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Numerics of Implied Binomial Trees

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Numerics of Implied Binomial Trees *

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Abstract

Market option prices in last 20 years confirmed deviations from the Black and Scholes (BS) models assumptions, especially on the BS implied volatility. Implied binomial trees (IBT) models capture the variations of the implied volatility known as "volatility smile". They provide a discrete approximation to the continuous risk neutral process for the underlying assets. In this paper, we describe the numerical construction of IBTs by Derman and Kani (DK) and an alternative method by Barle and Cakici (BC). After the formation of IBT we can estimate the implied local volatility and the state price density (SPD). We compare the SPD estimated by the IBT methods with a conditional density computed from a simulated diffusion process. In addition, we apply the IBT to EUREX option prices and compare the estimated SPDs. Both IBT methods coincide well with the estimation from the simulated process, though the BC method shows smaller deviations in case of high interest rate, particularly.

Keywords: Implied Tree Models; Implied Volatility; Local Volatility; Option Pricing
JEL classification: G12; G13; C13

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10 Numerics of Implied Binomial Trees

Wolfgang Härdle and Alena Myšičková

For about 20 years now, discrepancies between market option prices and Black and Scholes (BS) prices have widened. The observed market option price showed that the BS implied volatility, computed from the market option price by inverting the BS formula varies with strike price and time to expiration. These variations are known as "the volatility smile (skew)" and volatility term structure, respectively.

In order to capture the dependence on strike and time to maturity, various smile-consistent models (based on an arbitrage-free approach), have been proposed in the literature. One approach is to model the volatility as a stochastic process, see Hull and White (1987) or Derman and Kani (1998); another works with discontinuous jumps in the stock price, see Merton (1976). However, these extensions cause several practical difficulties such as the violation of the risk-neutrality or no-arbitrage. In contrast, more recent publications proposed by Rubinstein (1994), Derman and Kani (1994), Dupire (1994), and Barle and Cakici (1998) have introduced a locally deterministic volatility function that varies with market price and time. These models independently construct a discrete approximation to the continuous risk neutral process for the underlying assets in the form of binomial or trinomial trees. These deterministic volatility models have both practical and theoretical advantages: they are easily realisable and preserve the no-arbitrage idea inherent in the BS model.

The implied binomial tree (IBT) method constructs a numerical procedure which is consistent with the smile effect and the term structure of the implied volatility. The IBT algorithm is a data adaptive modification of the Cox, Ross and Rubinstein (1979)(CRR) method where the stock evolves along a risk neutral binomial tree with constant volatility.

The following three requirements should be minimally satisfied by an IBT:

- correct reproduction of the volatility smile
- node transition probabilities lying in $[0, 1]$-intervall only
- risk neutral branching process (forward price of the underlying asset equals the conditional expected value of itself) at each step.

The last two conditions also guarantee no-arbitrage; should the stock price fall below or above its corresponding forward price, the transition probability would exceed the $[0, 1]$-interval.

The basic aim of the IBT is the estimation of implied probability distributions, or state price densities (SPD), and local volatility surfaces. Furthermore, the IBT may evaluate the future stock price distributions according to the BS implied volatility surfaces which are calculated from observed daily market European option prices.

In this chapter, we describe the numerical construction of the IBT and compare the predicted implied price distributions. In Section 10.1, a detailed construction of the IBT algorithm for European options is presented. First, we introduce the Derman and Kani (1994) (DK) algorithm and show its possible drawbacks. Afterwards, we follow an alternative IBT algorithm by Barle and Cakici (1998) (BC), which modifies the DK method by a normalisation of the central nodes according to the forward price in order to increase its stability in the presence of high interest rates. In Section 10.2 we compare the SPD estimations with simulated conditional density from a diffusion process with a non-constant volatility. In the last section, we apply the IBT to a real data set containing underlying asset price, strike price, time to maturity, interest rate, and call/put option price from EUREX (Deutsche Börse Database). We compare the SPD estimated by real market data with those predicted by the IBT.

10.1 Construction of the IBT

In the early 1970s, Black and Scholes presented the Geometric Brownian Motion (GBM) model, where the stock price S_t is a solution of the stochastic differential equation (SDE):

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t \, , \tag{10.1}$$

with a standard Wiener process W_t and the constant instantaneous drift μ. The constant instantaneous volatility function σ measures the return variability around its expectation μ. Using a risk neutral measure Q, see Fengler (2005), the BS pricing formulae for european call and put options are:

$$C_t = e^{-r\tau}\mathsf{E}_{\mathsf{Q}}\{\max(S_T - K, 0)\} \tag{10.2}$$
$$P_t = e^{-r\tau}\mathsf{E}_{\mathsf{Q}}\{\max(K - S_T, 0)\} \, . \tag{10.3}$$

Under these relations the underlying at the expiration date follows a conditional lognormal distribution with density:

$$q(S_T|S_t, r, \tau, \sigma) = \frac{1}{S_T\sqrt{2\pi\sigma^2\tau}} \exp\left[-\frac{\left\{\log\left(\frac{S_T}{S_t}\right) - (r - \frac{\sigma^2}{2})\tau\right\}^2}{2\sigma^2\tau}\right] . \qquad (10.4)$$

In the upper equations T is the expiration date, S_t is the stock price at time t, $\tau = T - t$ is time to maturity, K is the strike price and r is the riskless interest rate.

Looking at a general SDE for an underlying asset price process:

$$\frac{dS_t}{S_t} = \mu(S_t, t)dt + \sigma(S_t, t, \cdot)dW_t \ , \qquad (10.5)$$

we can differentiate the following three concepts of volatility, see Fengler (2005):

Instantaneous volatility $\sigma(S_t, t, \cdot)$

- measures the instantaneous standard deviation of $\log S_t$
- depends on the current level of the asset price S_t, time t and possibly on other state variables denoted with '$\cdot$'.

Implied volatility $\widehat{\sigma}_t(K, T)$

- the BS option price implied measure of volatility, the instantaneous standard deviation of $\log S_t$
- the volatility parameter corresponds to the BS price and a particular observed market option price
- depends on the strike K, the expiration date T and time t.

Local volatility $\sigma_{K,T}(S_t, t)$

- expected instantaneous volatility conditional on a particular level of the asset price $S_T = K$ at $t = T$
- In a deterministic model we can write $\sigma_{K,T}(S_t, t) = \sigma(K, T)$.

The CRR binomial tree is constructed as a discrete approximation of a GBM process with a constant instantaneous volatility $\sigma_t(S_t, t) = \sigma$. Analogously, the IBT can be viewed as a discretization of an instantaneous volatility model:

$$\frac{dS_t}{S_t} = \mu_t dt + \sigma(S_t, t)dW_t, \qquad (10.6)$$

where $\sigma(S_t, t)$ depends on both the underlying price and time. The purpose of the IBT is to construct a discrete implementation of the extended BS model based on the observed option prices yielding the variable volatility $\sigma(S_t, t)$. In addition, the IBT may reflect a non-constant drift μ_t. After the construction of the IBT, we are able to estimate a local volatility from underlying stock prices and transition probabilities.

In the IBT construction, only observable data (market option prices, underlying prices, interest rate) are used, it is therefore nonparametric in nature. Several alternative studies based on the kernel method, A¨it-Sahalia and Lo (1998), or nonparametric constrained least squares, Yatchew and Härdle (2006), and curve-fitting methods, Jackwerth and Rubinstein (1996) have been published in recent years.

10.1.1 The Derman and Kani Algorithm

In the DK IBT approach, stock prices, transition probabilities and Arrow-Debreu prices (discounted risk neutral probabilities) are calculated iteratively level by level, starting in the level zero.

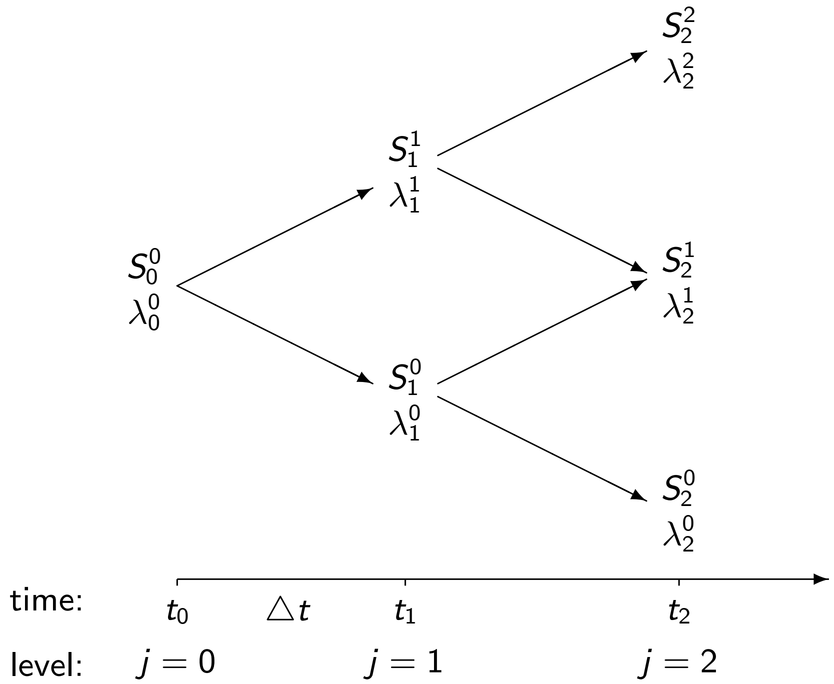

Figure 10.1. Construction of an implied binomial tree.

Figure 10.1 illustrates the construction of the first two nodes of an IBT. We build the IBT on the time interval $[0, T]$ with $j = 0, 1, 2, \ldots, n$ equally spaced levels, $\triangle t$ apart. We start at zero level with $t = 0$, here the stock price equals the current price of the underlying: $S_0^0 = S$. There are $n+1$ nodes at the nth level of the tree, we indicate the stock price of the ith node at the nth level by S_n^i, and the forward price at level $n+1$ of S_n^i at level n by $F_n^i = e^{r\triangle t} S_n^i$. The conditional probability $p_{i+1}^n = \mathrm{P}(S_{n+1} = S_{n+1}^{i+1} | S_n = S_n^i)$ is the transition probability of making a transition from node (n, i) to node $(n+1, i+1)$.

The forward price $F_{n,i}$ is required to satisfy the risk neutral condition:

$$F_n^i = p_{i+1}^n S_{n+1}^{i+1} + (1 - p_{i+1}^n) S_{n+1}^i \,. \tag{10.7}$$

Thus we obtain the transition probability from the following equation:

$$p_{i+1}^n = \frac{F_n^i - S_{n+1}^i}{S_{n+1}^{i+1} - S_{n+1}^i} \,. \tag{10.8}$$

The Arrow-Debreu price is the price of an option which pays 1 unit payoff if the stock price S_t at time t attains the value S_n^i, and 0 otherwise. The Arrow-Debreu price in the state i at level n can be computed as the expected discounted value of its payoff: $\lambda_n^i = \mathsf{E}[e^{-rt} \mathbf{1}(S_t = S_n^i) | S_0 = S_0^0]$. In general, Arrow-Debreu prices can be obtained by the iterative formula, where $\lambda_0^0 = 1$ as a definition.

$$\begin{aligned} \lambda_{n+1}^0 &= e^{-r\triangle t} \left\{ \lambda_n^0 (1 - p_1^n) \right\} \\ \lambda_{n+1}^{i+1} &= e^{-r\triangle t} \left\{ \lambda_n^i p_{i+1}^n + \lambda_n^{i+1} (1 - p_{i+2}^n) \right\}, \quad 0 \le i \le n-1 \\ \lambda_{n+1}^{n+1} &= e^{-r\triangle t} \left\{ \lambda_n^n p_{n+1}^n \right\} \end{aligned} \tag{10.9}$$

To illustrate the calculation of the Arrow-Debreu prices, we provide an example with a construction of a CRR binomial tree. Let us assume that the current value of the underlying $S = 100$, time to maturity $\tau = T = 2$ years, $\triangle t = 1$ year, constant volatility $\sigma = 10\%$, and riskless interest rate $r = 0.03$. The Arrow-Debreu price tree shown in the Figure 10.3 can be calculated from the stock price tree in the Figure 10.2.

Using the CRR method, the stock price at the lower node at the first level equals $S_1^0 = S_0^0 \cdot e^{-\sigma \triangle t} = 100 \cdot e^{-0.1} = 90.52$, and at the upper node $S_1^1 = S_0^0 \cdot e^{\sigma \triangle t} = 110.47$. The transition probability $p_1^0 = 0.61$ is obtained by the formula (10.8) with $F_0^0 = S_0^0 e^{0.03} = 103.05$. Now, we calculate λ_1^i for $i = 0, 1$, according to the formula (10.9): $\lambda_1^0 = e^{-r \triangle t} \cdot \lambda_0^0 \cdot (1 - p_1^0) = 0.36$ and $\lambda_1^1 = e^{-r \triangle t} \cdot \lambda_0^0 \cdot p_1^0 = 0.61$. At the second level, we calculate the stock prices according to the corresponding nodes at the first level, for example: $S_2^0 = S_1^0 \cdot e^{-\sigma \triangle t} = 81.55$, $S_2^1 = S_0^0 = 100$ and $S_2^2 = S_1^1 \cdot e^{\sigma \triangle t} = 122.04$.

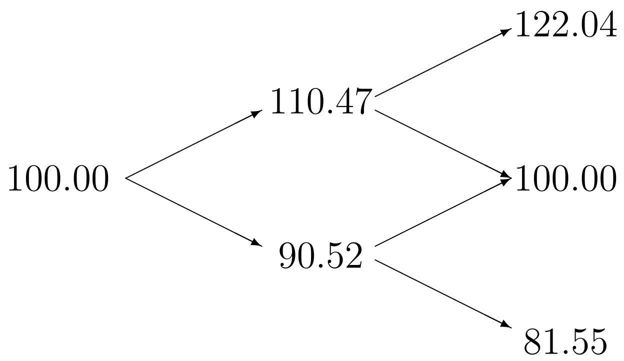

Figure 10.2. CRR binomial tree for stock prices with $T = 2$ years, $\triangle t = 1$, $\sigma = 0.1$ and $r = 0.03$. XFGIBT01

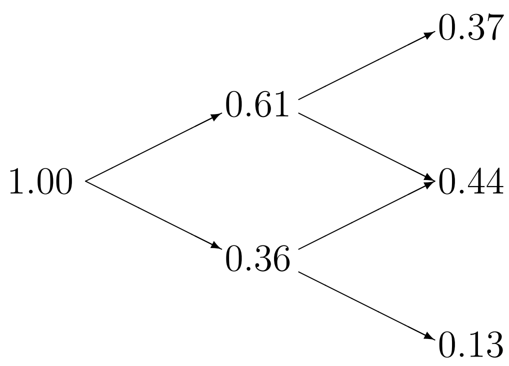

Figure 10.3. CRR binomial tree for Arrow-Debreu prices with $T = 2$ years, $\triangle t = 1$, $\sigma = 0.1$ and $r = 0.03$. XFGIBT01

The corresponding Arrow-Debreu prices λ_2^i for $i = 0, 1, 2$ are obtained by the substitution in the formula 10.9:

$$\begin{aligned}
\lambda_2^0 &= e^{-r\triangle t} \cdot \lambda_1^0 \cdot (1 - p_1^1) = 0.13 \\
\lambda_2^1 &= e^{-r\triangle t} \cdot \{\lambda_1^0 \cdot p_1^1 + \lambda_1^1 \cdot (1 - p_2^1) = 0.44\} \\
\lambda_2^2 &= e^{-r\triangle t} \cdot \lambda_1^1 \cdot p_2^1 = 0.37 \,.
\end{aligned}$$

In the BS model with the state price density (SPD) from 10.4, the option prices are given by:

$$C(K,\tau) = e^{-r\tau} \int_0^{+\infty} \max(S_T - K, 0)\, q(S_T|S_t, r, \tau) dS_T, \tag{10.10}$$

$$P(K,\tau) = e^{-r\tau} \int_0^{+\infty} \max(K - S_T, 0)\, q(S_T|S_t, r, \tau) dS_T\ , \tag{10.11}$$

where $C(K,\tau)$ and $P(K,\tau)$ denote call option price and put option price respectively, and K is the strike price. In the IBT, option prices are calculated in discrete time intervals $\tau = n\triangle t$ using the Arrow-Debreu prices,

$$C(K, n\triangle t) = \sum_{i=0}^{n} \lambda_{n+1}^{i+1} \max(S_{n+1}^{i+1} - K, 0)\ , \tag{10.12}$$

$$P(K, n\triangle t) = \sum_{i=0}^{n} \lambda_{n+1}^{i+1} \max(K - S_{n+1}^{i+1}, 0)\ . \tag{10.13}$$

Using the risk neutral condition (10.7) and the discrete option price calculation from (10.12) or (10.13), one obtains the iteration formulae to construct the IBT.

Let us assume the strike price is equal to the known stock price: $K = S_n^i = S$. Then the contribution from the transition to the first in-the-money upper node can be separated from the other contributions. Using the iterative formulae for the Arrow-Debreu prices (10.9) in the equation (10.12):

$$
\begin{aligned}
e^{r\triangle t}C(S,n\triangle t) &= \lambda_n^0(1-p_1^n)\max(S_{n+1}^0-S,0)+\lambda_n^n p_{n+1}^n\max(S_{n+1}^{n+1}-S,0)\\
&\quad+\sum_{j=0}^{n-1}\Big\{\lambda_n^j p_{j+1}^n+\lambda_n^{j+1}(1-p_{j+2}^n)\Big\}\max(S_{n+1}^{j+1}-S,0)\\
&= \left\{\lambda_n^i p_{i+1}^n+\lambda_n^{i+1}(1-p_{i+2}^n)\right\}(S_{n+1}^{i+1}-S)+\lambda_n^n p_{n+1}^n(S_{n+1}^{n+1}-S)\\
&\quad+\sum_{j=i+1}^{n-1}\left\{\lambda_n^j p_{j+1}^n+\lambda_n^{j+1}\left(1-p_{j+2}^n\right)\right\}(S_{n+1}^{j+1}-S)\\
&= \lambda_n^i p_{i+1}^n(S_{n+1}^{i+1}-S)\\
&\quad+\sum_{j=i+1}^{n-1}\lambda_n^j p_{j+1}^n(S_{n+1}^{j+1}-S)+\lambda_n^n p_{n+1}^n(S_{n+1}^{n+1}-S)\\
&\quad+\lambda_n^{i+1}(1-p_{i+2}^n)(S_{n+1}^{i+1}-S)+\sum_{j=i+2}^{n}\lambda_n^j(1-p_{j+1}^n)(S_{n+1}^j-S)\\
&= \lambda_n^i p_{i+1}^n(S_{n+1}^{i+1}-S)\\
&\quad+\sum_{j=i+1}^{n}\lambda_n^j\Big\{\left(1-p_{j+1}^n\right)(S_{n+1}^j-S)+p_{j+1}^n(S_{n+1}^{j+1}-S)\Big\}\ .
\end{aligned}
$$

Entering the risk neutral condition (10.7) in the last term, one obtains:

$$
e^{r\triangle t}C(S,n\triangle t)=\lambda_n^i p_{i+1}^n\left(S_{n+1}^{i+1}-S\right)+\sum_{j=i+1}^{n}\lambda_n^j\left(F_n^j-S\right)\ . \tag{10.14}
$$

Now, the stock price for the upper node can be rewritten in terms of the known Arrow-Debreu prices λ_n^i, the known stock prices S_n^i and the known forwards F_n^i:

$$
S_{n+1}^{i+1}=\frac{S_{n+1}^i\left\{C\left(S_n^i,n\triangle t\right)e^{r\triangle t}-\rho_u\right\}-\lambda_n^i S_n^i\left(F_n^i-S_{n+1}^i\right)}{C\left(S_n^i,n\triangle t\right)e^{r\triangle t}-\rho_u-\lambda_n^i\left(F_n^i-S_{n+1}^i\right)}\ , \tag{10.15}
$$

where ρ_u denotes the following summation term:

$$
\rho_u=\sum_{j=i+1}^{n}\lambda_n^j(F_n^j-S_n^i)\ . \tag{10.16}
$$

The transition from the nth to the $(n+1)$th level of the tree is defined by $(2n+3)$ parameters, i.e. $(n+2)$ stock prices of the nodes at the $(n+1)$th level, and $(n+1)$ transition probabilities (when the IBT starts at the zero-level). Suppose $(2n+1)$ parameters corresponding to the nth level are known, the stock prices S^i_{n+1} and transition probabilities p^n_{i+1} at all nodes above the centre of the tree corresponding to the $(n+1)$th level can be found iteratively using the equations (10.15) and (10.8) as follows:

We always start from the central nodes, if n is odd, define $S^i_{n+1} = S^0_0 = S$, for $i = (n+1)/2$. If n is even, we start from the two central nodes just below and above the centre of the level, S^i_{n+1} and S^{i+1}_{n+1} for $i = n/2$, and set $S^i_{n+1} = (S^i_n)^2/S^{i+1}_{n+1} = S^2/S^{i+1}_{n+1}$, which adjusts the logarithmic CRR centring spacing between S^i_n and S^{i+1}_{n+1} to be the same as that between S^i_n and S^i_{n+1}. Substituting this relation into (10.15) one gets the formula for the upper of the two central nodes for the odd levels:

$$S^{i+1}_{n+1} = \frac{S\left\{C\left(S, n\triangle t\right)e^{r\triangle t} + \lambda^i_n S - \rho_u\right\}}{\lambda^i_n F^i_n - e^{r\triangle t}C\left(S, n\triangle t\right) + \rho_u} \qquad \text{for } i = \frac{n}{2}\,. \tag{10.17}$$

Once we have the initial nodes' stock prices, according to the relationships among the different parameters, we can repeat the process to calculate those at higher nodes $(n+1, j)$, $j = i+2, \dots n+1$ one by one.

Similarly, we can calculate the parameters at lower nodes $(n+1, j)$, $j = i-1, \dots, 1$ at the $(n+1)$th level by using the known put prices $P(K, n\triangle t)$ for $K = S^i_n$.

$$S^i_{n+1} = \frac{S^{i+1}_{n+1}\left\{e^{r\triangle t}P(S^i_n, n\triangle t) - \rho_l\right\} - \lambda^i_n S^i_n(F^i_n - S^{i+1}_{n+1})}{e^{r\triangle t}P\left\{S^i_n, (n+1)\triangle t\right\} - \rho_l + \lambda^i_n(F^i_n - S^{i+1}_{n+1})}\,, \tag{10.18}$$

where ρ_l denotes the sum over all nodes below the one with price S^i_n:

$$\rho_l = \sum_{j=0}^{i-1} \lambda^j_n (S^i_n - F^j_n)\,. \tag{10.19}$$

Transition probabilities and Arrow-Debreu prices are obtained by (10.8) and (10.9), respectively.

$C(K, \tau)$ and $P(K, \tau)$ in (10.15) and (10.18) are the interpolated values for a call or put struck today at strike price K and time to maturity τ. In the DK construction, they are obtained by the CRR binomial tree with constant parameters $\sigma = \sigma_{imp}(K, \tau)$, calculated from the known market option prices. In practice, calculating interpolated option prices by the CRR method is computationally intensive.

10.1.2 Compensation

The transition probability p_i^n at any node should lie between 0 and 1, this condition avoids the riskless arbitrage: if $p_{i+1}^n > 1$, the stock price S_{n+1}^{i+1} would fall below the forward price F_n^i, similarly, if $p_{i+1}^n < 0$, the strike price S_{n+1}^i would fall above the forward price F_n^i. Therefore it is useful to limit the estimated stock prices by the neighbouring forwards from the previous level:

$$F_n^i < S_{n+1}^{i+1} < F_n^{i+1} \; . \qquad (10.20)$$

If the stock price does not fulfil the above inequality condition, we redefine it by assuming that the logaritmic difference between the stock prices at this node and its adjacent is equal to the logaritmic difference between the corresponding stock prices at the two nodes at the previous level, i.e., $\log(S_{n+1}^{i+1}/S_{n+1}^i) = \log(S_n^i/S_n^{i-1})$. Sometimes, the obtained price still does not satisfy inequality (10.20), then we substitute the stock price S_{n+1}^{i+1} by the average of F_n^i and F_n^{i+1}.

As used in the construction of the IBT in (10.12) or (10.13), the implied conditional distribution, the SPD $q(S_T|S_t, r, \tau)$, could be estimated at discrete time $\tau = n\triangle t$ by the product of the Arrow-Debreu prices λ_{n+1}^i at the $(n+1)$th level with the influence of the interest rate $e^{rn\triangle t}$. To fulfill the risk-neutrality condition (10.7), the conditional expected value of the underlying log stock price in the following $(n+1)$th level, given the stock price at the nth level is defined as:

$$M = \mathsf{E}_{\mathrm{Q}}\{\log(S_{n+1})|S_n = S_n^i\} = p_{i+1}^n \log(S_{n+1}^{i+1}) + (1-p_{i+1}^n)\log(S_{n+1}^i) \; . \qquad (10.21)$$

We can specify such a condition also for the conditional second moments of $\log(S_{n+1})$ at $S_n = S_n^i$, which is the implied local volatility $\sigma^2(S_n^i, n\triangle t)$ during the time period $\triangle t$:

$$\begin{aligned} \sigma^2(S_n^i, \triangle t) &= \mathsf{Var}_{\mathrm{Q}}\{\log(S_{n+1})|S_n = S_n^i\} \\ &= p_{i+1}^n\{\log(S_{n+1}^{i+1}) - M\}^2 + (1-p_{i+1}^n)\{\log(S_{n+1}^i) - M\}^2 \\ &= 2\log\left(\frac{S_{n+1}^{i+1}}{S_{n+1}^i}\right)\{p_{i+1}^n(1-p_{i+1}^n)\} \; . \end{aligned} \qquad (10.22)$$

After the construction of an IBT, all stock prices, transition probabilities, and Arrow-Debreu prices at any node in the tree are known. We are thus able to calculate the local volatility $\sigma(S_n^i, m\triangle t)$ at any level m.

In general, the instantaneous volatility function used in the diffusion model (10.6) is different from the local volatility function derived in (10.22), only in the BS model are they identical. Additional, the BS implied volatility

$\widehat{\sigma}(K, \tau)$, which assumes the Black-Scholes model at least locally, differs from the local volatility $\sigma(s, \tau)$, they describe different characteristics of the second moment using different parameters.

If we choose $\triangle t$ small enough, we obtain the estimated SPD at fixed time to maturity, and the distribution of local volatility $\sigma(S, \tau)$.

10.1.3 Barle and Cakici Algorithm

Barle and Cakici (1998) (BC) suggest an improvement of the DK construction. The first major modification is the choice of the strike price in which the option should be evaluated (as in 10.14). In the BC algorithm, the strike price K is chosen to be equal to the forward price F_n^i, and similarly to the DK construction, using the discrete approximation (10.12) we get:

$$\begin{aligned} e^{r\triangle t} C(F_n^i, n\triangle t) &= \sum_{j=0}^{n} \lambda_{n+1}^{j+1} \max(S_{n+1}^{j+1} - F_n^i, 0) \\ &= \left\{\lambda_n^i p_{i+1}^n + \lambda_n^{i+1}(1 - p_{i+2}^n)\right\} (S_{n+1}^{i+1} - F_n^i) + \lambda_n^n p_{n+1}^n (S_{n+1}^{n+1} - F_n^i) \\ &\quad + \sum_{j=i+1}^{n-1} \left\{\lambda_n^j p_{j+1}^n + \lambda_n^{j+1}\left(1 - p_{j+2}^n\right)\right\} (S_{n+1}^{j+1} - F_n^i) \\ &= \lambda_n^i p_{i+1}^n (S_{n+1}^{i+1} - F_n^i) \\ &\quad + \sum_{j=i+1}^{n} \lambda_n^j \left\{\left(1 - p_{j+1}^n\right)(S_{n+1}^j - F_n^i) + p_{j+1}^n (S_{n+1}^{j+1} - F_n^i)\right\} . \end{aligned}$$

Entering the risk neutral condition again (10.7) one obtains:

$$e^{r\triangle t} C(F_n^i, n\triangle t) = \lambda_n^i p_{i+1}^n \left(S_{n+1}^{i+1} - F_n^i\right) + \sum_{j=i+1}^{n} \lambda_n^j \left(F_n^j - F_n^i\right) . \tag{10.23}$$

Identify the upper sum as:

$$\varrho_u = \sum_{j=i+1}^{n} \lambda_n^j \left(F_n^j - F_n^i\right) , \tag{10.24}$$

and using the equation for the transition probability (10.8) we can write the recursion relation for the stock price in the upper node as follows:

$$S_{n+1}^{i+1} = \frac{S_{n+1}^i \left\{C\left(F_n^i, n\triangle t\right) e^{r\triangle t} - \varrho_u\right\} - \lambda_n^i F_n^i \left(F_n^i - S_{n+1}^i\right)}{C\left(F_n^i, n\triangle t\right) e^{r\triangle t} - \varrho_u - \lambda_n^i \left(F_n^i - S_{n+1}^i\right)} . \tag{10.25}$$

Analogous to the DK construction, we start from the central nodes of the binomial tree, but in contrast with the DK construction the BC construction takes the riskless interest rate into account. If $(n+1)$ is even, the price of the central node $S^i_{n+1} = S^0_0 e^{r\triangle t}$ for $i = (n+1)/2$. If $(n+1)$ is odd, the two central nodes must satisfy $S^i_{n+1} \cdot S^{i+1}_{n+1} = (F^i_n)^2$. Adding this condition to the equation (10.25) the lower central node can be calculated as:

$$S^i_{n+1} = F^i_n \frac{\lambda^i_n F^i_n - \{e^{r\triangle t} C(F^i_n, n\triangle t) - \varrho_u\}}{\lambda^i_n F^i_n + \{e^{r\triangle t} C(F^i_n, n\triangle t) - \varrho_u\}} \quad \text{for} \quad i = 1 + n/2, \tag{10.26}$$

the upper one is then: $S^{i+1}_{n+1} = (F^i_n)^2 / S^i_{n+1}$.

After stock prices of the central nodes are obtained, we repeat the recursion equation (10.25) to calculate the stock prices at higher nodes $(n+1, j), j = i+2, \ldots, n+1$. The transition probabilities and Arrow-Debreu prices are calculated through (10.8) and (10.9), respectively.

Similarly, an analogous recursion relation for the stock prices at lower nodes can be found by using put option prices at strike F^i_n:

$$S^i_{n+1} = \frac{S^{i+1}_{n+1}\{P(F^i_n, n\triangle t)e^{r\triangle t} - \varrho_l\}\lambda^i_n F^i_n (S^{i+1}_{n+1} - F^i_n)}{P(F^i_n, n\triangle t)e^{r\triangle t} - \varrho_l - \lambda^i_n (S^{i+1}_{n+1} - F^i_n)}, \tag{10.27}$$

where where ϱ_l denotes the lower sum:

$$\varrho_l = \sum_{j=0}^{i-1} \lambda^j_n (F^i_n - F^j_n) \ .$$

Notice that BC use the Black-Scholes call and put option prices $C(K,\tau)$ and $P(K,\tau)$, which makes the calculation faster than the interpolation technique based on the CRR method.

The balancing inequality (10.20), to avoid negative transition probabilities, and therewith the arbitrage is still used in the BC algorithm: they re-estimate S^{i+1}_{n+1} by the average of F^i_n and F^{i+1}_n, though the choice of any point between these forward prices is sufficient.

10.2 A Simulation and a Comparison of the SPDs

The following detailed example illustrates the construction of the tree from the smile, using the DK algorithm first, and the BC algorithm afterwards.

Let us assume that the current value of the underlying stock $S = 100$, with no dividend and the annually compounded riskless interest rate $r = 3\%$ per year for all time expirations. For the implied volatility function, we use a convex function:

$$\widehat{\sigma} = \frac{-0.2}{\{\log(K/S_t)\}^2 + 1} + 0.3 \;, \qquad (10.28)$$

taken from Fengler (2005). For simplicity, we do not model a term structure of the implied volatility. The BS option prices needed for growing the tree are calculated from this implied volatility function. We construct the IBTs with time to maturity $T = 1$ year discretized in five time steps.

10.2.1 Simulation Using the DK Algorithm

Using the assumption on the BS implied volatility surface described above, we obtain the one year stock price implied binomial tree (Figure 10.4), the upward transition probability tree (Figure 10.5), and the Arrow-Debreu price tree (Figure 10.6).

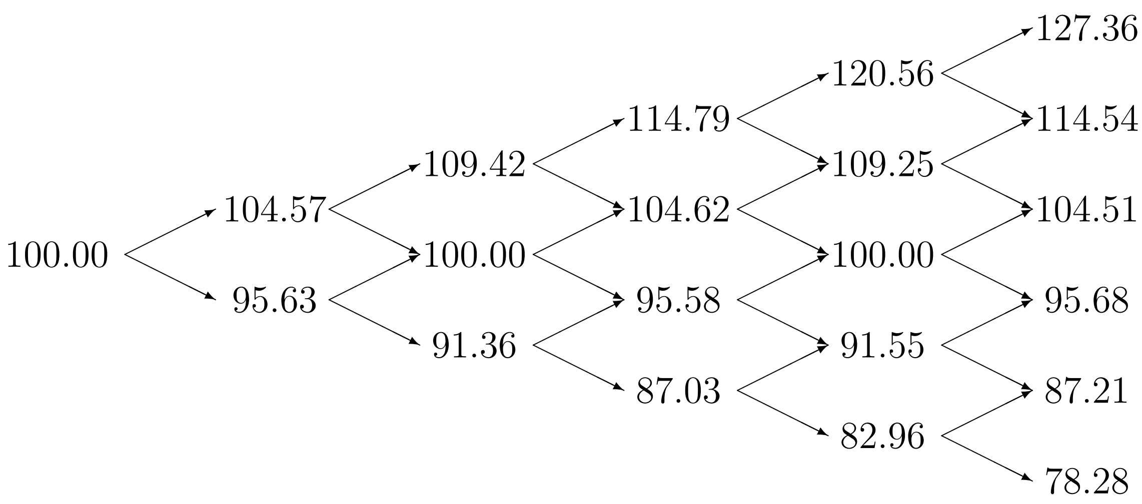

Figure 10.4. Stock price tree calculated with the DK algorithm with $S_0^0 = 100$, $r = 0.03$ and $T = 1$ year. XFGIBT01

All the IBTs correspond to time to maturity $\tau = 1$ year, and $\triangle t = 1/5$ year. Figure 10.4 shows the estimated stock prices starting at the zero level with $S_0^0 = S = 100$. The elements in the j-th column correspond to the $(j-1)$th level of the stock price tree. Figure 10.5 shows the transition probabilities, its element (n, j) represents the transition probability from the node $(n-1, j-1)$ to the node (n, j). The third tree displayed in Figure 10.6 contains the Arrow-Debreu prices. Its elements in the j-th column match the Arrow-Debreu

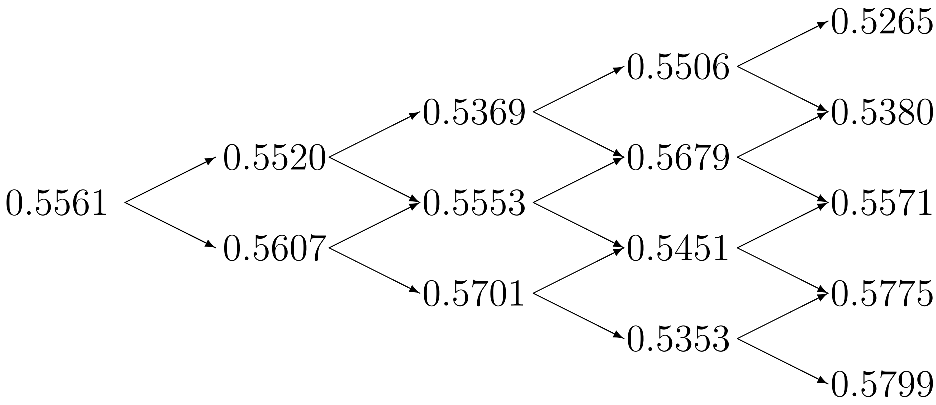

Figure 10.5. Transition probability tree calculated with the DK algorithm with $S_0^0 = 100$, $r = 0.03$ and $T = 1$ year. XFGIBT01

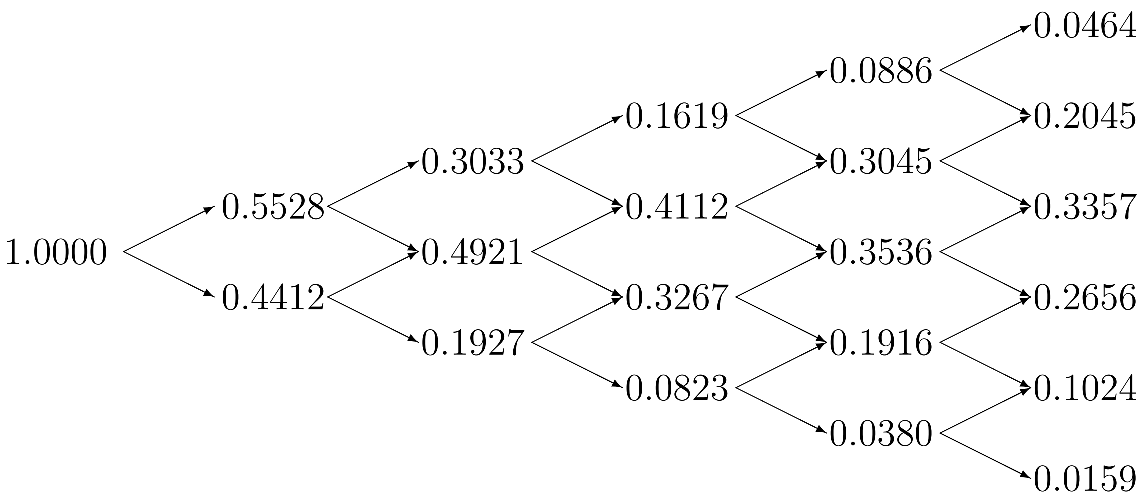

Figure 10.6. Arrow-Debreu price tree calculated with the DK algorithm with $S_0^0 = 100$, $r = 0.03$ and $T = 1$ year. XFGIBT01

prices in the $(j-1)$ th level. Using the stock prices together with Arrow-Debreu prices of the nodes at the final level, a discrete approximation of the implied price distribution can be obtained. Notice that by the definition of the Arrow-Debreu price, the risk neutral probability corresponding to each node should be calculated as the product of the Arrow-Debreu price and the factor $e^{rj\triangle t}$ in the level j.

Choosing the time steps small enough, we obtain more accurate estimation of the implied price distribution and the local volatility surface $\sigma(S, \tau)$. We still use the same implied volatility function from (10.28), and assume $S_0^0 = 100$, $r = 0.03$, $T = 5$ years.

SPD estimation arising from fitting the implied five-year tree with 40 levels is shown in Figure 10.7. Local volatility surface computed from the implied tree at different times to maturity and stock price levels is shown in Figure 10.8. Obviously, the local volatility captures the volatility smile, which decreases with the strike price and increases with the time to maturity.

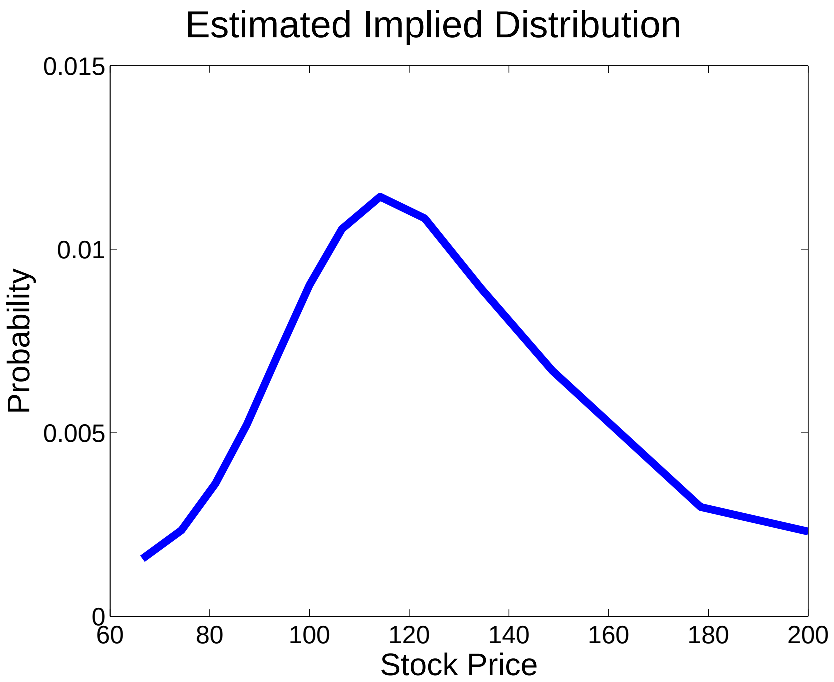

Figure 10.7. SPD estimation by the DK IBT computed with $S_0^0 = 100$, $r = 0.03$ and $T = 5$ years. XFGIBT02

10.2.2 Simulation Using the BC Algorithm

The BC algorithm can be applied in analogy to the DK technique. The computing part is replaced by the BC algorithm, we are using the implied volatility function from (10.28) as in the DK algorithm. Figures 10.9 - 10.11 show the one-year stock price tree with five steps, transition probability tree, and Arrow-Debreu tree. Figure 10.12 presents the plot of the estimated SPD by fitting a five year implied binomial tree with 40 levels using BC algorithm. Figure 10.13 shows the characteristics of the local volatility surface of the generated IBT, the local volatility follows the "volatility smile", which decreases with the stock price and increases with time.

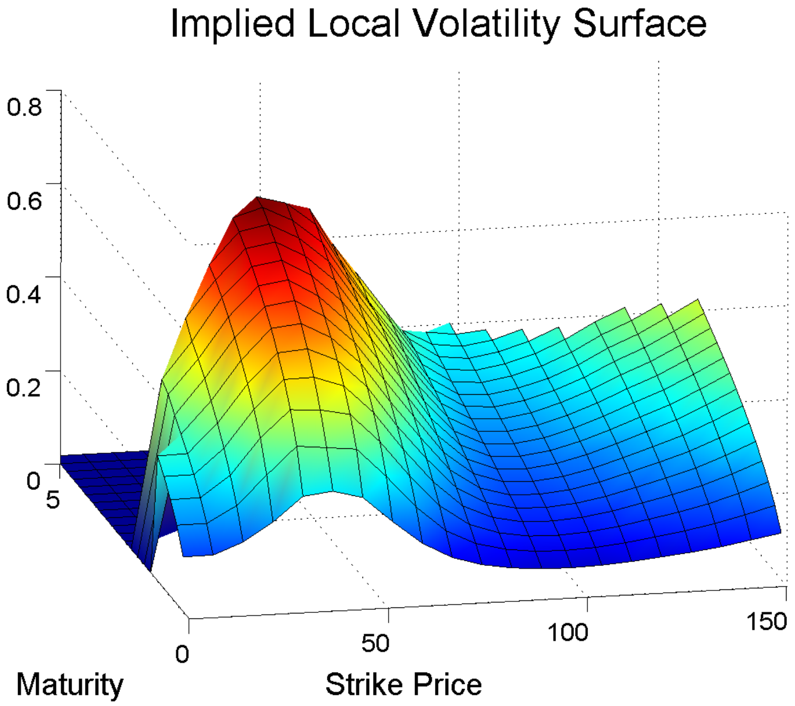

Figure 10.8. Implied local volatility surface estimated by the DK IBT with $S_0^0 = 100$, $r = 0.03$ and $T = 5$ years. XFGIBT02 .

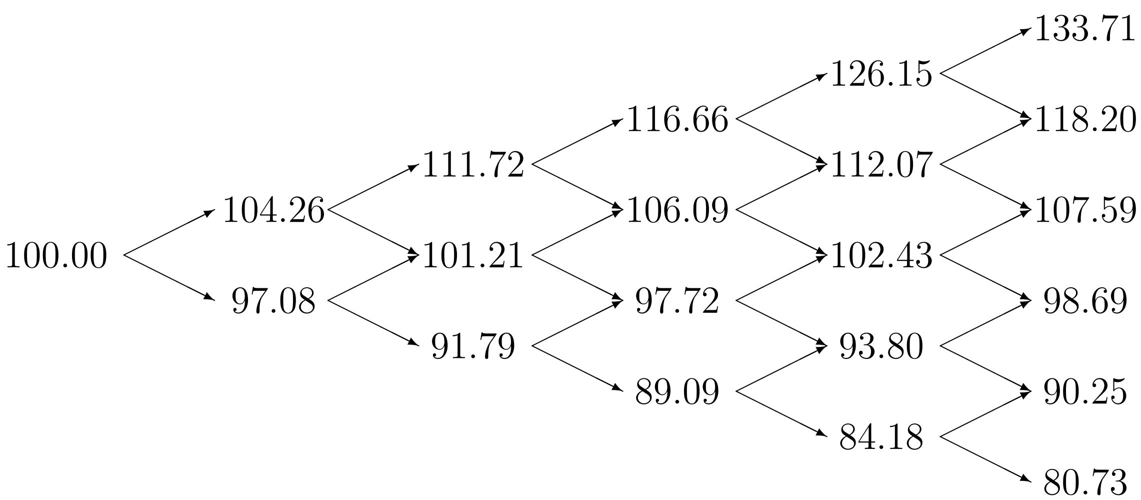

Figure 10.9. Stock price tree calculated with the BC algorithm with $S_0^0 = 100$, $r = 0.03$ and $T = 1$ year. XFGIBT01

10.2.3 Comparison with the Monte-Carlo Simulation

We now compare the SPD estimation obtained by the two IBT methods with the estimated density function of a simulated process S_t generated from

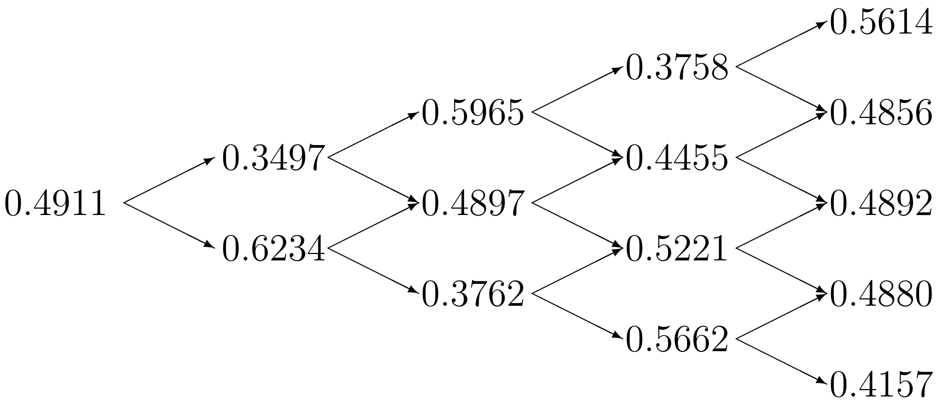

Figure 10.10. Transition probability tree calculated with the BC algorithm with $S_0^0 = 100$, $r = 0.03$ and $T = 1$ year.
XFGIBT01

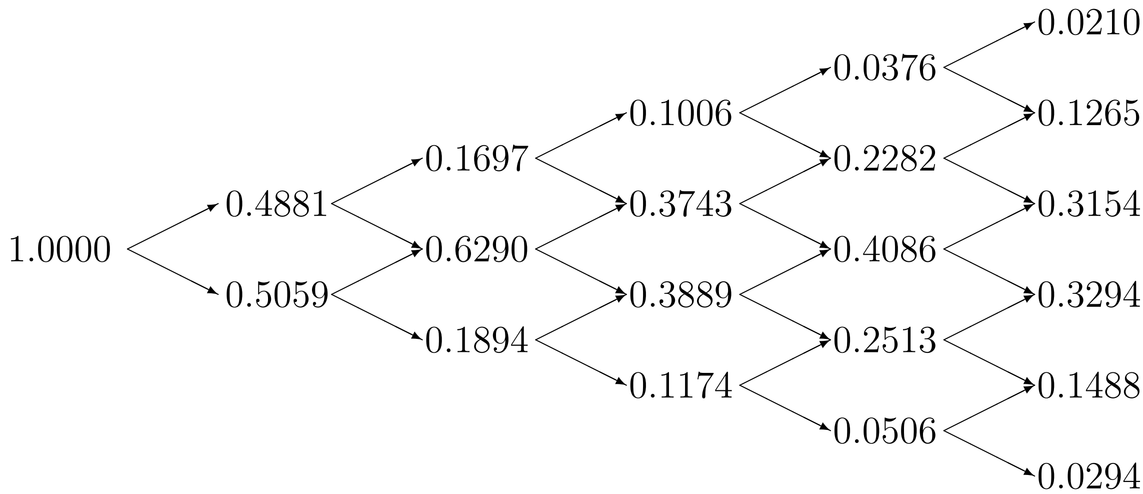

Figure 10.11. Arrow-Debreu price tree calculated with the BC algorithm with $S_0^0 = 100$, $r = 0.03$ and $T = 1$ year.
XFGIBT01

the diffusion process (10.6). To perform a discrete approximation of this diffusion process, we use the Euler scheme with time step $\delta = 1/1000$, the constant drift $\mu_t = r = 0.03$ and the volatility function $\sigma(S_t, t) = \left[\frac{-0.2}{\{\log(K/S_t)\}^2 + 1} + 0.3\right]$.

Compared to Sections 10.2.2 and 10.2.2 where we started from the BS implied volatility surface, here we construct the IBTs direct from the simulated option price function. In the construction of the IBTs, we calculate the option prices

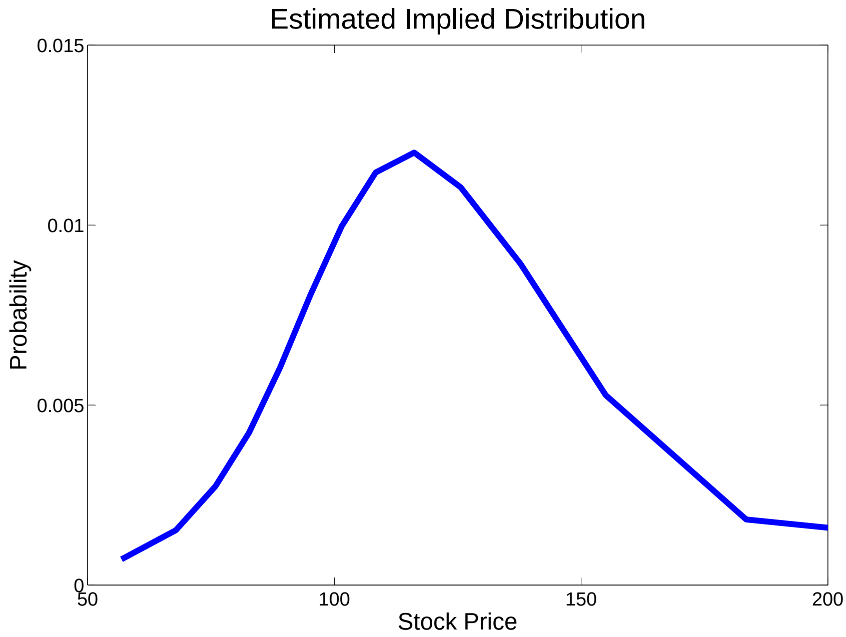

Figure 10.12. SPD estimation by the BC IBT computed with $S_0^0 = 100$, $r = 0.03$ and $T = 5$ years. XFGIBT02

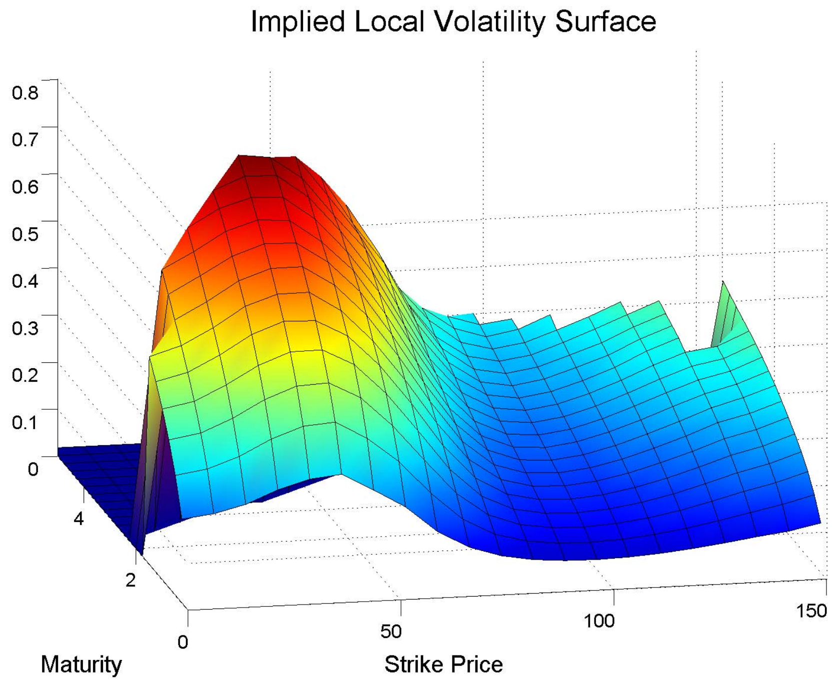

Figure 10.13. Implied local volatility surface estimated by the BC IBT with $S_0^0 = 100$, $r = 0.03$ and $T = 5$ years. XFGIBT02

corresponding to each node at the implied tree according to their theoretical definitions (10.3) and (10.3) from the simulated asset prices S_t. We simulate S_t for $t = i/4$ year, $i = 1, \ldots, 50$ in the diffusion model (10.6) with the

Monte-Carlo simulation method.

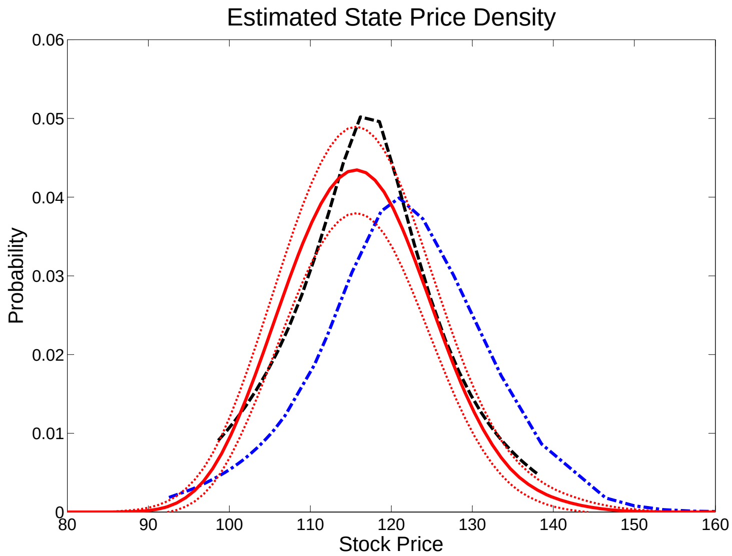

Figure 10.14. SPD estimation from the DK IBT (blue dashed line) and from the BC IBT (black dashed line) compared to the estimation by Monte-Carlo simulation with its 95% confidence band (red lines). Level = 50, T = 5 years, $\triangle t$ = 0.1 year. XFGIBT03

From the estimated distribution shown in Figure 10.14, we observe small deviations of the SPDs obtained from the two IBT methods from the estimation obtained by the Monte-Carlo simulation. The SPD estimation by the BC algorithm coincides substantially better with the estimation from the simulated process than the estimation by the DK algorithm, which shows a shifted mean of its SPD.

As above, we can also estimate the local volatility surface from the both implied binomial trees. Compare Figure 10.15 with Figure 10.16 and notice that some edge values cannot be obtained directly from the five-year IBT. However, both local volatility surface plots actually coincide with the volatility smile characteristic, the implied local volatility of the out-the-money options decreases with the increasing stock price, and increases with time.

10.3 Example – Analysis of EUREX Data

In the following example we use the IBTs to estimate the price distribution of the real stock market data. We use underlying asset prices, strike prices, time to maturity, interest rates, and call/put option prices from EUREX at 19

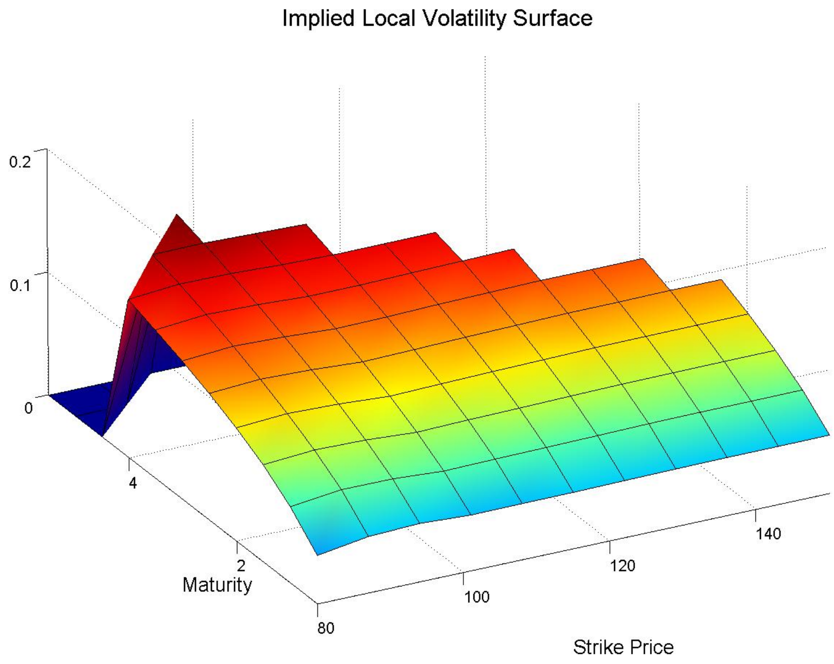

Figure 10.15. Implied local volatility surface of the simulated model, calculated from DK IBT. XFGIBTcdk

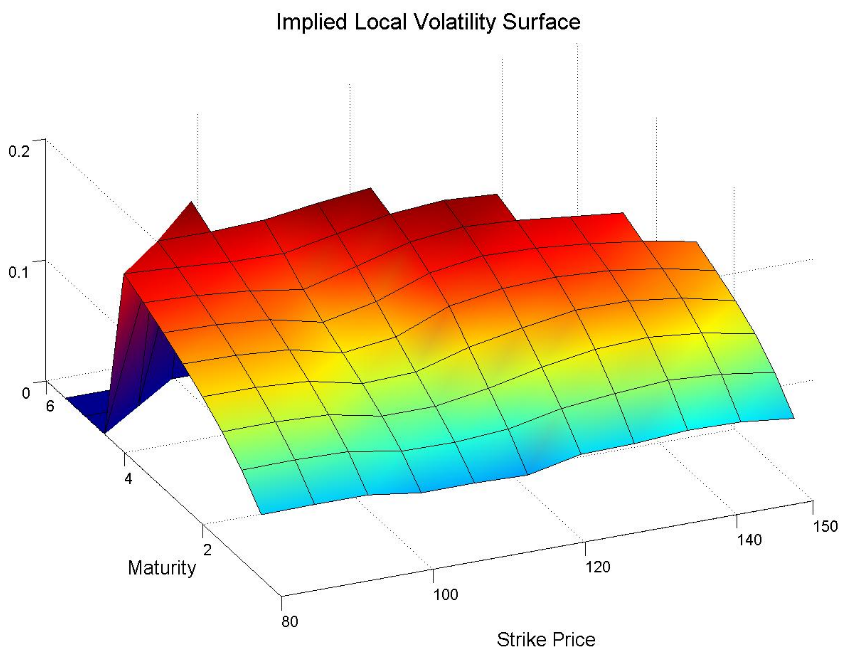

Figure 10.16. Implied local volatility surface of the simulated model, calculated from BC IBT. XFGIBTcbc

March, 2007, taken from the database of German stock exchange. First, we estimate the BS implied volatility surface from the data set with the technique

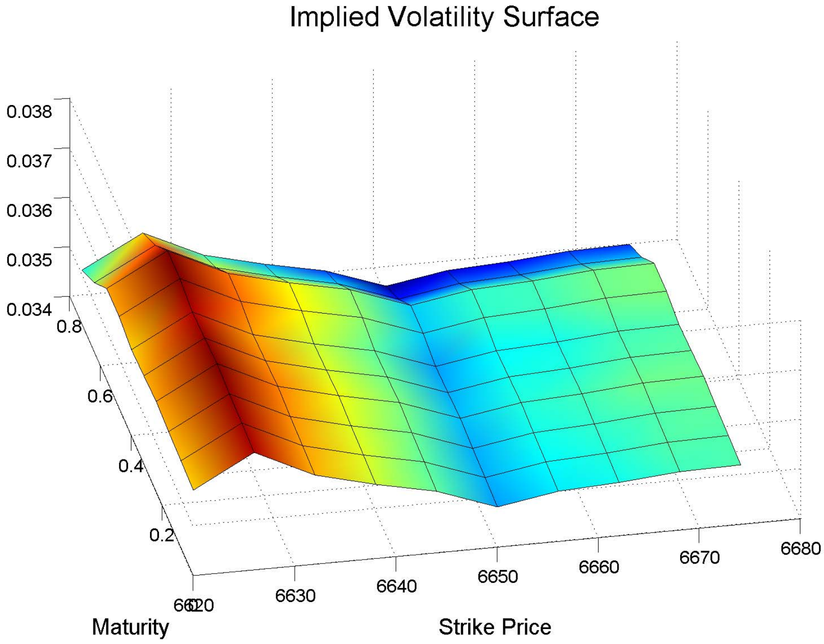

Figure 10.17. BS implied volatility surface estimated from real stock and option prices. XFGIBT05

of Fengler, Härdle and Villa (2003). Figure 10.17 shows the estimated implied volatility surface, which reflects the characteristics that the implied volatility decreases with the strike price and increases with time to maturity.

Now we construct the IBTs, where we calculate the interpolated option prices with the CRR binomial tree method using the estimated implied volatility. Fitting the function of option prices directly from the market option prices causes difficulties since the function approaches a value of zero for very high strike prices which would violate no-arbitrage conditions.

The estimated stock price distribution, obtained by the BC and the DK IBT with 40 levels, for $\tau = 0.5$ year, is shown in Figure 10.18. Obviously, the both estimated SPDs are nearly identical. The SPDs do not show any deviations from the log-normal characteristics according to their skewness and kurtosis.

From the simulations and real data example, we conclude that the implied binomial tree is a simple smile-consistent method to assess the future stock prices. Still, some limitations of the algorithms remain. With an increasing interest rate or with a small time step, negative transition probabilities occur more often. When the interest rate is high, the BC algorithm is a better choice. The DK algorithm cannot handle with higher interest rates such as $r = 0.2$, in this case the BC algorithm still can be used. In addition, the negative probabilities appear more rarely in the BC algorithm than in

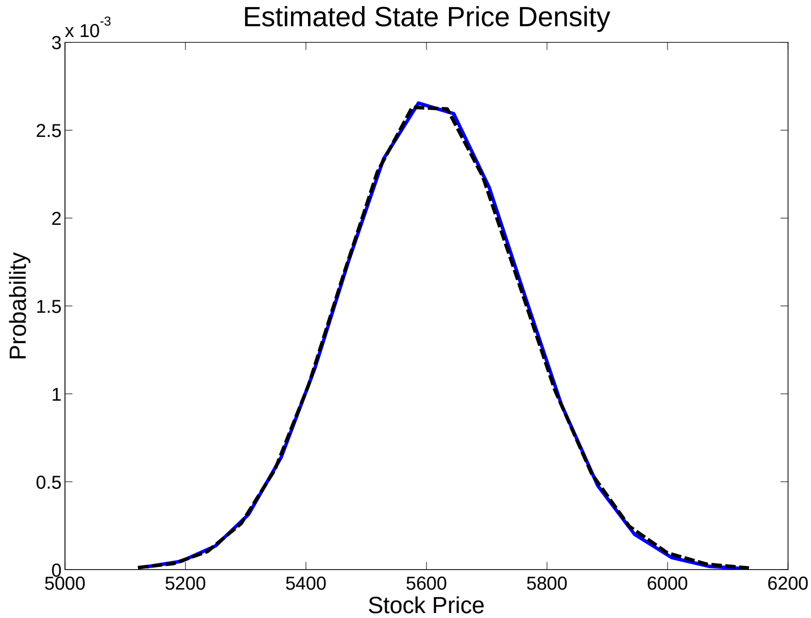

Figure 10.18. SPD estimation by the BC IBT (black dashed line) and by the DK IBT (blue solid line) from the EUREX data, $\tau = 0.5$ year, level = 25. XFGIBT05

the DK construction, even though most of them appear at the edge of the trees. But, by modifying these values we are effectively losing the information about the volatility behavior at the corresponding nodes. This deficiency is a consequence of our condition that continuous diffusion process is modeled as a discrete binomial process. Improving of this requirement leads to a transition to multinomial or varinomial trees which have a drawback of more complicated models with difficult realization.

Besides its basic function to price derivatives in consistency with market prices, IBTs are also useful for hedging, calculating local volatility surfaces or estimation of the future price distribution according to the historical data. In the practical application, the reliability of the approach depends critically on the quality of the dynamics estimation of the underlying process, such as of the BS implied volatility surface obtained from the market option prices.

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