\text{trend}} = dV_{\text{trend}} - \Delta dS_{\text{trend}} + q\Delta S_{\text{trend}} dt$$

where $qSdt$ is the amount required during a short time interval $dt$ to finance the holding. Combining the above equation with

$$d\Pi_{\text{trend}} = r\Pi_{\text{trend}}(0)\left(\exp\int_0^t r(\tau)d\tau\right)dt$$

yields

$$\Delta = \frac{\dot{V}_{\text{trend}} - r\Pi_{\text{trend}}(0)\left(\exp\int_0^t r(\tau)d\tau\right)}{\dot{S}_{\text{trend}} - qS_{\text{trend}}}$$

The derivation of the initial conditions $\Delta(0)$ and $\Pi_{\text{trend}}(0)$ remains unaltered.

# 3 Numerical simulations

## 3.1 Two examples of delta hedging

Take two derivative prices: one put (CFU9PY3500) and one call (CFU9CY3500). The underlying asset is the CAC 40. Figures 1-(a), 1-(b) and 1-(c) display the daily closing data. We focus on the 223 days before September $18^{th}$, 2009. Figures 2-(a) and 2-(b) (resp. 3-(a) and 3-(b)) present the stock prices and the derivative prices during this period, as well as their corresponding trends. Figure 3-(c) shows the daily evolution of the risk-free interest rate, which yields the tracking objective. The control variable $\Delta$ is plotted in Figure 3-(d).

## 3.2 Abrupt changes

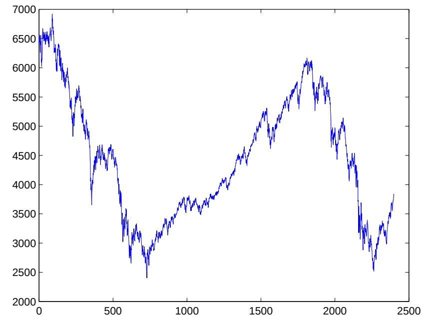
### 3.2.1 Forecasts

We assume that an abrupt change, *i.e.*, a jump, is preceded by "unusual" fluctuations around the trend, and further develop techniques from [13], and from [11, 12]. In Figure 4-(a), which displays forecasts of abrupt changes, the symbols $o$ indicate if the jump is upward or downward.

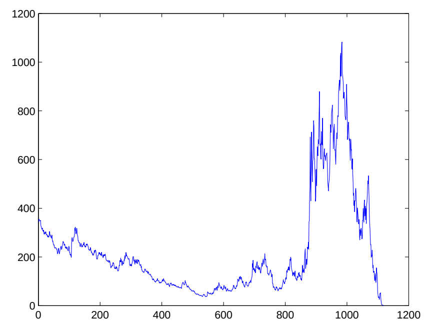
### 3.2.2 Dynamic hedging

Taking advantage of the above forecasts allows to avoid the risk-free tracking strategy (5), which would imply too strong variations of $\Delta$ and cause some type of market illiquidity. The Figures 4-(b,c,d) show some preliminary attempts, where other less "violent" open-loop tracking controls have been selected.

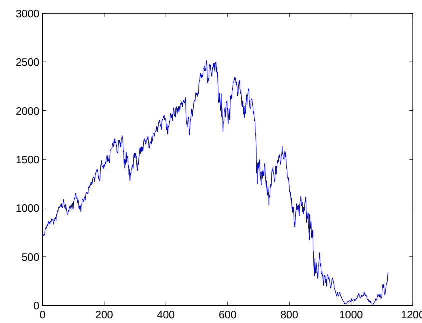
**Remark 3.1** *Numerous types of dynamic hedging have been suggested in the literature in the presence of jumps (see, e.g., [5, 22, 27] and the reference therein). Remember [7] moreover the well known lack of robustness of the BSM setting with jumps.*




*(a)* Underlying asset: daily values of the CAC from 28 April 2000 until 18 September 2009




*(b)* Option: CFU9PY3500 daily prices from 9 May 2009 until 18 September 2009




*(c)* Option: CFU9CY3500 daily prices from 9 May 2009 until 18 September 2009

Figure 1: Daily data

# 4 Conclusion

Lack of space prevented us from examining more involved options, futures, and other derivatives, than in Section 2.3. Subsequent works will do that, and also introduce several time scales thanks to the *nonstandard analytic* framework of the Cartier-Perrin theorem [4].

**Acknowledgement**. The authors would like to thank Frédéric HATT for stimulating discussions.

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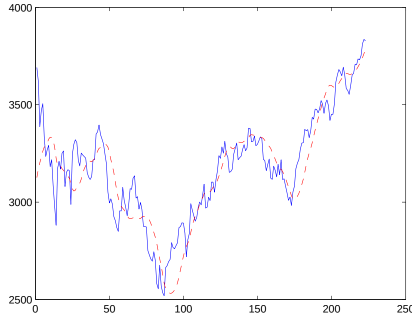
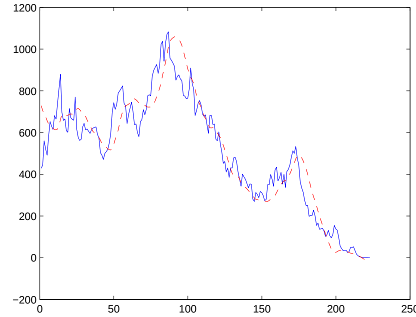
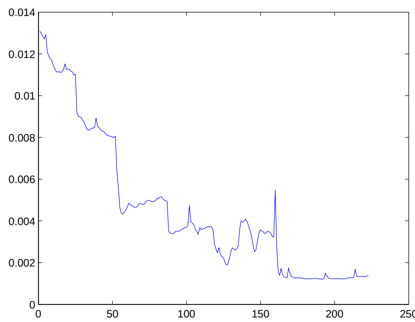


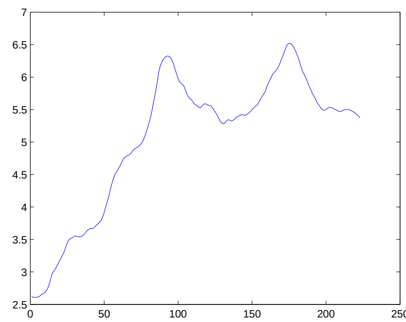

*(a)* Underlying asset: daily values during the last 223 days, and trend (- -)




*(b)* Option: daily values during the last 223 days, and trend (- -)




*(c)* Daily interest rate $r$




*(d)* $\Delta$ tracking

Figure 2: Example 1: CFU9PY3500

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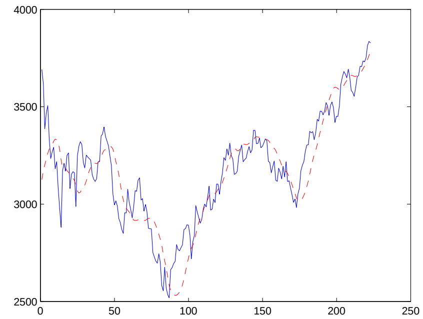
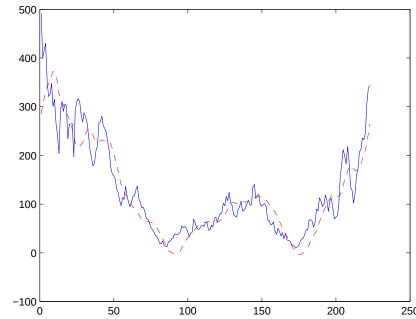
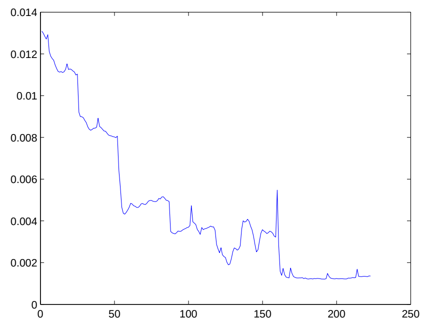


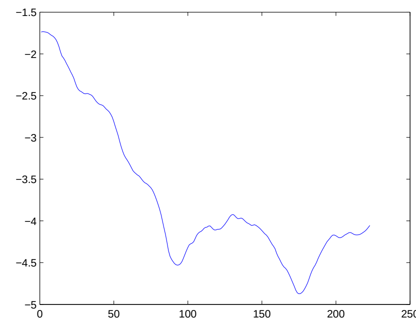

*(a)* Underlying asset: values during the last 223 days, and trend (- -)




*(b)* Option: values during the last 223 days, and trend (- -) (- -)




*(c)* Daily interest rate $r$




*(d)* $\Delta$ tracking

Figure 3: Example 2: CFU9CY3500

El Jai, L. Afifi, E. Zerrik (Eds), Presses Universitaires de Perpignan, 2009, pp. 43–62 (available at `http://hal.inria.fr/inria-00352834/en/`).

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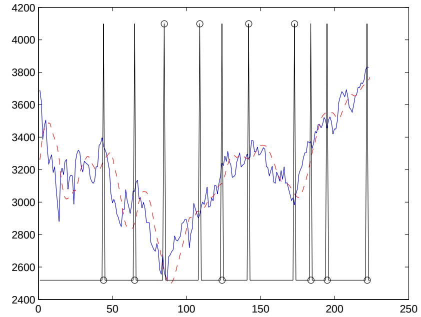
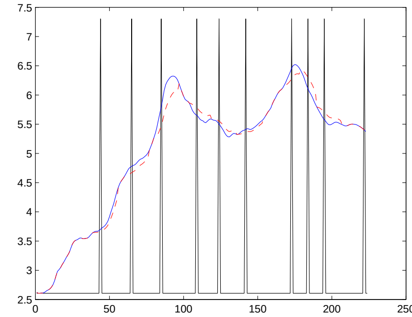
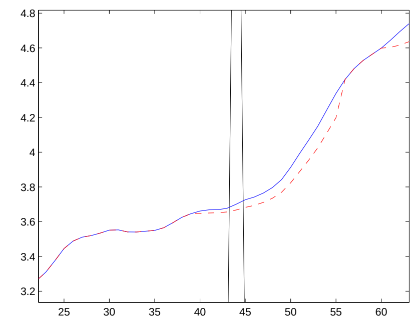


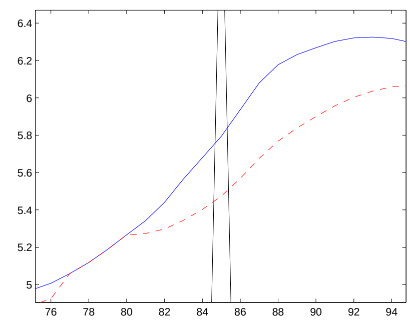

*(a)* Underlying (–), trend (- -), prediction of abrupt change locations (l) and their directions (o)




*(b)* Risk-free Δ tracking (–) and Δ tracking (- -), prediction of abrupt change locations (l)




*(c)* Zoom on (b)




*(d)* Zoom on (b)

Figure 4: Example 1 (continued): CFU9PY3500

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