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Oligopoly Games with Local Monopolistic Approximation

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Abstract. We propose a repeated oligopoly game where quantity setting firms have incomplete knowledge of the demand function of the market in which they operate. At each time step they solve a profit maximization problem by using a subjective approximation of the demand function based on a local estimate its partial derivative, computed at the current values of prices and outputs, obtained through market experiments. At each time step they extrapolate such local approximation by assuming a linear demand function and ignoring the effects of the competitors' outputs. Despite a so rough approximation, that we call "Local Monopolistic Approximation" (LMA), the repeated game may converge to a Nash equilibrium of the true oligopoly game, i.e. the game played under the assumption of full information. An explicit form of the dynamical system that describes the time evolution of oligopoly games with LMA is given for arbitrary differentiable demand functions, provided that the cost functions are linear or quadratic. Sufficient conditions for the local stability of Nash Equilibria are given. In the particular case of an isoelastic demand function, we show that the repeated game based on LMA always converges to a Nash equilibrium, both with linear and quadratic cost functions. This stability result is compared with "best reply" dynamics, obtained under the assumption of isoelastic demand (fully known by the players) and linear costs.

Key words: Oligopoly games, bounded rationality, subjective demand, Nash Equilibrium, dynamical systems, stability.

1 Introduction

The notion of Nash equilibrium in an oligopoly game is based on the assumption that each firm knows the market where it operates and knows what the other firms decide to do. In particular, each firm is assumed to know the entire demand curve for the good it produces. It is more likely, however, that real firms only use some local estimate of the demand function, obtained through market experiments, when they compute their strategic variables as solutions of a profit maximization problem. Some authors use terms like "estimated" or "perceived" or "subjective" demand function, in order to say that the demand function that the firms use to solve their profit maximization problem is obtained through market experiments or by some "rule of thumb" (Baumol and Quandt, 1964, Silvestre, 1977, Bonanno and Zeeman, 1985).

Many authors have recently investigated the possible outcomes of repeated oligopoly games where the players have a misspecified knowledge of the demand function. Leonard and Nishimura (1999) examined discrete time dynamic duopolies and illustrated how the steady states (that are no longer Nash equilibria of the true game) change their stability properties as the result of the incorrect assessment of the demand function, the misspecification being due to a multiplicative scale factor. This model has been further generalized in Chiarella and Szidarovszky (2003) in which the firms may also misspecify the shape of the demand function and not only its scale as it was assumed in the original model of Leonard and Nishimura. Bischi, Chiarella and Kopel (2004) propose a duopoly model where the players lack knowledge of the market demand and, differently from the model of Léonard and Nishimura, the assumption of decreasing reaction functions is relaxed. This implies that new steady states (that are not Nash Equilibria) may be created, when (one or both) players over- or underestimate the demand.

Interesting cases are obtained when an oligopoly game, repeatedly played by boundedly rational players that do not know the demand function, converges in the long run to a Nash equilibrium, i.e. to the same equilibrium that is reached in one-shot under full rationality. This may be seen as an evolutionary explanation of the outcome of a Nash equilibrium, and in the case of several Nash equilibria the repeated game may act as an equilibrium selection device. Of course, the more refined the decision-making process, the more expensive it is likely to be, and therefore, especially when a (single) decision is not of crucial importance, no more than an approximate solution may be justified. Some authors denote "optimally imperfect decisions" the decisions based on simple and inexpensive computations, wellsuited for frequent repetition (on this point see Baumol and Quandt, 1964).

This approach has been recently developed in a paper by Tuinstra (2004), where the subjective demand framework is used in a discrete-time price adjustment process. Following the seminal papers by Negishi (1961), Tuinstra assumes that at each time step price setting firms use subjective linear demand functions that only depends upon their own price and pass through the price-quantity combinations of the current state of the economy. Moreover, as in Silvestre (1977), the slope of the subjective demand curve is assumed to coincide with the slope of the objective (generally nonlinear) demand curve. The firms observe the amount the current price and they compute the slope of the true demand curve at that price. With this information they estimate a linear demand curve, and by using this estimation they set a new optimal price. In the case of linear cost functions, Tuinstra obtains an explicit discrete time dynamical system to describe the price adjustment process, and investigates its dynamical properties. He also gives sufficient conditions for stability, that involve cross-price effects and the curvature of the demand curve, and by using particular nonlinear demand functions, associated with linear cost functions, he shows the occurrence of non convergent trajectories and complicated dynamics.

In this paper we propose an similar adjustment mechanism for a repeated oligopoly game where quantity-setting firms solve a profit maximization problem by using a linear approximation of the demand function. Like in Tuinstra (2004) this approximation is based on the estimate of the partial derivative of the demand function computed in the current state of the market and ignoring the presence of the competitors. No efforts are made by the players to learn the true (and generally non linear) demand function[1]. Firms are just assumed to perform market experiments in a neighborhood of the current state of the market. Through these experiments each firm is assumed to get a correct estimate of the partial derivative of the demand function with respect to its own quantity (or price) variations. This estimate is then used to obtain a linear approximation (and extrapolation) of the demand function without any guess about the influence of the competitors (i.e. a monopolistic approximation). However, even if the firms solve their profit maximization problems on the basis of such a rough approximation, that we call "local monopolistic approximation" (LMA henceforth) the repeated game may converge to a Nash equilibrium of the true oligopoly game, i.e. the game played under the assumption of full information. In other words, the repeated game with LMA has the same (Nash) equilibria as the so called "best reply" dynamics, and it is interesting to compare the stability properties of a Nash equilibria under these two different adjustment mechanisms, based on different degrees of rationality and information sets. In order to make such a comparison, it is necessary to find suitable demand and cost functions that allow us to get an explicit discrete dynamical system. As we shall see, this is easily obtained for an arbitrary differentiable demand function, not only in the case of linear cost functions (like in the case considered by Tuinstra (2004)) but also in the case of quadratic cost functions. Instead, explicit dynamical systems that represent best reply dynamics are not easily obtained, as an explicit analytic expression of the reaction functions is rarely found. An exception is the Cournot oligopoly game with best reply obtained by using linear costs and an isoelastic demand function, as proposed by Puu (1991). So, in the case of homogeneous products and isoelastic demand function we can compare the stability of the Nash equilibrium under these two different kinds of profit maximizing output adjustment mechanism, the LMA and the best reply. In fact, the properties of best reply dynamics of Cournot duopoly games has been extensively studied by Puu (1991, 1996) see also Puu (2000) who showed the trajectories may not converge to the Cournot-Nash equilibrium and the outcome of complex trajectories is possible. In this paper we show that the repeated game based on LMA with an isoelastic demand function always converges to a Nash equilibrium. This implies that in this case the adjustment mechanism with LMA is more stable than the "best reply" dynamics, obtained under the assumption of full knowledge of the demand. In other words, in this case less rationality (and less information) leads to more stability. However, as we shall see, the stability ranking between these two different adjustment mechanisms, is not so immediate when we also consider the basins of attraction of the Nash equilibrium when it is stable under both the dynamic processes.

The paper is organized as follows. In section 2 we recall the setup and the notations of Cournot oligopoly games with best reply dynamics, and we briefly recall the results obtained by Puu (1991) for the Cournot duopoly with best reply and isoelastic demand function. In section 3 we introduce

[1] A model with a learning mechanism has recently been proposed by Bischi, Sbragia and Szidarovszky (2004). They consider Cournot oligopolies where players know that the demand and cost functions are linear, but while the firms are assumed to know the cost functions, they misspecify the slope of the demand function, and they try to learn the true slope on the basis of the discrepancy they observe, at any repetition of the game, between the expected price and the realized one.

the adjustment mechanism based on LMA, we argue about the information set required to perform it, and we give some general stability conditions. In section 4 we study the existence and stability of Nash equilibrium under LMA with an isoelastic demand function and linear cost functions, and we compare these results with those obtained under the best reply dynamics applied to the same economic context. In section 5 we consider an aggregate model that describes the dynamics of the total output of the oligopoly system when homogeneous products, linear costs and isoelastic demand functions are considered. In section 6 we prove the stability of the Nash equilibrium under the LMA dynamics with isoelastic demand and quadratic costs. Section 7 concludes and indicates some possible directions and developments of future researches.

2 Best Reply dynamics

Let us consider an industry where n firms, indexed by $i = 1, ..., n$, produce differentiated products, with production levels q_i, $i = 1, ..., n$, respectively. Strategic interaction arises because the demanded quantity of a given product depends on all the prices according to the demand functions

$$q_i = D^i \left(p_i, \mathbf{p}_{-i} \right), \quad i = 1, ..., n \tag{1}$$

where $\mathbf{p}_{-i}$ represents the set of prices p_j with $j \neq i$. If $C_i(q_i)$ denotes the cost function of producer i, then the profit at time period t is

$$\pi_i(t) = p_i(t)q_i(t) - C_i \left(q_i(t) \right) \tag{2}$$

The producers are assumed to be price takers and quantity setting, i.e. at each time t they decide the next-period productions $q_i(t+1)$ by maximizing the profit expected at the next period $t+1$:

$$q_i(t+1) = \arg\max_{q_i} \pi_i^e(t+1) = \arg\max_{q_i} \left[p_i^e(t+1)q_i - C_i \left(q_i \right) \right] \tag{3}$$

where $p_i^e \left(t+1 \right)$ represents the price expected by player i for period $(t+1)$. In the traditional Cournot game *players are assumed to know the demand function*, so each player i expresses the expected price $p_i^e(t+1)$ by using the inverse demand functions

$$p_i^e(t+1) = f^i \left(q_i(t+1), q_{-i}^e \left(t+1 \right) \right) \tag{4}$$

where $q_{-i}^e \left(t+1 \right)$ represents the output decisions of other players as expected by player i.

The Cournot optimization problem becomes:

$$q_i(t+1) = \arg\max_{q_i} [f^i \left(q_i, q_{-i}^e \left(t+1 \right) \right) q_i - C_i \left(q_i \right)] \tag{5}$$

In the simplest (and lucky) cases one can uniquely express q_i as functions of q_{-i}^e

$$q_i(t+1) = r_i \left(q_{-i}^e(t+1) \right) \tag{6}$$

where r_i are the *reaction functions*. The Nash equilibria correspond to the fixed points of the map (6), i.e. are located at the intersections of the reaction curves. If players correctly forecast the competitors' decisions, i.e. $q_{-i}^e(t+1) = q_{-i}(t+1)$, then the Nash equilibria can be directly computed, in one-shot. Instead, in a bounded rationality setting, players may not know beforehand the competitors' choices, and consequently they formulate some reasonable forecast, on the basis of their information set. The simplest assumption, proposed by Cournot (1838), is that of *naive expectations*, $q_{-i}^e(t+1) = q_{-i}(t)$, i.e. each firm expects that the production of the other firms will remain the same as in current period[2]. Under this assumption (6) gives rise to a discrete-time dynamical system (*Best Reply Dynamics*)

$$q_i(t+1) = r_i(q_{-i}(t)) \tag{7}$$

Every Nash equilibrium is also an equilibrium of the Best Reply Dynamics, because the intersections of the reaction curves are the fixed points of (7). However, such equilibria are not reached in one shot. They may be reached asymptotically, in the long run, if they are stable under the best reply dynamics. This may be seen as an evolutionary explanation of the outcome of a Nash equilibrium, and in the case of several Nash equilibria the repeated game (7) may act as an equilibrium selection device. However, the dynamical system (7) may not converge to a Nash equilibrium, and it may exhibit asymptotic convergence to periodic or chaotic attractors (see e.g. Rand, 1978, Dana and Montrucchio, 1986, Puu, 1991, 1998, Bischi, Mammana and Gardini, 2000). In particular, Puu (1991) considered a Cournot duopoly game with an isoelastic demand (with unitary elasticity)

$$p = f(q_1, q_2) = \frac{1}{q_1 + q_2} \tag{8}$$

together with linear cost functions $C_i = c_i q_i$, $i = 1, 2$. He proved that unimodal reaction functions are obtained, and the best reply dynamics can give rise to complex trajectories that do not converge to a Nash equilibrium. These reaction functions can be easily obtained by solving the optimization problem:

$$q_i(t+1) = \arg\max_{q_i} \frac{q_i}{q_i + q_{-i}^e(t+1)} - c_i q_i$$

whose first order conditions become

$$\frac{q_{-i}^e(t+1)}{\left[q_i + q_{-i}^e(t+1)\right]^2} - c_i = 0$$

from which q_i is obtained as a solution of a second degree algebraic equation, namely $q_i = -q_{-1}^e \pm \sqrt{q_{-1}^e / c_i}$. Only the solution with "+" represents a maximum, hence the dynamical system obtained

[2] Other kinds of expectations mechanisms can be used, such as adaptive expectations, see e.g. Szidarovszky and Okuguchi (1988), Bischi and Kopel (2001).

by Puu assuming Cournot expectations is given by

$$q_1(t+1) = -q_2(t) + \sqrt{\frac{q_2(t)}{c_1}} \tag{9}$$
$$q_2(t+1) = -q_2(t) + \sqrt{\frac{q_1(t)}{c_2}}$$

A unique Nash equilibrium exists, given by

$$\mathbf{q}^* = (q_1^*; q_2^*) = \left(\frac{c_2}{(c_1+c_2)^2}; \frac{c_1}{(c_1+c_2)^2} \right) \tag{10}$$

whose stability properties are given by Puu (1991) in terms of the ratio between the marginal costs c_1/c_2. First of all, feasible (i.e. bounded and non negative) trajectories of the best reply dynamics are obtained provided that $c_1/c_2 \in [4/25, 25/4] = [0.16, 6.25]$. Moreover, the Nash equilibrium (10) is stable if and only if $c_1/c_2 \in (3-2\sqrt{2}, 3+2\sqrt{2}) \simeq (0.17, 5.83)$. If c_1/c_2 exits this interval then the Nash equilibrium loses stability via a period doubling bifurcation. Indeed, Puu (1991) shows that if c_1/c_2 falls outside the interval $(3-2\sqrt{2}, 3+2\sqrt{2})$ then the asymptotic dynamics may converge to periodic cycles of even exhibit chaotic motion around the Nash equilibrium, as shown in fig. 1a, where a chaotic trajectory is shown, together with the reaction curves, obtained with $c_1 = 1$ and $c_2 = 0.161$. In fig. 1b, obtained with $c_1 = 1$ and $c_2 = 0.7$, the stable equilibrium (10) is shown, together with its basin of attraction, represented by the white region, whereas the grey region represent the set of initial conditions that generate unfeasible trajectories.

Insert fig. 1 a,b

3 The model with local and monopolistic approximation

The best reply dynamics are obtained under the assumptions that, at each time, the firms have a global knowledge of the demand function and know their current production and the current production of the competitors as well, but they are not able to get a correct forecast of the competitors' choices for the next period. In this work we also consider another kind of information lack: We relax the assumption that each firm knows the corresponding demand function. However we assume that, through brief market experiments, at any time period each firm is able to get a correct estimate of the partial derivative

$$f_i^i(t) := \frac{\partial f^i(q_i(t), q_{-i}(t))}{\partial q_i} \tag{11}$$

Then, each firm i uses this estimate to obtain a "rule of thumb" computation of the expected price

$$p_i^e(t+1) = p_i(t) + f_i^i(t)(q_i(t+1) - q_i(t)) \tag{12}$$

where $p_i(t) = f^i(q_1(t), ..., q_n(t))$ and $f_i^i(t)$ is defined in (11).

Of course, the approximation (12) is easier to be obtained than a global knowledge of the demand function (that involves values of p_k or q_k that may be very different from the current ones). Indeed, the estimate of $f_i^i(t)$ at time t may be obtained by computing the effects of small price or quantity variations. For example, introducing at time t a small output variation Δq_i, firm i can compute

$$\frac{f^i(q_i(t) + \Delta q_i, q_{-i}(t)) - f^i(q_i(t), q_{-i}(t))}{\Delta q_i} \tag{13}$$

and we assume that this allow firm i to get a correct estimate of $f_i^i(t)$. It is worth to note that an estimate of $f_i^i(t)$ can also be obtained through small price variations Δp_i, that allow firm i to compute

$$\frac{D^i(p_i(t) + \Delta p_i, p_{-i}(t)) - D^i(p_i(t), p_{-i}(t))}{\Delta p_i} \tag{14}$$

and consequently a correct estimate of the partial derivative $D_i^i(t)$, from which $f_i^i(t) = 1/D_i^i(t)$ can be computed.

Notice that (12) is not a linear approximation of f^i. In fact, as firm i cannot obtain an estimate of $f_j^i(t)$, with $j \neq i$, it simply neglects the influence of the competitor's production in the computation of the expected price. Of course, this is a very rough approximation. However, many authors claim that this is not far from reality, see e.g. Kirman (1975) on this point. Moreover, as we shall see below, even if the firms neglect the influence of competitors' outputs in the computation of the expected price, the dynamic process generated by such a repeated game can converge to the same equilibria as the best reply dynamics.

In fact, if the producer i uses (12) to compute the expected price, then the first order conditions for the optimization problem (3) become

$$\frac{\partial}{\partial q_i(t+1)} \left[q_i(t+1) \left(p_i(t) + f_i^i(t)(q_i(t+1) - q_i(t)) \right) - C_i(q_i(t+1)) \right] = 0 \quad i = 1, ..., n$$

i.e.

$$p_i(t) + 2f_i^i(t)q_i(t+1) - f_i^i(t)q_i(t)) - C_i'(q_i(t+1)) = 0 \qquad i = 1, .., n \tag{15}$$

where, again, $p_i(t) = f^i(q_1(t), ..., q_n(t))$ and C_i' denoted the derivative of the cost function.

Notice that, in order to compute $q_i(t+1)$, at time t firm i needs the following *information set*:

(i1) Its current output $q_i(t)$;

(i2) The current price of its good, $p_i(t)$;

(i3) The partial derivative $f_i^i(t)$;

(i4) Its own cost function $C_i(q_i)$.

The following proposition states that even if players use a linear and monopolistic approximation of the demand function, the equilibria are the same as in the oligopoly game with full information. So, even for oligopoly games where the firms use a so rough approximations of the inverse demand functions, the steady states are the Nash equilibria of the true game.

Proposition 1. *The steady states of the optimization problem with local monopolistic approximation (15) are the Nash equilibria of the Cournot game (5) with best reply and perfect knowledge of the inverse demand function.*

A proof of this proposition is given in the Appendix.

A study of the dynamic properties of the adjustment process (15), based on the local monopolistic approximation of the demand function, is possible if the implicit equation (15) can be written in the form of an explicit discrete time dynamical system, i.e. if one can uniquely compute $q_i(t+1)$ from the knowledge the state variables at time t. This can be obtained if we consider suitable cost functions, such as the following two commonly used cost functions

i) Linear cost functions

$$C_i = c_{i0} + c_i q_i, \qquad c_{i0} \geqslant 0, c_i > 0. \tag{16}$$

With this cost function we have $C_i^{'}(q_i(t+1)) = c_i$ and (15) gives

$$q_i(t+1) = \frac{1}{2}q_i(t) + \frac{c_i - p_i(t)}{2f_i^i(t)} \qquad\qquad i = 1, ..., n \tag{17}$$

where, $p_i(t) = f^i\left(q_1(t), ..., q_n\left(t\right)\right)$ and $f_i^i(t)$ is defined in (11).

ii) Quadratic cost function:

$$C_i = c_{im} + c_i {q_i}^2, \quad c_{im} \geqslant 0, c_i > 0 \tag{18}$$

With this cost function we have $C_i^{'}(q_i(t+1)) = 2c_i q_i(t+1)$, and (15) gives

$$q_i(t+1) = \frac{q_i(t)f_i^i(t) - p_i(t)}{2\left[f_i^i(t) - c_i\right]} \qquad\qquad i = 1, ..., n \tag{19}$$

where, again, $p_i(t) = f^i\left(q_1(t), ..., q_n\left(t\right)\right)$ and $f_i^i(t)$ is defined in (11).

Notice that in the price adjustment process proposed by Tuinstra (2004) only the case of linear cost functions gives rise to an explicit dynamical system. Instead, in our quantity-setting framework we can obtain an explicit expression of the dynamical system with quadratic costs. This may be important in several applications. Just to quote an important case, in fishery models the harvesting costs depend on the square of harvested quantity (see Clark (1990), Szidarovszky and Okuguchi (1998), where this quadratic cost function is derived from a “production function” of the Cobb-Douglas type with fishing effort and fish stock as the two inputs).

So, if we consider linear or quadratic cost functions and we assume several different kinds of nonlinear differentiable demand functions, the adjustment mechanisms (17) and (19) allow us to study several different dynamical systems obtained by using different nonlinear demand functions. It is interesting obtain some ranking about the stability properties of the different models, based on the comparison of the regions of stability of Nash equilibria in the space of the parameters, or on the comparison of their basins of attraction, under different assumptions on the demand functions.

The following propositions, proved in the Appendix, gives sufficient conditions for the stability of a Nash equilibrium under the two adjustment dynamics (17) and (19) respectively.

Proposition 2. *Let $\mathbf{q}^*$ be a Nash equilibrium for the oligopoly game defined by (5) with a linear cost function (16), and let the inverse demand functions $f^i(q_1, ..., q_n)$ be twice differentiable. Then $\mathbf{q}^*$ is a steady state of (17) and a sufficient condition for local stability of $\mathbf{q}^*$ under (17) is*

$$q_i^* \left|f_{ii}^i(\mathbf{q}^*)\right| + \sum_{j \neq i} \left|f_j^i(\mathbf{q}^*) + q_i^* f_{ij}^i(\mathbf{q}^*)\right| \leqslant 2 \left|f_i^i(\mathbf{q}^*)\right| \qquad \text{for all } i = 1, ..., n \tag{20}$$

where $f_{ij}^i(q^*) = \frac{\partial^2 f^i}{\partial q_i \partial q_j}$

Proposition 3. *Let $\mathbf{q}^*$ be a Nash equilibrium for the oligopoly game defined by (5) with a quadratic cost function (18), and let the inverse demand functions $f^i(q_1, ..., q_n)$ be twice differentiable. Then $\mathbf{q}^*$ is a steady state of (19) and a sufficient condition for local stability of $\mathbf{q}^*$ under (19) is*

$$q_i^* \left|f_{ii}^i(\mathbf{q}^*)\right| + \sum_{j \neq i} \left|f_j^i(\mathbf{q}^*) + q_i^* f_{ij}^i(\mathbf{q}^*)\right| \leqslant 2 \left|f_i^i(\mathbf{q}^*) - c_i\right| \qquad \text{for all } i = 1, ..., n \tag{21}$$

where $f_{ij}^i(q^*) = \frac{\partial^2 f^i}{\partial q_i \partial q_j}$

These stability conditions are similar to the one given in Tuinstra (2004), where some intuitive interpretation is given in terms of diagonal dominance in the matrix of substitution effects and curvature of the demand functions.

In the following we shall focus our attention on the comparison between the stability properties of the Nash equilibria under the two different kinds of adjustment processes: the best reply dynamics and the dynamics with LMA. In order to do such a comparison both these dynamic adjustments must be expressed as explicit dynamical systems. As we have argued above, in the case of LMA this can be easily obtained for every differentiable demand function, provided that the cost functions are linear or quadratic. Instead, when using nonlinear demand functions, it is not easy to obtain an explicit form of the reaction functions. In fact, the optimization problem (5) may have a non unique solution, and even in the case of uniqueness the expression of the corresponding reaction functions may be quite involved. As we have shown in section 2, one of the lucky cases where nonlinear reaction functions can be easily obtained starting from a suitable nonlinear reaction function is obtained by using an isoelastic demand function associated with linear costs.

In order to make such a comparison, in the following we shall focus our attention on duopoly games with isoelastic demand functions.

4 Cournot game with an isoelastic demand function and linear costs

In this section we consider a duopoly model with the following isoelastic demand function

$$p = f(q_1, q_2) = \frac{1}{(q_1 + q_2)^\alpha}, \qquad \alpha > 0 \tag{22}$$

that for $\alpha = 1$ reduces to the case studied by Puu (1991). The model (17) with $n = 2$ and demand function (22) becomes a two dimensional dynamical system, defined by iterated map

$$\begin{aligned} q_1(t+1) &= \frac{1}{2}q_1(t) - \frac{1}{2\alpha}\left(q_1(t) + q_2(t)\right)\left(c_1\left(q_1(t) + q_2(t)\right)^\alpha - 1\right) \\ q_2(t+1) &= \frac{1}{2}q_2 - \frac{1}{2\alpha}\left(q_1(t) + q_2(t)\right)\left(c_1\left(q_1(t) + q_2(t)\right)^\alpha - 1\right) \end{aligned} \tag{23}$$

The following proposition holds (the proof is in the Appendix)

Proposition 4. *If $\alpha < 2$ the dynamical system (23) has a unique nonvanishing equilibrium, given by $q^* = (q_1^*, q_2^*)$, with*

$$\begin{aligned} q_1^* &= \frac{1}{\alpha}\left(\frac{2-\alpha}{c_1+c_2}\right)^{\frac{1}{\alpha}}\left(\frac{c_2 + c_1(1-\alpha)}{c_1+c_2}\right) \\ q_2^* &= \frac{1}{\alpha}\left(\frac{2-\alpha}{c_1+c_2}\right)^{\frac{1}{\alpha}}\left(\frac{c_1 + c_2(1-\alpha)}{c_1+c_2}\right) \end{aligned}$$

which is positive if $\alpha > 1 - \frac{\min(c_1,c_2)}{\max(c_1.c_2)}$. This equilibrium is always locally asymptotically stable.

According to proposition 1, a positive fixed point of (23) is also a Nash equilibrium of the Cournot duopoly game with best reply. Of course, for $\alpha = 1$ the positive equilibrium coincides of the positive equilibrium (10) of the duopoly model with best reply studied by Puu. Proposition 4 states that such equilibrium is always stable under the local monopolistic adjustment mechanism. This contrasts with the results on best reply dynamics discussed by Puu (1991, 1998), see also Puu (2000), and described above. So, in this particular case we can conclude that even if less rational and less informed players introduce rough approximations and correct their decision process every period, they converge to the optimal outcome (Nash equilibrium) for a wider range of parameters than in a game with players that know the true nonlinear demand and at each time step play the best reply. Indeed, in the case of isoelastic demand and linear costs, the convergence to the Nash equilibrium of the process with LMA is always ensured, whereas for certain sets of parameters the best reply dynamics does not converge to it. So, in this case we may guess that *less rationality (and information) implies more stability.*

However, we want to stress that this statement is obtained through a comparison of the stability region in the space of parameters (c_1, c_2), in the sense that the Nash equilibrium $\mathbf{q}^*$ is stable in each points of the plane of the parameters (c_1, c_2) for the model with LMA, whereas the stability only holds in the subset $c_1/c_2 \in (3 - 2\sqrt{2}, 3 + 2\sqrt{2})$ in the case of best reply adjustment. Different conclusions are obtained if we compare the basins of attraction. In fact, with cost parameters such that the Nash equilibrium is stable under both the adjustment mechanisms, larger basins of attraction can be observed for the model with best reply. This can be seen, for example, by a comparison of fig. 2 and fig. 1b, obtained with the same parameters $c_1 = 1$ and $c_2 = 0.7$, where fig. 2 represents the basin of attraction (white region) of the stable Nash Equilibrium of the model (23) with LMA.

Insert Fig. 2

5 Case of isoelastic demand, linear costs and n players

A question which is often discussed in the literature on oligopoly games concerns the effect on stability of the number of players. In general this issue is not an easy task, because an increase in the number of players implies an increase in the dimensions of the dynamical system. To obtain some insight into this question, let us consider an oligopoly game with LMA in the form (17) and let us assume homogeneous products. Then the inverse demand function assumes the form

$$p = f(Q)$$

where $Q = \sum_{i=1}^{n} q_i$ is the total output in the oligopoly market. In this case the model (17) becomes

$$q_i(t+1) = \frac{1}{2} q_i(t) + \frac{c_i - f(Q(t))}{2f'(Q(t))} \quad i = 1, ..., n \tag{24}$$

being $f_i^i = f'(Q(t))$ for each i. This n-dimensional dynamical system in the dynamic variables q_i can be reduced to a one-dimensional dynamical system in the total quantity $Q(t)$ by summing up the equations (24)

$$Q(t+1) = \frac{1}{2} Q(t) + \frac{\gamma - nf(Q(t)}{2f'(Q(t))} \tag{25}$$

where $\gamma = \sum_{i=1}^{n} c_i$. The dynamic equation (25) of the aggregate production includes the number of players $n \in \mathbb{N}$ as a parameter. So, we can investigate the effects of this parameter on the dynamics of the global production.

It is trivial to see that if $(q_1^*, ..., q_1^*)$ is a steady state of the disaggregated dynamical system (24), then $Q^* = \sum_{i=1}^{n} q_i^*$ is a steady state of the aggregated dynamical system (25). In particular, if $(q_1^*, ..., q_1^*)$ is a Nash equilibrium, then it is a fixed point of (24) and consequently it corresponds to a fixed point of 25). However, the converse is not true in general, because a fixed point Q^* of (25) can correspond to several different arrangements of $(q_1, ..., q_n)$, that do not correspond to fixed points of (24).

We now consider the model (17) with n firms and homogeneous products in the case of isoelastic demand function

$$p = f(Q) = \frac{1}{Q} \tag{26}$$

Puu (1996), Agiza et al. (1998), Agliari, Puu and Gardini (2000) considered some particular cases with 3 or 4 competitors that repeatedly play the oligopoly game according to the best reply dynamics, given by the dynamic equations

$$q_i\,(t+1) = \frac{1}{c_i}\sqrt{\sum_{j=1,j\neq i}^{n} q_j} - \sum_{j=1,j\neq i}^{n} q_j \quad , \quad i = 1,...,n \tag{27}$$

and showed several kinds of dynamic situations where the convergence to a Nash equilibrium does not occur. Indeed, quite complex dynamic scenarios may arise, characterized by periodic, quasi-periodic or chaotic motion.

If we assume LMA the dynamical system (24) with n players and inverse demand function (26) becomes

$$q_i(t+1) = \frac{1}{2}\left[q_i(t) + Q(t) - c_i Q^2(t)\right] \quad , \quad i = 1,...,n \tag{28}$$

and the one-dimensional map (25) that describes the time evolution of the aggregated output $Q(t)$ becomes

$$Q(t+1) = \frac{1}{2}\left[1 + n - \gamma Q(t)\right] Q(t) \tag{29}$$

where $\gamma = \sum_{i=1}^{n} c_i$. This is a quadratic one-dimensional map, topologically conjugate to the standard logistic map $x(t+1) = \mu x(t)\,(1 - x(t))$ through the linear homeomorphism $Q = x(1+n)/\gamma$ and with the parameters related by

$$\mu = \frac{1+n}{2}.$$

So, the time evolution of the aggregated production can be deduced from the well known properties of the logistic map (see e.g. Devaney, 1989). In particular, we are interested in the role of the integer parameter n.

First of all, we notice that the dynamics of (29) converge to the positive steady state $Q^* = (1+n-2)/\gamma$ provided that $n \leqslant 5$, corresponding to the well known condition $\mu \leqslant 3$. The convergence is monotone if $n \leqslant 3$, whereas it exhibits damped oscillations if $4 \leqslant n \leqslant 5$. With 6 players we have $\mu = 3.5$, hence we have stable oscillations of period 4 being $\mu > 1 + \sqrt{6}$. The case of 7 competitors gives fully developed chaos, as it corresponds with $\mu = 4$.

Hence, stability is obtained for a limited number of oligopolists, namely $n \leqslant 5$, and instability occurs increasing the number of players.

6 Duopoly with isoelastic demand and quadratic costs

In this section we consider the duopoly model with isoelastic demand function (8) and quadratic costs (18). The best reply dynamics cannot be expressed by a simple dynamical system. In fact, the profit of player i is $\pi_i = q_i/\left(q_1 + q_2\right)^2 - c_{im} - c_i q_i^2$, and the first order conditions for profit maximization give rise to a third degree algebraic equations. For example, the condition for the reaction function of player 1 becomes

$$2c_1q_1^3 + 4c_1q_2q_1^2 + 2c_1q_2^2q_1 - q_2 = 0.$$

So, even if it is easy to see that a unique positive solution $q_1 = r_1(q_2)$ exists, its expression is not easy to be handled. Instead, if we consider the dynamics with LMA (19) with isoelastic demand (8) and $n = 2$, a simple two-dimensional dynamical system is get, represented by the following two-dimensional iterated map:

$$\begin{aligned} q_1(t+1) &= \frac{2q_1(t) + q_2(t)}{2(1 + c_1(q_1(t) + q_2(t))^2)} \\ q_2(t+1) &= \frac{q_1(t) + 2q_2(t)}{2(1 + c_2(q_1(t) + q_2(t))^2)}. \end{aligned} \tag{30}$$

The following proposition holds

Proposition 5. *The dynamical system (30) has a unique nonvanishing equilibrium, given by* $\mathbf{q}^* = (q_1^*, q_2^*)$, *with*

$$\begin{aligned} q_1^* &= \frac{\sqrt{c_2}}{\sqrt{c_1} + \sqrt{c_2}} \frac{1}{\sqrt{2\sqrt{c_1c_2}}} \\ q_2^* &= \frac{\sqrt{c_1}}{\sqrt{c_1} + \sqrt{c_2}} \frac{1}{\sqrt{2\sqrt{c_1c_2}}} \end{aligned} \tag{31}$$

This equilibrium is always locally asymptotically stable.

The proof of this proposition is given in the Appendix.

According to Proposition 1, $\mathbf{q}^*$ is a Nash equilibrium, located at the intersection of the reaction functions. As we do not know an explicit expression of the reaction functions it is impossible to study its stability under best reply. However, the Nash equilibrium reveals a strong stability under LMA. Simulations show that even the basins of attraction are large if compared with the case of the model with LMA and linear costs.

7 Summary and further developments

In this paper we proposed a repeated oligopoly game where we assume that the players do not know the demand function of the market in which they operate, and at each time step they solve a profit maximization problem by using a linear approximation of the demand function, based on the local estimate of its partial derivative, and neglecting the outputs of the competitors. A similar adjustment mechanism has been recently proposed by Tuinstra (2004) for a price a price adjustment process, and an explicit dynamical system is obtained for linear cost functions. Instead, our quantity-setting local monopolistic approximation (LMA) gives rise to explicit discrete-time dynamical systems for any differentiable demand function provided that linear or quadratic cost functions are considered.

As the oligopoly game obtained under the assumption of LMA has the same equilibria of the best reply dynamics (i.e. Nash Equilibria) we tried to compare the stability properties of such equilibria under these two different kinds of dynamic adjustments. Such a comparison has been performed by using one of the simplest nonlinear demand functions, the isoelastic one, and linear cost functions. In fact, this is one of the lucky cases where the reaction functions can be analytically computed (Puu, 1991) and consequently the dynamical system that gives the best reply dynamics can be explicitly written. The results obtained for this particular example show that the adjustment process based on LMA is more stable than the best reply dynamics, even if it is characterized by a lower degree of information and rationality. Furthermore, we showed that the stability of Nash equilibrium also holds when quadratic costs are considered.

In the case of homogeneous products with an isoelastic demand function and linear costs we also analyzed the effects of increasing the number of competitors in the market, and we showed that more players may lead to periodic and chaotic motion.

As the dynamic adjustment mechanism proposed in this paper allows us to get an explicit dynamical system for any arbitrary nonlinear and differentiable demand function, provided that the cost functions are linear or quadratic, a plethora of dynamic model can be studied to get a comparison of stability and dynamic properties under several economic assumptions on nonlinear demand functions that characterizes the economy. In particular, the robustness of the statement "less rationality (and information) implies more stability" given in this paper on the basis of the example with an isoelastic demand function, can be analyzed by considering other examples obtained with different kinds of demand functions.

Another remark about future work, that may be done in the framework of oligopoly games with LMA, concerns the presence of denominator in the maps (17) and (19). Indeed, some interesting dynamic phenomena can be expected, related to the peculiar properties of maps with a denominator that vanishes in a zero-measure subset of the state space, as stressed in Bischi, Gardini and Mira (1999, 2003).

Of course, a similar adjustment mechanism can be applied to Bertrand oligopoly games, and even in the mixed case where some players play a Cournot game and some other ones play a Bertrand game (see e.g. Bylka and Komar, 1976, Cheng, 1985, Onozaky and Matsumoto, 2003).

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Appendix.

Proof of proposition 1

Necessary conditions for (5) are expressed by the first order conditions

$$\frac{\partial \pi_i^e}{\partial q_i(t+1)} = \frac{\partial f^i(q_i(t+1), q_{-i}^e(t+1))}{\partial q_i(t+1)} q_i(t+1) + f^i(q_i(t+1), q_{-i}^e(t+1)) - C_i'(q_i(t+1)) = 0 \quad (32)$$

At a steady state we have $q_i(t+1) = q_i(t) = q_i^*$ and $q_{-i}^e(t+1) = q_{-i}(t) = q_{-i}^*$. So, the conditions (32), that define the best reply dynamics, at a steady state become

$$q_i^* f_i^i(q_i^*, q_{-i}^*) + f^i(q_i^*, q_{-i}^*) = C_i'(q_i^*) \ , \ i = 1, ..., n.$$

This is the equation whose solutions are the Nash equilibria of the Cournot oligopoly game. The first order conditions of the game with LMA (15), at a steady state, become

$$f_i^i q_i + f_i(q_1, q_2) = C_i'(q_i) \ , \ i = 1, ..., n.$$

So, we obtain the same equations at the equilibria both in the case either firms have a perfect knowledge of the demand function, or when they use the local monopolistic approximation. □

To prove the propositions 2 and 3 we follow arguments similar to those given in Tuinstra (2004). These are based on the following result (see e.g. Atkinson, 1989): Let λ be an eigenvalue of the matrix $\mathbf{A}$ and let $\|\cdot\|$ be any matrix norm. Then $|\lambda| \leqslant \|\mathbf{A}\|$. Now, if we consider $\|\mathbf{A}\|_\infty = \max_i \sum_{j=1}^n |\mathbf{A}_{ij}|$ then a sufficient condition for the eigenvalues of $\mathbf{A}$ be inside the unit circle is

$$\sum_{j=1}^n |\mathbf{A}_{ij}| \leqslant 1 \ \text{ for all } i = 1, ..., n \quad (33)$$

Proof of proposition 2.

Let $\mathbf{J}$ denote the Jacobian matrix of (17) computed at $\mathbf{q}^*$. Its diagonal entries are

$$J_{ii} = -\frac{\left(c_i - f^i\left(\mathbf{q}^*\right)\right) f_{ii}^i\left(\mathbf{q}^*\right)}{2\left(f_i^i\left(\mathbf{q}^*\right)\right)^2}$$

and the off-diagonal entries are

$$J_{ij} = -\frac{f_j^i\left(\mathbf{q}^*\right) + \left(c_i - f^i\left(\mathbf{q}^*\right)\right) f_{ij}^i\left(\mathbf{q}^*\right)}{2\left(f_i^i\left(\mathbf{q}^*\right)\right)^2}$$

So, the sufficient condition (33) becomes

$$\frac{\left|c_i - f^i\left(\mathbf{q}^*\right)\right|\left|f^i_{ii}\left(\mathbf{q}^*\right)\right|}{\left(f^i_i\left(\mathbf{q}^*\right)\right)^2} + \sum_{j\neq i} \frac{\left|f^i_j\left(\mathbf{q}^*\right) + \left(c_i - f^i\left(\mathbf{q}^*\right)\right) f^i_{ij}\left(\mathbf{q}^*\right)\right|}{\left(f^i_i\left(\mathbf{q}^*\right)\right)^2} \leqslant 2 \qquad \text{for all } i = 1, ..., n \tag{34}$$

According to Proposition 1, a Nash equilibrium is a fixed point of the map (17), hence $\mathbf{q}^* = (q^*_1, ..., q^*_n)$ satisfies the steady-state equations

$$c_i - f^i(\mathbf{q}^*) = q^*_i f^i_i(\mathbf{q}^*)$$

by which (34) is transformed into (20) □.

Proof of proposition 3.

Let $\mathbf{J}$ denote the Jacobian matrix of (19) computed at $\mathbf{q}^*$. Its diagonal entries are

$$J_{ii} = \frac{f^i_{ii}\left(\mathbf{q}^*\right)\left(f^i\left(\mathbf{q}^*\right) - c_i q^*_i\right)}{2\left(f^i_i\left(\mathbf{q}^*\right) - c_i\right)^2}$$

and the off-diagonal entries are

$$J_{ij} = \frac{f^i_j\left(\mathbf{q}^*\right)\left(c_i - f^i_i\left(\mathbf{q}^*\right)\right) + f^i_{ij}\left(\mathbf{q}^*\right)\left(f^i\left(\mathbf{q}^*\right) - c_i q^*_i\right)}{2\left(f^i_i\left(\mathbf{q}^*\right) - c_i\right)^2}$$

So, the sufficient condition (33) becomes

$$\frac{\left|f^i\left(\mathbf{q}^*\right) - c_i q^*_i\right|\left|f^i_{ii}\left(\mathbf{q}^*\right)\right|}{\left(f^i_i\left(\mathbf{q}^*\right) - c_i\right)^2} + \sum_{j\neq i} \frac{\left|f^i_j\left(\mathbf{q}^*\right)\left(c_i - f^i_i\left(\mathbf{q}^*\right)\right) + f^i_{ij}\left(\mathbf{q}^*\right)\left(f^i\left(\mathbf{q}^*\right) - c_i q^*_i\right)\right|}{\left(f^i_i\left(\mathbf{q}^*\right) - c_i\right)^2} \leqslant 2 \quad \text{for all } i = 1, ..., n \tag{35}$$

According to Proposition 1, a Nash equilibrium is a fixed point of the map (19), hence $\mathbf{q}^* = (q^*_1, ..., q^*_n)$ satisfies the steady-state equations

$$q^*_i f^i_i(\mathbf{q}^*) = 2q^*_i c_i - f^i(\mathbf{q}^*)$$

that can be written as

$$f^i(\mathbf{q}^*) - q^*_i c_i = q^*_i (c_i - f^i_i(\mathbf{q}^*))$$

by which (35) is transformed into (21) □.

Proof of proposition 4

The equation to find the fixed points, obtained by setting $q_i(t+1) = q_i(t)$ in (23) becomes

$$\begin{array}{l} q_1 + \frac{1}{\alpha}\left(q_1(t) + q_2(t)\right)\left(c_1\left(q_1(t) + q_2(t)\right)^\alpha - 1\right) = 0 \\ q_2 + \frac{1}{\alpha}\left(q_1(t) + q_2(t)\right)\left(c_1\left(q_1(t) + q_2(t)\right)^\alpha - 1\right) = 0 \end{array} \tag{36}$$

After adding the two equations, we get

$$(q_1 + q_2)^\alpha = \frac{2-\alpha}{c_1 + c_2}$$

This equilibrium condition shows us that a non vanishing steady state exists only if $\alpha < 2$. We can use this equation to substitute, for example, $q_2 = -q_1 + \left(\frac{2-\alpha}{c_1+c_2}\right)^{1/\alpha}$ in one of the equations (36), from which we get q_1^* as in proposition 4. From the expression of q_1^* we have that $q_1^* > 0$ if $\alpha > 1 - c_2/c_1$. As α is a positive parameter, this condition restricts the range of α only in the case $c_2 < c_1$. Analogously, $q_2^* > 0$ if $\alpha > 1 - c_1/c_2$, and this condition restricts the range of α only in the case $c_1 < c_2$.

The study of the stability of the equilibrium (q_1^*, q_2^*) is particularly easy. In fact, the Jacobian matrix for the map (23)

$$J(q_1, q_2) = \begin{bmatrix} \frac{1}{2} - \frac{1}{2\alpha}\left[(\alpha+1)c_1(q_1+q_2)^\alpha - 1\right] & -\frac{1}{2\alpha}\left[c_1(\alpha+1)(q_1+q_2)^\alpha - 1\right] \\ -\frac{1}{2\alpha}\left[c_2(\alpha+1)(q_1+q_2)^\alpha - 1\right] & \frac{1}{2} - \frac{1}{2\alpha}\left[(\alpha+1)c_2(q_1+q_2)^\alpha - 1\right] \end{bmatrix}$$

computed at the equilibrium

$$J(q_1^*, q_2^*) = \begin{bmatrix} \frac{1}{2} - \frac{1}{2\alpha}\left[(\alpha+1)c_1\frac{2-\alpha}{c_1+c_2} - 1\right] & -\frac{1}{2\alpha}\left[c_1(\alpha+1)\frac{2-\alpha}{c_1+c_2} - 1\right] \\ -\frac{1}{2\alpha}\left[c_2(\alpha+1)\frac{2-\alpha}{c_1+c_2} - 1\right] & \frac{1}{2} - \frac{1}{2\alpha}\left[(\alpha+1)c_2\frac{2-\alpha}{c_1+c_2} - 1\right] \end{bmatrix}$$

gives the simple characteristic equation

$$\lambda^2 - \frac{1+\alpha}{2}\lambda + \frac{\alpha}{4} = 0.$$

Hence the eigenvalues are $\lambda_1 = 1/2$ and $\lambda_2 = \alpha/2$. This implies that the equilibrium (q_1^*, q_2^*) is locally stable for each α in the range $0 < \alpha < 2$ $\square$.

Proof of proposition 5

The equations to find the fixed points, obtained by setting $q_i(t+1) = q_i(t)$ in (30), become

$$\begin{aligned} 2c_1 q_1 (q_1+q_2)^2 &= q_2 \\ 2c_2 q_2 (q_1+q_2)^2 &= q_1 \end{aligned} \tag{37}$$

from which $\frac{c_1 q_1}{c_2 q_2} = \frac{q_2}{q_1}$ and consequently $q_1 = \sqrt{\frac{c_2}{c_1}}q_2$. Substituting this into one of the equations (37) we get the equilibrium (31). Sufficient conditions for the stability of $\mathbf{q}^*$ are easily obtained from the computation of the Jacobian matrix $\mathbf{J} = (J_{ij})$ of (30) at the equilibrium $\mathbf{q}^*$. In fact, the diagonal entries are

$$J_{11} = -\frac{q_1^* f_{11}^1(\mathbf{q}^*)}{2\left(f_1^1(\mathbf{q}^*) - c_1\right)} = \frac{3c_1\sqrt{c_2}}{3c_1\sqrt{c_2} + c_1\sqrt{c_1} + 2c_2\sqrt{c_1}}$$

$$J_{11} = -\frac{q_2^* f_{22}^2(\mathbf{q}^*)}{2\left(f_2^2(\mathbf{q}^*) - c_2\right)} = \frac{3c_2\sqrt{c_1}}{3c_2\sqrt{c_1} + c_2\sqrt{c_2} + 2c_1\sqrt{c_2}}$$

and the off-diagonal entries are

$$J_{12} = -\frac{q_1^* f_{12}^1(\mathbf{q}^*) + f_2^1(\mathbf{q}^*)}{2\left(f_1^1(\mathbf{q}^*) - c_1\right)} = \frac{c_1\sqrt{c_2} - c_2\sqrt{c_1}}{3c_1\sqrt{c_2} + c_1\sqrt{c_1} + 2c_2\sqrt{c_1}}$$
$$J_{21} = -\frac{q_2^* f_{21}^2(\mathbf{q}^*) + f_1^2(\mathbf{q}^*)}{2\left(f_2^2(\mathbf{q}^*) - c_2\right)} = \frac{c_2\sqrt{c_1} - c_1\sqrt{c_2}}{3c_2\sqrt{c_1} + c_2\sqrt{c_2} + 2c_1\sqrt{c_2}}$$

Hence, the trace of the Jacobian matrix at the equilibrium is

$$Tr = \frac{3c_1\sqrt{c_2}}{3c_1\sqrt{c_2} + c_1\sqrt{c_1} + 2c_2\sqrt{c_1}} + \frac{3c_1\sqrt{c_2}}{3c_1\sqrt{c_2} + c_1\sqrt{c_1} + 2c_2\sqrt{c_1}}$$

and the determinant is

$$Det = \frac{c_1c_2(2\sqrt{c_1c_2} + c_1 + c_2)}{\left(3c_1\sqrt{c_2} + c_1\sqrt{c_1} + 2c_2\sqrt{c_1}\right)\left(3c_1\sqrt{c_2} + c_1\sqrt{c_1} + 2c_2\sqrt{c_1}\right)}$$

A set of sufficient conditions for the stability of $\mathbf{q}^*$, i.e. for the eigenvalues to be inside the unit circle of the complex plane, is given by

$$1 + Tr + Det > 0; \quad 1 - Tr + Det > 0; \quad Det < 1 \tag{38}$$

(see e.g. Medio and Lines, 2001, p.52). These conditions become trivial in our case. In fact, being Tr and Det both positive, the first condition is always satisfied. Moreover

$$1 - Tr + Det = \frac{4c_1c_2\left(c_2\sqrt{c_1} + c_1 + c_2\right)}{\left(3c_1\sqrt{c_2} + c_1\sqrt{c_1} + 2c_2\sqrt{c_1}\right)\left(3c_1\sqrt{c_2} + c_1\sqrt{c_1} + 2c_2\sqrt{c_1}\right)} > 0$$

and $Det < 1$ being $c_1c_2(2\sqrt{c_1c_2} + c_1 + c_2) < \left(3c_1\sqrt{c_2} + c_1\sqrt{c_1} + 2c_2\sqrt{c_1}\right)\left(3c_1\sqrt{c_2} + c_1\sqrt{c_1} + 2c_2\sqrt{c_1}\right)$.

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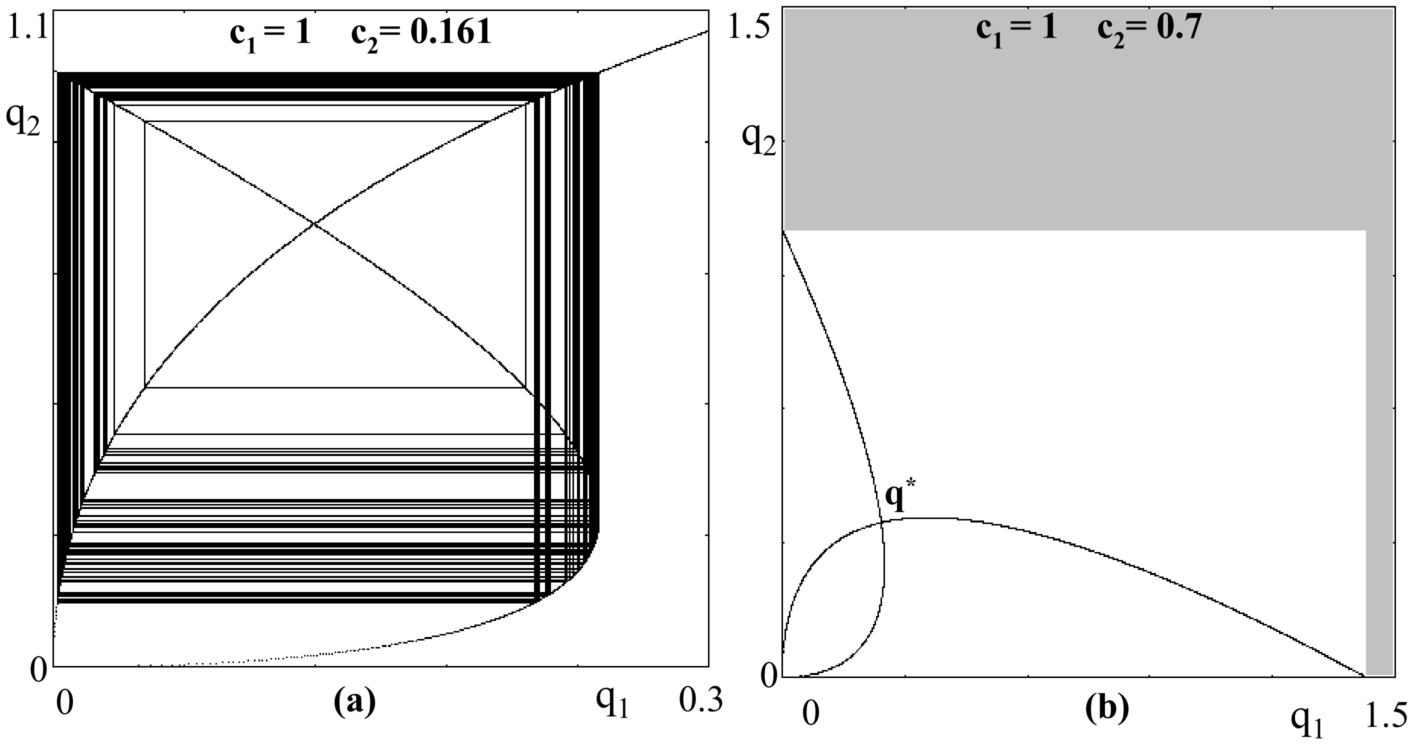
Figure Captions

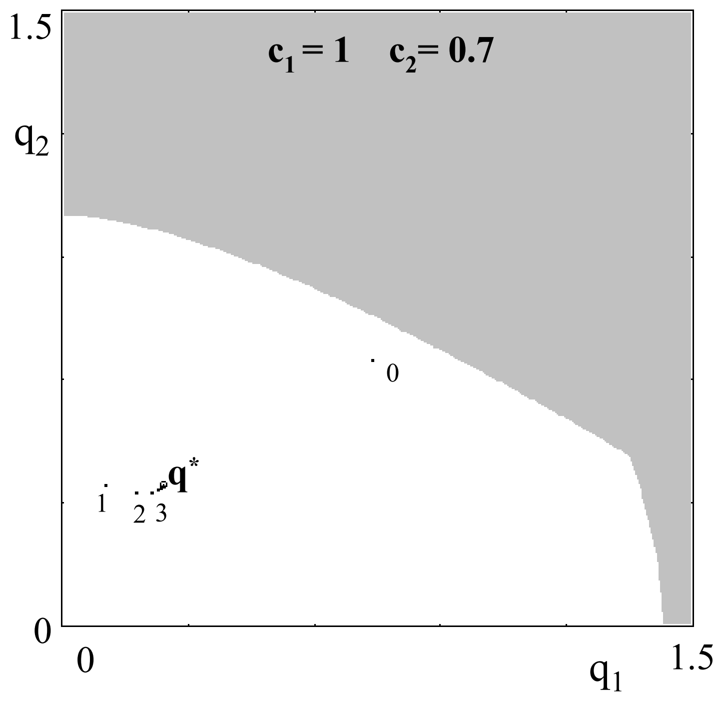
Fig. 1. Duopoly model with isoelastic demand (8), linear costs and best reply dynamics (a) A chaotic trajectory obtained with parameters $c_1 = 1$ and $c_2 = 0.161$. (b) For the same model with $c_1 = 1$ and $c_2 = 0.7$, the white region represents the basin of attraction of the stable Nash Equilibrium, located at the intersection of the reaction curves, whereas the grey region represent the set of initial conditions that generate unfeasible trajectories.

Fig.2. Duopoly model with LMA dynamics, obtained with isoelastic demand (8) and linear costs. The parameter values are the same as in fig. 1b. A typical trajectory that converges to the Nash equilibrium is also represented by dots labelled by $0, 1, ...$



1.1
c1 = 1 c2= 0.161
q2
0
0
(a)
q1
0.3
1.5
c1 = 1 c2= 0.7
q2
q*
0
0
(b)
q1
1.5

Fig. 1



1.5
c1 = 1 c2= 0.7
q2
0
1
2
3
q*
0
0
q1
1.5

Fig. 2