Statistics of Risk Aversion

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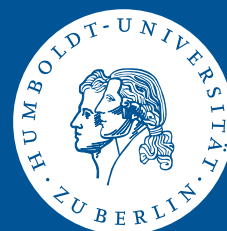
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1. Introduction

Option prices are a valuable source of information concerning risk assessments from investors about future financial payoffs. The information is summarized in the state price densities (SPD), the continuous counterpart (normalized by a constant) from Arrow-Debreu security prices. Under no arbitrage assumptions the state price densities - corresponding to a risk neutral measure Q - are derived from option prices as in Breeden and Litzenberger (1978). In contrast to the state price density, the historical density $p(x)$ describes the random variations of the underlying price.

In a utility based framework, investors facing financial risk have preference-indifference relations represented by a utility function. Standard economic theory enforces that the utility function is concave or equivalently, that investors are risk averse. Equilibrium and non-arbitrage arguments, as in Merton (1973), show that preferences from investors are related to the state price and historical densities, allowing to conclude the functional form of one given the functional form of the remaining two. Part of this relation is given by the pricing kernel (PK).

In this paper we investigate pricing kernels from DAX and ODAX data in a *time varying* approach and consider their *term structure*. In order to approximate and analyse the complex dynamic structure from pricing kernels and risk preferences across different maturities, we use *dynamic semiparametric factor models* (DSFMs). The obtained PKs exhibit risk proclivity in certain regions of returns and maturities. The dynamics of risk aversion and proclivity from investors preferences is investigated through sensitivity analysis with respect to the basis functions.

In the sequel pricing kernels are defined (section 2), their relation to utility functions and the DSFM estimation method are described (section 3). In the empirical part (section 4) pricing kernels are estimated from DAX and ODAX data sets and sensitivity analysis based on estimated loading coefficients and basis functions is performed.

2. Pricing Kernels

A flexible approach in a complete market is to assume that the price of a security follows a

diffusion process described by

$$(1) \quad \frac{dS_t}{S_t} = \mu(S_t, t)dt + \sigma(S_t, t)dB_t$$

where $t \in [0, T]$, B_t is a standard Brownian motion under measure P. Defining

$$(2) \quad \lambda_t = \frac{\mu(S_t, t) - r}{\sigma(S_t, t)}$$

as the market price of risk, the risk neutral measure Q is obtained by $\frac{dQ}{dP}\Big|_{\mathcal{F}_t} = \zeta_t$ where with (2)

$$\zeta_t = \exp\left(-\int_0^t \lambda_u dB_u - \frac{1}{2}\int_0^t \lambda_u^2 du\right)$$

The pricing kernel, or stochastic discount factor is defined for maturity $\tau = s - t$, $0 \le t \le s \le T$ as:

$$(3) \quad M_{t,\tau} = e^{-r\tau}\frac{\zeta_s}{\zeta_t}$$

Assuming the existence of a representative investor with utility function u, constant interest rates r and defining a wealth $\{W_s\}$ and a consumption process $\{C_s\}$, $C_s = 0$, the investor adjusts the amounts $\{\xi_s\}$ invested in the asset S_s at times s and consumes all his wealth at T, $C_T = W_T$. That means, he chooses ξ_s via the Merton optimization problem:

$$\max_{\{\xi_s, t \le s \le T\}} E[u(W_T)]$$

subjected to $W_s \ge 0$ and $dW_s = \{rW_s + \xi_s(\mu - r)\}ds + \xi_s \sigma dB_s$. In equilibrium all wealth is invested in the asset, at the end all wealth is consumed $C_T = W_T = S_T$. In the Merton model the PK is path independent and (3) is equal to the marginal rate of substitution:

$$(4) \quad \frac{u'(S_T)}{u'(S_t)} = M_{t,\tau} = e^{-r\tau}\frac{q_t(S_T)}{p_t(S_T)}$$

Here q_t, p_t denote the risk neutral and historical density at time t. The functional relation between the utility u and the densities q_t, p_t is given by

$$(5) \quad u(S_T) = e^{-r\tau}u'(S_t)\int \frac{q_t(S_T)}{p_t(S_T)}dS_T$$

3. PK Estimation with Dynamic Semiparametric Factor Models

Breeden and Litzenberger (1978) showed how the SPD $q_t(S_T)$ may be obtained from option prices. Ait-Sahalia and Lo (1998) used the estimate:

$$(6) \quad \widehat{q}_t(S_T) = e^{r\tau}\frac{\partial^2 C_{t,BS}\{S_t, K, \tau, r_t, \widehat{\sigma}_t(\kappa, \tau)\}}{\partial K^2}\Bigg|_{K=S_T}$$

where $C_{t,BS}$ is the Black-Scholes price at time t and $\widehat{\sigma}_t(\kappa, \tau)$ is a nonparametric estimator for the implied volatility.

Implied volatilities may be estimated from option prices. On each day $i = 1, \ldots, I$ there are J_i options traded. Each intra-day trade $j = 1, \ldots, J_i$ corresponds to an implied volatility $\sigma_{i,j}$, and a
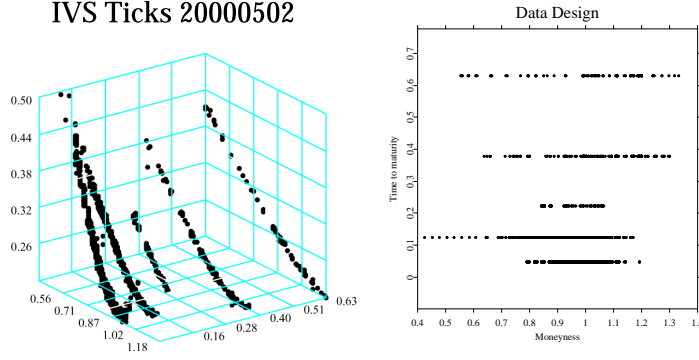
Figure 1: Call and put implied volatilities observed (left), data design (right), ODAX on 20000502

pair of moneyness and maturity $X_{i,j} = (\kappa_{i,j}, \tau_{i,j})^\top$. Moneyness is defined as $\kappa_{i,j} = \frac{K}{F(t_{i,j})}$ where K is strike and $F(t_{i,j}) = S_{t_{i,j}} \exp(r_{\tau_{i,j}} \tau_{i,j})$ are futures prices. Figure 1 depicts the implied volatilities corresponding to trades on ODAX in day 20000502 (dates are written as year, month, day).

Dynamic semi-parametric factor models (DSFM), Fengler et al. (2007), employ the time series structure of implied volatilities regressing log implied volatilities $Y_{i,j} = \log \sigma_{i,j}$ on $X_{i,j}$ using smooth basis functions m_l, $l = 0, \ldots, L$ weighted with factor loadings $z_{i,l}$:

$$(7) \quad Y_{i,j} = \sum_{l=0}^{L} z_{i,l} m_l(X_{i,j}) + \varepsilon_{i,j}$$

where $\varepsilon_{i,j}$ is noise and $z_{i,0} \equiv 1$. Following Borak et al. (2007), the basis functions are expanded using a series estimator for functions $\psi_k : \mathbb{R} \to \mathbb{R}$ and coefficients $\gamma_{l,k} \in \mathbb{R}$

$$m_l(X_{i,j}) = \sum_{k=1}^{K} \gamma_{l,k} \psi_k(X_{i,j})$$

Defining the matrices $Z = (z_{i,l})$, $\Gamma = (\gamma_{l,k})$ for $i = 1, \ldots, I$, $k = 1, \ldots, K$ and $l = 0, \ldots, L$ we obtain the least square estimators as

$$(\widehat{\Gamma}, \widehat{Z}) = \arg \min_{\Gamma \in \mathcal{G}, Z \in \mathcal{Z}} \sum_{i=1}^{I} \sum_{j=1}^{J} \left\{ Y_{i,j} - z_i^\top \Gamma \psi(X_{i,j}) \right\}^2$$

where $z_i = (z_{i,0}, \ldots, z_{i,L})^\top$, $\psi(x) = \{\psi_1(x), \ldots, \psi_K(x)\}^\top$, $\mathcal{G} = \mathcal{M}(L+1, K)$, $\mathcal{Z} = \{Z \in \mathcal{M}(I, L+1) : z_{i,0} \equiv 1\}$ and $\mathcal{M}(a, b)$ is the set of all $(a \times b)$ matrices. The estimators for the basis functions in (7) are $\widehat{m}_l(x) = \widehat{\gamma}_l^\top \psi(x)$ where $\gamma_l = (z_{l,1}, \ldots, z_{l,K})^\top$. Denoting $\widehat{m} = (\widehat{m}_0, \ldots, \widehat{m}_L)^\top$, the implied volatility at time i is estimated as $\widehat{\sigma}_i(\kappa, \tau) = exp\left\{ \widehat{z}_i^\top \widehat{m}(\kappa, \tau) \right\}$.

Using (6), the state price density may be approximated by

$$(8) \quad \widehat{q}_t(\kappa, \tau, \widehat{z}_t, \widehat{m}) = \varphi(d_2) \left\{ \frac{1}{K \widehat{\sigma}_t \sqrt{\tau}} + \frac{2d_1}{\widehat{\sigma}_t} \frac{\partial \widehat{\sigma}_t}{\partial K} + \frac{K \sqrt{\tau} d_1 d_2}{\widehat{\sigma}_t} \left(\frac{\partial \widehat{\sigma}_t}{\partial K} \right)^2 + K \sqrt{\tau} \frac{\partial^2 \widehat{\sigma}_t}{\partial K^2} \right\} \Bigg|_{K=S_T}$$

where $\varphi(x)$ is the standard normal pdf, $d_1 = \frac{\log\left(\frac{S_t}{K}\right) + (r + \frac{1}{2} \widehat{\sigma}_t^2) \tau}{\widehat{\sigma}_t \sqrt{\tau}}$ and $d_2 = d_1 - \widehat{\sigma}_t \sqrt{\tau}$. As in Ait-Sahalia and Lo (2000) we define an estimate $\widehat{M}_t(\kappa, \tau)$ of the PK as the ratio between the estimated SPD and
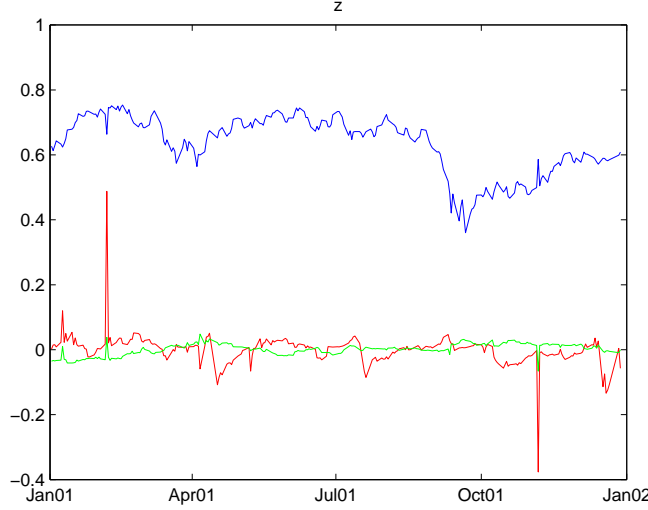
Figure 2: Loading factors $\widehat{z}_{t,l}$, $l = 1, 2, 3$ **from the top**

the estimated p:

$$\widehat{M}_t(\kappa, \tau, \widehat{z}_t, \widehat{m}) = e^{-r_t \tau} \frac{\widehat{q}_t(\kappa, \tau, \widehat{z}_t, \widehat{m})}{\widehat{p}_t(\kappa, \tau)}$$

Here $\widehat{p}_t$ is estimated by a GARCH(1,1) model. It is our interest in this paper to examine the dynamic structure of (9).

4. Empirical Results

Here SPDs and PKs are estimated from intraday DAX and ODAX data. The options data ranges from 20010101 to 20020101 corresponding to 253 trading days. The SPD $\widehat{q}_t$ (8) is estimated with 4 basis functions. The historical density $\widehat{p}_t$ is estimated with GARCH(1,1) from the last 240 observations. The time series of loading factors $\{\widehat{z}_{t,l}\}$ and basis functions $\widehat{m}_l$, $l = 0, \ldots, 3$, $t = 1, \ldots, 253$ are in figures 2 and 3. The summary statistics from loading factors series are in table 1. From (8) and (9) we obtain a sequence of 253 SPDs and PKs moving on time over a grid of moneyness and maturities, figure 4 shows one shot of this sequence, at day 20010710.

Risk averse utilities u are concave. Hence, (4) implies that under risk aversion pricing kernels are monotone decreasing in moneyness. Figure 4 shows that DAX PKs - similarly to S&P PKs, Ait-Sahalia and Lo (2000) - present risk proclivity at certain moneyness levels (PKs are increasing between 0.9 and 1) and these levels vary across maturities, hence we verify the *empirical pricing kernel paradox*.

In order to analyse the dynamics of risk aversion and proclivity implied in PKs we focus on the SPDs, as these contain information about preferences from investors. In figure 5 we can see that the mean of $\widehat{q}_t(S_T)$ and the factor loadings $\widehat{z}_{t,1}$ from the basis function $\widehat{m}_1$ are correlated. We investigate the influence of the basis functions $\widehat{m}_l$ on the mean, variance and skewness of SPDs by sensitivity analysis to changes in the loadings $\{\widehat{z}_{t,l}\}$: the SPDs are recalculated with new loadings, where one of the coefficients varies linearly and the remaining are hold fixed.

We define *scenario loadings* W^l corresponding to a linear increase on factor l in N steps from

Table 1: Loading factors, descriptive statistics

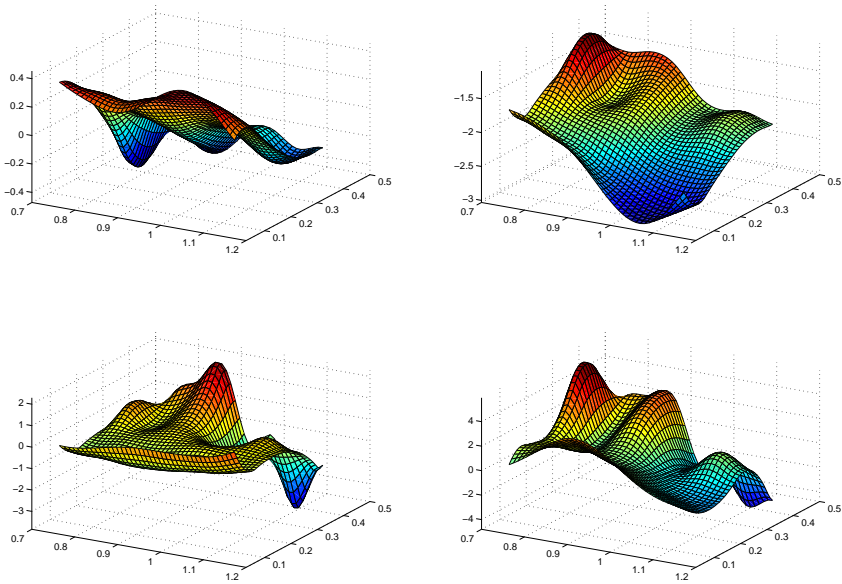|	min	max	median	mean	std.dev.
$\widehat{z}_{t,1}$	0.36	0.75	0.66	0.63	0.09
$\widehat{z}_{t,2}$	-0.37	0.49	0.01	0.00	0.05
$\widehat{z}_{t,3}$	-0.07	0.05	0.00	0.00	0.02

Figure 3: Basis functions $\widehat{m}_l$, $l = 0, \ldots, 3$ clockwise
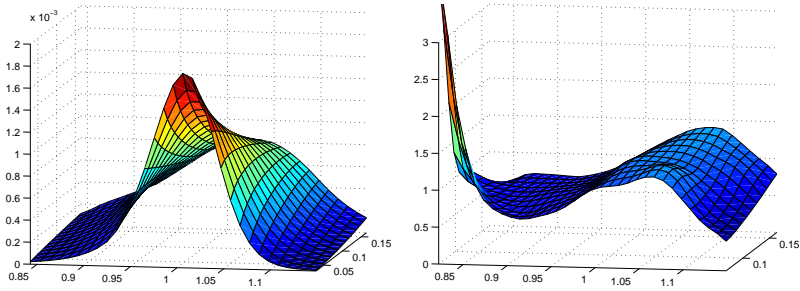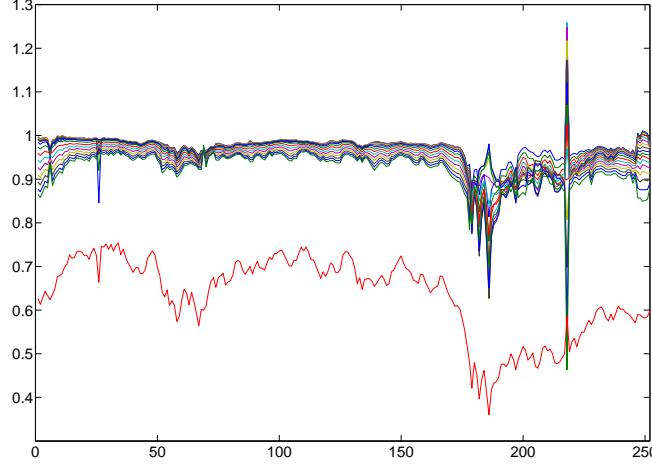

Figure 4: Estimated SPD (left) and PK (right) across κ and τ at $t = 20010710$

Figure 5: Mean from SPD for $\tau = 25, \ldots, 75$ days (above), $\widehat{z}_{t,1}$ (below)

levels d_l to u_l. The remaining coefficients are held constant at median of estimated values. The matrices $W^l = (w_{n,j}^l)$, $l, j = 0, \ldots, 3$, and $n = 1, \ldots, N$ contain the scenario loadings to factor l as

$$w_{n,j}^l = \left\{ d_j + \frac{n-1}{N-1}(u_j - d_j) \right\} \mathbf{1}(j = l) + med(\widehat{z}_{t,j})\mathbf{1}(j \neq l)$$

with levels given by $d_l = \min \widehat{z}_{t,l} - 0.5|\min \widehat{z}_{t,l}|$ and $u_l = \max \widehat{z}_{t,l} + 0.5|\max \widehat{z}_{t,l}|$. SPDs estimated with loadings W^l are under influence of variations in factor l while the remaining factor are constant at a typical level (median). The observed changes in mean, variance and skewness may be considered as a typical effect of variation in factor l.

Figure 6 shows the mean, variance and skewness of SPD obtained with W^l, for $l = 1, 2, 3$, $N = 50$ and three different maturities (25, 40 and 75 days) plotted against n. From the middle column, we verify that increasing factor 2 decreases mean while increasing variance and skewness for short maturities (full lines) and increases mean while decreasing variance and skewness for long maturities (dashed lines). From the left column, we verify that increasing factor 1 increases mean while decreasing variance and skewness for all maturities.

Figure 7 plots PKs obtained with W^l, for $l = 1, 2, 3$, $N = 50$ and two maturities, short (20) and long (75 days), against moneyness. The pricing kernels are calculated with weak ($n = 1$) medium (25) and strong (50) loadings from each factor. The influence of factor 1 is represented by the left pair in figure 7: for both maturities weak loadings result in risk proclivity for $\kappa > 1$ while strong loadings in risk proclivity for $\kappa < 1$.

In the middle pair we see that factor 2 has the opposite influence at short maturities: weak loadings result in risk proclivity for $\kappa < 1$, strong loadings for $\kappa > 1$. This effect is not clear at long maturities. Increasing loadings of the factor 3 shifts the risk proclivity region to higher values of κ in both maturities, as observed in the right pair of figure 7.
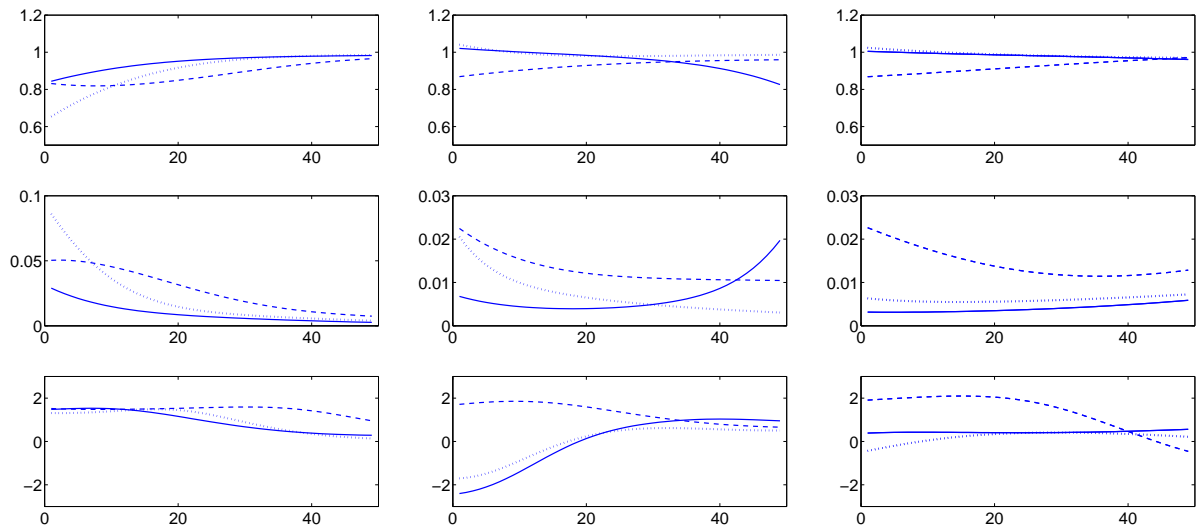
Figure 6: Mean, variance and skewness from SPD (from the top) plotted against n. For W^1, W^2 and W^3 (from the left), $\tau = 25$ (full), 40 (dotted) and 75 (dashed) days, $N = 50$
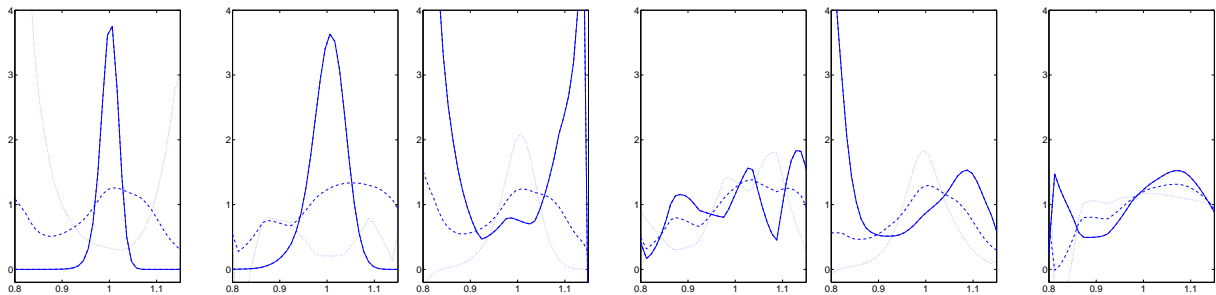

Figure 7: PK obtained with loadings W^1 (left pair), W^2 (middle pair) and W^3 (right pair), $\tau = 25$ (first) and 75 (second in each pair) days, $n = 1$ (dotted), 25 (dashed) and 50 (full)

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ABSTRACT

Information about risk preferences from investors is essential for modelling a wide range of quantitative finance applications. Valuable information related to preferences can be extracted from option prices through pricing kernels. In this paper, pricing kernels and their term structure are estimated in a time varying approach from DAX and ODAX data using dynamic semiparametric factor model (DSFM). DSFM smooths in time and space simultaneously, approximating complex dynamic structures by basis functions and a time series of loading coefficients. Contradicting standard risk aversion assumptions, the estimated pricing kernels indicate risk proclivity in certain levels of return. The analysis of the time series of loading coefficients allows a better understanding of the dynamic behaviour from investors preferences towards risk.

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