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1. Introduction

Until recently, technological change was treated in innovation models mainly as an exogenous force. This approach made it possible to study some consequences of the introduction of new technologies without considering their diffusion mechanism. However, the relevance of an endogenous theory of economic evolution was recognized many years ago. Back in 1939, J. Schumpeter (see, e.g., [1]) developed a conceptual framework for the diffusion of innovations which divided the mechanism of technological change into two components: development of new technologies (innovation proper) and adoption of new technologies (imitation). Little is known about the more delicate first component, despite considerable statistical material which points to certain regularities governing the development of major technological changes. The imitation process has been studied in greater detail and on a more rigorous level. Imitation is the basic element of the diffusion of innovations from innovative firms to all other firms.

It is natural to assume that the rate of diffusion is proportional to the fraction of firms that have not adopted the innovation. Very fruitful is the simple thesis that the corresponding proportionality coefficient increases linearly with the increase of the fraction of firms that have adopted the new technology. This thesis leads to the conclusion that the fraction of firms that have adopted the innovation should vary over time according to the logistic curve. Empirical studies (see, e.g., [2-7]) have indeed shown that for most innovations the adoption

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curves are S-shaped and close to the logistic curve; in some cases, the logistic curve provides a good fit to the statistical data. A model based on entirely different considerations which also leads to a S-shaped diffusion curve was proposed in [8].

Another observation of fundamental importance for our purposes is that technologies characterized by different efficiencies coexist at any given time in the economy, even in single-commodity industries. This fact is the basis of the highly original theory of production functions traceable to Houthakker and Johansen [9] and further developed in [10] (see also [11,12]). The distribution curves of capacity by efficiency levels in a given industry are found to be similar at different time moments. In many cases, these curves are unimodal. Moreover, there is also a certain similarity between the capacity distribution curves of different industries.¹ This contradicts the standard micro-economic theory which claims that investments should be restricted only to the most efficient (most profitable) technologies, and the fraction of low-return industrial capacities should be negligible. From the classical viewpoint, the existence of technologies of different efficiencies is evidence of economic disequilibrium.

The purpose of this paper is to show how both these facts—the “logistic” character of the diffusion curves and the stable form of capacity distribution curves by efficiency levels—are the consequence of a “dynamic equilibrium” between innovation and imitation. The problem was first considered by Iwai [13] (see also [14]), whose paper has had a major influence on the development of our ideas. However, the solution technique developed in [13] is not fully satisfactory in our opinion. Iwai characterizes each efficiency level by the corresponding unit cost and assumes that the minimum unit cost decreases over time exponentially. Initially, when a certain minimum unit cost is achieved, it characterizes only one particular firm, but then imitation begins. It is also assumed that all firms have equal opportunities of moving to any higher technological level. Because of this and some additional assumptions, the rate of change of the distribution function at any point is determined only by its values at that very point, and its variation over time can be found. In the diffusion curves obtained in this way, the current time and the initial time of diffusion are functions of the corresponding minimum unit costs. Thus, Iwai derives a distribution of firms by the ratio of the unit cost of a given level to the current minimum unit cost.

In our opinion, the assumption of equiprobable “jumps” to any higher level is too strong. It is more plausible to postulate an alternative “extreme” assumption that allows transition only to the next higher level.² Even less satisfactory is the assumption of exogenous changes of the unit costs, which largely predetermines the rate of tech-

nological evolution. It is desirable to explain the observed facts by examining the interaction of innovation and imitation. The alternative model proposed in this paper is largely free from these shortcomings.

Section 2 presents a differential-difference equation which describes in continuous time the evolution of the distribution curve of firms by discrete efficiency levels. An explicit solution of this equation is obtained for arbitrary finite initial conditions (Section 3) and we show that a one-parametric family of waves—functions of a linear combination of both arguments—satisfies this solution. These waves represent all the possible shifts of the logistic probability distribution moving with constant velocity.

Section 4 presents the main theorem which asserts that an arbitrary solution exponentially converges to one of the waves. This theorem explains the S-shape of observed diffusion curves and the stable form of the distribution of firms by efficiency levels. Section 5 briefly reviews some possible generalizations, modifications, and applications of the proposed evolution equation. Section 6 contains the proofs of all the propositions.

2. The evolution equation

Consider a production system (e.g., an industry) which includes several firms ordered by efficiency levels. The concept of efficiency level admits different definitions. Thus, in [14] the measure of efficiency is the cost per unit added value, in [10] the specific wage, in [15,16] the degree of automation and mechanization of production as determined by a special classification. For our purposes, the particular efficiency criterion is immaterial, and it is only relevant that each firm attempts to move to a level with a higher index.

Denote by $F_n(t)$ the fraction of firms that at the moment t are on levels with index not higher than n . Here $t \in [0, \infty)$, and n may take any nonnegative integer value, $n=0, 1, \dots$. The symbol F_n stands for the corresponding time function. The sequence of F_n treated as a time function describes the evolution of the distribution curve of firms by efficiency levels. We assume that this evolution obeys the following rule.

Main thesis. A firm may move only to the next highest level. The number of firms passing in unit time from level n to level $n+1$ is proportional to the number of firms on level n at the given time, and the proportionality coefficient is a linear function of the fraction of firms on all levels higher than n .

Since the total number of firms is assumed fixed, we easily see that by the main thesis the function F_n is nonincreasing for any n . The rate

of decrease is determined by the rate of transition of firms from level n to level $n+1$ (the fraction of firms on level n is $F_n - F_{n-1}$). Thus,

$$dF_n/dt = -(\alpha + \beta(1 - F_n(t))(F_n(t) - F_{n-1}(t)), \quad (1)$$

where α and β are positive constants. The right-hand side of (1) can be represented as the sum of two terms, $\alpha(F_n - F_{n-1})$ and $\beta(1 - F_n)(F_{n-1} - F_n)$. The first term characterizes the innovation process proper, i.e., spontaneous invention and operationalization. Its rate is proportional to the fraction (or, equivalently, the number) of n -th level firms with proportionality coefficient α . The second term makes a significant contribution only if a substantial part of firms are on efficiency levels higher than n . It determines the rate of imitation—the process of adoption of technologies and experience. Because of imitation, the production techniques operationalized by more innovative firms “diffuse” and are adopted by the rest of the firms. Both processes distort the distribution curve, moving it toward higher efficiency. The evolution of this curve described by the equation (1) is the main topic of our paper.

The equation (1) has a simple probabilistic interpretation. Assume that the probability of transition of a firm from level n to level $n+1$ in unit time is α in the innovation process and $\beta(1 - F_n)$ in the imitation process, and both processes are independent. Then the transition probability is $\alpha + \beta(1 - F_n)$, where $\beta = \beta(1 - \alpha)$. Thus, the right-hand side of (1) is interpreted as the mean rate of change of the fraction of firms on levels up to n .

3. Solutions of the evolution equation

We use the following initial conditions and a natural boundary condition:

$$0 \leq F_n(0) \leq 1, \quad 1 \leq n < \infty; \quad \lim_{n \rightarrow \infty} F_n(0) = 1; \quad F_0(t) = 0 \quad \text{for all } t \geq 0. \quad (2)$$

In this case, the equation (1) can be solved in explicit form.

Introduce new variables z_n , $0 \leq n < \infty$, such that

$$F_n = \beta^{-1}(\mu - z_{n-1}/z_n), \quad 1 \leq n < \infty, \quad z_0 = e^{\mu t}, \quad \mu = \alpha + \beta. \quad (3)$$

The substitution (3) reduces the original equation (1) to the form³

$$dz_n/dt = z_{n-1}, \quad 1 \leq n < \infty, \quad z_0 = e^{\mu t},$$

or

$$d^n z_n / dt^n = e^{\mu t}.$$

Therefore

$$z_n = \mu^{-n} e^{\mu t} + \sum_{k=0}^{n-1} c_{n-k} t^k / k!,$$

where c_n are arbitrary constants. Clearly, $c_n = z_n(0) - \mu^{-n}$, and thus

$$z_n = \mu^{-n} h_n + \sum_{k=0}^{n-1} z_{n-k}(0) t^k / k!, \quad 1 \leq n < \infty, \quad (4)$$

where

$$h_n(t) = \sum_{k=n}^{\infty} (\mu t)^k / k!. \quad (5)$$

Moreover, from (3) we obtain

$$z_n(0) = \prod_{k=1}^n w_k(0), \quad (6)$$

where $w_k = (\mu - \beta F_k(0))^{-1}$. It is helpful to introduce the notation

$$\xi_n = \frac{z_{n-1} - \alpha z_n}{\mu z_n - z_{n-1}} \quad (7)$$

From (3) we obtain

$$F_n = (1 + \xi_n)^{-1} \quad (8)$$

and substituting (4) in (7) we obtain

$$\xi_n = (\alpha/\mu)^n (h_n + q_n) / r_n \quad (9)$$

where

$$q_n = \mu^n \sum_{k=0}^{n-1} z_{n-k}(0) (1 - F_{n-k}(0)) t^k / k!, \quad (10)$$

$$r_n = \alpha^n \sum_{k=0}^{n-1} z_{n-k}(0) (F_{n-k}(0)) t^k / k!. \quad (11)$$

The formulas (5)-(11) are the solution of the given Cauchy problem (1), (2).

Of particular interest are initial conditions containing only a finite number N of values different from 1:

$$0 < F_k(0) < 1, \quad 1 \leq k \leq N; \quad F_k(0) = 1, \quad k \geq N+1. \quad (12)$$

In this case,

$$\mu^{-1} < w_n(0) < \alpha^{-1}, \quad 1 \leq n \leq N; \quad w_n(0) = \alpha^{-1}, \quad n > N, \quad (13)$$

$$z_n(0) = \alpha^{-n} P, \quad n > N; \quad P = \alpha^N \prod_{k=1}^N w_k(0), \quad (14)$$

so that the formulas (10), (11) can be refined.

Now consider the equation (1) on the entire time axis. A direct check shows that it has the family of solutions

$$F_n^*(t, A) = \left[1 + A(\alpha/\mu)^n e^{\beta t} \right]^{-1}, \quad (15)$$

where A is an arbitrary parameter. Indeed, let

$$v_n = \beta t - n \ln(\mu/\alpha) + \ln A.$$

Then $F_n^* = (1 + e^{v_n})^{-1}$ and it is easy to show that

$$dF_n^*/dt = -\beta(1 - F_n^*)F_n^*, \quad F_n^* - F_{n-1}^* = \beta(1 - F_n^*)F_n^*/(\mu - \beta F_n^*),$$

which gives the sought result.⁴ The function F_n^* tends to zero as $n \rightarrow -\infty$ and to 1 as $n \rightarrow \infty$ for any fixed t . It depends only on the difference $t - \gamma_0 n$,

$$\gamma_0 = \beta^{-1} \ln(\mu/\alpha), \quad (16)$$

and defines a wave moving uniformly along the axis n with constant velocity $1/\gamma_0$; during the time γ_0 , it advances one level to the right. For a fixed t , it describes a so-called logistic probability distribution, and for a fixed n it defines a decreasing logistic curve describing the variation over time of the fraction of firms on efficiency level n . Changing the parameter A shifts the entire distribution. For $t = t(n) = \gamma_0 n - \beta^{-1} \ln A$, the maximum fraction of firms are concentrated on level n .

The family (15) is central to the analysis of the asymptotic behavior of solutions.

4. Asymptotic behavior of solutions

Before tackling the main problem of this section—analysis of the behavior of solutions as $t \rightarrow \infty$ —we should note two simple but important facts.⁵

Proposition 1. If $F_n(0) \geq F_{n-1}(0)$ for all n , then $F_n(t) \geq F_{n-1}(t)$ for all n, t .

Thus, equation (1) indeed defines the evolution of the distribution curve.

Let

$$B(t) = \prod_{k=1}^{\infty} \left[1 + (\beta/\alpha)(1 - F_k(t)) \right]. \quad (17)$$

Proposition 2. Let $F_n, 1 \leq n < \infty$, be a solution of the problem (1), (2), and

$$\sum_{k=1}^{\infty} (1 - F_k(0)) < \infty.$$

Then for any $t \geq 0$ we have the equality

$$e^{-\beta t} B(t) = B(0) < \infty.$$

In particular, Proposition 2 holds if the initial conditions satisfy (12). The expression $e^{-\beta t} B(t)$ is a first integral for this class of solutions.

The following two theorems are the main results of this paper.

Theorem 1. Let $F_n, 1 \leq n < \infty$, be a solution of the problem (1), (2) and $B(0) < \infty$. For any $\epsilon > 0$ there is a number $T(\epsilon)$ such that for $A = B(0)$ we have the inequalities

$$|F_n(t) - F_n^*(t, A)| \leq \epsilon$$

for all $t \geq T(\epsilon)$ and all $n \geq 0$. Here the function F_n^* is defined by formula (15).

Theorem 2. Let $F_n, 1 \leq n < \infty$, be a solution of the problem (1), (2), (12). For $A = B(0)$ we have the bound⁶

$$|F_n(t) - F_n^*(t, A)| \leq \lambda e^{-\gamma t}, \quad 0 \leq n < \infty, \quad t \geq T_0,$$

where λ, T_0 are constants dependent on $\alpha, \beta, B(0), N, \gamma = \gamma(\alpha, \beta) = \min\{\gamma_1, \gamma_2\}$, and the constants γ_1, γ_2 are the unique solutions of the equations

$$\frac{\beta + \mu}{\beta + \gamma_1} \ln(\mu/\alpha) + \ln(\beta + \gamma_1) = 1 + 2 \ln \mu - \ln \alpha + \ln \ln(\mu/\alpha), \quad (18)$$

$$\frac{\alpha - \beta}{\beta + \gamma_1} \ln(\mu/\alpha) + \ln(\beta + \gamma_2) = 1 - \ln \mu + 2 \ln \alpha + \ln \ln(\mu/\alpha),$$

in the intervals

$$0 < \gamma_1 < \mu \ln(\mu/\alpha) - \beta \quad \text{and} \quad 0 < \gamma_2 < \beta - \alpha \ln(\mu/\alpha). \quad (19)$$

Remark. For $\beta \ll \alpha$ and for $\beta \gg \alpha$, γ_1 is greater than γ_2 . However, for $\beta \approx \alpha$, we have $\gamma_1 < \gamma_2$.

Note that (14), (6), and (17) lead to the equality $B(0) = 1/P$. Theorem 1 will be proved as a corollary of Theorem 2 in the last section.

If the initial distribution is $F_n(0) = F_n^*(0, A)$, then it is easy to see that

$$e^{-\beta t} B^*(t) = A + e^{-\beta t}$$

The expression in the right-hand side depends on t , because in this case $F_0(0) > 0$ but it nevertheless tends to A as $t \rightarrow \infty$. This explains to a certain extent the appearance of the constant A in Theorems 1 and 2.

Let us consider the substantive meaning of these propositions. Assume that initially the firms are arbitrarily distributed over a finite number of efficiency levels. Innovation shifts the distribution curve to the right. The rate of transition to more efficient technologies is increased by imitation, which distorts the distribution curve. By Theorem 1, due to interaction of the two processes, the "rate of increase of efficiency" eventually approaches a fixed value $\beta/\ln(\mu/\alpha)$ (see (16)) which depends on constants of innovation and imitation, and the distribution approaches the logistic distribution. The fraction of firms on any given level decreases over time according to a logistic curve. Theorem 2 strengthens Theorem 1, asserting that an arbitrary initial distribution converges to the logistic distribution at an exponential rate, i.e., fairly fast. Thus, both facts noted in the Introduction—the S-shaped diffusion curves and stability of the distribution curves by efficiency levels—are explained by the proposed model.⁷

5. Some generalizations and modifications

Let us consider some natural extensions of the proposed theory. It can be shown that an analog of Theorem 1 holds also for much more general equations than equation (1):

$$dF_n/dt = \varphi(F_n)(F_{n-1} - F_n), \quad (20)$$

where φ is a continuous positive decreasing function on the interval $[0, 1]$. This will be proved in a separate paper. Even if φ depends on n , the solutions of (20) that correspond to different initial conditions but have the same first integral apparently converge to one another.

In the derivation of the basic equation, we assumed that firms may

only move to the next higher level as a result of innovation and imitation. A diametrically opposite assumption allows transition with equal probability to any higher level as a result of imitation. If the innovation mechanism remains as before, we obtain the equation

$$dF_n/dt = -\alpha(F_n - F_{n-1}) - \beta(F_n)(1 - F_n), \quad (21)$$

whose solutions apparently also converge to some wave.

As a result of asset depreciation, some firms may move to lower efficiency levels. The introduction of this factor gives rise to terms dependent on F_{n+1} in the right-hand side of the equations. For example, the simplest modification of the equation (20) in this case has the form

$$dF_n/dt = q(F_n)(F_{n-1} - F_n) + \mu(F_{n+1} - F_n), \quad (22)$$

The model (1), as well as its modifications, does not describe economic growth. We hope that in many relevant cases the study of growth models with an interaction of innovation and imitation will be reduced, by an appropriate substitution of variables, to the investigation of the above equations or their perturbations. We have actually developed some examples of such models. Note that in this case the equation (1) acquires a somewhat different interpretation: it describes the distribution of capacity (and not firms) by efficiency levels.

Theorems 1 and 2 are related to a wide class of results obtained for dynamic processes in physics, biology, chemical kinetics, and some other areas [17, 18]. The equation (1), being apparently the first example of a nonlinear economic equation with a stable single wave, is asymptotically close to the Burgers equation (see Hopf's theorem in [17]), but unlike the latter it is a differential-difference equation. Because of this special feature, it does not rely on any assumptions of transitions between "infinitely close" levels. Yet the interaction has an infinite propagation velocity in this case: it is easy to see that at any nonzero time moment there are firms on arbitrarily high efficiency levels in model (1). The fraction of these firms rapidly diminishes, so that the distribution may be approximately regarded as concentrated on an interval of fixed length. Nevertheless, elimination of the infinite propagation velocity of interactions appears to provide another plausible direction for improving the model.

6. Proofs

Proof of Proposition 1. Let the functions $F_n(t)$ satisfy the equation (1) under the conditions (2) and let $F_n(0) \geq F_{n-1}(0)$ for all n . Set $f_n =$

$F_n - F_{n-1}$. Then from (1) we obtain

$$df_n/dt = -(\mu - \beta F_n)f_n + (\mu - \beta F_{n-1})f_{n-1}. \quad (23)$$

The proof is by induction. Clearly, $f_1(t) \geq 0$ for all t , and without loss of generality we may take $f_1(0) > 0$. Then $f_1(t) > 0$ for all t . Indeed, if t_0 is the first time moment such that $f_1(t_0) = 0$, then for $t \in [0, t_0)$ we have $df_1/dt = -\mu f_1$. Thus, $0 = f_1(t_0) \geq f_1(0)e^{-\mu t_0} > 0$, a contradiction.

Now assume that $f_k(t) > 0$ for $t > 0$ for any $k \leq n-1$, but $f_n(t_0) = 0$, $t_0 > 0$. We may assume that $f_n(t) \geq 0$, $t \leq t_0$. On the other hand, for t close to t_0 , by (23) and the inequality $f_{n-1} > 0$, we obtain df_n/dt , so that this case is ruled out. Therefore, $f_n(t) > 0$ for all $t > 0$. Q.E.D.

Proof of Proposition 2. The equation (1) is equivalent to the equation

$$\frac{d}{dt} (1/\beta) \ln((\mu - \beta F_n)/\alpha) = F_n - F_{n-1}. \quad (24)$$

Summing over n , we obtain after simple manipulations

$$\frac{d}{dt} \ln e^{-\beta t} \prod_{k=1}^n V_k = \beta(F_n - 1), \quad (25)$$

where $V_k = 1 + (\beta/\alpha)(1 - F_k)$. Integration gives

$$\ln e^{-\beta t} \prod_{k=1}^n V_k(t) = \ln \prod_{k=1}^n V_k(0) + \int_0^t \beta(F_n(\tau) - 1) d\tau. \quad (26)$$

Take a fixed t . By assumption, the right-hand side of (26) is bounded, and therefore the product $\prod_{k=1}^n V_k(t)$ is convergent. Therefore $F_n(t) \rightarrow 1$ as $n \rightarrow \infty$. As this is true for any $\tau \in [0, t]$, the integral in the right-hand side of (26) tends to zero as $n \rightarrow \infty$. Thus, (26) leads to the sought proposition. Q.E.D.

Proof of Theorem 1 using Theorem 2. Let

$$B_n(t) = \prod_{k=1}^n \left[1 + (\beta/\alpha)(1 - F_k(t)) \right], \quad B_0(t) = 1. \quad (27)$$

Take a fixed $\epsilon > 0$. The condition $A = B(0) = \lim_{n \rightarrow \infty} B_n(0)$ implies the existence of $N = N(\epsilon)$ such that

$$0 \leq A - B_N(0) < A\epsilon \quad \text{and} \quad \sup_{k \geq N+1} (1 - F_k(0)) < \epsilon. \quad (28)$$

Take G , $0 \leq G \leq 1$, such that

$$B_N(0) \left[1 + (\beta/\alpha)(1 - G) \right] = A. \quad (29)$$

From (28) and (29) we obtain the inequality

$$1 - G < (\alpha/\beta)\epsilon. \quad (30)$$

Let $\tilde{F}_0(t)$, $n=0, 1, \dots$, be a solution of the equation (1) with the boundary and initial conditions

$$\begin{aligned} \tilde{F}_0(t) &= 0, \quad \tilde{F}_k(0) = F_k(0), \quad 1 \leq k \leq N, \\ \tilde{F}_{N+1}(0) &= G, \quad \tilde{F}_k(0) = 1, \quad k \geq N+2. \end{aligned} \quad (31)$$

The conditions (29), (31) ensure the equalities

$$\lim_{n \rightarrow \infty} \tilde{B}_n(0) = \lim_{n \rightarrow \infty} \prod_{k=1}^n \left[1 + (\beta/\alpha)(1 - \tilde{F}_k(0)) \right] = \tilde{B}_{n+1}(0) = A. \quad (32)$$

This and Theorem 2 lead to the bound

$$|\tilde{F}_n(t) - F_n^*(t, A)| \leq \epsilon, \quad 0 \leq n < \infty, \quad (33)$$

for all $t \geq T(\epsilon)$.

We will now prove that

$$|\tilde{F}_n(t) - F_n(t)| = 0(\epsilon). \quad (34)$$

Introduce the variables z_n and \tilde{z}_n , $0 \leq n < \infty$, such that

$$F_n = \beta^{-1}(\mu - z_{n-1}/z_n), \quad \tilde{F}_n = \beta^{-1}(\mu - \tilde{z}_{n-1}/\tilde{z}_n), \quad z_0 = \tilde{z}_0 = e^{\mu t}. \quad (35)$$

Substituting (35) in (27), we obtain

$$B_n(t) = \alpha^{-n} \prod_{k=1}^n z_{k-1}/z_k = \alpha^{-n} z_0/z_n = \alpha^{-n} e^{\mu t}/z_n.$$

Hence

$$z_n = \alpha^{-n} e^{\mu t}/B_n(t) \quad (36)$$

Similarly,

$$\tilde{z}_n = \alpha^{-n} e^{\mu t}/\tilde{B}_n(t).$$

Now let

$$\Delta_n(t) = z_n(t)/\tilde{z}_n(t).$$

From (32), (36) and Proposition 2 we obtain the equality

$$\lim_{n \rightarrow \infty} \Delta_n(t) = \lim_{n \rightarrow \infty} (\tilde{B}_n(t)/B_n(t)) = \tilde{B}_{N+1}(0)/B(0) = 1. \quad (37)$$

Now use the equations (see Sec. 3)

$$dz_n/dt = z_{n-1} \quad \text{and} \quad d\tilde{z}_n/dt = \tilde{z}_{n-1}. \quad (38)$$

These equations lead to the following equation for Δ_n :

$$\Delta_n^{-1} d\Delta_n(t)/dt = (\tilde{z}_{n-1}/\tilde{z}_n)(\Delta_{n-1}(t)/\Delta_n(t) - 1) = (\mu - \beta\tilde{F}_n)(\Delta_{n-1}/\Delta_n - 1). \quad (39)$$

Using (39), we will prove the bound

$$\sup_{n \geq 0, t \geq 0} \Delta_n(t) \leq \sup_n \Delta_n(0). \quad (40)$$

Take a fixed $T > 0$. By (37), $\sup_{n \geq 0, t \leq T} \Delta_n(t)$ is either equal to 1, and then (40) holds, or is greater than 1 and is attained for some pairs (n, t) , $n \geq 0$, $t \leq T$. Take the pair (n^*, t^*) with the least n . Clearly, $n^* > 0$, because $\Delta_0(t) = 1$. Moreover, by the choice of n^* , we have

$$\Delta_{n^*-1}(t^*) < \Delta_{n^*}(t^*). \quad (41)$$

To prove the bound (40), it suffices to show that $t^* = 0$. Assume that this is not so. Let $T \geq t^* > 0$. From (39) and (41) it follows that $d\Delta_{n^*}(t^*)/dt < 0$. However, this inequality and the condition $t^* \neq 0$ contradict the fact that t^* is a maximum point of the function $\Delta_{n^*}(t)$ on the interval $[0, T]$. Hence, $t^* = 0$. This proves the bound (40).

We similarly prove the bound

$$\inf_{n \geq 0, t \geq 0} \Delta_n(t) \geq \inf_{n \geq 0} \Delta_n(0). \quad (42)$$

From (36) it follows that

$$\Delta_n(0) = \tilde{B}_n(0)/B_n(0).$$

By (31), $\Delta_n(0) = 1$ for $n \leq N$, and for $n > N$ we have

$$\tilde{B}_n(0) = A; \quad A \geq B_n(0) \geq B_N(0) > (1-\epsilon)A.$$

Here the first relationship follows from (29) and (31), and the rest follow from the definition of B_n and (28). Thus, for any n ,

$$1 \leq \Delta_n(0) \leq 1/(1-\epsilon). \quad (43)$$

Hence, by (40) and (42), we obtain similar inequalities for $\Delta_n(t)$ for any n, t , which leads to the relationship (34), because by (35)

$$\tilde{F}_n - F_n = \beta^{-1}(\mu - \beta\tilde{F}_n)(\Delta_{n-1}/\Delta_n - 1).$$

Q.E.D.

The proof of Theorem 1 relies on some auxiliary lemmas. Consider the relationships

$$\mu\psi_1^{-1}\ln(\mu/\alpha - \beta) = \mu\psi_1^{-1}(\psi_1 - 1 - \ln\psi_1), \quad 1 < \psi_1 < (\mu/\beta)\ln(\mu/\alpha), \quad (44)$$

$$\beta - \alpha\psi_2^{-1}\ln(\mu/\alpha) = \alpha\psi_2^{-1}(\psi_2 - 1 - \ln\psi_2), \quad (\alpha/\beta)\ln(\mu/\alpha) < \psi_2 < 1. \quad (45)$$

Lemma 1. There exist (unique) values ψ_1, ψ_2 that satisfy (44) and (45).

Proof. Clearly, $\ln(\mu/\alpha) = \int_{\mu/\alpha}^{\mu} dx/x > \beta/\mu$, so that the inequalities (44) define a nonempty set of numbers. The equation (44) may be rewritten as $\ln\psi_1 = (1 + \beta/\mu)\psi_1 - 1 - \ln(\mu/\alpha)$.

For $\psi_1 = 1$, the left-hand side is greater than the right-hand side, and conversely for $\psi_1 = (\mu/\beta)\ln(\mu/\alpha)$, because

$$\ln\left[(\mu/\beta)\ln(\mu/\alpha)\right] = \int_1^{(\mu/\beta)\ln(\mu/\alpha)} dx/x < (\mu/\beta)\ln((\mu/\alpha) - 1).$$

Hence follows solvability of (44). Solvability of (45) is proved similarly.

Introduce the new variables

$$\gamma_1 = \mu\psi_1^{-1}\ln(\mu/\alpha - \beta), \quad \gamma_2 = \beta - \alpha\psi_2^{-1}\ln(\mu/\alpha). \quad (46)$$

In these variables, the relationships (44), (45) take the form (18), (19). This proves existence (and uniqueness) of the solutions of (18), (19).

To avoid introduction of new notation, we will use $\psi_1, \psi_2, \gamma_1, \gamma_2$ in what follows for the numbers that satisfy (44)-(46).

Let $\tau = t/n$.

Lemma 2. If $\tau \geq \psi_1\mu^{-1}$, then for some $\lambda_1 = \lambda_1(\alpha, \beta)$

$$h_n(t) = \sum_{k=n}^{\infty} (k!)^{-1}(\mu t)^k \geq e^{\mu t}(1 - \lambda_1 t^{-1/2} e^{\gamma_1 t}). \quad (47)$$

Proof. We have

$$g_n(t) = e^{\mu t} - h_n(t) = \sum_{k=1}^{n-1} (\mu t)^k / k!$$

$$\leq \frac{(\mu t)^{n-1}}{(n-1)!} (1 + (n-1)/\mu t + ((n-1)/\mu t)^2 + \dots).$$

Summing and using Stirling's inequality, we obtain

$$\begin{aligned} g_n(t) &\leq \frac{(\mu t)^n n e^n}{n^n (2\pi n)^{1/2} (\mu t - n)} \\ &= \frac{\sqrt{\tau} e^{\mu t}}{(2\pi t)^{1/2} (\mu \tau - 1)} e^{(t/\tau)(1 - \mu\tau + \ln \mu\tau)}. \end{aligned}$$

It is easy to check that the right-hand side is a decreasing function of the parameter τ (because $\mu\tau > 1$). Substituting the minimum value $\tau = \psi_1 \mu^{-1}$ and using (46), (44), we obtain (47) for

$$\lambda_1 = (\psi_1)^{1/2} (2\pi\mu)^{-1/2} (\psi_1 - 1)^{-1}.$$

Lemma 3. If $\tau \leq \psi_2 \alpha^{-1}$, $t \geq T_0 = \psi_2 N / \alpha (1 - \psi_2)$, then $n > N$ and for some $\lambda_2 = \lambda_2(\alpha, \beta, N)$

$$d_n(t) = \sum_{k=0}^{n-N-1} (\alpha t)^k / k! \geq e^{\alpha t} (1 - \lambda_2 t^{-1/2} e^{\gamma_2 t}). \quad (48)$$

Proof. Under the assumptions of the lemma, it is easy to check that $\alpha t \leq (n-N)/\psi_2$, $n > N(1 - \psi_2)^{-1}$. Therefore, each term of the series

$$\mu_n = e^{\alpha t} - d_n(t) = \sum_{k=n-N}^{\infty} (\alpha t)^k / k!$$

does not exceed $\psi_2^{(k-n)/2} (\alpha t)^{n-N} (n-N)!$. Summing and using Stirling's formula and the inequality $(n/(n-N))^{n-N} \leq e^N$, we obtain

$$\begin{aligned} u^n &\leq \frac{e^n (\alpha \tau)^{n-N}}{(1 - \psi_2) (2\pi \alpha t)^{1/2}} \\ &= \frac{e^{\alpha t}}{(1 - \psi_2) (2\pi \alpha t)^{1/2}} \exp\{(t/\tau)(1 - \alpha\tau + \ln \alpha\tau) - N \ln \alpha\tau\}. \end{aligned}$$

The function in the right-hand side of this inequality is increasing in τ for any $t \geq T_0$. Substituting the maximum value $\tau = \psi_2 \alpha^{-1}$ and using (46), (45), we obtain the sought bound

$$u_n \leq (1 - \psi_2)^{-1} (2\pi \alpha t)^{-1/2} e^{-N \ln \psi_2} e^{(\alpha - \psi_2) t}.$$

Let

$$z_n^* \mu^{-n} e^{\mu t} + \alpha^{-n} P e^{\alpha t}. \quad (49)$$

In what follows we assume that $A = 1/P$ (see (14)). Then from (15) and (49) we have

$$F_n^* = \beta^{-1} (\mu - z_{n-1}^* / z_n^*).$$

Since $z_n(0) \leq \alpha^{-n}$ (see (6)), we obtain from (4) and (49)

$$z_n \leq z_n^*, \quad n \geq 1. \quad (50)$$

Proof of Theorem 2. Let the initial values $F_k(0)$ satisfy the conditions (2), (12). Consider the variables z_n defined in (3). Substituting (14) in (4), we can represent them in the form

$$z_n = \mu^{-n} h_n + \alpha^{-n} P d_n + b_n, \quad n > N, \quad (51)$$

where $d_n(t)$ is defined in (48),

$$b_n(t) = \sum_{k=n-N}^{n-1} z_{n-k}(0) t^k / k! \leq \alpha^{-n} \sum_{k=n-N}^{n-1} (\alpha t)^k / k!. \quad (52)$$

In what follows we assume that $t \geq T_0$ (see Lemma 3).

Let $\delta_n = |F_n - F_n^*|$, $\tau = t/n$. The proof is conducted separately for each of the three ranges of τ .

a) Let $\psi_1 \mu^{-1} < \tau < \psi_2 \alpha^{-1}$. By (49)-(51) and by Lemmas 2 and 3, we have

$$\begin{aligned} 0 \leq z_n^* - z_n &\leq \mu^{-n} \lambda_1 t^{-1/2} e^{(\mu - \gamma_1) t} + \alpha^{-n} P \lambda_2 t^{-1/2} e^{(\alpha - \gamma_2) t} \\ &\leq \lambda_3 t^{-1/2} e^{-\gamma t} z_n^*, \end{aligned} \quad (53)$$

where $\gamma = \min\{\gamma_1, \gamma_2\}$, $\lambda_3 = \max\{\lambda_1, \lambda_2\}$. Let

$$K_1 = (1 - \lambda_3 T_0^{-1/2} e^{-\lambda T_0})^{-1}. \quad (54)$$

From (3) and (53) we obtain that

$$\delta_n = \frac{1}{\beta} \frac{z_{n-1}^*}{z_n^*} \left| \frac{z_n^* - z_{n-1}^*}{z_n^*} - \frac{z_{n-1}^* - z_{n-2}^*}{z_{n-1}^*} \right| \frac{z_n^*}{z_n} \leq \frac{2\mu}{\beta} K_1 t^{-1/2} e^{-\gamma t}, \quad (55)$$

because $z_{n-1}^*/z_n^* \leq \mu$ (see (49)). The sought bound is obtained after \sqrt{t} is replaced with $\sqrt{T_0}$.

b) Let $\tau \leq \psi_1 \mu_{-1}$. From (3) and (51) we obtain

$$1 - F_n = (z_{n-1} - \alpha z_n)/z_n \leq (\mu^{-n+1} h_{n-1} + b_{n-1})/\alpha^{-n} d_n,$$

because $d_n > d_{n-1}$. From (52) we obtain $b_{n-1} \leq \alpha^{-n} (e^{\alpha t} - d_n)$. Applying Lemma 3, we obtain

$$1 - F_n \leq K_1 (\mu/P) (\alpha/\mu)^n e^{\beta t} + K_1 \mu^{-1} \lambda_2 t^{-1/2} e^{-\gamma_2 t}, \quad (56)$$

where K_1 is defined in (54). Substituting for n and \sqrt{t} in (56) their minimum values $\mu t \psi_1^{-1}$ and $T_0^{1/2}$ and noting that $1 - F_n^* \leq e^{-\gamma_1 t}$, we obtain the sought inequality for δ_n .

c) It remains to consider the case $\tau \geq \psi_2 \alpha^{-1}$. Unlike a) and b), we may have $n \leq N(1/\psi_2)^{-1}$ in this case. By (3), (4), noting that $z_n(0) \leq \alpha^{-n}$ and $h_n < h_{n-1}$, we obtain

$$F_n = \frac{\mu z_n - z_{n-1}}{z_n} \leq \frac{\mu \alpha^{-n} e^{\alpha t}}{\mu^{-n} h_n} \leq (\mu/\alpha)^n \mu K_1 e^{-\beta t}. \quad (57)$$

The last inequality follows from Lemma 2. Since $n \leq \alpha \psi_2^{-1} t$, then by (57) $F_n \leq \mu K_1 e^{-\gamma_2 t}$. It is easily verified that $F_n^* \leq e^{-\gamma_2 t}$, and so $\delta_n \leq (1 + \mu K_1)^{-\gamma_2 t}$. Thus, we always have $\delta_n \leq \lambda e^{-\gamma t}$, where λ is a constant.

Remark. After this paper had been submitted for publication, we learned of the existence of [19], which introduces a certain class of differential-difference equations reducible to linear equations by a substitution of the type (3). The equation (1) belongs to this class. However, the central question of the asymptotic behavior of solutions was not considered in [19].

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Notes

1. In an empirical study, the notions of "capacity" and "efficiency level" of course should be defined more precisely. Thus, Sato [10] considered the distribution of added value by the ratio of wages to added value. The corresponding histograms were constructed for more than 300 industries, and 75-80% of them were found to be unimodal [10, p. 186].

2. A remark in [13] suggests that Iwai is willing to accept this position.

3. Indeed, take the derivative dF_n/dt of (3) and equate it to the right-hand side of (1), replacing $\mu - \beta F_n$ with z_{n-1}/z_n . After obvious manipulations, we obtain the relationship

$$(1/z_n)(dz_n/dt) = (1/z_{n-1})(dz_{n-1}/dt) - \beta(F_n - F_{n-1}).$$

Noting that $(1/z_0)dz_0/dt = \mu$, $F_0 = 0$ and again using (3), we obtain the sought relationship.

4. The existence of the family of wave solutions (15) and of the substitution (3) linearizing the original equation, came as a total surprise to us.

5. All proofs are collected in the last section.

6. It would be interesting to show that this bound is unimprovable in the order of the exponential function.

7. Empirical studies mostly deal with the distribution of capacities, and not the number of firms. Our analysis implicitly assumes that the average relative capacity of a firm is independent of the efficiency level (in [13] it is claimed that the shape of the distribution curve of the number of firms is also stable over time).